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$V = L$ AND INTUITIVE PLAUSIBILITY IN SET THEORY. A CASE STUDY

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Abstract. What counts as an intuitively plausible set theoretic content (notion, axiom or theorem) has been a matter of much debate in contemporary philosophy of mathematics. In this paper I develop a critical appraisal of the issue. I analyze first R. B. Jensen's positions on the epistemic status of the *axiom of constructibility*. I then formulate and discuss a view of intuitiveness in set theory that assumes it to hinge basically on mathematical *success*. At the same time, I present accounts of set theoretic axioms and theorems formulated in non-strictly mathematical terms, e.g., by appealing to the *iterative concept* of set and/or to overall methodological principles, like *unify* and *maximize*, and investigate the relation of the latter to success in mathematics.

§1. Introduction. $V = L$ is the *axiom of constructibility*. It is a statement about the set theoretic universe. It says that the universe is a cumulative hierarchy of sets indexed by the ordinals, with the first stage L_0 being the null set $\emptyset$, $L_{\alpha+1}$ being given by all the subsets of L_α definable over the latter,[1] and union being taken at limit stages. Formally, $V = L = \cup_{\alpha \in Ord} L_\alpha$. V is defined as a cumulative hierarchy of sets with the null set $\emptyset$ being V_0, the full power set of V_α being $V_{\alpha+1}$, and union being taken at limit stages. Formally, $V = \cup_{\alpha \in Ord} V_\alpha$.

$V = L$ has a controversial status within the set theoretic community. The prevailing attitude among set theorists is that it should be rejected on the ground that it clashes with the *maximum iterative concept* of set. The latter is supposed to express an intuitive understanding of sets on which set theory is based and to which it should be faithful. As a result the axiom of constructibility is discarded as counter-intuitive and false.

However, although this position on $V = L$ is the dominant one, it is not so obvious as might seem at the outset. Purely methodological reasons may play a more important role than $V = L$'s alleged incompatibility with the maximum iterative concept in its being rejected as a set theoretic axiom.

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[1] A set X is definable over $(L_\alpha, \in)$ if there is a formula ϕ of the first-order language of set theory and some $a_1, \ldots, a_n \in L_\alpha$ such that $X = \{x \in L_\alpha : (L_\alpha, \in) \models \phi[x, a_1, \ldots, a_n]\}$.

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Moreover, there are leading set theorists, most notably R. B. Jensen, who are of a very different mind on the axiom of constructibility. They do not think it should be denied as implausible or as opposed to basic insights about sets.

Minority views on $V = L$ like those set out in [22] have not been carefully analysed in the literature. Thus a critical appraisal of them is still lacking. This paper tries to fill this gap. Section 2 describes the current state of the debate on the axiom of constructibility. In particular Jensen's and other minority positions are presented, and their implications with regard to the philosophy of set theory are made explicit. Sections 3 through 6 focus on the issue of intuitive plausibility in set theory and analyse it in light of some of Jensen's remarks on the axiom of constructibility.

Let me say in advance that a quite unorthodox philosophical scenario emerges from the analyses that follow. Traditional dichotomies, most notably the opposition *intrinsic necessity* vs. *extrinsic evidence*, collapse and the maximum iterative concept is no longer understood as a basic motivation for set theoretic axioms. Questions arise too, the most remarkable being about the nature of *success* in mathematics, a detailed analysis of which, however, in spite of their relevance to the topic of this contribution, must be postponed to another occasion. I hope, though, that this work succeeds in emphasizing epistemic phenomena whose bearing on intuitiveness in set theory has mostly been ignored, thus making Jensen's views more intelligible and shedding new light on the general issue of what counts as an intuitively plausible set theoretic content.

§2. What is the status of $V = L$? Arguments against, arguments for. In this section I briefly outline the debate surrounding $V = L$ within the set theoretic community, and I conclude by making explicit some philosophical questions raised by that debate. Let me say in advance that the aim of this section is to present an overview of several major, diverse positions on the axiom of constructibility, not to conduct a criticism of any of them, open to objections as they may be. A critical appraisal of some of the views reported here is developed in sections 3 through 5.

There seems to be no agreement among set theorists either as to the epistemic status of $V = L$ or as to its ontological presuppositions, although some views are more popular than others. In fact the prevailing attitude towards the axiom of constructibility among set theorists is a negative one. Maddy makes the point in [26].

> Does $V = L$? Is the Axiom of Constructibility true? Most people with an opinion would answer no. But on what grounds?

