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OPEN QUESTIONS IN REVERSE MATHEMATICS

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Abstract. We present a list of open questions in reverse mathematics, including some relevant background information for each question. We also mention some of the areas of reverse mathematics that are starting to be developed and where interesting open question may be found.

§1. Introduction. The objective of this paper is to provide a source of open questions in reverse mathematics and to point to some of the areas where there could be interesting developments. The questions I discuss are mostly known and come from the literature. My objective was to compile them in one place and discuss them in the context of related work. The list is definitely not comprehensive, and my choice of questions and topics is undoubtedly affected by my personal taste and my own research. The idea to write this paper came about after the last two workshops in reverse mathematics: the one in Banff in December 2008, organized by Cholak, Csima, Lempp, Lerman, Shore, and Slaman (`www.birs.ca/events/2008/5-day-workshops/08w5019`), and the one in Chicago in November 2009, organized by Dzhafarov and Hirschfeldt (`www.math.uchicago.edu/~drh/workshop.html`). Each of these workshops had a session on open questions where people suggested problems they liked. Then Shore and Dzhafarov compiled the respective lists of questions into files that are now available either online or by request. Many of the questions posed here come from those listings. Another paper on open questions in reverse mathematics was written ten years ago by Friedman and Simpson [FS00]. There are still some unanswered questions from that paper, and I will cite a few here.

This paper is not intended to describe the subject or explain its motivations. For the motivations as to why we do reverse mathematics and the types of results we get, I highly recommend the recent articles by Simpson

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[Sim10] and Shore [Sho10]. These articles are written for a general logic audience, and each motivates the subject from its own viewpoint. For general background and extensive results in reverse mathematics, the standard reference is Simpson's book [Sim09].

The objective of reverse mathematics, as described by Friedman and Simpson, is to classify the theorems of mathematics according to the set existence axioms needed for their proofs, or, as some of us also view it, according to the types of constructions needed in their proofs. In the last couple decades, there has been a lot of work in this area, classifying many theorems from all over mathematics. Many theorems are still waiting to be analyzed, and there are still some areas of mathematics that have barely been looked at by reverse mathematicians. There is still a lot of work to be done in this direction. The work of this type that has been done has been very fruitful; for instance, it has led us to the conclusion that most theorems in mathematics are equivalent to one of the *big five* systems over RCA_0. (RCA_0 referes to the Recursive Comprehension Axiom scheme, and the rest of the big five are Weak König's Lemma WKL_0, the Arithmetic Comprehension Axiom scheme ACA_0, Arithmetic Transfinite Recursion ATR_0, and the Π^1_1-Comprehension Axiom scheme Π^1_1-CA_0.) Lately, researchers have been more interested in finding theorems which are not equivalent to any of the big five systems. Even though we now we know of many theorems that are not equivalent to any of the big five systems, we would still claim that the great majority of the theorems from classical mathematics are equivalent to one of the big five. This phenomenon is still quite striking. Though we have some sense of why this phenomenon occurs, we really do not have a clear explanation for it, let alone a strictly logical or mathematical reason for it. The way I view it, gaining a greater understanding of this phenomenon is currently one of the driving questions behind reverse mathematics.

To study the big five phenomenon, one distinction that I think is worth making is the one between robust systems and non-robust systems. A system is *robust* if it is equivalent to small perturbations of itself. This is not a precise notion yet, but we can still recognize some robust systems. All the big five systems are very robust. For example, most theorems about ordinals, stated in different possible ways, are all equivalent to each other and to ATR_0. Apart from those systems, weak weak König's Lemma (WWKL_0) is also robust, and we know no more than one or two other systems that may be robust.

Another important question is whether the following conjecture holds. We know many examples of theorems from mathematics which are incomparable in strength over RCA_0. However, if we look at their consistency strength, they all seem to be linearly ordered, or at least we have not been able to prove the existence of a counterexample. This also occurs if we look at the relation of interpretability between systems. (Friedman showed that one theory is interpretable in another if and only if its consistency can be proved from the consistency of the latter theory in a somewhat effective way; see [Smo85, §5].) Friedman and Simpson [FS00] proposed the following conjecture, which they call the *interpretability conjecture*: Let X, Y be any finite sets of actual mathematical theorems in the published literature, which can be stated in second-order arithmetic. Then either RCA_0+X is interpretable in RCA_0+Y, or RCA_0+Y is interpretable in RCA_0+X.

In this paper, when I refer to the *strength* of a theorem, I mean proof-theoretic strength as used in the reverse mathematics literature (i.e., measured by comparing the sets of implications of the theorem), and not consistency strength, which is more commonly used in proof theory or set theory. Also, when I ask about implications or equivalences between statements, I mean over the base system RCA_0.

§2. Ramsey's theorem.

Combinatorics seems to be the area of mathematics where we have found the greatest number of theorems escaping the big five. This is probably why there are so many open questions regarding the strengths of theorems from combinatorics. Ramsey-like theorems have particularly attracted the attention of reverse mathematicians.

2.1. Ramsey's theorem for pairs. Both Ramsey's theorem and König's lemma are important combinatorial tools used all over mathematics. Weak König's lemma, WKL_0, has turned out to be equivalent to many theorems from various branches of mathematics. Ramsey's theorem for pairs, however, has not. While it is true that compactness arguments (i.e., arguments using WKL_0) are much more common than combinatorial arguments using Ramsey's theorem for pairs (denoted by RT^2_2), the number of theorems that have been proved equivalent to RT^2_2 seems disproportionately small. However, a good many theorems are known to be implied by RT^2_2, or to be very close to it. The main difference, I believe, seems to be that WKL_0 is a very robust system, while RT^2_2 is not.

Let me start by stating the classical Ramsey theorem.

RT^n_k: Every coloring of the n-tuples of natural numbers with k colors has an infinite homogeneous set.

RT^n: For every k, RT^n_k.

For $n \geq 3$ and $k \geq 2$, we know that RT^n_k and RT^n are both equivalent to ACA_0 (which follows from Jockusch [Joc72]). It is for $n = 2$ that the open questions arise. We know that WKL_0 cannot imply RT^2_2 (because, using the low-basis theorem [JS72], we can build an ω-model of WKL_0 that contains only low sets, but by results of Jockusch [Joc72], every ω-model of RT^2_2 contains some non-Δ^0_2 set). It is unknown whether the converse holds. This is one of the most well-known open questions in the field.

QUESTION 1. Does $\mathsf{RCA}_0+\mathsf{RT}^2_2$ imply WKL_0?

