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SECOND ORDER LOGIC OR SET THEORY?

JOUKO VÄÄNÄNEN

Abstract. We try to answer the question which is the "right" foundation of mathematics, second order logic or set theory. Since the former is usually thought of as a formal language and the latter as a first order theory, we have to rephrase the question. We formulate what we call *the second order view* and a competing *set theory view*, and then discuss the merits of both views. On the surface these two views seem to be in manifest conflict with each other. However, our conclusion is that it is very difficult to see any real difference between the two. We analyze a phenomenon we call *internal categoricity* which extends the familiar categoricity results of second order logic to Henkin models and show that set theory enjoys the same kind of internal categoricity. Thus the existence of non-standard models, which is usually taken as a property of first order set theory, and categoricity, which is usually taken as a property of second order axiomatizations, can coherently coexist when put into their proper context. We also take a fresh look at complete second order axiomatizations and give a hierarchy result for second order characterizable structures. Finally we consider the problem of existence in mathematics from both points of view and find that second order logic depends on what we call *large domain assumptions*, which come quite close to the meaning of the axioms of set theory.

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§**1. Introduction.** Two views of the nature of mathematics seem to be in utter conflict with each other. One is what we will call the *second order view* which takes the concept of structure as the basic notion in mathematics and stipulates that mathematics is simply the study of the higher order properties of such structures. According to this view it is immaterial and even misleading to discuss what the elements of such structures are, as the structures are taken up to isomorphism only. The apparently opposing view is the *set theory view* which takes mathematical objects (i.e., sets) and their membership relation as the basic notion and builds all of mathematics from this basic concept, not only up to isomorphism but to identity.

Both views have their appeal. Let us first discuss the appeal of the second order view. It is indeed immaterial in mathematics what kind of objects for example the real numbers are, as long as they satisfy the axioms of completely ordered fields. We know that there is, up to isomorphism, only one such field, namely $\mathbb{R}$, and giving preference to one construction of the reals over another seems unfounded. The same applies to the natural numbers $\mathbb{N}$, the complex numbers $\mathbb{C}$, the Euclidean spaces $\mathbb{R}^n$, the free group of countably many generators, the Banach spaces ℓ_p, and so on. Typically mathematical research takes place in one of these classical structures making reference to elements, subsets, and relations on the structure, in some cases also to families of subsets, all well handled by second (or higher) order logic on the particular structure. There is no need for a universal theory of mathematical objects which would show how all these structures and their properties are reduced to some more basic objects ("sets") and their properties ("ZFC-axioms"). This mode of thinking is sometimes called structuralism as it emphasizes structures; as it finds the search for a universal foundation of mathematics unnecessary, it is sometimes thought of as anti-foundationalist.

Let us then consider the other side, the set theory view. According to this view it is an important achievement that most if not all of mathematics can be reduced to the concept of set, whose properties have an intuitively appealing axiomatization. If there is ever doubt concerning a mathematical argument, one only needs to reduce it to set theory and if this can be done, the argument can be declared correct. In this reduction it may turn out that strong principles like the Axiom of Choice are invoked and this may deserve a special mention in the result, and raise the question whether the use was essential. As compared with the second order view, this approach is foundationalist[1] in spirit. However, set theorists do not claim that mathematical objects really are sets, only that they can be thought of as sets. The fact that set theorists define the ordered pair (a, b) as $\{\{a\}, \{a, b\}\}$ does not mean that set theorists claim that ordered pairs have to be defined in this way, or in any way for that matter. The operation (x, y) could be taken as an undefined basic concept in addition to $\in$. However, there is certain beauty in having as few basic concepts as possible, and taking only $\in$ is a kind of record—hard to beat. Since variables in set theory are thought to range over sets, that is, elements of the universe, set theory is usually thought of as a first order theory.

The conflict between the two views is obvious: The second order view says that building everything from one ingredient (sets) is not necessary and leads to questions that cannot really be answered and which touch only very lightly on "core mathematics" if at all. The second order view also points out that the first order axioms ZFC of set theory have non-standard models, while the second order axiomatizations of the classical structures are categorical. The set theory view maintains that, contrary to the second order view, in order to know the second order properties of infinite structures one needs some axioms, be they axioms of second order logic or set theory, and these axioms have non-standard models in both cases. Even if one takes separate axioms for each structure there is a common core in these axioms, and this, according to the set theory view, is the foundation on which second order logic rests. Here then is the foundationalist/anti-foundationalist divide: is there a unitary concept of mathematical truth, or are there separate notions of truth each based on its own structure?

This paper ends with an investigation of the existence of mathematical objects. For second order logic this means the existence of structures. We point out, and give some technical results to this effect, that one cannot give evidence in the form of proofs from the axioms of second order logic for the existence of a mathematical structure, unless we already know the existence of at least one structure of at least the same cardinality. It is not at all clear, and we leave it as an open problem, how to formulate an axiom of second order logic that would remedy this weakness. Any axiom that states the largeness of the universe would be false in all structures of smaller size and therefore cannot be called an axiom. So we would seem to need an axiom that refers to the "outside" of a structure. If we could state the existence of large structures "outside" our domain, as superstructures in a sense, then we would have some way of solving this problem. In Section §6 we discuss extensions of second order logic, such as sort logic and higher order logics, which offer some ways to refer to a superstructure. We also discuss how set theory solves this problem.

[1]By foundationalism I mean here the position that all mathematics can be reduced to one concept (here the concept of a set) and to axioms governing this concept (such as the ZFC axioms of set theory).

We conclude that the second order view suffers from a weakness that the set theory view solves easily. It is tempting to adopt the set theory view as the primary view and then formulate the second order view as a secondary view which appeals to set theory for the existence of structures, but this ruins the autonomy of the second order view with respect to set theory.

§**2. The second order view.** In the early years of the 20th century the basic classical structures $\mathbb{N}, \mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n$, and so on, of number theory, geometry, algebra and analysis were axiomatized by Peano, Dedekind, Cantor, Hilbert, Veblen and Huntington (see e.g., [2]). These axiomatizations were second order and their first order versions (axiom schemata) were introduced only later.

The second order view sees mathematics as formulated most intelligibly in second order logic. According to this view the propositions of mathematics are of the form

$$\mathfrak{A} \models \phi, \tag{1}$$

where $\mathfrak{A}$ is one of the classical structures and ϕ is a second (or higher) order sentence. Both $\mathfrak{A}$ and ϕ have some finite vocabulary, which we assume to be a first order vocabulary, although higher order vocabularies are needed, for example in topology. The nature of the vocabulary is, however, not important for our discussion. The meaning of (1) is that whatever ϕ asserts about the elements, subsets, relations, etc of $\mathfrak{A}$, is true in $\mathfrak{A}$. So a mathematician working in number theory takes $\mathfrak{A} = \mathbb{N}$, and if he works in analytic number theory, he works with $\mathfrak{A} = \mathbb{R}$ or $\mathbb{C}$. If he uses algebraic methods, he may use $\mathfrak{A} = \mathbb{C}^n$. Naturally, a mathematician moves smoothly from one structure to another always appealing to the second order properties, such as induction or completeness, of the relevant structure, paying no attention to the fact that the vocabulary changes. It is assumed that there are canonical translations of the smaller structures into the bigger ones.

Not all mathematics is however of the form (1), for we sometimes establish universal truths, such as "every compact Hausdorff space is normal", or "every subgroup of a free group is free". So the second order view includes the provision that some mathematical propositions are of the form

$$\models \phi, \tag{2}$$

where ϕ is a second (or higher) order sentence. The meaning of (2) is that whichever structure $\mathfrak{A}$ of the vocabulary of ϕ we consider and whatever ϕ says about the elements, subsets, relations etc of this structure $\mathfrak{A}$, holds in $\mathfrak{A}$. So if ϕ talks about groups, the meaning of (2) is that every group, be it one of the "known" groups or just an abstract group, satisfies (2). The difference with (1) is that (2) is not a proposition about any particular structure, as is (1), but rather about the universe of all structures of the type that ϕ talks about. We prove below that there is no $\mathfrak{A}$ and no effective translation $\phi \mapsto \phi^*$ such that (2) can be reduced to $\mathfrak{A} \models \phi^*$, so we cannot discard (2).

Some propositions of the form (2) need third or even higher order logic. For example if we want to say that every linear order can be extended to a complete order, and similar mathematical facts, we have to go beyond the cardinality of the model. This can be done in third and higher order logics. This detail does not affect the main point of this paper.

2.1. More about second order characterizable structures. We will now sharpen (1) by specifying what a "classical structure" means. We stipulate that this refers simply to a structure that has a categorical second order definition, that is, we mean a structure $\mathfrak{A}$ such that there is a second order sentence $\theta_{\mathfrak{A}}$ such that the following two conditions hold:

$$\mathfrak{A} \models \theta_{\mathfrak{A}} \tag{3}$$

$$\forall \mathfrak{B}\, \forall \mathfrak{C}((\mathfrak{B} \models \theta_{\mathfrak{A}} \wedge \mathfrak{C} \models \theta_{\mathfrak{A}}) \rightarrow \mathfrak{B} \cong \mathfrak{C}). \tag{4}$$

We call such structures $\mathfrak{A}$ *second order characterizable*. Note that (3) is of type (1) and (4) is of type (2) (see (10) below). The classical structures $\mathbb{N}, \mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n$ are certainly second order characterizable in this sense.

Second order characterizable structures have the following pleasant property: If $\mathfrak{A}$ is second order characterizable and ϕ is second order, then (1) is equivalent to

$$\models \theta_{\mathfrak{A}} \rightarrow \phi. \tag{5}$$

At first sight (1) may look like a proposition about the relationship between an infinite object $\mathfrak{A}$ and a finite object ϕ, but in the equivalent form (5) it looks like a property of the finite string of symbols $\theta_{\mathfrak{A}} \rightarrow \phi$. So we seem to have a reduction of something which is infinitistic to something which is finitistic. The beauty of this reduction is marred by the symbol "$\models$" in (5), which brings in a genuinely infinitistic element. Still one cannot deny the virtue of dealing with the finite string $\theta_{\mathfrak{A}} \rightarrow \phi$ rather than with the infinite structure $\mathfrak{A}$. Maybe we can see by merely inspecting $\theta_{\mathfrak{A}} \rightarrow \phi$ that indeed, ϕ does follow from $\theta_{\mathfrak{A}}$. This may even be something that a computer can detect by looking at $\theta_{\mathfrak{A}}$ and ϕ very carefully.