Moschovakis' view that:

> the key argument against accepting $V = L$ is that [it] appears to restrict unduly the notion of arbitrary set of integers; there is no

> a priori ground why every subsets of ω should be definable ([34], p. 610),[2]

is considered by Maddy the most widely shared line of argument against the axiom of constructibility. Some authors, however, go further than Moschovakis in arguing against $V = L$. Whereas he appeals to the lack of an a priori ground for restricting the notion of an arbitrary set of integers, Wang gives positive arguments as to why one should not suppose that every subset of ω is definable. To this end, he appeals to the maximum iterative concept of set. According to the latter, sets should be thought of as "arbitrary multiplicities regardless of how or if they can be defined."[3] Hence, when developing a theory of sets one should not put too many restrictions on what counts as a set. In particular, when defining the cumulative hierarchy one should take a full power set.

The maximum iterative concept is mostly understood as a thought "about the nature of sets", which "can be described in a rough, but informative way, without first stating the theory the thought is behind";[4] a thought that "strikes people as entirely natural, free from artificiality, not at all ad hoc, one they might perhaps have formulated themselves."[5] As a result, set theoretic contents that are compatible with the maximum iterative concept, as well as overall maxims for developing set theory that can be traced back to it, are regarded as perfectly clear and well understood, natural and obvious. Equivalently, they are said to possess *intuitive plausibility*. Vice versa $V = L$, which conflicts with the view of the universe implied by the maximum iterative concept, is supposed to be incompatible with what we mean by 'set' and hence to be false and counter-intuitive. This is the conclusion drawn by Wang in [46].

> [The axiom of constructibility] is not generally accepted as true. Indeed, it is likely to be false according to the iterative concept of set. [...] By intention we view sets as arbitrary multiplicities regardless of how or if they can be defined. ([46], p. 196.)

Views like these are often expressed in the literature. To give a recent example, in [16] the axiom of constructibility is called 'suspicious' for reasons that resemble those adduced by Wang. In [16], however, besides the incompatibility of the axiom with the maximum iterative concept, mathematical results are appealed to in the case *contra* $V = L$. Specifically Hauser considers the fact that $V = L$ is inconsistent with (large) large

[2] [26], p. 16. ω (*omega*) is the set of all (set theoretic representatives for the) natural numbers.

[3] [46], p. 196. The process of set formation is illustrated in [46] in terms of idealized mental acts. I shall return on this point in section 6.

[4] [5], p. 219.

[5] [5], p. 218.

cardinals.[6] He takes the conflict between the two to be decided in favour of (large) large cardinals not only because the latter, but not $V = L$, are compatible with the maximum iterative concept,[7] but also because of the mathematical fact that the large cardinals inconsistent with $V = L$ furnish a combinatorial analysis of the transcendence of the universe in which they are supposed to hold (i.e., V), over L.[8]

> By contrast [to the axiom of foundation, which is 'implied' by the maximum iterative concept], if V [according to this concept] is to encompass all possible acts of collections and their iterations, then $V = L$ must look suspicious. One manifestation of this is that only a short initial segment of the large cardinal hierarchy is 'realized' in L. Moreover, the large cardinals not found in L furnish a combinatorial analysis of the transcendency of V over L [. . .] many set theorists construe this as evidence that $V = L$ is false. ([16], p. 103.)

Note that the combinatorial analysis to which Hauser refers amounts to a study of the universe L, for which ZFC + $V = L$ is true, "from within *the* universe" (i.e., V supposed to be different from L), a study that sheds light on the ways the former differs from the latter.[9] Thus under (large) large cardinal hypotheses, L turns out to be a sub-universe of the universe that can be investigated without assuming V to be L. Analogously, due to the fact that ZFC + $V = L \models \phi$ iff ZFC $\models \phi^L$, the theory developed

[6] $V = L$ is incompatible with the axiom stating the existence of a measurable cardinal (MC). $MC \models V \neq L$ was shown in the sixties by Scott, who commented on his result in the following way: "It did turn out that the existence of measurable cardinals was inconsistent with $V = L$, but so much the worse for the unnatural constructible sets" (Foreword to [3], p. xii). Note that the expression 'large large cardinals' was introduced precisely to distinguish between large cardinal axioms that are consistent with $V = L$, which are sometimes called 'small large cardinal axioms', and those that are not. See [24] on large cardinal axioms and their relations to the axiom of constructibility.

[7] More precisely with implications of this concept as to the 'length' and the 'thickness' of the universe. See [16] for details.

[8] Consider, e.g., the results, implied by (large) large cardinal assumptions, concerning the existence of Silver's *indiscernibles* for L and $0^{\#}$ (zero sharp), a non-constructible subset of ω that codes up information on how L differs from V (see [20] and [24] for technical details).