Whether RT^2 implies WKL_0 is just as interesting and also unknown. Even if RT^2_2 turns out to be incomparable with WKL_0, we already know that, in terms of first-order consequences, RT^2_2 lies strictly between WKL_0 and ACA_0. Let $(\varphi)^1$ denote the set of first-order consequences of $\mathsf{RCA}_0+\varphi$. Using results of Harrington; Cholak, Jockusch, and Slaman [CJS01]; and Paris and Kirby [KP77] we know that

$$\begin{aligned}(\mathsf{RCA}_0)^1 = (\mathsf{WKL}_0)^1 \subsetneq (\mathsf{B}\Sigma^0_2)^1 \subseteq (\mathsf{RT}^2_2)^1 \subseteq (\mathsf{I}\Sigma^0_2)^1 \subsetneq \\ (\mathsf{B}\Sigma^0_3)^1 \subseteq (\mathsf{RT}^2)^1 \subseteq (\mathsf{I}\Sigma^0_3)^1 \subsetneq \mathsf{PA} = (\mathsf{ACA}_0)^1.\end{aligned}$$

Here, $\mathsf{B}\Sigma^0_k$ refers to the bounding principle for Σ^0_k formulas, and $\mathsf{I}\Sigma^0_k$ to the induction principle for Σ^0_k formulas. We do know that $(\mathsf{B}\Sigma^0_2)^1 \subsetneq (\mathsf{I}\Sigma^0_2)^1$ and that $(\mathsf{B}\Sigma^0_3)^1 \subsetneq (\mathsf{I}\Sigma^0_3)^1$ (Kirby and Paris [KP77]). However, it is unknown where $(\mathsf{RT}^2_2)^1$ lies between $(\mathsf{B}\Sigma^0_2)^1$ and $(\mathsf{I}\Sigma^0_2)^1$, and where $(\mathsf{RT}^2)^1$ lies between $(\mathsf{B}\Sigma^0_3)^1$ and $(\mathsf{I}\Sigma^0_3)^1$, even if we restrict ourselves to the set of Π^0_2 consequences. We also do not know what the consistency strength of RT^2_2 is. All of these are very interesting questions. Let me highlight the following related open questions.

QUESTION 2. Does RT^2_2 prove that the Ackermann function is total? Does RT^2_2 prove that ω^ω is well-ordered? (ω^ω is presented as $\mathbb{N}^{<\mathbb{N}}$ where the strings are ordered first by length and then lexicographically.)

We note that $\mathsf{I}\Sigma^0_2$ proves both that the Ackermann function is total and that ω^ω is well-ordered, but neither is implied by $\mathsf{B}\Sigma^0_2$. Thus both statements are implied by RT^2 and the questions are open only for RT^2_2.

2.2. Statements below RT^2_2. There has been a lot of work on statements surrounding RT^2_2. There are many statements that are very close to RT^2_2 but that are not equivalent to it, or not known to be equivalent to it. This is why we say that RT^2_2 is a non-robust statement.

2.2.1. *Stable Ramsey theorem.* One common proof of RT^2_2 in ACA_0 has two steps: First transform the coloring into a stable coloring, and then deal with the stable coloring. Each of these two steps has an associated statement of second-order arithmetic, providing a splitting of RT^2_2 as the conjunction of two apparently simpler statements. A coloring $f\colon [\mathbb{N}]^2 \to \{0,1\}$ is *stable* if for every $a \in \mathbb{N}$ there exists a color $i \in \{0,1\}$ such that $f(a,b) = i$ for all sufficiently large b.

SRT^2_2: Every stable 2-coloring of pairs of natural numbers has an infinite homogeneous set.

COH: For every sequence of sets $\{A_0, A_1, \dots\}$, there exists an infinite set B such that, for every i, either $B \setminus A_i$ or $B \cap A_i$ is finite.

We know that RT^2_2 is equivalent to $\mathsf{SRT}^2_2+\mathsf{COH}$: For the right-to-left direction, given a 2-coloring $f\colon [\mathbb{N}]^2 \to 2$, apply COH to $A_i = \{b \in \mathbb{N} \mid f(i,b) = 1\}$ to get a stable 2-coloring $f \restriction [B]^2$, and then apply SRT^2_2 [CJS01]. That RT^2_2 implies SRT^2_2 is trivial, and that it implies COH requires a bit of work (see [CJS01, CJS09] and [Mil04]). It was also shown in [CJS01] that COH does not imply RT^2_2. However, the following question, which has been tried by many people, remains open.

QUESTION 3. Does $\mathsf{RCA}_0+\mathsf{SRT}^2_2$ imply RT^2_2?

One of the original motivations for this question is that it would give an example of a natural statement that can be non-trivially split into two somewhat natural statements. Hirschfeldt and Shore [HS07] have exhibited examples of this behavior below RT^2_2. They proved that the statements below are each properly split into a stable version and COH.

ADS: Every infinite linear ordering has either an ascending or a descending sequence.

CAC: Every infinite partial ordering contains an infinite set that is either a chain or an anti-chain.

A few other statements are considered in [HS07], and many questions are left open. For example, they show that CAC implies ADS but leave the reversal open.

QUESTION 4. Does ADS imply CAC?

Whether stable-ADS implies stable-CAC is also open (see [HS07] for definitions). On the cohesive side, it is not known whether cohesive-ADS implies COH. (Cohesive-ADS says that every infinite linear ordering has a subset of type either ω, ω^* or $\omega+\omega^*$.)

2.2.2. *The tournament statement.* A *tournament* is a binary relation T on a set P such that for all $a, b \in P$ with $a \neq b$, exactly one of $T(a,b)$ and $T(b,a)$ holds. Erdös and Moser [EM64] considered a finitary version of the following statement:

EM: Every infinite tournament has an infinite, transitive subtournament.

It was observed by Bovykin and Weiermann [BW] that RT^2_2 is equivalent to $\mathsf{ADS}+\mathsf{EM}$ over RCA_0. We know that ADS follows from CAC, which is strictly weaker than RT^2_2 [HS07].

QUESTION 5. Is EM strictly weaker than RT^2_2?

2.2.3. *Free-set and thin-set theorems.* An *infinite coloring of a set S of exponent n* is just a function $f\colon [S]^n \to \mathbb{N}$. A subset $A \subseteq S$ is said to be *free for f* if for every $\{x_1,\dots,x_n\} \in [A]^n$, either $f(x_1,\dots,x_n) \notin A$ or $f(x_1,\dots,x_n) \in \{x_1,\dots,x_n\}$. A subset $T \subseteq S$ is said to be *thin for f* if $f([T]^n) \subsetneq \mathbb{N}$. The following theorems were first considered by Friedman.

$\mathsf{FS}(n)$: Every infinite coloring of $\mathbb{N}$ of exponent n admits an infinite free set.

