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IN MEMORIAM: GREGORY HJORTH
1963–2011

Greg Hjorth suddenly and unexpectedly passed away on January 13, 2011 in Melbourne, at the age of 47, due to a heart attack. He was a remarkable person, a chess prodigy who competed internationally at a high level until his early 20s, then devoted himself to the study of philosophy and mathematics and went on to become a leading figure in the field of mathematical logic and its applications.

Hjorth was born in Melbourne on June 14, 1963, the son of Dr. Robert Hjorth, a neurologist, and Noela Hjorth, an artist. His sister Dr. Larissa Hjorth is a lecturer in the School of Media and Communication at the Royal Melbourne Institute of Technology. He went to school in London, while his father was working there, and then in Melbourne, where he attended Preshil School (roughly grades 4–8) and then St. Leonards College (roughly grades 9–12).

In his early teens, Hjorth became (in his own words) "madly obsessed with chess" and went on to compete in Australia and internationally over the next decade or so. At age 16 he got 2nd place in the 1979–80 Australian Championship and started his international career. In 1980 he played against Gary Kasparov, the later world chess champion (also aged 17 at that time), in Dortmund, in what was described as a hard fought game, which he eventually lost.

In 1982, 1985 and 1987 he won the annual Doeberl Cup, a major Australian Chess Tournament, and in 1983 he tied for first in the British Commonwealth Chess Championship. For a while he was ranked number three in Australia, with only two professional chess players ahead of him. Greg represented Australia in three World Chess Olympiads in the 1980's and gained the International Master title in 1984. It is widely believed that if he had continued he would have inevitably achieved the title of International Grand Master. As far as I can tell, his highest FIDE rating was 2440, which would be among the highest ever achieved, at least in recent decades, by a professional research mathematician.

His friend and chess colleague Guy West described him as follows:

> "Greg was a chess player with a deep appreciation of the artistic side of the game and he played games of great beauty and subtlety."

One can also say that this characterizes much of his mathematics.

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Hjorth is supposed to have said about chess players "If you are not in the top 100 by 21, get out", so apparently following his own advice he started shifting away from chess and enrolled at the University of Melbourne, where he studied Philosophy and Mathematics and obtained his BA (Hons) degree in 1988 with combined honours in these fields, receiving subject prizes not only in these subjects but also in classics.

In 1988 he started graduate school at the University of California at Berkeley, in the group of Logic and Methodology of Science. His Ph.D. thesis supervisor was Hugh Woodin, who describes him as a "remarkable student, more of a colleague than a student." In 1993 he received his Ph.D. with a thesis for which he was awarded the first Sacks Prize from the Association for Symbolic Logic (ASL).

From 1993 to 1995 Hjorth was a Bateman Research Instructor at the California Institute of Technology (Caltech) and then was appointed Assistant Professor of Mathematics in 1995 at UCLA, where he advanced to the rank of Associate Professor in 1997 and Full Professor in 2001. Since 2006, he was also appointed to an ARC Australian Professorial Fellowship at the Department of Mathematics and Statistics of the University of Melbourne.

Hjorth had 8 Ph.D. students at UCLA, in chronological order: Su Gao, Lyman Chaffee, Michael Oliver, Asger Törnquist, John Kittrell, Ioannis Souldatos, Alex Thompson and Inessa Epstein, whose thesis also won the Sacks Prize in 2008. He was moreover a co-advisor of a Ph.D. student at Caltech, Xuhua Li, and a Ph.D. student at the University of Münster, Philipp Schlicht, and also had several students working with him in Melbourne at the time of his death. Hjorth was an excellent advisor who deeply cared for and was devoted to his students. But beyond his own students, his work and his always exciting lectures at seminars and conferences around the world, as well as his private conversations, have inspired and influenced many other young logicians over the years.

Greg Hjorth made a series of far reaching contributions to mathematical logic and its applications to other areas of mathematics, especially various aspects of dynamical systems. His work, part of which I will describe below, included the development of entirely new theories as well as solutions of very hard problems, and was characterized with a striking originality and technical wizardry. It has been recognized by many honors, including a Sloan Foundation Fellowship in 1998, an invitation to the International Congress of Mathematicians in 1998, the Karp Prize of the ASL in 2003 and, just last year, the invitation to deliver one of the major lecture series in logic, the Alfred Tarski Lectures at UC Berkeley.

It is impossible, in a limited space, to really do justice to his mathematical work, which spans a very diverse range of subjects. I can only hope to give a glimpse of his remarkable achievements by highlighting a few of his main results.

Hjorth's early work was largely concerned with problems in the interface of descriptive set theory, large cardinals and inner model theory, in the tradition that developed from the foundational work of what is often referred to as the "Cabal School of set theory" in the 1970's and 1980's. Hjorth introduced many new ideas in this intensely studied field and in the mid 1990's he achieved breakthroughs in several difficult problems that had remained open for two decades or so.

One such result was the following: In the constructible from the reals universe $L(\mathbb{R})$, assuming it satisfies the Axiom of Determinacy, there is no sequence of $\aleph_2$ distinct $\mathbf{\Sigma}^1_2$ sets. This completely settled a long standing conjecture on which various partial results had been obtained by among others Jackson and Martin.