The reduction of (1) to (5) demonstrates that the "particular truth" represented by (1) can be reduced to the "universal truth" represented by (2). So in this sense it would suffice to study (2) only, but then the problem emerges of determining the meaning of the universal quantification over all structures in (2).

For second order characterizable structures we can completely overlook the question what kind of objects the elements of these structures are. This is part of the second order view. A second order characterizable structure is nothing in particular but just any structure that satisfies the axiomatization.

We need not, it seems, worry about the question how these structures are constructed or how they come into being, although this is not uncontroversial. One possibility is to say that if the axioms define something uniquely, and they are consistent, then this unique structure exists. The early researchers took it for granted that the axiomatization of, say the ordered field of the real numbers, is consistent, since the ordered field of reals itself satisfied it. However, it was later recognized that it makes sense to try to establish consistency, however difficult it turned out to be, without assuming first that we already have a model.

2.2. Second order truth. Let us then consider the question, what are the grounds under which a mathematician can assert (1), communicating thereby that his or her knowledge now covers (1). We assume that $\mathfrak{A}$ exists and that (1) is meaningful. To make progress in mathematics it is not enough to know that the proposition (1) has a truth-value—we should also determine what the truth-value is. For example, we can assume ϕ is false in $\mathfrak{A}$ and try to derive a contradiction, allowing us to conclude that ϕ is true in $\mathfrak{A}$. Or we can perhaps prove ϕ in $\mathfrak{A}$ by induction. Centuries of efforts have equipped mathematicians with tools to prove propositions of the form (1).

We want to emphasize the difference between knowing that ϕ has a truth value in $\mathfrak{A}$ and knowing what the truth value is. There is a marked difference between

$$\text{“}\mathfrak{A} \models \phi \text{ or } \mathfrak{A} \models \neg\phi\text{” is known,} \tag{6}$$

and

$$\text{“}\mathfrak{A} \models \phi \text{ is known” or “}\mathfrak{A} \models \neg\phi \text{ is known”.} \tag{7}$$

If we have given a mathematical definition of a formalized language and a mathematical definition of truth for that language, we can give a mathematical argument that every sentence of the language has a truth value in every structure. Then we have established (6), but from this it does not follow for any particular ϕ that we have established (7).

We maintain in this paper that the criterion for asserting a proposition in mathematics is having a proof for it. This means neither that we are limiting ourselves to constructive logic, nor that we give up the existence of mathematical objects and truth values of mathematical statements. The position adopted here is that it is essential to use classical logic in the analysis of infinite mathematical objects. A comparison of second order logic and set theory in the constructive context would be perfectly meaningful but is not the approach of this paper.

We have argued that the evidence we can give for asserting (1) is a proof of ϕ from $\theta_{\mathfrak{A}}$, a proof that follows standard mathematical rigor. It would be very surprising if such a rigorous proof could not be written, albeit with a lot of work, in one of the standard inference systems CA of second order logic consisting of Comprehension Axioms and the Axioms of Choice

[4, Ch. IV §1]. Indeed, if there were a rigorous argument that could not be so formalized, the immediate question would be, what *is* the argument based on. Can we use semantical inference, that is, justify (1) by showing that every model of $\theta_{\mathfrak{A}}$ satisfies ϕ? Yes, and this can even be shortened to the idea of showing that $\mathfrak{A}$ satisfies ϕ, and this is exactly what we are trying to do. So reference to semantic inference in this case means saying that $\mathfrak{A} \models \phi$ because $\mathfrak{A} \models \phi$, and this is not very helpful. So we have not committed any errors, but we have not made any progress either.

Recently there has been discussion concerning Fermat's Last theorem, in particular the question what is needed to formalize the proof. Usually the higher order axioms which are needed can be pin-pointed with some work. Many such results have been obtained in so-called *reverse mathematics*.

2.3. Second order characterizable structures and internal categoricity. Above we discussed grounds for asserting (1). The same discussion applies to (2). We can assert (2) if we have a proof of ϕ, typically from CA. But we should also ask ourselves, how do we recognize second order characterizable structures? After writing down $\theta_{\mathfrak{A}}$ we have to give grounds why (3) and (4) hold. In the light of the above discussion we would give evidence for asserting (3) by giving a proof of $\theta_{\mathfrak{A}} \to \theta_{\mathfrak{A}}$, which is not evidence for anything. So how can we ever assert (3)? We can perhaps prove

$$\exists R_1, \dots, R_n \theta_{\mathfrak{A}}, \tag{8}$$

from the CA axioms, where $R_1, \dots, R_n$ are the predicate symbols occurring in $\theta_{\mathfrak{A}}$. But then we would have given evidence for

$$\models \exists R_1, \dots, R_n \theta_{\mathfrak{A}}, \tag{9}$$

which is not what we want. We are not trying to show that every structure whatsoever permits relations that constitute a copy of $\mathfrak{A}$. Surely this cannot be true in structures that have a different cardinality than $\mathfrak{A}$. It seems that the only possibility is to simply assume (3), rather than trying to present evidence for it. This is the well-known problem of consistency of formal systems raised by Hilbert and settled in the negative by Gödel. If we are working in a stronger framework, we may "read off" $\mathfrak{A} \models \theta_{\mathfrak{A}}$, or anyway the existence of a model of $\theta_{\mathfrak{A}}$, from the existence of some larger structure, but obviously this only raises the question where did the larger structure come from? So (3) has to be taken on faith. After all, we have written $\theta_{\mathfrak{A}}$ so that it is true in $\mathfrak{A}$, so we may take the correctness of the process of writing down $\theta_{\mathfrak{A}}$ as the grounds for (3) even if this cannot be substantiated. It is in the spirit of the second order view that we simply assume the existence of $\mathfrak{A}$, or anyway a structure satisfying $\theta_{\mathfrak{A}}$. In set theory this problem is solved by assuming the existence of at least one infinite set and then working from there onwards by means of the operations of power-sets, unions, separation and replacement. We return to the problem of existence at the end of this paper.

Evidence for asserting (4) is clear. We may simply present a proof—informal or formal—for $\forall\mathfrak{B}\,\forall\mathfrak{C}((\mathfrak{B} \models \theta_{\mathfrak{A}} \wedge \mathfrak{C} \models \theta_{\mathfrak{A}}) \to \mathfrak{B} \cong \mathfrak{C})$. For a formal proof we have to translate this into second order logic. Suppose the vocabulary of $\theta_{\mathfrak{A}}$ is $\{R_1, \ldots, R_n\}$. Let $\{R'_1, \ldots, R'_n, U, U'\}$ be new predicate symbols so that the arity of each R'_i is the same as the arity of R_i. Let $\theta'_{\mathfrak{A}}$ be $\theta_{\mathfrak{A}}$ with each R_i replaced by R'_i. Let

$$\mathrm{ISOM}(U, R_1, \ldots, R_n, U', R'_1, \ldots, R'_n)$$

be the second order sentence saying that there is a bijection $U \to U'$ which maps each R_i onto R'_i. Any proof of

$$\begin{aligned}\forall U\,\forall U'\,\forall R_1 \ldots R_n R'_1 \ldots R'_n(((\theta_{\mathfrak{A}})^{(U)} \wedge (\theta'_{\mathfrak{A}})^{(U')})\\ \to \mathrm{ISOM}(U, R_1, \ldots, R_n, U', R'_1, \ldots, R'_n)) \quad (10)\end{aligned}$$

from the axioms of second order logic can be rightfully used as evidence for asserting (4). Here $\phi^{(U)}$ means the relativization of ϕ to the unary predicate U. For the classical structures this can be readily done.

Suppose we have given evidence for (1) by exhibiting a proof of ϕ from $\theta_{\mathfrak{A}}$ and the axioms CA of second order logic, or of (4) by exhibiting a proof of (10) from CA. In fact we have then proved more than was asked. We can define the concept of "Henkin model" of the CA axioms of second order logic. These are non-standard or "false" models in the same sense as a Klein Bottle is an "unreal" surface or Gödel's rotating universe is an "unreal" solution to Einstein's field equations. The ordinary "real" models are "full" Henkin models, because in them the range of second order variables includes all subsets and relations. The Henkin models are like a cloud around the real models. By proving ϕ from $\theta_{\mathfrak{A}}$ and the axioms CA we have shown that ϕ holds even in the non-standard Henkin models that CA has. More exactly, we have shown that ϕ holds in a whole class of structures, a class that has $\mathfrak{A}$ as a member. In particular, we have shown that ϕ holds in $\mathfrak{A}$. The fact that we proved that ϕ holds in more structures than we wanted should in no way lessen our faith in ϕ holding in $\mathfrak{A}$. Most likely it is just easier to justify (1) in this way.

But does the above discussion not contradict (4)? The answer to this riddle is revealed by an inspection of the vocabularies. The vocabulary of $\theta_{\mathfrak{A}} \wedge \theta_{\mathfrak{A}'}$ is

$$L = \{R_1, \ldots, R_n, R'_1, \ldots, R'_n\}.$$

Let us consider a Henkin model $\mathfrak{C}$ of $\theta_{\mathfrak{A}} \wedge \theta_{\mathfrak{A}'}$. Let $\mathfrak{C}_0$ be the reduct of $\mathfrak{C}$ to $\{R_1, \ldots, R_n\}$, and $\mathfrak{C}_1$ the reduct to $\{R'_1, \ldots, R'_n\}$. Since we assume that $\mathfrak{C}$ satisfies the CA axioms, and we have established (10), we may conclude that $\mathfrak{C}_0 \cong \mathfrak{C}_1$. If we start from two arbitrary Henkin models $\mathfrak{C}_1$ and $\mathfrak{C}_1$, which do not arise from a common expansion $\mathfrak{C}$, there is no way to conclude from (10) that $\mathfrak{C}_0 \cong \mathfrak{C}_1$. We call this phenomenon *internal categoricity*, meaning that any two models even in the general sense of Henkin models that cohere by having a common expansion to a model of CA, are isomorphic. Note that internal categoricity implies categoricity in the ordinary sense: if we take two "real models" of the CA axioms, that is, Henkin models in which all subsets and relations are in the range of the second order variables, then the models do cohere because they have a common expansion that satisfies CA, namely the "full" Henkin model of the union of the vocabularies. So internal categoricity is indeed a particularly strong form of categoricity.