[9] It emerges from the arguments reported here that in set theory the cumulative hierarchy V plays a conceptual role different from that of any other model for ZFC (+ additional axioms). In fact what axioms, beyond the system ZFC, hold in V is not fixed at the outset (e.g., due to a corpus of meta-mathematical results). It is the axioms that one wishes to accept as definitive set theoretic achievements, i.e., as inevitable starting points with which to begin in set theory, that are supposed to hold in V, not vice versa. On the other hand, as long as it is not specified what axioms hold in it (beyond ZFC), V remains vague. Under these circumstances, it seems reasonable to say that in contemporary set theory appealing to V amounts to endorsing a specific *heuristic* assuming the overall goal of set theory to be the achievement of an *ultimate* extension of ZFC, which, once available, would be true for *the* universe (i.e., V). For more on the role of V as a distinguished heuristic in contemporary set theory, see [1].

under ZFC + $V = L$ can be obtained under ZFC + (large) large cardinals (taking ϕ^L instead of ϕ).[10] So from the mathematical point of view nothing seems to be lost concerning L by working in V under (large) large cardinals. Indeed new insights are gained on the constructible hierarchy through the comparative study of it and *the* universe, when the latter is supposed not to be L. Steel makes a similar point.

> The language of set theory as used by the believer in $V = L$ can certainly be translated into the language of set theory as used by the believer in measurable cardinals, via the translation $\phi \to \phi^L$. There is no translation in the other direction. While it is true that adopting $V = L$ enables one to settle new formal sentences, this is in fact a completely sterile move, because one settles ϕ by giving it the same interpretation as ϕ^L, which can be settled in anyone's theory. ([43] p. 423.)[11]

Arguments like these against $V = L$ appear to be motivated by *methodological* concerns. It is recommended not to cut off interesting set theoretic structures. The suggestion of preferring a unique all-inclusive theory of sets over two alternative ones, i.e., ZFC + $V = L$ and ZFC + (large) large cardinals, is also implicit in the proposal to reject ZFC + $V = L$ and work under ZFC + (large) large cardinals. Both admonitions appear to be dictated by the goal of formulating a single comprehensive theory of sets expressing the "broadest point of view".[12] This is a goal similar to the one I assume to be pursued by those, like Wang, who intend to be faithful to the maximum iterative concept. Here, however, the aim of formulating a univocal theory of sets that represents the "broadest point of view" is taken as reasonable in itself, whether or not it is the maximum iterative concept that suggests it. It works as a purely methodological ground for discarding $V = L$.

The widespread rejection of $V = L$ among set theorists is also explained in purely methodological terms by Maddy.[13] She is very explicit in stating the goals pursued by set theory. The latter should furnish a foundation for the rest of mathematics by providing a framework broad enough to supply instances for all structures of classical mathematics. Whence derives the twofold methodological admonition to set theorists that they should *unify* and *maximize*: they should strive for a single fundamental theory of sets which does not impose any restrictions on the range of available isomorphism types. ZFC + $V = L$ rules out that the extra isomorphism type available from "ZFC + $0^\#$" exists.[14] Hence it is restrictive and should be rejected.

[10] ϕ^L is the *relativization* of ϕ to L. See [20] for details.

[11] Indeed, the claim that "no translation in the other direction" exists may be challenged; see section 5.

[12] To use Steel's words. See [43], p. 422.

[13] See [27], [28].

[14] See footnote 8 above on $0^\#$.

Again, the admonition to maximize is not motivated here by the maximum iterative concept. $V = L$, in turn, is viewed as restrictive because of its mathematical consequences and not on a priori grounds.[15]

Thus far I have reported arguments to the effect that $V = L$ should not be adopted as a set theoretic axiom.[16] However, they do not exhaust the entire range of positions on the axiom of constructibility held within the set theoretic community. Sy Friedman, for example, in [10], gives an argument contra $V = L$ of a nature quite different from those mentioned above. Sy Friedman also has methodological concerns. His purpose is to find a canonical interpretation or standard model of ZFC. This leads him to reject L as the ultimate picture of the universe. But he does not reject L, and the axiom of constructibility that is true in it, on the ground that they are 'limitative'. It is the "ease with which it [L] can be modified by forcing" that goes wrong with the universe L. Moreover it is recognized that L is preferable to any cumulative hierarchy defined in terms of full power set alone, which is not "a canonical description, in light of the vagueness of the power set operation".[17] In L the opacity of the power set operation is replaced by the *fine structure* theory of Jensen.[18] Put briefly: although L goes wrong, it is something determinate with which to begin the search for a canonical model for ZFC.