$\mathsf{TS}(n)$: Every infinite coloring of $\mathbb{N}$ of exponent n admits an infinite thin set.

Friedman and Simpson [FS00] asked about the strengths of these theorems, and then Cholak, Giusto, Hirst, and Jockusch [CGHJ05] studied them in more detail. A whole list of open questions can be found in [CGHJ05, §7]; I will cite just one.

QUESTION 6. Does either $\mathsf{FS}(2)$ or $\mathsf{TS}(2)$ imply RT^2_2?

Note that we do not know if RT^n_2 implies $\mathsf{FS}(n)$, which does imply $\mathsf{TS}(n)$ over RCA_0.

At the Chicago workshop, Joe Miller proposed the study the statements $\mathsf{RT}^n_{k,j}$ which say that for every k-coloring of n-tuples there exists an infinite set that uses only j colors. These statements have already been analyzed by combinatorists and set theorist, both in the countable and uncountable case, but not from a reverse mathematics viewpoint. Lempp, Miller, and Ng observed that $\mathsf{RT}^3_{3,2}$ implies RT^2_2, but not much else is known about these statements.

2.2.4. *Tree Ramsey theorem.* Let $2^{<\mathbb{N}}$ be the full binary tree, and let $[2^{<\mathbb{N}}]^n$ denote the set of n-tuples of comparable nodes of $2^{<\mathbb{N}}$.

TT^n_k: For every coloring of $[2^{<\mathbb{N}}]^n$ with k colors, there exists $S \subseteq 2^{<\mathbb{N}}$ such that S is order isomorphic to $2^{<\mathbb{N}}$ and $[S]^n$ is monochromatic.

The tree Ramsey theorem was first analyzed by McNicholl [McN95] and Chubb, Hirst, and McNicholl [CHM09]. Further work was done by Corduan, Groszek, and Mileti [CGM]. It is easy to see that TT^n_k implies RT^n_k, and for $n \geq 3$ it can be shown that they are equivalent. For exponent 2, this question is still open:

QUESTION 7. Is TT^2_2 strictly stronger than RT^2_2?

For $n = 1$, however, we know that $\forall k \mathsf{TT}^1_k$ is strictly stronger than $\forall k \mathsf{RT}^1_k$ [CGM]. It was already known [CHM09] that $\forall k \mathsf{TT}^1_k$ follows from $\mathsf{RCA}_0 + \mathsf{I}\Sigma^0_2$, but whether it implies $\mathsf{I}\Sigma^0_2$ is still open.

2.2.5. *Polarized Ramsey's theorem.* With the intention of finding statements that could help separate RT^2_2 from SRT^2_2, Dzhafarov and Hirst [DH09] introduced a few additional versions of Ramsey's theorem. We mention only one, but the reader should consult [DH09, DLH10, CLY10] for other statements and questions.

IPT^n_k: For every coloring of $[\mathbb{N}]^n$ with k colors, there exist infinite sets $H_1, \dots, H_n$ such that $\{(x_1, \dots, x_n) \in H_1 \times \cdots \times H_n \mid x_1 < x_2 < \cdots < x_n\}$ is monochromatic.

They showed that IPT^2_2 follows from RT^2_2 and implies SRT^2_2.

QUESTION 8. Is IPT^2_2 equivalent to RT^2_2, to SRT^2_2, to both, or to neither?

2.3. Hindman's theorem. Another well-known open question concerns the strength of Hindman's theorem.

HT: For every coloring of $\mathbb{N}$ with finitely many colors, there is an infinite set A such that the set of numbers which can be written as a sum of distinct elements of A is monochromatic.

Blass, Hirst, and Simpson [BHS87] showed that HT can be proved in ACA_0^+ and that it implies ACA_0 (where ACA_0^+ is $\mathsf{RCA}_0+\forall X(X^{(\omega)}$ exists)).

QUESTION 9. Is HT equivalent to ACA_0^+, or to ACA_0, or does it lie strictly between them?

There are various proofs of Hindman's theorem. One of them, due to Glazer, uses ultrafilters on the natural numbers; Hirst [Hir04] showed that HT is equivalent to the statement about ultrafilters (restricted to countable Boolean algebras) used in Glazer's proof, and to other combinatorial statements. Other open questions regarding HT can be found in [Bla05]; see also section 3.5 below.

2.4. Dual Ramsey theorem. For $k \leq \omega$, let $(\omega)^k$ be the set of partitions of $\mathbb{N}$ into k pieces. If $X \in (\omega)^\omega$, let $(X)^k$ be the set of $Y \in (\omega)^k$ which are coarser than X, in the sense that every piece in Y is a union of pieces of X. Carlson and Simpson [CS84] proved the following theorem, which we call the *dual Ramsey theorem*.

DRT^k: If $(\omega)^k$ is colored with finitely many colors in a Borel way, then there exists $X \in (\omega)^\omega$ such that $(X)^k$ is monochromatic.

Slaman [Sla] showed that DRT^k can be proved in Π^1_1-CA_0. J. Miller and Solomon [MS04] showed that it is not provable in WKL_0 if $k \geq 3$ and that it implies ACA_0 for $k \geq 4$.

QUESTION 10. What else can we say about the strength of DRT^k?

Let me now describe the main combinatorial lemma in [CS84]. An *infinite variable word* on a finite alphabet A is an ω-sequence W of elements of $A \cup \{x_i \mid i \in \mathbb{N}\}$ in which all variables occur and all occurrences of x_i come before any occurrence of x_{i+1}. Given $\bar{a} = a_0a_1 \dots a_{k-1} \in A^{<\omega}$, we let $W(\bar{a})$ denote the finite A-string obtained by replacing x_i with a_i in W and then truncating the result just before the first occurrence of x_k.

CS: If $A^{<\mathbb{N}}$ is colored with finitely many colors, there exists an infinite variable word W such that $\{W(\bar{a})\colon \bar{a} \in A^{<\mathbb{N}}\}$ is monochromatic.

Miller and Solomon [MS04] proved that CS is not probable in WKL_0.

QUESTION 11. What is the strength of CS?

Another statement that would be very interesting to analyze is a theorem of Carlson, which provides a natural generalization of a large part of qualitative Ramsey theory, including RT, DRT, HT, the Galvin–Prikry theorem, etc., all in one theorem. This theorem says that $\mathcal{S}(L, 1)$ is a Ramsey space, but we refer the reader to [Car88] for definitions and background.

§3. Algebra, analysis, and topology.