2.4. Summary of the second order view. The second order view presented above is based on the belief in the meaningfulness of propositions like (1) and (2) and on them being true or false, on the belief that the second order variables of ϕ really range over all subsets and relations on the domain of the model $\mathfrak{A}$, and on the belief that $\theta_{\mathfrak{A}}$ characterizes $\mathfrak{A}$ up to isomorphism. Where the above second order view may diverge from the view of some supporters of second order logic is that the justification for asserting (1), (2) and (4) is secured by means of proofs. We were driven to this by contemplating the evidence that we could possibly give. We pointed out that the fact that the proof systems, when formalized, permit countable and other non-standard models, does not in itself cast doubt on the bound variables of ϕ in (1) ranging over *all* subsets and relations on the domain. The existence of non-standard models reveals the strength—not the weakness—of the relevant proofs, indicating that (1), (2) and (4) are special cases of more general results.

§3. The set theory view. Cantor introduced set theory in the context of studying sets of reals and their properties. He then went on to introduce arbitrary sets probably not realizing, but certainly not claiming, that his theory could be used as the foundation for all of mathematics; rather he had to defend the idea that set theory is mathematics at all.

Originally set theory had so-called urelements, that is, elements that are not sets and also have themselves no elements. Subsequently it turned out that mathematics can be developed in set theory without urelements. Despite this fact, it would be most natural and perfectly in harmony with everything that is done in set theory to include natural numbers, real numbers and so on as urelements and let the sets built "on top" of the urelements dictate the necessary properties of these numbers. Indeed, we could build a close relationship between the second order view and set theory by taking for every second order characterizable structure a set of urelements from which the structure is built. Although this approach would not change anything essential in our account of set theory, we abandon it in favor of the more standard approach of disregarding urelements.

Set theory is based on the idea that there is a universe of sets and all of mathematics can be embedded into this universe giving mathematics a uniform framework. The prevailing view in set theory is that the universe of sets has the structure of a hierarchy, called the *cumulative hierarchy*:

$$\begin{aligned} V_0 &= \emptyset, \\ V_{\alpha+1} &= \mathcal{P}(V_\alpha), \\ V_\nu &= \bigcup_{\beta<\nu} V_\beta, \text{ if } \nu \text{ is a limit ordinal.} \end{aligned}$$

It is held that every set is an element of some V_α. This conviction, which may appear somewhat arbitrary, is simply based on the fact that nothing else seems necessary. It is also customary to denote the union of all the sets V_α by V and what we have just said amounts to saying that V is the universe of set theory. Of course, V is not a set, but what is called a proper class.

It is common in set theory to identify natural numbers with finite ordinals:

$$n = \{0, \dots, n-1\}$$

and $\mathbb{N}$ with ω. Then we can construct integers as equivalence classes of pairs of natural numbers, so $\mathbb{Z} \in V_{\omega+3}$, rational numbers as equivalence classes of integers, so $\mathbb{Q} \in V_{\omega+5}$, and real numbers as equivalence classes of sets of rational numbers, so $\mathbb{R} \in V_{\omega+7}$. The classical structures, so important in second order logic, can all be constructed explicitly as elements of, say $V_{\omega+12}$. Since second order logic can be readily interpreted in set theory one may easily check that the constructed structures all satisfy their second order characterizations, so we have constructed, up to isomorphism, the same structures as was done in second order logic. The difference with second order logic is that we now accept these structures as individual structures rather than as particular equivalence classes of the isomorphism relation. This decision is of no consequence mathematically, it is just a curiosity and part of set-theoretical thinking.

According to what we are calling the set theory view mathematical propositions are of the form

$$\Phi(a_0, \dots, a_{n-1}), \tag{11}$$

where $\Phi(x_0, \dots, x_{n-1})$ is a first order formula of the vocabulary $\{\in\}$ and $a_0, \dots, a_{n-1}$ are some specific sets. The meaning of (11) is that the sets $a_0, \dots, a_{n-1}$ have the property $\Phi(x_0, \dots, x_{n-1})$. Note that $\Phi(x_0, \dots, x_{n-1})$ will most likely have quantifiers, and the variables bound by these quantifiers range over the entire universe V of sets. In other words, the meaning of (11) is that $\Phi(x_0, \dots, x_{n-1})$ is true in the proper class size model $(V, \in)$ under the assignment $x_i \mapsto a_i$. But one should not think that this is a definition of the meaning of (11). It is impossible to actually *define* the meaning of (11) because truth is undefinable. However, for any fixed m the truth of (11) for formulas $\Phi(x_0, \dots, x_{n-1})$ of quantifier rank $\leq m$ (and hence, a fortiori, for formulas with at most m symbols) can be defined by a formula of that quantifier rank (roughly speaking, because for any m set theory has a so called universal Σ_m-formula, see [8]).

By *specific* mathematical objects we mean definable objects, that is, sets a for which there is a first order formula $\theta(x)$ of the vocabulary $\{\in\}$ such that the following two conditions hold:

$$\phi(a) \tag{12}$$

$$\forall x((\phi(x) \wedge \phi(y)) \rightarrow x = y) \tag{13}$$

Note the resemblance to (3) and (4). Like truth, also definability is not itself definable. But for each fixed quantifier rank m we can define the concept of definability by a formula of quantifier rank $\leq m$.

As the above remarks show, set theory is open ended in the sense that we cannot once and for all secure the meaning of (11), for two reasons: neither truth nor definability is definable. If we limit ourselves to a bounded quantifier rank, then suddenly (11) becomes expressible on that quantifier rank.

Familiar mathematical objects such as

$$\mathbb{N}, \mathbb{Q}, \mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n, \pi, \sqrt[3]{2}, \log, \sin, \text{etc.}$$

are all definable on quantifier rank ≤ 2. To find a definable mathematical object not definable on quantifier rank 2 one has to go deep into set theory. Likewise, in mathematical practice one has to think hard to find examples of (11) where $\Phi(x_0, \ldots, x_{n-1})$ could not be be written by a formula of quantifier rank ≤ 2.

There are, intuitively speaking, only countably many definable sets so most sets are undefinable. For example, most real numbers are in this sense undefinable. Also, there is no reason to believe that complex things like well-orderings of the reals have to be definable. These objects are needed for the general workings of set theory but there is no reason to think one is able to put one's finger on them. This is completely in harmony with the general philosophy of the set theory view of mathematics, which is geared towards definable objects but allows undefinable objects in the background as, after all, everything cannot be definable. To assume that everything is definable would mean giving up the power-set axiom, which however is necessary for constructing such basic structures as $\mathbb{R}$. So the set theory view maintains coherence by allowing some (most) sets to be undefinable, i.e., sets we cannot talk about explicitly but which still occur in the range of bound variables. In modern set theory (see e.g., [6]) one can actually prove from large cardinal assumptions that there is no well-ordering of the reals which is projective (i.e., Σ^1_n for some n).

We can easily express both (1) and (2) in the form (11), so the set-theoretic framework is in this sense at least as powerful as the second order framework.

Note that we think of the quantifiers in (11) as ranging over all sets of the universe, including all subsets of the parameters $a_0, \ldots, a_{n-1}$.

There is a *bounded set theory view* which is seemingly weaker than the above original set theory view. It is the view that mathematical propositions are of the form

$$(V_\alpha, \in) \models \Phi(a_0, \ldots, a_{n-1}), \tag{14}$$

where α is a definable ordinal large enough for V_α to contain $a_0, \ldots, a_{n-1}$, the sets $a_0, \ldots, a_{n-1}$ are definable over $(V_\alpha, \in)$, and $\Phi(x_0, \ldots, x_{n-1})$ is a first order formula of set theory. The meaning of (14) should be clear: we can use the ordinary Tarski truth definition for first order logic. We can express (1) and (2) in this form. For (1) in most cases $\alpha \leq \omega + 7$ suffices. For (2) it is more difficult to determine an upper bound for α and quite large α may be needed. So this modified set theory view is sufficient to account for practically all of mathematics outside set theory (and perhaps category theory) itself and also to account for mathematics in the sense of the second order view. The disadvantage of the bounded view is that it does not cover all of set theory, except by increasing α as soon as it is needed.[2] So the original set theory view is more stable, needing no adjustment from proposition to proposition. On the other hand, the original set theory view cannot be understood inside set theory itself (because of the undefinability of truth) except on a case by case basis or by bounding the quantifier rank.

The bounded set theory view allows us to focus on the interesting and confusing question whether set theory is first order or higher order. In (14) the bound variables of $\Phi(x_0, \ldots, x_{n-1})$ range over V_α. If $a_0, \ldots, a_{n-1} \in V_\beta$ and $\beta < \alpha$, then all subsets of $a_0, \ldots, a_{n-1}$ are in the range of those bound variables, so we can express second order properties of $a_0, \ldots, a_{n-1}$. In particular, if $\mathfrak{A}$ is a structure such that $A \in V_\alpha$, then $\mathcal{P}(A) \subseteq V_\alpha$, so first order logic over $(V_\alpha, \in)$ can express any second order properties of $\mathfrak{A}$. If $A \in V_\beta, \beta < \alpha$, then the same is true of third order properties of $\mathfrak{A}$. So is $(V_\alpha, \in) \models \Phi(a_0, \ldots, a_{n-1})$ first order or not? It is first order from the perspective of $(V_\alpha, \in)$, the bound variables of $\Phi(a_0, \ldots, a_{n-1})$ ranging over *elements* of V_α. At the same time, $(V_\alpha, \in) \models \Phi(a_0, \ldots, a_{n-1})$ is higher order from the perspective of the sets $a_0, \ldots, a_{n-1}$ in the following sense. Since $a_0, \ldots, a_{n-1} \in V_\alpha$, $P(a_0) \cup \cdots \cup \mathcal{P}(a_{n-1}) \subseteq V_\alpha$. So the bound variables of $\Phi(a_0, \ldots, a_{n-1})$ have all subsets of each a_i in their range. If α is a limit ordinal, this is true of subsets of subsets of each a_i, subsets of subsets of subsets of each a_i, etc. We conclude: the set theory view includes the higher order view in this sense.

What are the grounds under which a mathematician can assert something like (11)? Note that unless $a_0, \ldots, a_{n-1}$ were definable we could not assert

[2]By the Levy Reflection principle ([6] Theorem 12.14) any true sentence with set parameters is true in some V_α.

(11) at all. Intuitively speaking, we can assert (11) if we somehow know that (11) is how $a_0, \dots, a_{n-1}$ sit in the entire universe of sets. There are obvious difficulties in stating precisely what such knowledge could consist of and we have alluded to this already in connection with second order logic. The most obvious solution is again to rely on proofs. Justification for asserting (11) is a formal or informal proof of $\Phi(a_0, \dots, a_{n-1})$ from first principles, such as the ZFC axioms. We need not go into the question whether the ZFC axioms are acceptable as first principles, because in most cases of mathematics outside set theory one can use weaker axioms such as Zermelo's set theory (or even weaker). However, it is part and parcel of the philosophy of the set theory view that one does not shy away from the strongest first principles, even large cardinal axioms. There are other foundational positions such as predicativism, finitism, quasi-intuitionism, constructivism, intuitionism, and so on where the weaker axiom systems are relevant.