Another impressive result is the following: In the early 1970's the problem of the invariance of the inner model $L[T^n]$ was proposed by Moschovakis. Working within Projective Determinacy, T^n is the tree of a Π^1_n-scale on a complete Π^1_n-set of reals, when n is odd, and the tree of Δ^1_{n+1}-scale on a complete Π^1_n-set of reals, when n is even. The case $n = 1$ was quickly resolved and the first odd case $n = 3$ became one of the Victoria Delfino Problems. The odd case was finally settled affirmatively by Becker and Kechris in the early 1980's but no progress was made in the even case for many years. A breakthrough was finally achieved in 1996, when Hjorth established the first even case $n = 2$. The higher even case $n \geq 4$ is still open.

Hjorth always had a strong interest in set theoretic aspects of model theory. Here are two of his most interesting results in that area. Shelah proved in 1978 that for each countable cardinal κ, there is a complete countable theory with exactly κ *minimal models* and asked whether, assuming the failure of the Continuum Hypothesis (CH), it was possible to have exactly $\aleph_1$ many minimal models. In 1995 Hjorth found a beautiful answer to this problem by showing that such a theory fails to exist iff $\aleph_1$ is inaccessible in every constructible from a real model.

Julia Knight showed in 1977 that there is countable model which characterizes $\aleph_1$ (i.e., whose Scott sentence has a model of that cardinality but no higher one) and raised the question of whether this is true for each $\aleph_n$, n finite. Laskowski and Shelah provided a positive answer for $n = 2$. In 2002 Hjorth solved completely Knight's problem by proving something much stronger, namely that it is actually true for each $\aleph_\alpha, \alpha < \omega_1$, which is the optimal result in ZFC.

The *Vaught Conjecture* (VC) and its generalization, the *Topological Vaught Conjecture* (TVC), are famous open problems in mathematical logic. The original form, due to Vaught, is exactly 50 years old and asserts that any first-order theory has either countably many or else perfectly many countable models up to isomorphism (i.e., it is a strong form of the Continuum Hypothesis for the cardinality of the set of countable models of a theory).

The TVC, formulated by D. E. Miller in the 1970's, is a natural extension of this conjecture to a topological dynamics context: if a Polish group G acts continuously on a Polish space X, then any invariant Borel subset of X contains either countably many or perfectly-many orbits. Hjorth made two crucial contributions to this problem.

The TVC is true for locally compact Polish groups by classical theorems and in the 1970's Sami showed that it is also true for abelian groups, but nothing more was known beyond that for the next two decades until Hjorth and Solecki proved around 1995 that the TVC holds for all nilpotent Polish groups and also those admitting a 2-sided invariant metric. A couple of years later Becker beautifully completed this line of work showing that the TVC holds for all Polish groups admitting a left-invariant complete metric.

The TVC was known to fail if one replaces invariant Borel by invariant analytic sets, even in a classical model theory context, which corresponds essentially to actions of the infinite symmetric group S_∞. It follows also that TVC for analytic sets also fails for any group G that can be mapped homomorphically and continuously onto S_∞, i.e., any G which has S_∞ as a factor. Remarkably, Hjorth showed in 1998 that a Polish group G fails to satisfy the TVC for analytic sets *exactly* when S_∞ is a factor of G. This has also the following striking consequence: if the VC fails (which is widely believed but not proved yet), then the groups that satisfy the TVC are *exactly* the ones that do not have S_∞ as a factor.

The theory of Borel and analytic equivalence relations is a very active area of research in set theory today and serves as the foundation for the development of a theory of complexity of classification problems in mathematics. The global structure of Borel equivalence relations is guided by a series of dichotomies, the earliest instances of which are the Silver Dichotomy (1980) and the general Glimm–Effros Dichotomy (Harrington–Kechris–Louveau, 1990). Hjorth and his collaborators have substantially advanced our understanding of this structure by establishing further dichotomy results that serve as the background for some sweeping conjectures that delineate the overall structure.

I will next describe some further contributions of Hjorth in this area. Countable Borel equivalence relations, i.e., those generated by Borel actions of countable groups, play a central role in this area. An important subclass are the so-called *treeable ones*, i.e., those for which the equivalence classes are endowed, in a uniform way, with the structure of an acyclic, connected graph. Up to Borel bi-reducibility, these can be also viewed as the ones induced by a free Borel action of a free group. It was known that there are at least two non-trivial examples of such equivalence relations, the *hyperfinite* and the *universal* one (again up to bi-reducibility), and it was a major problem to show that others exist. This was finally solved by Hjorth, first by proving a far reaching orthogonality theorem, concerning Bernoulli vs profinite actions, which implied that there are intermediate ones, which are both non-universal and non-hyperfinite, and finally in one of his latest (still unpublished) papers by proving that there are in fact uncountably many incomparable ones.