I now turn to the views of Jensen. Although in [22], he does not provide an explicit characterization of the axiom of constructibility as a true or intuitively plausible statement, he does describe the epistemic features of $V = L$ in quite the same terms as others describe the status of axioms that are said to be true or intuitive because they are 'implied' by the maximum iterative concept. The axiom of constructibility is described by Jensen as clear in content and perfectly plausible, and even attractive. Besides the axiom's mathematical fruitfulness, its conformity with extra mathematical insights (specifically, the 'Ockham's razor' methodological maxim) is what makes it appealing to Jensen.

> This axiom [$V = L$] makes a clear statement about the nature of the mathematical universe. It is mathematically fruitful in that it solves many problems and leads to interesting new concepts and

[15]In [26], however, the author gives a priori arguments against $V = L$. Maddy analyzes the reasons why the axiom was viewed with suspicion even before Scott's theorem. But in [26] the maximum iterative concept plays no role. It is $V = L$'s being inspired by a discredited methodological maxim—what Maddy calls *definibilism*—that rules out $V = L$ as an axiom of set theory (see [26] for details).

[16]At least it should not be adopted as an axiom belonging to the "broadest point of view", advocated by Steel in [43], which "is meant to exclude from attention the temporary adoption of restrictive assumptions as a convenient device for avoiding irrelevant structures" ([43], p. 422).

[17][10], pp. 166–167.

[18]Jensen's *fine structure* theory is further discussed in section 4.

> theories. It is philosophically attractive for adherents of "Ockham's razor", which says that one should avoid superfluous existence assumptions. I personally find it a very attractive axiom. ([22], p. 398.)

$V = L$'s merits are no reason for Jensen to discard rival views of the universe. Indeed, he sees not only $V = L$ but also (large) large cardinal assumptions incompatible with it as justified on both mathematical and extra-mathematical grounds.

> The axiom of constructibility on the one hand and large cardinals and determinacy on the other embody two radically different conceptions of the universe of all sets. How can these conceptions be justified? Most proponents of $V = L$ and similar axioms support their belief with a mild version of Ockham's razor. L is adequate for all of mathematics; it gives clear answers to deep questions, it leads to interesting mathematics. Why should one assume more? The proponents of strong axioms of infinity usually call themselves Platonists and use this to support their belief. ([22], p. 400.)

Finally, different methodological attitudes explain the opposition in today's set theoretic community between those who prefer working in universes for which (large) large cardinals axioms are true and those who prefer working in L. A position reminiscent of the methodological attitude which [26] supposes to be discredited (i.e., *definabilism*), is expressed. Moreover, one may infer from Jensen's words that not only working under $V = L$ but also studying multiple universes realizing incompatible axioms are perfectly reasonable options in contemporary set theory.[19] The following long passage is worth quoting in full here:

> Could it be that the duality in modern set theory is nothing but a new manifestation of an ancient conflict between two points of view, I almost want to say two emotional states, which have always existed in mathematics? I'll call them the arithmetical and the geometrical point of view. I'll call also the first one the Pythagorean point of view [. . .] In other words, every mathematical structure can be interpreted in the natural structure of the positive integers; it gives to all of mathematics the intuitive clarity of the natural numbers [. . .] The geometric point of view has been dominant since the rise of analysis. Thus I also call it the Newtonian point

[19]Some practitioners indeed adopt a multiple-universe perspective as the broadest point of view on sets. An extreme form of this view is explicitly endorsed by Shelah in [41], where the author suggests merging different pictures of the universe. "A typical universe of set theory is the parallel of Mr. John Smith, the typical American; my typical universe is quite interesting (even pluralistic), it has long intervals where GCH holds, but others in which it is violated badly, many λ's such that λ^+-Aronszajn is special, and it may have a lot of measurables [. . .]" ([41], p. 6).

> of view. The Newtonian directs his gaze to the real rather than the natural numbers. He is less impressed by their clarity than by their boundless multiplicity. [...] If one accepted set theory, than there is no doubt that Cantor has refused Pythagoreanism in the strongest form showing that there are more real than natural numbers. But Cantor also introduced the ordinal numbers, which are in every sense the transfinite continuation of the natural numbers. They share much of the intuitive clarity of the natural numbers. Thus Cantor, who refuted the old Pythagoreanism, made possible a new Pythagoreanism in which the ordinal numbers take over the role of the natural numbers. In this sense Gödel's axiom of constructibility seems to me to embody a coherently Pythagorean picture of the world. And this picture cannot be refuted for Gödel showed that $V = L$ is consistent if the other axioms are. But this axiom provides—modulo the ordinal numbers—a complete description of all sets and is therefore unacceptable for a Newtonian. For him there must be a real number not generated by constructible processes. Since one cannot prove in set theory that such a number exists, one must seek new axioms. Thus, the ancient conflict is fought in a new arena. ([22], pp. 400–401.)