Friedman and Simpson [FS00] assert that, even then in the year 2000, most of the theorems that are part of the core curriculum in undergraduate or graduate programs in mathematics had already been analyzed and proved equivalent to one of the big five systems (and a few to WWKL_0). If we include theorems that are part of those core subjects but may be a bit beyond the core curriculum, we start finding theorems which have not been fully analyzed or have not yet been proved equivalent to any of the big five systems. I will mention a few of these examples in this section.

3.1. Algebra. There has been a lot of work on the reverse mathematics of algebra. Almost all the theorems from algebra that have been analyzed have been proved equivalent to one of the big five systems, and mostly to RCA_0, WKL_0, or ACA_0. But there are still many theorems waiting to be analyzed. There are still also a few open questions.

First, here are two questions that were already asked in [FS00]. When studying countable torsion Abelian groups, the main tool is the analysis their Ulm sequences. Ulm sequences have ordinal length, which is why ATR_0 turned out to be the right place to work with reduced Abelian p-groups (see [Fri, GM08]). If a group is not reduced, it takes Π^1_1-CA_0 to prove that it can be expressed as a sum of a divisible part and a reduced part [FSS83], and this is why most of the results about torsion Abelian groups involve Π^1_1-CA_0.

QUESTION 12. Are the following statements equivalent to Π^1_1-CA_0?

- Let G, H be countable torsion abelian groups, where $G + G$ and $H + H$ are isomorphic. Then G, H are isomorphic.
- Let G and H be countable torsion Abelian groups such that each is a direct summand of the other. Then G and H are isomorphic.

These results were considered by Kaplansky [Kap69]. Friedman [Fri] conjectured that the answers to the questions above are positive, and also that if we restrict ourselves to reduced torsion Abelian groups, then the statements should be equivalent to ATR_0.

Recently, Chris Conidis [Con10] studied the strength of the classical theorem that every Artinian Abelian ring is Noetherian. He showed that this statement is implied by ACA_0 and that it implies WKL_0.

QUESTION 13. Where between ACA_0 and WKL_0 does the statement that every Artinian Abelian ring is Noetherian lie?

3.2. General topology. In second-order arithmetic there is a natural way to encode complete separable metric spaces by taking a countable dense subset and representing the points as fast-converging Cauchy sequences of points in the dense set (see [Sim09, §II.5]). However, if we want to talk about general topology, the situation is much less clear. Mummert and Simpson [MS04] have proposed the study of general second-countable topological spaces using MF spaces. Every partial ordering P determines an MF space $\mathrm{MF}(P)$ as follows: The points of $\mathrm{MF}(P)$ are the maximal filters in P, and a basis for the topology is composed of the sets $N_p = \{F \in \mathrm{MF}(P) \mid p \in F\}$ for $p \in P$. They analyzed the strength of the Urysohn's metrization theorem and showed the following striking result: the statement that says that an MF space is metrizable if and only if it is regular (i.e., that points and closed sets can be separated by open sets) is equivalent to Π^1_2-CA_0 over Π^1_1-CA_0.

At the Chicago workshop, Mummert asked the following question:

QUESTION 14. How strong is Alexandroff's one-point compactification theorem for MF spaces? The one-point compactification theorem says that to every space we can add a point in a way that makes the whole space compact.

An important question we need to answer is whether this is a natural way of coding topological spaces, and if not, we need to develop other ways of representing these spaces. It seems that the right approach to general topology would be to study the algebra of open sets of topological spaces. MF spaces were defined with this in mind. However, the most problematic aspect of this representation is that being a point of an MF space (i.e., a maximal filter of P) is a Π^1_1 property. This is exploited in Mummert and Simpson's proof that the metrization theorem implies Π^1_2-CA_0 over Π^1_1-CA_0: They build a computable P such that $\mathrm{MF}(P)$ is regular and $\{(p,q) \in P^2 \colon N_p \subseteq N_q\}$ is Π^1_2-complete, and then they use the fact that, on a complete separable metric space, deciding whether one open set is included in another is Π^1_1.

We say that a partial ordering P defines a *proper MF space* $\mathrm{MF}(P)$ if for all $p, q \in P$, we have that $p \leq q$ if and only if $N_p \subseteq N_q$, and that if $p|$, there exists $r \in P$ with $N_r = N_p \cap N_q$. The problem with these types of spaces is that deciding whether a partial ordering P defines a proper MF-space is Π^1_2-complete (which follows from [MS04]). On the other hand, they could still aid in our understanding of where the complexity in the metrization theorem comes from.

QUESTION 15. What is the strength of the statement that a proper MF space is metrizable if and only if it is regular?

3.3. Real analysis and topology. Here is another question from [FS00] that has not yet been solved.

The *strong Tietze theorem* says that if X is a compact metric space, K is a closed subset of X, and f is a continuous function on K, then f can be extended to a continuous real-valued function on all of X. There are standard ways to present compact metric spaces and continuous functions (see [Sim09, §II.5 and §II.6]). There are various ways of representing closed sets. The strong Tietze theorem was studied by Giusto and Simpson [GS00] for several of the definitions of closed sets. Here we represent a closed set K as the complement of an open set. For this definition of closed set, they showed that the strong Tietze theorem lies between WKL_0 and DNR_0.

QUESTION 16. Is the strong Tietze theorem equivalent to WKL_0?

See [AS06] for a study of the fundamental aspects of the theory of metric, Hilbert, and Banach spaces in the context of subsystems of second-order arithmetic. There are also some interesting open questions mentioned in the last section of [AS06].

3.4. Randomness and measure theory. Effective randomness can be used to study the strengths of theorems of the form "almost every real has property P." Knowing how random a real needs to be to have property P should allow us to compare the strengths of such theorems. For instance, Brattka, Miller, and Nies [BMN] showed that x is computably random if and only if $f'(x)$ exists for each computable non-decreasing function f on $[0, 1]$. Gács, Hoyrup, and Rojas [GHR10] studied Birkhoff's pointwise ergodic theorem in connection with Schnorr randomness. Pathak [Pat09] studied the Lebesgue differentiation theorem, and proved that it holds at Martin-Löf random reals, but left the reversal open.

This should translate to reverse mathematics in some way. For instance, the system of weak weak König lemma WWKL_0 (introduced by Simpson and Yu [YS90]) has been shown to be equivalent to many basic theorems about measure theory and hence also randomness. It says that binary trees of positive measure have paths. This system is equivalent (at least on ω-models; see [ASKHLS04, Lemma 1.3]) to the existence of Martin-Löf random reals relative to any given set (that is already in the model).

Going back to Pathak's result above, Simpson noticed it implies that the Lebesgue differentiation theorem can be proved in WWKL_0, and he asked about the reversal.