How can we recognize what the definable sets referred to in (11) are? As in the case of second order logic there is no way to give evidence for (12) apart from making sure we have written $\phi(x)$ so that it faithfully reflects our understanding of a. However, we can prove

$$\exists x\, \phi(x) \tag{15}$$

in ZFC and declare that a is the unique set given by (13)-(15). This corresponds in second order logic to deriving (8) from $\theta_{\mathfrak{B}}$ for a bigger $\mathfrak{B}$.

Just as in the case of second order logic, giving a proof for $\Phi(a_0, \dots, a_{n-1})$ from ZFC tells us much more than what (11) maintains. For (11) to be true it is enough that $\Phi(a_0, \dots, a_{n-1})$ is true in $(V, \in)$, while if $\Phi(a_0, \dots, a_{n-1})$ is provable from ZFC, it is true in every universe where the ZFC axioms hold. One important such universe is the universe L of constructible sets due to Gödel. Other universes, called inner models, are known in great numbers but even to prove $V \neq L$ (and not just its consistency) one has to use principles, such as large cardinal axioms, that go beyond ZFC. It is part of the set theory view that we believe in the existence of countable transitive sets M for which $(M, \in) \models \text{ZFC}$. Although we cannot present evidence for this assertion on the basis of ZFC only, it is part of the set theory view. One can use large cardinal assumptions as evidence. If we have a proof of $\Phi(a_0, \dots, a_{n-1})$ from ZFC, then $\Phi(a_0, \dots, a_{n-1})$ is true in each such countable transitive model $(M, \in)$ as well as in all the generic extensions of such models obtained by Cohen's method. All this emphasizes what deep consequences the assertion of (11) with a proof from ZFC as evidence has. In such a case we should not think that the fact that $\Phi(a_0, \dots, a_{n-1})$ holds in numerous artificial universes of sets in any way undermines our conviction that we have established the truth of $\Phi(a_0, \dots, a_{n-1})$ in the actual set-theoretical universe. We have just established more than was asked. Practically, it is is easier to give evidence for a proposition holding in a large number of different kinds of universes, one of which is the "real" one, than it is to derive the proposition in just the one that we are considering.

As in second order logic, we can prove the internal categoricity of set theory: If set theory is formalized with two $\in$-relations, say $\in_1$ and $\in_2$, and the ZFC axioms[3] are adopted in the common vocabulary $\{\in_1, \in_2\}$, let us call it ZFC($\in_1, \in_2$), then one can prove in ZFC($\in_1, \in_2$) that the equation

$$y \in_2 F(x) \iff \exists z\,(y = F(z) \wedge z \in_1 x) \tag{16}$$

defines a class function F which is an isomorphism between the $\in_1$-sets and the $\in_2$-sets. In this sense set theory, like second order logic, has *internal categoricity*. If we look at the formalized ZFC($\in_1$) and ZFC($\in_2$) inside ZFC($\in_1, \in_2$), assuming the consistency of ZFC, then we have non-isomorphic models $(M, \in_1^M)$ and $(N, \in_2^M)$ of ZFC, but these two models cannot be put together into one model of ZFC($\in_1, \in_2$). As in second order logic, this demonstrates that the non-standard models really have to be constructed from the outside. The *maximalist* intuition that the universe of sets has really *all* the sets, and respectively the intuition that the second order variables of second order logic really range over *all* subsets of the domain, corresponds to the idea that our language is so rich that whatever exists can already be referred to by a term of our language.[4] When this intuition is combined with internal categoricity we get ordinary categoricity. But note that we get ordinary categoricity only in an informal sense, because the maximalist intuition is an informal principle. This is in harmony with the overall approach of treating second order logic or set theory as *the* foundation of mathematics rather than a *formalization* of the foundation. When second order logic or set theory is conceived of as *the* foundation of mathematics, both have an equal amount of (internal) categoricity, and when they are formalized and looked at as mathematical objects, both have an equal amount of categoricity if only "full" models of the formalizations are considered and an equal amount of non-categoricity if the proof methods of the formalizations are considered.

Since the universe of sets is somewhat elusive, there is a temptation to forget about it and study only the artificial universes $(M, \in)$, which happen to satisfy the ZFC axioms. One may even go further and deny the coherence of the set-theoretical universe and maintain that any justified assertion of truth of a sentence in the set-theoretical universe is merely assertion of truth of the sentence in all models of ZFC. All models are then equal, one is not favored

[3] In particular, formulas in the Replacement and Separation Schemata can involve both $\in_1$ and $\in_2$.

[4] In the case of set theory we noted above that if we have several epsilon-relations and they satisfy the ZFC axioms even in the larger vocabulary which contains all of them, we get the result that they are all isomorphic to each other. We can interpret this by saying that the set theoretical universe is unique as far as we talk about alternatives that can be *named*.

above the others. Let us give in to this temptation for a moment. What are the propositions of mathematics, if not of the form (11)? Are they of the form (1) and (5), in which case we are working within the second order view? Are they of the form $(M, \in) \models \Phi(a_0, \ldots, a_{n-1})$, where $(M, \in) \models$ ZFC? What is $\in$ here? Are they of the form $(M, E) \models \Phi(a_0, \ldots, a_{n-1})$, where $a_0, \ldots, a_{n-1} \in M$, $E \subseteq M \times M$, and $(M, E) \models$ ZFC? So what is M here? How can we assert $(M, E) \models \Phi(a_0, \ldots, a_{n-1})$ if we do not know how to refer to M and $a_0, \ldots, a_{n-1}$? Or are they of the form

$$\forall M\, \forall E \subseteq M \times M((M, E) \models \mathrm{ZFC} \to (M, E) \models \Phi(t_0, \ldots, t_{n-1})),$$

where $t_0, \ldots, t_{n-1}$ are definable terms of the language of set theory? If this is the form of a mathematical proposition, it would be simpler to give it in the form

$$\mathrm{ZFC} \vdash \Phi(t_0, \ldots, t_{n-1}),$$

because then we would be dealing with just finite proofs and we would not have to answer the question, what are the ranges of the quantifiers $\forall M\, \forall E \subseteq M \times M$? We have arrived at the position that propositions of mathematics are existence claims of proofs from the ZFC axioms. This is appealing because we have just argued that proofs are the evidence for asserting truth anyway. So why not just stick to the evidence and forget what it is evidence for? If we take this line because we believe only in the existence of *finite* mathematical objects, then we can really only justify the use of *constructive* logic in proofs, and then we have abandoned classical mathematics. On the other hand if we believe in the existence of *infinite* mathematical objects, we have not explained what we mean by propositions about them.

§**4. Second order characterizable structures.** The concept of a second order characterizable structure makes perfect sense in the context of the set theory view (see Section 3). Let us then adopt the set theory view and spend a moment investigating what can be said about such structures. Since the set theory view as formulated here is a foundationalist position it is particularly suited to an attempt to understand the general concept of a second order characterizable structure.

4.1. A hierarchy of second order characterizable structures. We show that second order characterizable structures form a hierarchy. Upon closer inspection this hierarchy reveals some essential features of second order logic.

Suppose $\mathfrak{A}$ is a second order characterizable structure in a relational vocabulary $L = \{R_1, \ldots, R_n\}$ and $|A| = \kappa$. Then κ is second order characterizable as a structure of the empty vocabulary by the sentence $\theta_\kappa = \exists R_1 \ldots \exists R_n \theta_{\mathfrak{A}}$, that is, a structure B of the empty vocabulary satisfies θ_κ if and only if $|B| = \kappa$. Thus the cardinalities of second order characterizable structures are all second order characterizable.

The concept of being second order characterizable is definable in set theory, so we can consider without difficulty the countable set of all second order characterizable cardinal numbers. It starts with the finite numbers, then come $\aleph_0, \aleph_1, \ldots$ until we reach $\aleph_\omega$. Then follow $\aleph_{\omega+1}, \aleph_{\omega+2}, \ldots$. In fact it is not so easy to see where this simple pattern breaks. This has been studied by S. Garland [3]. The following observation shows that the second order characterizable cardinals extend all across the set-theoretical universe, apart from very large cardinals.

PROPOSITION 1. *The first inaccessible* (*Mahlo, weakly compact, Ramsey*) *cardinal is second order characterizable. If κ is the first measurable cardinal, then 2^κ is second order characterizable. All second order characterizable cardinals are below the first strong cardinal.*[5]

PROOF. Let ϕ_0 be the conjunction of the finitely many ZFC axioms written in second order logic. Models of ϕ_0 are, up to isomorphism, of the form $(V_\kappa, \in)$, κ inaccessible. Let ϕ_1 be the conjunction of ϕ_0 and a first order sentence saying that every limit cardinal is singular. Then ϕ_1 characterizes up to isomorphism $(V_\kappa, \in)$, where κ is the first inaccessible. Let ϕ_2 be the conjunction of ϕ_0, the second order sentence saying that every closed unbounded class of ordinals has a regular element, and a first order sentence saying that every limit cardinal is non-Mahlo. Then ϕ_2 characterizes up to isomorphism $(V_\kappa, \in)$, where κ is the first Mahlo. Let ϕ_3 be the conjunction of ϕ_0, the second order sentence saying that every class size tree with set size levels has a cofinal branch, and a first order sentence saying that there are no weakly compact cardinals. Then ϕ_3 characterizes up to isomorphism $(V_\kappa, \in)$, where κ is the first weakly compact cardinal. Let ϕ_4 be the conjunction of ϕ_0, the second order sentence saying that every coloring of finite subsets of the universe by two colors has a class size set which is for each n homogeneous for finite subsets of size n of its elements, and a first order sentence saying that there are no Ramsey cardinals. Then ϕ_4 characterizes up to isomorphism $(V_\kappa, \in)$, where κ is the first Ramsey cardinal. Let ϕ_5 be the conjunction of the relativization of ϕ_0 to the unary predicate M, the second order sentence saying that every subset of M is in the universe, the second order sentence saying that a subset U of the universe is an ultrafilter on M which is complete with respect to subsets of M which are elements of M, and a first order sentence saying that there are no measurable cardinals in M. Then ϕ_5 characterizes up to isomorphism $(\mathcal{P}(V_\kappa), V_\kappa, \in)$, where κ is the first measurable cardinal. Finally, let κ be strong. Suppose λ is second order characterizable. Suppose $j \colon V \to M$ is an elementary embedding with critical point κ such that $\mathcal{P}(\lambda) \in M$. Since $\mathcal{P}(\lambda) \subseteq M$,

[5]The result concerning inaccessible cardinals is due to Zermelo. The result concerning measurable cardinals is due to D. Scott. The result concerning strong cardinals is due to M. Magidor. Large cardinals are started above ω.