Sy Friedman's and Jensen's views on $V = L$ are clear. We are confronted with the alternative of either regarding them as extravagancies of no philosophical significance or taking them seriously, as calling for a revision of popular positions in set theory and its philosophy. That L appears more canonical to Sy Friedman than any cumulative hierarchy defined in terms of full power set, and that $V = L$ sounds perfectly clear to Jensen, may mean that, contrary to widely held views, being 'implied by' or being 'compatible with' either the maximum iterative concept or the methodological principle *maximize* is not the only way in which set theoretic contents are natural and/or intuitively plausible to practitioners. Likewise, that different, even incompatible, views of the universe are seen as all acceptable and regarded as able to coexist may mean that *unify* is not a methodological desideratum that all set theorists can or should share. More generally, in light of Jensen's Pythagoreanism vs. Newtonianism dichotomy, one might think that a wide variety of methodological attitudes are simultaneously at work in contemporary set theory and that they convey diverging views as to what counts as an intuitively plausible set theoretic content.

I see no reason for doubting Sy Friedman's and Jensen's views on $V = L$. Indeed, I shall take them seriously here, thereby suggesting that popular views on the issue 'what does intuitiveness amount to in set theory?' should be revised.

§3. **Intuitiveness in set theory.** Naturalness and clarity in content, together with closeness to the meaning of 'set', are epistemic features usually ascribed to set theoretical axioms and theorems if they are interpretable in terms of the maximum iterative concept. In fact, this is what their intuitiveness is commonly held to consist of. By contrast, the thesis that I shall defend here is that $V = L$ can be legitimately regarded as natural, clear, and even closely bound up with what we mean by 'set'. I contend that there is more than interpretability in terms of the maximum iterative concept to intuitiveness in set theory.

Before arguing for this thesis I shall meet a possible objection to my above claim that Friedman's and Jensen's views on $V = L$ call for a revision of the received view on intuitiveness in set theory. The axiom of constructibility, the objection runs, has the status of a working hypothesis with many interesting implications. It is for this reason that it might be found attractive, rather than because its content is natural and clear or closely bound up with what we mean by 'set'.[20] The expression 'extrinsic evidence' was introduced into the literature by Gödel to characterize the persuasive force of axioms that are fruitful in consequences and "shed light" on an entire field.[21] Gödel instead called 'intrinsic necessity' the epistemic status of axioms and principles that are implied by the maximum iterative concept, their content being clear, natural and close to what we mean by 'set', i.e., intuitively plausible.[22] The objection is that extrinsic evidence, i.e., attractiveness due to mathematical success, is all there is to the epistemic status of $V = L$ and that it suffices to explain Sy Friedman's and Jensen's attitudes to the axiom of constructibility.

In reply, I point out that the words of Jensen quoted in section 2 appear to exclude that $V = L$'s fruitfulness in consequences is the only reason why it may be found attractive. Adherence to extra-mathematical insights seems to play a role as well. However let us assume that attractiveness due to success is all there is to the epistemic status of the axiom of constructibility. If this were the case, I would see it as no reason for excluding a priori that $V = L$ might be legitimately described as having a natural and clear content close to what we mean by 'set'. For on closer analysis it might turn out that these are just the epistemic features which constitute extrinsic evidence. Put otherwise, if cautiously analysed, evidence due to success, far from being the opposite of intuitive plausibility in set theory, may prove to be a mode of it, though one different from interpretability in terms of the maximum iterative concept. In sections 4 through 6 I give arguments for this view.

[20]Indeed, among the reasons why Jensen finds $V = L$ a very attractive axiom is that "it is mathematically fruitful in that it solves many problems and leads to interesting new concepts and theories"([22], p. 398). These "concepts and theories", e.g., the fine structure theory and the notion of inner model for large cardinals, are briefly described in [22]. See [32] and [44] for a more detailed description.

[21][12], p. 183.

[22]See [12], p. 181 ff.

My thesis in the rest of this paper can then be restated as follows. Besides interpretability in terms of the maximum iterative concept, there is a mode of being intuitive in set theory that hinges on mathematical success. In the literature it is called 'extrinsic evidence' and it is usually not viewed as a way of being natural, clear and close to the meaning of 'set' for set theoretic axioms. But, if my analysis is correct, this might be called into question. Moreover, interpretability in terms of the maximum iterative concept may be coherently understood as an implication of success. For me, therefore, Gödel's distinction between extrinsic evidence and intrinsic necessity ceases to be exclusive. Instead I regard these epistemic features of set theoretic axioms as continuous with one another.