LDT: If f is an L_1-function on $[0, 1]^d$, represented as a Cauchy limit of polynomials, then for almost all $x \in [0, 1]^d$, we have that $f(x) = \lim_{Q \searrow x} \frac{\int_Q f}{\mu(Q)}$, where Q ranges over the open cubes that contain x.

QUESTION 17. Does LDT imply WWKL_0?

3.5. Dynamics. The Auslander–Ellis theorem from topological dynamics can be used as a tool to give yet another proof of Hindman's theorem, as shown by Furstenberg and Weiss. (See section 2.3 above for Hindman's theorem.)

AET: Let X be a compact metric space and let $T\colon X \to X$ be continuous, defining a dynamical system. For every $x \in X$, there exists $y \in X$ that is uniformly recurrent and proximal to x. That is, $\forall \varepsilon > 0 \exists m \forall n \exists k < m(d(T^{n+k}(y), y) < \varepsilon)$ and $\forall \varepsilon > 0 \exists^\infty n(d(T^n(x), T^n(y)) < \varepsilon)$.

Blass, Hirst and Simpson [BHS87] proved that AET follows from ACA_0^+. Simpson (see [FS00, page 138]) conjectured that it is equivalent to ACA_0.

Question 18. Is AET equivalent to ACA_0?

There are a few other results on the reverse mathematics of theorems from topological dynamics and ergodic theory. Researchers seem to agree there should be many more interesting theorems to be analyzed in these areas. For instance, Avigad and Towsner analyzed the Furstenberg–Zimmer structure theorem from ergodic theory. This theorem is hard to state briefly, so for background information we refer the reader to the survey paper [Avi09] (where it is referred to as the Furstenberg structure theorem). Avigad and Towsner conjectured that it is equivalent to Π^1_1-CA_0. They had ideas on how to prove both directions, but now they believe they know how to prove only the right-to-left direction. Avigad had claimed the equivalence in [Avi09, Theorem 5.3] but now says the question is open.

Question 19. Does the Furstenberg–Zimmer structure theorem imply Π^1_1-CA_0?

§4. Well-orders and well-quasi-orders.

Well-quasi-orderings occur naturally in many areas of mathematics. For instance, trees, linear orderings, sequences, finite graphs, etc. are all well-quasi-ordered under certain embeddability relations. Many proofs about them are of unusual strength, and that is why they they are particularly attractive to reverse mathematicians and proof theorists. For a recent survey of results on well-quasi-orders and better-quasi-orders in reverse mathematics, see [Mar05].

Definition 4.1. A quasi-ordering (or pre-ordering) P is a *well-quasi-ordering* (*wqo*) if for every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq P$, there exists $i < j$ such that $x_i \leq_P x_j$.

The definition above, which we take as our working definition, is equivalent to saying that P has no infinite descending sequences and no infinite antichains. (The strengths of the equivalences between various other definitions of wqo are analyzed in [CMS04], where some questions are left open.)

The main two tools used in working with wqo's are length calculations and better-quasi-ordering (bqo) theory.

4.1. Linearizations and lengths. A linearization of a partial ordering $(P, \leq_P)$ is a relation $\leq_L$ on P extending $\leq_P$ such that $(P, \leq_L)$ is a linear ordering. Let us start by considering the following statement.

$\mathsf{EXT}(\omega^*)$: Every well-founded partial ordering has a well-ordered linearization.

Kierstead and Rosenstein showed that every well-founded computable partial ordering has a well-ordered computable linearization. However, this does not mean that $\mathsf{EXT}(\omega^*)$ holds in RCA_0: Rosenstein and Statman showed that not every computably well-founded computable partial ordering has a computably well-ordered computable linearization. (See [Ros84] for these results.) Downey, Hirschfeldt, Lempp, and Solomon [DHLS03] proved that $\mathsf{EXT}(\omega^*)$ can be proved in ACA_0 and that it is strictly stronger than WKL_0.

QUESTION 20. Is $\mathsf{EXT}(\omega^*)$ equivalent to ACA_0?

When P is a wqo, it is not hard to see that every linearization of P is well-ordered (this is actually equivalent to P being a wqo).

DEFINITION 4.2. The *length* of a wqo W is the supremum of the order types of all the linearizations of W. We denote it by $o(W)$.

The supremum is actually achieved by a single linearization of W (de Jongh and Parikh [dJP77]). This is why the length of a wqo W is sometimes also called the *maximal order type* of W. This notion is an extremely useful tool when working with wqo's, as we will see below. A monograph that presents many length calculations is [Sch79].

4.2. Fraïssé's conjecture. Fraïssé conjectured in [Fra48] that the class of scattered linear orderings is well-quasi-ordered under embeddability, and Laver proved it in [Lav71]. Since there is only one non-scattered countable linear ordering, in the countable case we refer to the following statement as *Fraïssé's conjecture*:

FRA: The class of (countable) linear orderings is well-quasi-ordered under the embeddability relation.

Laver's original proof of FRA works in Π^1_2-CA_0. Since it is a true Π^1_2 statement, FRA cannot imply Π^1_1-CA_0. Shore [Sho93] proved that $\mathsf{RCA}_0+\mathsf{FRA}$ implies ATR_0, but the following question has been open for more than twenty years.

QUESTION 21. Is FRA equivalent to ATR_0 over RCA_0? Is it even implied by Π^1_1-CA_0?

Marcone and Montalbán have done extensive work on this question [Mar05, Mon07]. From Montalbán's results in [Mon06a], we get that FRA

is a robust statement, and that it is necessary and sufficient to work with linear orderings and the embeddability relation on them. These results make the Question 21 much more interesting: A negative answer would give a new robust system.

Recently, Marcone and Montalbán started analyzing this question from the bottom up. Given an ordinal α, let $\mathbb{L}_\alpha$ be the partial ordering obtained by considering the class of linear orderings of Hausdorff rank less than α, modulo the relation of equimorphism (bi-embeddability), and ordered by embeddability. (For computable α, $\mathbb{L}_\alpha$ is countable and can be presented computably [Mon05].) It is not hard to show that FRA is equivalent to the statement that for every ordinal α, $\mathbb{L}_\alpha$ is a wqo. Marcone and Montalbán [MM09] considered $\mathbb{L}_\omega$, the class of linear ordering of finite Hausdorff rank. They showed that $\mathbb{L}_\omega$ has length $\varepsilon_{\varepsilon_{\varepsilon_{\dots}}}$, the first fixed point of the epsilon ordinal function, and proved that the well-quasi-orderedness of $\mathbb{L}_\omega$ is equivalent to the well-orderedness of $\varepsilon_{\varepsilon_{\varepsilon_{\dots}}}$ over ACA_0^+. (We note that $\varepsilon_{\varepsilon_{\varepsilon_{\dots}}}$ is the proof-theoretic ordinal of ACA_0^+, and hence its well-orderedness is not provable in ACA_0^+.)