$M \models \exists\alpha < j(\kappa)((\theta_\lambda)^{(\alpha)})$. Thus $V \models \exists\alpha < \kappa((\theta_\lambda)^{(\alpha)})$. Let $\alpha < \kappa$ with $\alpha \models \theta_\lambda$. Then $\alpha = \lambda$, so $\lambda < \kappa$. ⊣

The second order characterizable cardinals come in clusters in the following sense:

Lemma 2. *If κ is second order characterizable, then so are κ^+, 2^κ, $\aleph_\kappa$ and $\beth_\kappa$. More generally, if κ and λ are second order characterizable, then so is κ^λ.*

Proof. Let ϕ_0 be the second order sentence saying that $<$ is a well-order in which every initial segment satisfies θ_κ but the whole universe itself does not satisfy θ_κ. Then ϕ_0 is satisfied, up to isomorphism, only by $(\kappa^+, <)$. Let ϕ_1 be the second order sentence saying that $<$ is a well-order in which no initial segment satisfies θ_κ but the whole universe itself satisfies θ_κ. Then ϕ_1 is satisfied, up to isomorphism, only by $(\kappa, <)$. Let ϕ_2 be the conjunction of the relativization of ϕ_1 to a unary predicate U, the second order sentence saying that $<$ is a well ordering of the universe with each initial segment of smaller cardinality than the universe, the second order sentence saying that elements of U are "cardinals" in $<$ that is, points a whose every initial segment has fewer elements than a in $<$. Then ϕ_2 is satisfied, up to isomorphism, only by $(\aleph_\kappa, <)$. Let ϕ_3 be the conjunction of the relativization of ϕ_1 to a unary predicate U, the second order axioms of Zermelo's set theory, and the first order axiom which says that U is the class of all cardinal numbers. Then ϕ_4 is satisfied, up to isomorphism, only by $(V_\kappa, \in)$, the cardinality of which is $\beth_\kappa$. For the last claim, let ψ be the conjunction of the relativization of θ_λ to a unary predicate P, the relativization of θ_λ to the unary predicate Q, the first order sentence saying that for every x the function $z \mapsto F(x,z)$ is a function $Q \to P$, and the second order sentence saying that every function $Q \to P$ is the function $z \mapsto F(x,z)$ for some x. Then, up to isomorphism, the only model of $\exists F\psi$ is $({}^\lambda\kappa, \lambda \cup \kappa, \lambda, \kappa)$. ⊣

Thus the first non-second order characterizable cardinal is a singular strong limit cardinal. The smallest singular strong limit cardinal is $\beth_\omega$, but this cardinal is second order characterizable by the above. In fact it immediately follows from Lemma 2 that if κ is the supremum of second order characterizable cardinals, then $\kappa = \aleph_\kappa = \beth_\kappa$.

If we put any particular second order characterizable cardinal κ under the microscope we can immediately see that there are countably infinitely many non-isomorphic second order characterizable structures of that cardinality. First, there are certainly infinitely many non-isomorphic ones because each structure $(\alpha, <)$, $\kappa \leq \alpha \leq \kappa + \omega$ is second order characterizable. On the other hand there are only countably many second order characterizable structures overall.

How far from each other are the second order characterizable structures in a given cardinality? Let us say that a structure is *Turing-reducible* to another structure if the second order theory of the first is Turing-reducible (see e.g., [9, §9.4]) to that of the second. Two structures are *Turing-equivalent* if they are Turing-reducible to each other.

PROPOSITION 3. *If $\mathfrak{A}$ and $\mathfrak{B}$ are second order characterizable structures such that $|A| \leq |B|$, then $\mathfrak{A}$ is Turing reducible to $\mathfrak{B}$. In particular, all second order characterizable structures of the same cardinality are Turing-equivalent.*

PROOF. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are second order characterizable structures such that $|A| \leq |B|$. Let $L = \{R_1, \dots, R_n\}$ be the relational vocabulary of $\mathfrak{A}$. Now for any ϕ in the vocabulary L

$$\mathfrak{A} \models \phi \iff \mathfrak{B} \models \forall P \forall R_1 \dots \forall R_n((\theta_{\mathfrak{A}} \to \phi)^{(P)}),$$

where $\psi^{(P)}$ denotes the relativization of ψ to the unary predicate P. ⊣

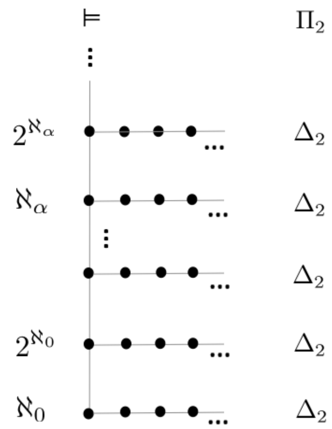

FIGURE 1. The hierarchy of second order characterizable structures.

The picture (Figure 1) that emerges now about infinite second order characterizable structures is the following: There is a countable hierarchy of second order characterizable cardinals that extends very high in the scale of cardinal numbers. At each cardinal there are countably many non-isomorphic structures and also structures of different vocabularies. Whatever the vocabulary L (apart from trivial cases), in each infinite cardinality κ there are $\aleph_0$ non-isomorphic second order characterizable L-structures of that cardinality. All we need to do is take one predicate symbol $R \in L$ and consider structures with universe κ and R of different finite sizes. However, all these $\aleph_0$ second order characterizable structures of a fixed cardinality have the same second order theory up to Turing-equivalence. Knowing truth in one, means knowing truth in any of them, as well as in any structure on a lower level. Thus when we go down in size, the complexity goes down, or at least does not increase. What about going up? Does the complexity increase every time we go up in the cardinality of the model?

PROPOSITION 4. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are any infinite second order characterizable structures such that* $2^{|A|} \leq |B|$, *then* $\mathfrak{B}$ *is not Turing reducible to* $\mathfrak{A}$.

PROOF. The proof is a standard undefinability of truth argument. Let $\kappa = |A|$ and $\lambda = |B|$. Note that $(\kappa, <)$ and $(\lambda, <)$ are second order characterizable, and therefore also $\mathfrak{B}' = (\lambda \cup \mathcal{P}(\kappa), \lambda, <, \mathcal{P}(\kappa), \kappa, \pi, \mathbb{N})$ and $\mathfrak{A}' = (\kappa, <, \pi, \mathbb{N})$, where π is a bijection of $\kappa \times \kappa$ onto κ. It suffices to show that $\mathfrak{B}'$ is not Turing-reducible to $\mathfrak{A}'$. Let L be the vocabulary of $\mathfrak{B}'$ and $L' \subset L$ that of $\mathfrak{A}'$. Suppose for all second order L-sentences ϕ

$$\mathfrak{B}' \models \phi \iff \mathfrak{A}' \models \phi^*$$

with some recursive function $\phi \mapsto \phi^*$ from L-sentences to L'-sentences. We use $\underline{n}$ to denote the definable term which in $\mathfrak{B}'$ and $\mathfrak{A}'$ denotes the natural number n. Using standard methods one can write a second order L-sentence $\Theta(x, y)$ such that for all L'-formulas $\phi(x)$ and any $n \in \mathbb{N}$

$$\mathfrak{B}' \models \Theta(\underline{\ulcorner \phi \urcorner}, \underline{n}) \iff \mathfrak{A}' \models \phi(\underline{n}).$$

Thus

$$\mathfrak{A}' \models \phi(\underline{n}) \iff \mathfrak{A}' \models \Theta(\underline{\ulcorner \phi \urcorner}, \underline{n})^*.$$

Let $\psi(x)$ be the L'-formula which says $\neg\Theta(\underline{a}, \underline{a})^*$ for the "natural number" a, in the formal sense, that is the value of x. Let $k = \ulcorner \psi(x) \urcorner$. Now

$$\mathfrak{A}' \models \Theta(\underline{k}, \underline{k})^* \iff \mathfrak{A}' \models \psi(\underline{k}) \iff \mathfrak{A}' \models \neg(\Theta(\underline{k}, \underline{k}))^*,$$

a contradiction. ⊣

Let us now revisit the picture (Figure 1) of second order characterizable structures. When we go up in the cardinalities of the models, we obtain more and more complex theories. We can interpret this as a form of *anti-foundationalism* in the sense that there is no individual second order characterizable structure $\mathfrak{A}$ such that the truth of any other second order characterizable structure is Turing-reducible to truth in $\mathfrak{A}$. This is even true if we only consider empty vocabularies, that is, structures with a universe only and no structure whatsoever. If we try to use brute force by letting P be the set of Gödel numbers of second order sentences ϕ such that (2) holds, then we do get a structure $(\mathbb{N}, +, \cdot, P)$ such that truth in any second order characterizable structure is Turing-reducible to truth in $(\mathbb{N}, +, \cdot, P)$, but the price we pay is that the structure $(\mathbb{N}, +, \cdot, P)$ itself is not second order characterizable.

4.2. Definability of second order characterizable structures. The second order characterizable structures are by definition definable in second order logic but here we examine in what sense they are definable in set theory.

The Levy-hierarchy [8] of Σ_n- and Π_n-formulas is useful in estimating the set-theoretical complexity of mathematical concepts. Most concepts in mathematics outside set theory are Σ_n- or Π_n-definable with $n \leq 2$. Since a concept may have several equivalent formulations it is important to specify which axioms are used to obtain Σ_n- or Π_n-definability. Accordingly we say that a property of sets is Σ_n-definable if there is a Σ_n-formula that defines the property. The concept of Π_n-definability is defined similarly. Finally, a property of sets is Δ_n-definable if it is both Σ_n- and Π_n-definable. For example, finiteness is Δ_1-, countability is Σ_1- and "x is the power-set of y" is Π_1-definable.

Since a second order characterizable structure is specified up to isomorphism only, it does not make sense to ask if an individual second order characterizable structure is definable in set theory. The whole isomorphism class is obviously definable and we now show it is actually Δ_2-definable.