Finally a word of caution. The analyses reported below focus on specific examples of successful mathematics in connection with ZFC and its extensions. For reason of space, no general investigation of mathematical success is conducted in this paper, relevant though this may be to the argument. Nevertheless, success is nowhere identical to logical strength here, the latter being for me not sufficient to the former—and to intuitiveness due to success. In fact, it is axioms that turn out to be fruitful in consequences, i.e., logically implying, and not being merely consistent with, a wide range of set theoretic statements that I suppose to be successful. Moreover, I suggest that in order for an axiom to be successful, its consequences need be *relevant* in distinct ways. For example, they may be relevant in that they convey a comprehensive description of some *general* mathematical phenomenon, e.g., the *infinite combinatorics* obtained by assuming $V = L$ or the theory of *projective sets* of real numbers implied by *PD*, the axiom of *Projective Determinacy*.[23] Alternatively, they may furnish answers to unresolved questions of set theory, like the *Continuum Hypothesis*, which, under $V = L$, becomes a special case of a general fact of cardinal arithmetic (the *Generalized Continuum Hypothesis*), or the *Lebesgue measurability* of projective sets of reals, addressed as an extremely difficult question in the 1920s, later proved to be independent of ZFC and, finally (in 1964), answered in the positive (under *PD*).[24] Or they may make it possible to integrate formerly unrelated

[23]More is said in section 4 on the infinitary combinatorics implied by ZFC + $V = L$. As to *PD*, it was originally formulated in the context of *descriptive set theory*, the study of sets of real numbers having a 'simple' description, under which the hierarchy of *projective sets* falls. *PD* states the existence of a winning strategy in infinite two-person games of perfect information played on integers, when the pay-off set is a projective set (i.e., it states the *determinacy* of all games with projective pay-off sets). *PD* extends to the sets of the whole projective hierarchy regularity results (being Lebesgue measurable, having the property of Baire and, for uncountable sets, having a perfect-subset) provided by ZFC for its first two levels. In the late 1980s *PD* was proved to be implied by (large) large cardinal axioms. See [34], [25], as well as [20] and [24], for details on the results cited here.

[24]The *Generalized Continuum Hypothesis*, GCH, states that, for every α, the cardinality of $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ (the *Continuum Hypothesis*, CH, formulated by Cantor, is the same sentence for $\alpha = 0$). See also [22] and [47] on these issues.

mathematical facts—the proof of *PD* under (large) large cardinal hypotheses being an example, as well as the comparison in terms of *consistency strength*, via (equi-)consistency results involving (large) large cardinals, of otherwise unrelated propositions independent of ZFC.[25]

These few remarks about 'success' should suffice to establish the following. Some axioms have been successful that turned out to be mutually inconsistent, like $V = L$ and (large) large cardinal hypotheses. Moreover for an axiom success need not be a matter of all-or-nothing (although $V = L$ is successful for, e.g., the infinitary combinatorics it implies, it fails to be so for, e.g., the theory of projective sets of reals derived from it).[26] Finally, intuitiveness due to success is not to be confused with the familiarity one develops towards *whatever* system in which one intensively works, and with the confidence one consequently has towards distinguished hypotheses, whose implications one has investigated at length. This kind of familiarity seems to be related to cognitive processes that may occur independently of success. At the same time it is likely to be a disposition that concerns the *individual* mathematician. Instead, success as conceived here is *public* in character. It is acknowledged within a community of research, being measured against some corpus of established results, in terms of the advances achieved by a new piece of mathematics with respect to that corpus. Accordingly, the attitudes one has towards successful axioms are commonly shared by many. How this is compatible with the fact that Jensen's and Friedman's views on $V = L$ are in the minority, is an issue that, together with others suggested by these few notes on 'success', I address in the following sections.

§4. Intuitiveness and success in set theory. I now argue for the view that mathematical success is sufficient to produce intuitiveness in set theory understood as naturalness, clarity in content, and closeness to the meaning of 'set'.[27] To this end let me observe first that the meaning of 'set' need

[25] See also [43] on the point. Note that this understanding of success is close to Gödel's. In [12], he characterizes 'success' as "fruitfulness in consequences and, in particular, in 'verifiable' consequences, i.e., consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs" ([12], p. 182). Besides being abundant in verifiable consequences, successful axioms are also described as "shedding [...] light upon a whole discipline, and furnishing [...] powerful methods for solving given problems (and even resolving them, as far as that is possible, in a constructivistic way) [...]" ([12], p. 183).

[26] See, e.g., [43] on the issue, to which I return in section 4.