They now believe that answering the following question will lead to a solution to Question 21.

Question 22. Given an ordinal α, what is the length of $\mathbb{L}_\alpha$?

4.3. Transfinite sequences with finite range. Here is a wqo result, whose strength we do not know: Given a wqo Q and an ordinal α, let $S^r_\alpha(Q)$ be the set of *restricted α-sequences from Q*, that is, the set of functions $s\colon \alpha \to Q$ with finite range. Given $s, t \in S^r_\alpha(Q)$, we let $s \leq t$ if there exists a monotone map $f\colon \alpha \to \alpha$ such that $\forall i (s(i) \leq_Q t(f(i)))$.

Question 23. How strong is the statement that if α is well-ordered and Q is a wqo, then $(S^r_\alpha(Q), \leq)$ is a wqo?

The statement in Question 23 was proved by Nash-Williams [NW65]. I expect the answer to this question will be determined by an analysis of the length of the wqo involved. The analysis of the lengths of these wqo's given in [Sch79] is erroneous, and nobody has fixed it yet.

4.4. Better-quasi-orderings. The main tool used in dealing with wqo's of infinite objects is Nash-Williams' notion of a bqo [NW68]. The classical definition of a bqo, which is the one we use as our working definition, is a bit technical, so we refer the reader to [Mar05] for it. The following is an equivalent definition (essentially due to Simpson [MW85]), which might be easier to understand.

Definition 4.3. A quasi-ordering Q is a *better-quasi-ordering* (*bqo*) if for every continuous function $f\colon [\mathbb{N}]^\mathbb{N} \to Q$ there exists an $X \in [\mathbb{N}]^\mathbb{N}$ such that $f(X) \leq_Q f(X^-)$ (where $[\mathbb{N}]^\mathbb{N}$ is the space of infinite subsets of $\mathbb{N}$, X^- is X without its smallest element, and where Q has the discrete topology).

Laver proved a stronger statement than FRA; we refer to the following statement as *Laver's theorem*:

LAV: If Q is a bqo, then so is $\mathbb{L}(Q)$, where $\mathbb{L}(Q)$ is the class of countable linear orderings whose elements are labeled with elements of Q and embeddings need to map elements to elements with larger than or equal labels.

All we know about the strength of LAV is that it can be proved in Π^1_2-CA_0 (using Laver's original proof), and that it implies FRA and hence also ATR_0.

Another important statement, weaker than LAV and similar to the statement in Question 23, is Nash-Williams' theorem:

NWT: If α is well-ordered and Q is a bqo, then $S_\alpha(Q)$ is also a bqo, where $S_\alpha(Q)$ is the set of α-sequences from Q.

Marcone [Mar96] showed that Π^1_1-$\mathsf{CA}_0 \vdash \mathsf{NWT}$, and it follows from Shore's result [Sho93] that NWT implies ATR_0. Marcone also showed that, over ATR_0, NWT is equivalent to the generalized Higman theorem that says that if Q is a bqo, then so is $Q^{<\omega}$.

Question 24. What are the strengths of LAV and NWT?

The following question shows how little we know about the logical strength of better-quasi-orderedness. For $n \in \omega$, let n represent the partial ordering consisting of n incomparable elements. That 2 is a bqo can be proved in RCA_0 [Mar05]. But so far, the only proof we know that 3 is a bqo uses the clopen Ramsey theorem, which is equivalent to ATR_0.

Question 25. Is there a system weaker than ATR_0 that can prove that 3 is a bqo?

4.5. Well-order-preserving operators. Here are two questions related to the Howard–Bachmann ordinal notation system and to Carlson's notion of patterns of resemblance.

Let LO be the class of countable linear orderings, and let WO be the class of well-orderings. Given a functional $f\colon \mathrm{LO} \to \mathrm{LO}$, statements of the form $\forall \mathcal{X} \in \mathrm{LO}\ (\mathrm{WO}(\mathcal{X}) \Longrightarrow \mathrm{WO}(f(\mathcal{X})))$ have been studied for different functionals f coming from ordinal notations in proof theory such as $\mathcal{X} \mapsto \omega^{\mathcal{X}}$, $\mathcal{X} \mapsto \varepsilon_{\mathcal{X}}$, etc. and shown to have interesting strengths [Gir87, Hir94, FMW, AR10, RW, MM, Rat]. For a survey of recent results, see the Introduction of [MM].

Rathjen has conjectured (personal communication) that if we consider functionals F that map functionals to functionals that preserve well-orderedness, we should get statements equivalent to those of the form "every set belongs to a countably coded β-model of some theory T," where, of course, F has to do with the ordinal analysis of T. In second-order arithmetic, we cannot quantify over all functionals $f\colon \mathrm{WO} \to \mathrm{WO}$, and in many cases it would not even make sense to consider all such functionals. We will restrict ourselves to the class of *dilators*, introduced by Girard [Gir81], which form a nice class of functionals $f\colon \mathrm{WO} \to \mathrm{WO}$ that can be represented by reals. Each dilator f comes from a *system of ordinal denotations* as in [Gir81, §0.1]: If X is a well-ordering, then $f(X)$ is defined from X by building terms using functions symbols and rules to compare terms. We use DIL to denote the class of dilators.

Montalbán has circulated a draft note [Mon] with his conjectures of what statements of this form could be equivalent to Π^1_1-CA_0. Montalbán's conjectures are based on Rathjen's conjecture above.

QUESTION 26. Is the following statement equivalent to Π^1_1-CA_0?

$$\forall f \in \mathrm{DIL}\ \big(\vartheta(f(\Omega+1)) \text{ is well-ordered}\big).$$

This statement has to do with the Howard–Bachmann notation system. We are considering the version of this system from [RW93, Section 1], where they use the projection operator ϑ instead of ψ or θ. Recall that Ω is viewed as a large, "inaccessible" ordinal (such as $\aleph_1$ or ω_1^{CK}). In the definition of the ordinal notation for $f(\Omega+1)$ [RW93, Section 1], the sets $C(\alpha,\beta)$ should be closed not only under $+$ and ω^x but also under the other function symbols that generate the dilator f.