Proposition 5. *If $\mathfrak{A}$ is a second order characterizable structure, then the class $\{\mathfrak{B} \colon \mathfrak{B} \cong \mathfrak{A}\}$ is Δ_2-definable.*

Proof. Suppose L a finite vocabulary and $\mathfrak{A}$ is a second order characterizable L-structure. Suppose σ is the conjunction of a large finite part of ZFC. Let us call a model $(M, \in)$ of θ *supertransitive* if for every $a \in M$ every element and every subset of a is in M. Let $\mathrm{Sut}(M)$ be a Π_1-formula which says that M is supertransitive. Let $\mathrm{Voc}(x)$ be the standard definition of "x is a vocabulary". Let $\mathrm{SO}(L, x)$ be the set-theoretical definition of the class of second order L-formulas. Let $\mathrm{Str}(L, x)$ be the set-theoretical definition of L-structures. Let $\mathrm{Sat}(\mathfrak{A}, \phi)$ be the standard inductive truth-definition of second order logic written in the language of set theory. Let

$$P_1(z, x, y) = \mathrm{Voc}(z) \wedge \mathrm{Str}(z, x) \wedge \mathrm{SO}(z, y) \wedge \\ \exists M\, (z, x, y \in M \wedge \sigma^{(M)} \wedge \mathrm{Sut}(M) \wedge (\mathrm{Sat}(z, x, y))^{(M)})$$

and

$$P_2(z, x, y) = \mathrm{Voc}(z) \wedge \mathrm{SO}(z, y) \wedge \mathrm{Str}(z, x) \wedge \\ \forall M((z, x, y \in M \wedge \sigma^{(M)} \wedge \mathrm{Sut}(M)) \to (\mathrm{Sat}(z, x, y))^{(M)}).$$

Now $\mathrm{ZFC} \vdash \forall z\, \forall x\, \forall y (P_1(z, x, y) \leftrightarrow P_2(z, x, y))$ and if L is a vocabulary, $\mathfrak{B}$ an L-structure, then $\mathfrak{B} \cong \mathfrak{A} \iff P_1(L, \mathfrak{B}, \theta_{\mathfrak{A}})$. This shows that $\mathfrak{B} \cong \mathfrak{A}$ is a Δ_2 property of $\mathfrak{B}$ and L. ⊣

The following result, based on the idea of the proof of a related result of Ajtai [1] (see also [10], [5], [7]), demonstrates that even though the isomorphism class of a second order characterizable structure is definable in set theory, the question whether individual structures in the isomorphism class are definable in set theory is independent of the axioms of set theory:

Proposition 6. *Suppose $\mathfrak{A}$ is a second order characterizable structure. If $V = L$, then $\{\mathfrak{B} \colon \mathfrak{B} \cong \mathfrak{A}\}$ contains a Π_2-definable model. If* ZFC *is consistent, then it is consistent to have a countable second order characterizable structure $\mathfrak{A}$ such that $\{\mathfrak{B} \colon \mathfrak{B} \cong \mathfrak{A}\}$ contains no structures that are definable in set theory.*

PROOF. If $V = L$, then we can define the smallest element $\mathfrak{B}$ of $\{\mathfrak{B} \colon \mathfrak{B} \cong \mathfrak{A}\}$ in the canonical well-order of L: $\mathfrak{B}$ is the unique set satisfying

$$\mathrm{Str}(L, \mathfrak{B}) \wedge P_2(L, \mathfrak{B}, \theta_{\mathfrak{A}}) \wedge \forall x (P_1(L, x, \theta_{\mathfrak{A}}) \to \mathfrak{B} \leq_L x).$$

On the other hand if ZFC is consistent there is a forcing extension in which the set $\mathbb{R}$ has no well-ordering which would be definable in set theory with real parameters. However, let $<^*$ be a well-ordering of $\mathbb{R}$ (by the Axiom of Choice) in the order-type $|\mathbb{R}|$. The structure $(\mathbb{R}, <^*)$ is second order characterizable as $<^*$ is the unique well-ordering of the set $\mathbb{R}$ of the order-type $|\mathbb{R}|$. If $(\mathbb{R}, <^*)$ were definable in set theory, then we would get a definable well-order of $\mathbb{R}$. ⊣

The above Proposition shows that second order characterizable structures can be captured in set theory up to isomorphism, but if one wants to "pick out" any particular one there may be obstacles. This is in harmony with the general trend in set theory that one may not be able to choose elements from a definable class in a definable way. A good example is the set of well-orderings of the reals, used in the above proof. This set is of course a definable set, but there is no provably definable way of defining any particular such well-ordering. Rather the contrary, it is a consequence of large cardinal assumptions that no well-ordering of the reals can be definable on any level Σ^1_n of the projective hierarchy. Still it is consistent, relative to the consistency of large cardinals, that there is a supercompact cardinal and the reals have a definable well-order.

Perhaps the non-availability of specific definable structures in the isomorphism class of a second order characterizable structure gives credibility to the idea, seemingly part of the second order view, that we should consider structures up to isomorphism only, and not try to pinpoint any specific structure. In set theory all the structures isomorphic to a given one exist on an equal basis, none above others, and whether they are definable or not is an afterthought. The fact that we cannot pick a definable well-ordering of the field of reals does not mean that we could not *use* such (arbitrary) well-orderings.

We now investigate how complicated is the truth concept in the structures of the hierarchy (Figure 1) of second order characterizable structures. We already know that in terms of Turing-reducibility the complexity goes up as the size of the model goes up, at least if we make an exponential jump. We now show that the second order theories of all the second order characterizable structures are Δ_2-definable and this is the sharpest result at least in terms of the Levy-hierarchy. Moreover, we show that universal second order truth (2) is on the strictly higher level of Π_2-definability.

PROPOSITION 7. *The second order theory of any second order characterizable structure is Δ_2-definable. The second order theory of a second order characterizable structure of cardinality $\geq \beth_m$ cannot be Σ^m_n for any n.*

PROOF. We use the notation of Proposition 5. We showed already that the Σ_2-predicate $P_1(z,x,y)$ and the Π_2-predicate $P_1(z,x,y)$ are equivalent and hence Δ_2-definable. Now

$$\mathfrak{A} \models \phi \iff \exists x P_1(L,x,\theta_{\mathfrak{A}} \wedge \phi) \iff \forall x(P_1(L,x,\theta_{\mathfrak{A}}) \to P_2(L,x,\phi)).$$

If $P(x)$ is a Π^m_n property of natural numbers and $\mathfrak{A}$ has cardinality $\geq \beth_m$, then there is a second order θ in the vocabulary $L = \{R_1,\dots,R_k\}$ of number theory such that

$$P(a) \iff \mathfrak{A} \models \exists R_1 \dots \exists R_k(\theta \wedge P(a)).$$

Thus the second order theory of $\mathfrak{A}$ cannot be Σ^m_n. ⊣

The validity $\models \phi$, i.e., truth in *all* structures is more complicated than truth in any particular second order characterizable structure:

PROPOSITION 8. [12] *The predicate "ϕ has a model" is a Σ_2-complete predicate. Hence the predicate $\models \phi$ is a Π_2-complete property of (the Gödel number of) ϕ.*

PROOF. ϕ has a model if and only if there is a supertransitive set M such that it is true in $(M,\in)$ that ϕ has a model. Thus the predicate "ϕ has a model" is Σ_2. On the other hand, suppose $\exists x\,\forall y P(x,y,n)$ is a Σ_2-predicate. Let ϕ_n be a second order sentence the models of which are, up to isomorphism, exactly the models $(V_\alpha,\in)$, where $\alpha = \beth_\alpha$ and $(V_\alpha,\in) \models \exists x\,\forall y P(x,y,n)$. If $\exists x\,\forall y P(x,y,n)$ holds, we can find a model for ϕ_n by means of the Levy Reflection principle ([6] Theorem 12.14). On the other hand, suppose ϕ_n has a model. W.l.o.g. it is of the form $(V_\alpha,\in)$. Let $a \in V_\alpha$ such that $(V_\alpha,\in) \models \forall y P(a,y,n)$. Since in this case $H_\alpha = V_\alpha$, $(H_\alpha,\in) \models \forall y P(a,y,n)$, where H_α is the set of sets of hereditary cardinality $< \alpha$. By another application of the Levy Reflection Principle we get $(V,\in) \models \forall y P(a,y,n)$, and we have proved $\exists x\,\forall y P(x,y,n)$. ⊣

COROLLARY 9. *The second order theory of every second order characterizable structure is Turing reducible to the proposition "ϕ is valid" and to the proposition "ϕ has a model" but the latter propositions are not Turing reducible to the second order theory of any second order characterizable structure.*

Note that the proposition "ϕ is valid" is trivially Turing reducible to the second order theory of the structure $(\mathbb{N},<,P)$, where P is the set $\{\ulcorner\phi\urcorner : \models \phi\}$. Thus $(\mathbb{N},<,P)$ is an example of a structure that can be defined in set theory but is not second order characterizable. Note that the above Corollary is also an easy consequence of Proposition 4.

PROPOSITION 10. *The predicate "ϕ has at most one model up to isomorphism" is a Π_2-complete predicate.*

PROOF. We show that the predicate "ϕ has at least two models up to isomorphism" is a Σ_2-complete predicate. ϕ has at least two models if and only if there is a supertransitive set M such that it is true in $(M,\in)$ that ϕ has at least two models. Thus the predicate "ϕ has at least two models up to isomorphism" is Σ_2. On the other hand, suppose $\exists x\,\forall y P(x, y, n)$ is a Σ_2-predicate. Let ϕ_n be as, in the proof of Proposition 8, the second order sentence the models of which are, up to isomorphism, exactly the models $(V_\alpha, \in)$, where $\alpha = \beth_\alpha$ and $(V_\alpha, \in) \models \exists x\,\forall y P(x, y, n)$. If $\exists x\,\forall y P(x, y, n)$, then ϕ_n has two models of different cardinality, hence two non-isomorphic models. On the other hand, if ϕ_n has at least two models, it has a model and hence $\exists x\,\forall y P(x, y, n)$ holds. $\dashv$

PROPOSITION 11. *The predicate "ϕ is a second order characterization of a structure" is the conjunction of a Σ_2-complete and a Π_2-complete property of ϕ. This predicate is not Σ_2 or Π_2.*