[27] Mathematical success, I believe, is not only sufficient but also necessary for intuitiveness in set theory, a point I make in section 6. Consider for the moment that this notion of intuitiveness is far from intuitiveness as sometimes conceived in philosophy (including philosophy of set theory), where 'intuitive' refers to contents that are evocative of our implicit beliefs developed at a pre-reflective level, so that intuitiveness cannot be 'produced' (in time), and intuitive contents are supposed to appear so at the outset. I cannot rule out that this kind

not necessarily be regarded as a thought that precedes the formulation of a mathematical theory of sets and makes it possible by indicating what axioms should be adopted and what results should be regarded as desirable—where these axioms and results would deserve to be considered intuitive. Indeed, the rationale of set theory can be comprehended in terms of mathematical developments together with a meaning of 'set' that arises from the practice of set theory rather than being prior to it.

A look at the historical record seems to confirm this view:[28] for it was precisely problems left unresolved by real analysis that suggested taking infinite collections of real numbers as unitary mathematical objects at the end of the 19th century. The attempt to "articulate and solve the continuum problem" then produced "the whole transfinite landscape" as a result.[29] Thereafter it was Zermelo's desire to secure his well-ordering theorem from criticism that led to the first proposal of an axiom system for set theory at the beginning of the 20th century. Indeed, this was done "starting from set theory as it is historically given, to seek out principles required for establishing this mathematical discipline."[30] New developments followed the introduction of the axioms and their enlargement by Replacement and Foundation. Questions were asked and answers were given. Relations between known mathematical facts and newly discovered facts about sets were found. An endless procession of structures satisfying an axiomatization clearly recognizable as ZF was described in [49].[31] Hence a *successful* theory of sets was launched and delivered new mathematical insights.

By saying that the meaning of 'set' emerged from successful mathematics I am suggesting that it was exactly through the technical developments I

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of 'immediate' intuitiveness is ascribable to (some) set theoretic contents (notions, axioms, theorems). However, I point out that it is the maximum iterative concept that is mostly called upon in arguing for the 'immediate' intuitiveness of set theoretic notions, axioms and theorems, and that the maximum iterative concept, as I show in section 6, can be coherently understood as *no* pre-reflective belief of ours. On the other hand, I contend that even if 'immediate' intuitiveness may be correctly ascribed to some set theoretic contents, it would not be the dominant form of intuitiveness in set theory (see section 6 below).

[28][23] is a very comprehensive description of the mathematical developments of set theory from Cantor to Cohen, the thesis defended in it being that "from the beginning set theory actually developed through a progression of mathematical moves, whatever and sometimes in spite of what has been claimed on its behalf" ([23], p. 1). See also the Appendix to [24].

[29][23], p. 1.

[30][48], in [45], p. 200.

[31]ZF denotes the Zermelo–Fraenkel axiomatic system consisting of the following axioms: Extensionality, Pairing, Union, Infinity (which, however is omitted in [49]), Power Set, Foundation, Separation and Replacement. ZFC is ZF plus the Axiom of Choice (which in [49] was taken to belong to the underlying logic). Note that, since Foundation was taken as an axiom, the sets in the structures introduced in [49] were ranked into cumulative hierarchies.

have mentioned that theorists got an idea of what sets are in mathematics, which they then began to render also in informal and non-strictly-mathematical terms.[32] I also suggest that it was because they contributed to producing the mathematics from which the meaning of 'set' emerged that the standard axioms for set theory, i.e., the system ZFC, came to be viewed by those who knew what to do with them as closely bound up with what we mean by 'set', and as natural and clear assumptions about sets.[33]

Note that on this interpretation intuitiveness turns out to be an epistemic feature that was not attached to the ZFC axioms from their first formulation. These axioms had first to prove fruitful in consequences. Nor is one supposed to find the ZFC axioms intuitive at the outset: a certain degree of familiarity with them may be necessary. Finally, it is epistemic features of ZFC that are usually explained in terms of the maximum iterative concept that I account for here by appealing to mathematical success. In fact, that the evidence for ZFC is basically extrinsic is sometimes explicitly recognized in the literature. Consider, for instance, [31].

> [...] even for the ZFC axioms, the intrinsic evidence is not the main evidence. Consider, for example, the axiom of infinity. There is certainly intrinsic evidence for it. But there is intrinsic evidence against it that is at least as compelling: the intuitive idea that there can be no completed infinite totalities, for example. The reason for Infinity's acceptance is not that the intrinsic evidence for it is stronger than that against. The reason [...] is evidence that is predominantly extrinsic, lying in the depth, breadth and effectiveness of the subject it launched ([29], p. 759), (that is Cantorian set theory). Similar points can be made for other ZFC axioms and for weak large cardinal axioms such as that asserting the existence of an inaccessible cardinal. ([31], pp. 229–230.)

Of course ZFC is not the sole axiomatic system for sets that turned out to be successful. The same can be said of some extensions of it, e.g., ZFC+ $V = L$. This system proved to be fruitful in consequences, some of which provided an univocal answer to controversial set theoretic questions. In fact among

[32]In *conceptual terms*, to use the terminology of [19].