The simplest kinds of classification problems in this context are the so-called *concretely classifiable* ones. In this case one can classify the objects under discussion, under some notion of equivalence, by invariants that are fairly concrete, usually taking the form of real numbers, sequences of integers or more generally elements of some Polish space. There are classical techniques for showing that such classifications are impossible. However, once one goes beyond the concretely classifiable case things become much more complicated. The next level is what is called *classification by countable structures*. Here the invariants are isomorphism classes of countable structures. Such classifications occur often in mathematics, e.g., in ergodic theory, topological dynamics, operator algebras, etc.

No general method for showing non-classifiability in this wider sense existed until Hjorth developed his *theory of turbulence* that provides a powerful tool for dealing with such problems. Turbulence is a topological dynamics property of a continuous action of a Polish group on a Polish space that expresses the complexity of the "local orbits" of the action. The remarkable result of the theory is that turbulence prohibits classification by countable structures and in fact, under certain circumstances, a problem fails to admit classification by countable structures iff some turbulent action can be embedded in it (in a precise sense). This reduces the problem of non-classification to the discovery of appropriate turbulent actions.

This theory has been applied widely over the last few years by Hjorth and many others to show strong non-classifiability results in a variety of subjects, in topology, ergodic theory, operator algebras and group representations. For instance, Hjorth used these ideas to show that, as opposed to the classical classification theorems of Halmos-von Neumann and Ornstein, no classification by countable structures is possible for isomorphism of arbitrary ergodic, measure-preserving transformations. This line of research was continued with further remarkable results by Foreman, Rudolph and Weiss.

The classification of torsion-free abelian groups of finite rank, i.e., additive subgroups of $(\mathbb{Q}^n, +)$, is a classical problem in group theory. The case $n = 1$ was settled affirmatively by Baer in the 1930's but no substantial progress had been made since then towards the classification of rank 2 or higher groups.

Looking at this problem from the point of view of the preceding theory of complexity of classification, Hjorth made a major breakthrough by showing that the rank 2 case was, in a precise sense, strictly more complicated than the rank 1 case, using in a very ingenious way methods of ergodic theory. This was eventually completed by the beautiful work of Simon Thomas, who showed that for each n the complexity of the classification problem for rank n

groups is strictly less than that for rank $n + 1$ groups. Equivalently this can be stated as a rigidity result: the cardinality of the isomorphism classes of rank n groups, understood in the definable sense, determines the rank n. These results can be interpreted as meaning that no reasonable classification for rank 2 or higher groups exists.

Hjorth also made a significant contribution to the study of (countably) infinite rank, torsion-free abelian groups by showing that the isomorphism relation on such groups is not Borel. This was extended by Downey–Montalban to show that it is actually complete analytic. It is still open whether it has the maximum complexity among isomorphism problems for countable structures.

In recent years, Hjorth has also made several important contributions to various aspects of ergodic theory.

Orbit equivalence is a currently very active area of research in ergodic theory and its relation with the theory of operator algebras. Two free, ergodic, measure preserving actions (just actions in the sequel) of countable (infinite, discrete) groups are called *orbit equivalent* if they have essentially the same orbit spaces, more precisely the equivalence relations induced by these two actions are (measure theoretically) isomorphic. A famous theorem of Dye and Ornstein–Weiss asserts that any two actions of amenable groups are orbit equivalent. In the early 1980's Klaus Schmidt showed that any non-amenable group *without* Kazhdan's property (T) has at least two non-orbit equivalent actions and raised the question of whether this also holds for all non-amenable groups.

This was finally answered by Hjorth in 2003, when he showed that every non-amenable group that *does* have property (T) has actually continuum many distinct actions up to orbit equivalence. This gives the beautiful characterization: a group is amenable iff it has a unique action up to orbit equivalence. The question of whether any non-amenable group has indeed continuum many non-orbit equivalent actions was studied intensively over the next few years, with important breakthroughs by Gaboriau–Popa, who proved this for non-abelian free groups, and then by Ioana for all groups containing non-abelian free subgroups and finally culminated in the thesis of Epstein who showed that every non-amenable group has indeed continuum many such actions.

The theory of costs, originated by Levitt and developed extensively by Gaboriau, assigns to each action an important invariant (depending only on the equivalence relation it generates), called its *cost*, that can be used to distinguish such actions up to orbit equivalence. A fundamental result of Gaboriau's theory is that the cost of any action of the free group with n generators is exactly equal to n. This implies immediately that no two actions of free groups with different numbers of generators can be orbit equivalent.

Hjorth proved a deep converse to Gaboriau's theorem: any (countable, ergodic, measure preserving) equivalence relation which is treeable and has cost n can be induced by an action of the free group with n generators. Among other things this provides a powerful tool for generating actions of the free group. In that form it was used in a recent result of Gaboriau–Lyons, which established the so-called *measurable version of the von Neumann Conjecture*: every non-amenable group has an action the orbits of which contain the orbits of an action of the free group with 2 generators.

Greg Hjorth has made a lasting impact in mathematical logic and its applications. His influence on his colleagues and students, both at a professional and personal level, has been immense and the outpouring of sympathy that I have seen over the last few weeks from many mathematicians across the globe attest to the great loss that we all feel at the passing of such a brilliant colleague and a most gentle, generous and caring man.

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