QUESTION 27. Is the following statement equivalent to Π^1_1-CA_0?

$$\forall f \in \mathrm{DIL}\ \big(\exists \alpha \in \mathrm{WO}\ \big(\alpha <_1 f(\alpha+1)\big)\big).$$

The definition of $<_1$ is related to Carlson's notion of patterns of resemblance [Car01, Car09]. Given a dilator f, let L_f be the language which consists of the function symbols that generate f. Given an ordinal α, we have that $(\alpha, \leq, L_f)$ is a first-order structure. Following [Car01], we define a relation $\leq_1$ on ordinals by transfinite recursion: we let $\alpha \leq_1 \beta$ if and only if the structure $(\alpha; \leq, L_f, \leq_1)$ is a Σ_1-elementary substructure of $(\beta; \leq, L_f, \leq_1)$. Ideas from Wilken [Wil06, Wil07] may help to compare the strengths of the statements in the two questions above.

Neither of these statements has been analyzed and the precise definitions of the notions involved might have to be adjusted to get interesting results.

§5. Miscellaneous.

5.1. Determinacy. Logicians have been studying the strengths of the different levels of determinacy for many years, obtaining reversals that range from WKL_0 to large cardinal hypotheses. Recently, Montalbán and Shore found the exact amount of determinacy that is provable in second-order arithmetic [MS]: For each fixed n, it can be proved in second-order arithmetic that Boolean combinations of n Π^0_3-sets are determined, but it cannot be proved that all Boolean combinations of (any number of) Π^0_3-sets are determined.

QUESTION 28. Can we get a precise classification of the proof-theoretic strength of Π^0_3-Determinacy? What about determinacy for the finite levels of the difference hierarchy of Π^0_3 sets?

MedSalem and Tanaka have recently obtained such a precise classification for Δ^0_3-determinacy. Welch [Wel] has recently obtained partial results for the first question above, and Montalbán and Shore [MS] for the second.

Nemoto [Nem09](following Nemoto, MedSalem, and Tanaka [NOMT07]) studied the strength of determinacy for the classes of the Wadge hierarchy of Borel pointclasses on Cantor space. This hierarchy starts with the class of clopen sets, continues with the open sets, is followed by a sequence of classes of length ω_1 going through the difference hierarchy of open sets up to the class of Δ^0_2 sets, and keeps on going. Using old and new results, Nemoto obtained a fairly full picture of the strengths of determinacy for pointclasses ranging from Δ^0_1 to Δ^0_3 and systems ranging from WKL^*_0 to way beyond Π^1_1-TR_0 [Nem09, Table 2]. The strength of determinacy for a few classes are left open, such as the classes in the difference hierarchy of Σ^0_1 sets and the classes in the Wadge hierarchy between $\mathrm{Sep}(\Sigma^0_1, \Sigma^0_2)$ and $\mathrm{Bisep}(\Delta^0_2, \Sigma^0_2)$. A program that involves all these questions is the following.

QUESTION 29. Find the exact proof-theoretic strength of determinacy for each of the classes of the Wadge hierarchy on Cantor space.

An answer would provide a naturally defined spine of sub-systems of second-order arithmetic.

5.2. Hyperarithmetic analysis. We say that a theory T is a *theory of hyperarithmetic analysis* if all its ω-models are closed under hyperarithmetic reducibility and for every $Y \subseteq \omega$, $HYP(Y) \models T$. We say that a statement S is one *of hyperarithmetic analysis* if RCA_0+S is a theory of hyperarithmetic analysis. Many theories of hyperarithmetic analysis have been studied. Here are the main examples, from stronger to weaker:

Σ^1_1-dependent choice $\Rightarrow$ Σ^1_1-choice $\Rightarrow$ Π^1_1-separation $\Rightarrow$

Δ^1_1-CA_0 $\Rightarrow$ Jullien's indecomposability Theorem $\Rightarrow$

weak-Σ^1_1-choice $\Rightarrow$ the jump iteration statement.

Since the class of hyperarithmetic sets is such a natural class, one would expect that many theorems from mathematics are at this level (see, for instance, Friedman's ICM paper [Fri75]). Thus far, however, only one natural mathematical theorem has been found at this level, namely, Jullien's theorem on indecomposable linear orderings, which was analyzed first by Montalbán [Mon06b] and later by Neeman [Nee]. Another statement at this level is the *arithmetic Bolzano–Weierstrass theorem*, first mentioned in [Fri75] and analyzed by Conidis [Con]. This latter theorem uses the notion of arithmetic set of reals, which is not really a core-mathematics notion. (The question of how the arithmetic Bolzano–Weierstrass theorem compares with Π^1_1-separation is still open.) All the other theories at this level use concepts from logic. A general question would be the following.

QUESTION 30. Are there other natural theorems of mathematics which are statements of hyperarithmetic analysis?

In [Mon06b], Montalbán introduced statements about finitely terminating games that are also statements of hyperarithmetic analysis. The precise strengths of the statements that he called $\mathsf{DG\text{-}AC}_0$ and $\mathsf{CDG\text{-}AC}_0$ have not been deeply studied. Let me restate the latter of these statements with different terminology. An $L_{\omega_1,\omega}$ formula is a formula of arithmetic where infinite disjunctions and conjunctions are allowed. In second-order arithmetic we can represent them using well-founded trees, labeling the nodes with the usual logical connectors. A valuation of an $L_{\omega_1,\omega}$ formula φ is a function

$$v\colon \text{sub-formulas}(\varphi) \to \{T, F\}$$

where the obvious logical rules apply. Proving that such valuations always exist requires ATR_0. However, the following statement does not:

weak-$L_{\omega_1,\omega}$-CA: Let $\{\varphi_i : i \in \omega\}$ be a sequence of $L_{\omega_1,\omega}$ formulas for which valuations exist. Then there exists a set X such that $n \in X$ if and only if φ_n holds.

This follows from weak-Σ^1_1-AC_0 and is a statement of hyperarithmetic analysis (see Montalbán's [Mon06b] analysis of $\mathsf{CDG\text{-}AC}_0$).

QUESTION 31. Is weak-$L_{\omega_1,\omega}$-CA equivalent to weak-Σ^1_1-AC_0?

§6. Changing the setting.

6.1. Changing the base. A good base system for reverse mathematics needs to have two properties: It has to be weak enough to be able to distinguish theorems which should not be equivalent, and it has to be strong enough to prove the basic properties of the coding we are using to represent objects. RCA_0 has been solidly established as the standard base for reverse mathematics, but depending on the area of mathematics we are interested in, different bases might be more appropriate.