PROOF. The first claim follows from the previous two propositions. Suppose a Σ_2-predicate $\exists x\,\forall y P(x, y, n)$ is given. Let ψ_n be a second order sentence the models of which are, up to isomorphism, exactly the models $(V_\alpha, \in)$, where $\alpha = \beth_\alpha$, $(V_\alpha, \in) \models \exists x\,\forall y P(x, y, n)$, and $(V_\alpha, \in) \models \forall \beta(\beta \neq \beth_\beta \vee \neg\exists x\,\forall y P(x, y, n))^{(V_\beta)}$. Note that ψ_n has at most one model up to isomorphism. If $\exists x\,\forall y P(x, y, n)$ holds, we can find a model for ψ_n by using the Levy Reflection principle and taking the model of minimal rank. On the other hand, if ψ_n has a model, then clearly $\exists x\,\forall y P(x, y, n)$. This shows that the predicate "ϕ is a second order characterization of a structure" cannot be Π_2. To show that it cannot be Σ_2 either let $\exists x\,\forall y P(x, y, n)$ be again a Σ_2-predicate. let ϕ_n be as above, and let $\theta_n = \phi_n \vee \forall x\,\forall y(x = y \wedge x \in y)$. Note that θ_n has a model, whatever n is, and it has at least two non-isomorphic models if and only if ϕ_n has a model. Thus θ_n does not characterize a structure up to isomorphism if and only if ϕ_n has a model if and only if $\exists x\,\forall y P(x, y, n)$. This concludes the proof. $\dashv$

So recognizing whether a candidate second order sentence is a second order characterization of some structure is so complex a problem that it cannot (by Proposition 11 above) be reduced to truth $\mathfrak{A} \models \phi^*$ in any particular second order characterizable structure $\mathfrak{A}$. It encodes a solution to propositions of the type $\not\models \phi^*$. So in complexity it is above all the particular truths $\mathfrak{A} \models \phi^*$ and on a par with, but not equivalent to $\models \phi^*$. The whole framework of the second order view takes the concept of a second order characterizable structure as its starting point. In the case of familiar classical structures we can easily write the second order characterizations. But if we write down an arbitrary attempt at a a second order characterization, the problem of deciding whether we were successful is in principle harder than the problem of finding what is true in the structure, if the sentence indeed characterizes some structure.

Note that if ϕ is a second order characterization of $\mathfrak{A}$, then ϕ is complete, for if ψ is any second order sentence in the vocabulary of ϕ, then $\mathfrak{A} \models \psi$ implies $\phi \models \psi$ and $\mathfrak{A} \not\models \psi$ implies $\phi \models \neg\psi$.

PROPOSITION 12. *The property of ϕ being a (consistent) complete second order sentence is not Π_2. If $V = L$, then every consistent complete sentence characterizes some structure, but consistently some complete sentences have non-isomorphic models.* (*The second claim is due to* [10], *see also* [1] *and* [7]. *For stronger results see* [5].)

PROOF. The proof is similar to the proof of Proposition 6. $\dashv$

4.3. What if only second order characterizable structures exist? We already referred to the problem, what is the real meaning of $\models \phi$? Is it that every second order characterizable structure satisfies ϕ, or are we thinking of some larger category of structures, each of which satisfies ϕ? In the latter case the question arises, what are those structures that may not be second order characterizable? We first point out that if $V = L$, then there is no difference, but otherwise this is a real issue:

PROPOSITION 13. *Consider the conditions*:

(a) $\mathfrak{A} \models \phi$ *for every* $\mathfrak{A}$
(b) $\mathfrak{A} \models \phi$ *for every second order characterizable* $\mathfrak{A}$

If $V = L$, (a) and (b) are equivalent. On the other hand, if ZF is consistent, then it is consistent that (a) and (b) are not equivalent.

PROOF. We build on [1] and [10], see also [5] and [7]. Trivially (a) implies (b). Suppose $V = L$. Assume (a) fails. Suppose $\mathfrak{A}$ is a structure such that $\mathfrak{A} \not\models \phi$. Let $\mathfrak{A}$ be the $<_L$-smallest $\mathfrak{A}$ such that $\mathfrak{A} \models \neg\phi$. Now $\mathfrak{A}$ is second order characterizable by the sentence "I am isomorphic to the $<_L$-smallest model of $\neg\phi$". We contradict (b). For the second claim we start with $V = L$, then add a Cohen real G and construct the countable non-isomorphic models $F(G)$ and $F(-G)$ as in [1]. As proved by Ajtai [1], these are second order equivalent but not isomorphic. As pointed out by Solovay [10], there is a second order sentence θ which is true in both of them and which is complete. Let $L' = \{R_1, \dots, R_n\}$ be the relational vocabulary of θ. Let θ' be θ written in a new disjoint vocabulary $L' = \{R'_1, \dots, R'_n\}$. Let $\mathrm{Iso}(f, R_1, \dots, R_n, R'_1, \dots, R'_n)$ be the first order sentence which says that f is an isomorphism between the model determined by $R_1, \dots, R_n$ and the model determined by $\{R'_1, \dots, R'_n\}$. Consider ϕ which says in the vocabulary $L \cup L'$:

$$(\theta \wedge \theta') \to \exists f\, \mathrm{Iso}(f, R_1, \dots, R_n, R'_1, \dots, R'_n).$$

Let $(F(G), F(-G))$ be the $L \cup L'$-structure the L-reduct of which is $F(G)$ and the L'-reduct $F(-G)$. So ϕ and $(F(G), F(-G))$ violate (a). We now show that (b) holds for ϕ. Suppose $(\mathfrak{A}, \mathfrak{B})$ is second order characterizable $L \cup L'$-structure and satisfies $\theta \wedge \theta'$. Then $\mathfrak{A}$ and $\mathfrak{B}$ are second order equivalent, modulo a translation of the vocabulary. If ψ characterizes $(\mathfrak{A}, \mathfrak{B})$, then $\mathfrak{A}$ is second order characterizable by the sentence $\exists R'_1 \dots \exists R'_n \psi$, for suppose

$\mathfrak{A}' \models \exists R_1' \ldots \exists R_n' \psi$. Then $(\mathfrak{A}', \mathfrak{B}') \models \psi$, building $\mathfrak{B}'$ from $R_1', \ldots, R_n'$, whence $\mathfrak{A} \cong \mathfrak{A}'$. (b) is proved. ⊣

What about categoricity? When we defined the concept of a second order characterizable structure in (4) we used a quantifier ranging over all structures. To say that the quantifier ranges over second order characterizable structures leads to a circular definition. Still we obtain the following. We use the original definition of a second order characterizable structure and then make the following observation afterwards:

Proposition 14. *Suppose all models of θ have the same cardinality.*[6] *Consider*

(a) $(\mathfrak{B} \models \theta \wedge \mathfrak{C} \models \theta) \rightarrow \mathfrak{B} \cong \mathfrak{C}$ *for every* $\mathfrak{B}, \mathfrak{C}$.
(b) $(\mathfrak{B} \models \theta \wedge \mathfrak{C} \models \theta) \rightarrow \mathfrak{B} \cong \mathfrak{C}$ *for every second order characterizable* $\mathfrak{B}, \mathfrak{C}$.

If $V = L$, (a) and (b) are equivalent. On the other hand, if ZF is consistent, then it is consistent that (a) and (b) are not equivalent.

Proof. We build again on [1] and [10] (see also [5] and [7]). Trivially (a) implies (b). Assume $V = L$. Suppose (b) and suppose there are structures $\mathfrak{B}$ and $\mathfrak{C}$ such that $\mathfrak{B} \models \theta$, $\mathfrak{C} \models \theta$ but $\mathfrak{B} \not\cong \mathfrak{C}$. Let $(\mathfrak{B}, \mathfrak{C})$ be the $<_L$-smallest (mod $\cong$) $(\mathfrak{B}, \mathfrak{C})$ such that this holds. Now $(\mathfrak{B}, \mathfrak{C})$ is second order characterizable, hence both $\mathfrak{B}$ and $\mathfrak{C}$ are, and therefore (b) gives $\mathfrak{B} \cong \mathfrak{C}$. For the converse we add a Cohen real G and construct the countable non-isomorphic models $\mathfrak{B} = F(G)$ and $\mathfrak{C} = F(-G)$. As proved by Ajtai, these are second order equivalent. As pointed out by Solovay, there is a second order sentence θ which is true in both of them and which is complete. So we have a failure of (a). To see that (b) holds suppose $\mathfrak{B}', \mathfrak{C}'$ are second order characterizable, and $\mathfrak{B}' \models \theta$, $\mathfrak{C}' \models \theta$. Since θ is complete, $\mathfrak{B}'$ and $\mathfrak{C}'$ are second order equivalent. Hence $\mathfrak{C}' \models \theta_{\mathfrak{B}'}$, and $\mathfrak{B}' \cong \mathfrak{C}'$ follows. ⊣

The pattern is the same in both of the above results: If we assume $V = L$, then we can more or less dispense with non-second order characterizable structures, but otherwise we cannot. This shows that the set-theoretical assumption $V = L$ has analogues in second order logic, for example the assumption that every second order sentence with a model has a second order characterizable model. This is true if $V = L$, and false if we add a Cohen real, demonstrating that intrinsic properties of the mathematical universe according to the second order view play a similar role in second order logic as $V = L$ plays in set theory. The latter assumption solves virtually all otherwise unsolvable problems in set theory. Similarly the assumption that every consistent second order sentence has a second order characterizable model simplifies working with second order logic, because one need not worry about arbitrary structures. One can focus on the second order characterizable structures.

[6] For example, θ is a complete second order sentence.

4.4. Summary of second order structures. The second order characterizable structures form a hierarchy of increasing complexity. Second order truth is not expressible as truth in any particular second order characterizable structure. We can view this as a vindication of an anti-foundationalist position: there is no second order characterizable structure which "stands above" all others, rather each carries its own truth concept.

However, we observed that universal truth $\models \phi$ is more complex than any $\mathfrak{A} \models \phi^*$. In a sense,[7] universal truth provides a foundation for second order logic, albeit a very complex one, in particular more complex than is needed for any individual second order characterizable structure.

The situation is not unlike that prevailing in set theory, where truth in the whole universe is so complex that it is not definable at all. But for most practical purposes truth in some V_α for relatively small α, such as $\alpha = \omega+\omega$, suffices.

The universe of set theory is needed for general reasons: in order that the axioms could be spelled out in a simple and appealing way, in order not to have to decide how many different kinds of sets are in the end needed, and in order to have total freedom in set-theoretic constructions without "hitting the ceiling." If we tried to limit the set-theoretic operations we would raise the question, what is it that we have left out, and why?

In second order logic universal truth is needed to account for universal propositions (2). At the same time we can use it to give (1) the pleasant formulation (5).

We have used set theory to analyze second order characterizable structures and found set-theoretical concepts such as the cardinality of the model and the Levy-hierarchy useful tools. One can redefine these tools in the second order setting, but one would have to rely heavily on the concept of universal truth, and having done that, there is a temptation to see the whole of second order logic as the Δ_2-fragment of a more powerful framework, namely set theory.