6.1.1. *Strengthening the base.* The first step in arguing that a stronger base is needed in a particular case would be to find an equivalence that holds over a stronger base but not over RCA_0. There are very few examples where this holds. Hirschfeldt asked for more of such examples at the Banff workshop. One example is provided by Giusto and Marcone [GM98], where they show that WKL_0 is necessary to prove that certain statements about Compact, Lebesgue and Atsuji spaces are equivalent to ACA_0. These statements have the form "If a space has a certain property P, then it has property Q", and they show that WKL_0 is necessary to prove the existence of a space with property P and make such a statement non-trivial. For most of the other examples we know, the stronger base is RCA_0 plus some extra induction. When working in reverse mathematics, it is usually the case that if initially we use too much induction to prove a certain implication, the use of induction can be brought down later. A few examples where this is not the case have been found lately. See, for example, Neeman [Nee] and MedSalem and Tanaka [MT07]. The following are two examples where Σ^1_1 induction was used, but it is unknown whether it is necessary: Montalbán's proof that ATR_0 implies the extendability of η (i.e., that every poset without a linearly ordered dense subset has a linearization also without dense subsets)[Mon06a], and Conidis' proof that the arithmetic Bolzano–Weierstrass theorem implies weak-Σ^1_1-choice [Con].

6.1.2. *Weakening the base.* Buss' theory S^1_2 of *bounded arithmetic* (introduced in [Bus86]) has been widely studied. This theory attracted the interest of computer scientists because its provably total functions are exactly the polynomial-time-bounded computable functions, which leads to connections with the P vs NP problem. This is a first-order theory. If we want to work in second-order arithmetic (a.k.a. analysis), we have Ferreira's theory of *feasible analysis*, BTFA, where the provably total functions are also exactly the polynomial-time-bounded computable functions. (See [FF02].) This theory is, in a sense (but not exactly), Π^0_2-conservative over Buss' first-order theory S^1_2. Some results from analysis have already been analyzed over this new base system, yielding interesting results. This field is ripe for further developments.

Friedman and Simpson [FS00, Sec 10] proposed the study of RCA_* as a base, where, in RCA_* (introduced by Simpson and Smith [SS86]), Σ^0_1-induction is replaced by Σ^0_0-induction and the exponentiation function is assumed. Little work has been done on this. However, for example, Nemoto recently showed that most of the analysis of determinacy statements can be done over RCA_*, and she was able to separate two determinacy statements over RCA_*, both of which are equivalent to WKL_0 over RCA_0.

6.2. Higher-order reverse mathematics. Even if we can express most of mathematics in second-order arithmetic, there are many statements that one would like to consider but that either cannot be expressed at all in second order or require complicated and unnatural codings. This calls for the development of reverse mathematics for third-order or even higher-order arithmetic.

In [Koh05], Kohlenbach indicates that there is an interesting kind of reverse mathematics at higher orders. He proposes to use the language of arithmetic in all finite types. In this language there is a set $\boldsymbol{T}$ of types defined as follows: $\mathbb{N} \in \boldsymbol{T}$, and if $\rho, \tau \in \boldsymbol{T}$ then $\rho \to \tau \in \boldsymbol{T}$. (It is common to use 0 to denote the type $\mathbb{N}$ of natural numbers.) For each type in $\boldsymbol{T}$, we have variables that range over that type. For a base system, Kohlenbach defines

RCA_0^ω, which is based on a system previously considered by Feferman, and is conservative over RCA_0. Then he shows that the system he calls $(\exists^2)$, which is conservative over PA, is equivalent to various statements about continuous functions. This line of investigation was later continued by Sakamoto and Yamazaki [SY04], who analyzed other statements and systems. Also, Hunter [Hun08] has started the study of general topological spaces of any size.

There is a lot of room for further interesting work in this setting.

6.3. Strict reverse mathematics. Friedman has proposed the development what he calls *strict reverse mathematics* (SRM). The objective of SRM is to eliminate the following two possible criticisms of reverse mathematics: that we need to code objects in cumbersome ways (something that is not part of classical mathematics), and that some of the axiom schemes we use, such as comprehension and induction, are purely logical and do not come from mathematical practice.

As for the coding issue, Friedman says that, for each area X of mathematics, there will be a SRM for X. The basic concepts of X will be taken as primitives, avoiding the need for coding. This would also allow consideration of uncountable structures, thereby getting around this limitation of reverse mathematics.

As for axioms for the base theory for SRM for X, one would need to take purely natural mathematical statements from the practice of X. Friedman believes that, given X, it should be possible to find the right setting where one can find such axioms, though this might take some work. He has already done it in a few settings. To cite one, he considered a logical system with three sorts, corresponding to the natural numbers, the integers, and the set of functions from the naturals to the integers. He then composed some lists of "purely mathematical statements" (including the axioms of ordered semi-rings and of rings) and proved that they are equivalent (without the use of any base system) to RCA_0, or to WKL_0, etc.

For these proofs and additional background, see [Fri09].

6.4. Computable entailment. Shore [Sho] makes an explicit formalization of the intuition that "harder to prove" means "harder to compute". His formalization in the countable case is not new; in fact, this is what most of us computability theorists have in mind when we do reverse mathematics.

DEFINITION 6.1. Consider Π^1_2 formulas

$$\Psi \equiv \forall X \exists Y \Psi_1(X, Y),$$
$$\Phi \equiv \forall X \exists Y \Phi_1(X, Y).$$

We say that Ψ *computably entails* Φ if the following holds: If $\mathcal{C}$ is a class of sets which is closed under Turing reduction and the join operation, and has the property that for every $X \in \mathcal{C}$, there is $Y \in \mathcal{C}$ such that $\Psi_1(X, Y)$ holds, then we have that for every $X \in \mathcal{C}$, there is $Y \in \mathcal{C}$ such that $\Phi_1(X, Y)$ holds.

This essentially says that solutions to Φ can be computed from solutions to Ψ, perhaps by iterating Ψ a few times. Note that this is equivalent to saying that Φ is true in every ω-model in which Ψ is true. The two main advantages of this approach are that it provides a different expository route into the subject which may be more suitable for a mathematical audience that intuitively understands computability but may find formal proof systems foreign or less appealing, and that it provides an opportunity to deal with uncountable structures and higher-order statements that are out of reach of standard proof-theoretic methods.

Let me explain this second claim. Notice that, in the definition above, we can use any notion of reducibility in place of Turing reducibility and still get a notion of computable entailment. For instance, on uncountable sets one can use α-recursion, Blum–Shub–Smale computability, Borel reduction, or whatever works for the specific problem. Shore suggests the following general program: Develop a computability-theoretic type of reverse mathematical analysis of mathematical theorems on uncountable structures using whichever generalized notion of computability seems appropriate to the subject being analyzed. He has already gotten started on this program; in [Sho] he develops analogues of ACA_0 and WKL_0 that use α-recursion, and compares them to a few theorems such as the existence of basis for vectors spaces and the existence of prime ideals on rings.

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