§5. Set theory from a second order point of view. How does the set theory view appear from the perspective of the second order view? The structure $(V_\kappa, \in)$ (or V_κ for short), where κ is the first inaccessible cardinal $> \omega$, is second order characterizable, as Zermelo observed. It is fair to say that any mathematical proposition outside certain areas of set theory itself is of the form $V_\kappa \models \phi$, where ϕ is a first order sentence of the vocabulary $\{\in\}$. In particular, if $\mathfrak{A}$ is a second order characterizable structure in V_κ, such as any of the classical structures, then the proposition $\mathfrak{A} \models \phi$, where ϕ is second order, can be readily translated into $V_\kappa \models$ "$\mathfrak{A} \models \phi$", and thereby, using the notation of (3), into

$$\models \theta_{V_\kappa} \rightarrow \text{“}\mathfrak{A} \models \phi\text{”}.$$

[7]Recall that (1) can be reduced to (2).

The idea now is that from the understanding of what θ_{V_κ} means we should be able to deduce that what $\mathfrak{A} \models \phi$ means is true. The foundationalism of set theory manifests itself here inside of the second order view: the single sentence θ_{V_κ} encodes almost all of mathematics. From the anti-foundationalist point of view one may find it unreasonable that a single θ_{V_κ} would encode so much information that all questions $\mathfrak{A} \models \phi$ would be solvable. But we are working inside the second order position and conceived θ_{V_κ} in the second order framework as any other second order characterization. In this position we accept every second order characterizable structure on an equal basis. We did not stipulate that some structures are more important than others but merely that the second order characterizable structures are the important ones. So on what grounds should we abandon the poor devil V_κ who knows too much?

If we have a mathematical proposition of the form $\mathfrak{A} \models \phi$ and $\mathfrak{A} \in V_\kappa$, then we can use V_κ as follows. We want to assert $\mathfrak{A} \models \phi$ as true. We know that it is enough to assert the truth of $\models \theta_{V_\kappa} \rightarrow$ "$\mathfrak{A} \models \phi$". How can we justify this assertion? According to the above discussion the only method available is to give an informal (or formal) proof of $\theta_{V_\kappa} \rightarrow$ "$\mathfrak{A} \models \phi$" from the axioms CA of second order logic in the vocabulary $\{\in\}$. It would be very surprising if this would be anything but a proof of "$\mathfrak{A} \models \phi$" in ZFC, although theoretically the proof based on the second order language and the comprehension axioms is slightly stronger. So unless we have grounds to rule out V_κ as a legitimate second order characterizable structure, we can do full-fledged set theory inside the second order view. Only arguments involving cardinals larger than the first inaccessible are ruled out. But the first inaccessible can be replaced above by the second inaccessible, the third Mahlo, the fourth weakly compact, the fifth inaccessible above the third measurable, etc. So the limitation to κ can be eliminated case by case. In short, (practically) every set theoretic argument can be recast as an argument in second order logic.

Does doing set theory inside the second order view, however contrary to the anti-foundationalism of the second order view it is, give us anything more than what the set theory view does? It can be suggested that because θ_{V_κ} characterizes V_κ categorically, we have gained something. Let us take the CH as an example. We know that CH is true or false in V_κ and therefore

$$\models \theta_{V_\kappa} \rightarrow CH \text{ or } \models \theta_{V_\kappa} \rightarrow \neg CH.$$

In the set theory view we also know that CH is true or false in V_κ, that is

$$V_\kappa \models CH \text{ or } V_\kappa \models \neg CH.$$

In fact, the former does not give us anything more than the latter. In both cases we know that the means we have for giving evidence either way have been proved by Gödel and Cohen to be insufficient. If a second order principle concerning V_κ emerged that solves CH it would be immediately recognized in set theory in the same vein. A central criterion for accepting such new principles is the somewhat vague demand for naturality. It is conceivable that some principles would be more natural in the second order context and some others in the set theory context. For example, the assumption of large cardinals, which is used in set theory to prove that there is no projective well-order of the reals, has arisen in the set-theoretic framework and seems natural there. Formulating large cardinal assumptions in second order logic is uncharted territory.

§6. Large domain assumptions. We now return to the question of giving evidence for a proposition of the form "ϕ is a second order characterization of a structure". This proposition is the conjunction of "ϕ has a model" and "models of ϕ are isomorphic". The predicate "ϕ has a model" is a Σ_2-complete property of ϕ. Thus it can neither be reduced to the question $\mathfrak{A} \models \phi^*$ for some second order characterizable structure $\mathfrak{A}$ nor to $\models \phi^*$ for some second order ϕ^* obtained effectively from ϕ. This means that "ϕ has a model" is a proposition of a new kind. It is a proposition of a kind that we do not know how to give evidence for. This is a weakness in the second order view, as presented so far. It is not at all clear, and we have to leave it as an open problem, how to formulate an axiom of second order logic that would remedy this weakness.

One possible approach is the following: Suppose we already know the existence of *some* second order characterizable structure $\mathfrak{B}$ of the same cardinality as $\mathfrak{A}$. We can then give evidence for (8) by proving

$$\theta_{\mathfrak{B}} \to \exists R_1, \dots, R_n \theta_{\mathfrak{A}}$$

from the CA axioms. But from where do we obtain $\mathfrak{B}$? We can assume without loss of generality that $\mathfrak{B}$ has empty vocabulary. If we use relativization to a new predicate R we only need a second order characterizable structure $\mathfrak{B}$ of the empty vocabulary which is as large as $\mathfrak{A}$. Then we could give evidence for (8) by proving

$$\theta_{\mathfrak{B}} \to \exists R\,(\exists R_1, \dots, R_n \theta_{\mathfrak{A}})^{(R)}$$

from the CA axioms. What we need, in short, is a *large domain assumption $\theta_{\mathfrak{B}}$*. Just as with the *large cardinal assumptions* in set theory, the large domain assumptions cannot be proved from the CA axioms of second order logic.

Trying to manage without large domain assumptions amounts to talking about a bigger universe inside a smaller universe. What we seem to need are logical means to refer to the "outside" of a structure. Such logical means are the heart of the extension of second order logic called *sort logic* (see [11]). Alternatively, in higher order logics we can build an "outside" from the higher type objects, but we would need higher and higher types with no end. This problem does not arise in set theory because the axioms are designed for the very purpose of producing sets of higher and higher cardinalities and if larger sets are needed than provided by the axioms, large cardinal assumptions (inaccessible cardinals, measurable cardinals, etc) are added to the axioms.

Most of mathematics can be done in set theory without large cardinals, but in second order logic one seems to need large domain assumptions in order to give evidence for the construction of any infinite structure: we cannot give evidence for a characterization of the natural numbers without assuming the existence of a second order characterizable infinite structure; we cannot give evidence for a characterization of the real numbers without assuming the existence of a structure of at least continuum size; we cannot give evidence for the existence of a structure built from all mappings from reals to reals unless we assume the existence of a structure with that many elements. As we proceed we need more and more assumptions about the largeness of the universe. This phenomenon is familiar from Gödel's Incompleteness Theorem and is remedied in set theory by a combination of the axioms of infinity, power-set and replacement.

With this method we can give evidence for the second order characterizability of a structure if we already have a second order characterizable structure of the same or bigger cardinality, and we can give evidence for a sentence ϕ being a second order characterization of some structure if we already have evidence for the second order characterizability of another structure of the same or bigger cardinality. The idea comes to mind to assume the existence of as large second order characterizable structures as we may ever want. We might then ask what would be a second order characterization of a structure which is sufficiently large? A natural candidate is Zermelo's V_κ, where κ is the first strongly inaccessible. We have thus created for ourselves a copy of ZFC set theory.

We conclude that the second order view suffers from a weakness—the need to keep making new large domain assumptions one after another—that the set theory view solves succinctly by assuming that the universe is as large as possible. It is tempting to adopt the set theory view as the primary view and then formulate the second order view as a secondary view which appeals to set theory for the existence of structures, but this ruins the claim of the second order view as giving an autonomous explanation of mathematics.

§7. Conculsion. Second (and higher) order logic is comparable to set theory on the level of Π_2-formulas. Individual second order characterizable structures are organized into a hierarchy on the lower level of Δ_2-definability.

Second (and higher) order logic has its foundation, not in second order logic itself but in set theory, because the truth of Π_2-formulas can be defined in set theory. Respectively, set theory in full generality does not have a truth definition in set theory itself, but one can organize it into the hierarchy of

Π_n-definability, $n \in \mathbb{N}$, each level being definable on the next higher level. As was mentioned, higher order logic corresponds to the case $n \leq 2$.

Compared to each other, second order logic and set theory are not in total synchrony because of the different nature of the formalizations, namely what is the language and what is taken as an axiom. When we consider each as a description of what it is that mathematicians do, the differences of formalization all but disappear.

In particular, it is misleading to say that second order logic captures mathematical structures up to isomorphism while set theory is marred by the weakness of expressive power of first order logic manifested in the existence of non-standard models of basic foundational theories. Rather, both second order logic and set theory either manifest a high degree of categoricity or alternatively permit a plethora of non-standard models, depending on the perspective. Categoricity results when models are assumed to be "full", which in the case of formalized second order logic means that every subset and relation of the domain is in the range of the second order variables, a criterion that can be conveniently formulated in either informal set theory or in informal second order logic.

In the case of set theory the "fullness" of the models of the formalized set theory means that every subset of every set is in the range of first order variables, a criterion that can be conveniently formulated either in informal set theory itself or in informal second order logic. Much confusion arises if formal and informal are mixed up. If they are kept separate, the informal being what mathematicians are doing and the formal being our attempt to make the informal intelligible, second order logic and set theory fare rather equally.

Set theory gives a smoother approach but one has to be prepared to allow sets to be built up in an uninhibited way combining power-set, unions and applications of replacement. Second order logic settles for the more modest approach involving only one (or a couple of) application(s) of the power-set operation at a time, but the price is that one needs the concept of universal truth which turns out to encode a tremendous amount of set theory, and one cannot prove from the (comprehension) axioms the existence of second order characterizable structures without *large domain assumptions*.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
P.O. BOX 68
FI-00014 UNIVERSITY OF HELSINKI, FINLAND
and
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION
UNIVERSITY OF AMSTERDAM
P.O. BOX 94242
1090 GE AMSTERDAM, THE NETHERLANDS
E-mail: jouko.vaananen@helsinki.fi