[33]I am referring to matters of meaning and intuitiveness in set theory here. I do not consider the question whether the approach I propose might work for other mathematical areas. At any rate, consider that Parsons in [39] stresses the fact that also in arithmetic the consequences one draws from an axiom may affect its epistemic status in a backward fashion. "In mathematical thought and practice, the axioms of arithmetic are embedded in a rather dense network, in which their primitives may mostly play the role of primitives but do not always do so. Moreover, the process by which they have come to be as evident as they are includes also the deduction of consequences from them, consequences which in some cases were at the time when the axioms were formulated found quite as evident as the axioms themselves"([39], pp. 305–306). See also footnote 27 above.

the implications of ZFC + $V = L$ are the *Axiom of Choice* (AC) and the *Generalized Continuum Hypothesis* (GCH), together with numerous results in infinite combinatorics, like the principles $\diamond$ (*Diamond*) and $\square_\kappa$ (*square*$_\kappa$) for every uncountable ordinal κ, or the existence of *morasses*.[34] Moreover, these consequences of ZFC + $V = L$ have been derived along with results displaying overall regularity features of L. For instance, it is the existence of a definable well-order of L that makes AC a theorem of ZFC + $V = L$, whereby CGH and $\diamond$ are implied by the *Condensation Lemma*. $\square_\kappa$ (for every uncountable κ) and *morasses*, instead, depend on *fine-structural* results, the *fine-structure theory* developed by Jensen being the detailed description of how new sets regularly arise in the construction of L. All this turned *constructibility* into a well-known mathematical phenomenon—and the meaning of 'set' emerging from it into an exactly determinate one. Constructibility proved also to be a *general* phenomenon, with most of the results holding under ZFC + $V = L$, and the techniques through which they are obtained applying to L-like structures that are models for (large) large cardinals.[35] This is not to deny that some of the consequences of ZFC + $V = L$ have been regarded as arguments *against* rather than *for* it. I discuss some of them in section 5. Unaffected, however, is the fact that the overall character of ZFC + $V = L$ is that of a very successful system, a system to which, owing to its success, one may be reasonably inclined to ascribe the same epistemic features as to ZFC. In fact, some practitioners do so. This is a perfectly intelligible state of affairs, if meaning and intuitiveness are 'produced' in the way described here. Accordingly, behind Sy Friedman's and Jensen's views there lies a perfectly legitimate rationale, however in the minority these views may be.

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[34]See [33] on AC and its status and [12] on Cantor's Continuum Hypothesis. As to the other results mentioned, the following holds. The principle $\diamond$ states the existence of a sequence $\langle S_\alpha \colon \alpha < \omega_1 \rangle$ such that: $S_\alpha \subset \alpha$ and for all $X \subset \omega_1$, the set $\{\alpha < \omega_1 \colon X \cap \alpha = S_\alpha\}$ is *stationary* in ω_1, thus enabling one to 'guess', so to speak, closed unbounded subsets of ω_1 from below. A sequence $\langle S_\alpha \colon \alpha < \omega_1 \rangle$ with these properties is called a $\diamond$-*sequence*. The principle $\square_\kappa$ states that, for an uncountable κ, there exists a sequence $\langle C_\alpha \colon \alpha \in Lim(\kappa^+) \rangle$ (called $\square$-*sequence*) such that: C_α is a closed unbounded subset of α; if $\beta \in Lim(C_\alpha)$, then $C_\beta = C_\alpha \cap \beta$ and if $cf(\alpha) < \kappa$, then $Card(C_\beta) < \kappa$. Since the definition of *morasses* is somewhat complex, I give only an informal description here. A morass is an infinite combinatorial structure used to approximate 'larger' structures (e.g., some level L_{κ^+} of the constructible hierarchy) by directed systems of 'smaller' structures (e.g., levels L_β for β of a smaller cardinality than κ). Morasses are defined at (uncountable regular) cardinals and characterized by gap-parameters. A morass at the (uncountable regular) cardinal κ of gap 1, i.e., a $(\kappa, 1)$-*morass*, is a system approximating the levels L_α for $\alpha \in (\kappa, \kappa^+]$ by means of levels L_β, for $\beta < \kappa$, and maps $f_{\beta\beta'}$ between them. A $(\kappa, 2)$-*morasses* at κ is an 'higher gap' morass approximating the structure $L_{\kappa^{++}}$ in like manner as a $(\kappa, 1)$-*morass* approximates L_{κ^+} (and so on). See [21], [8], [20] and [40] for a detailed description of the mathematical phenomena at issue and the results mentioned in the above paragraph.

[35][22] presents an overview of L-like universes. For technical details see [20], [32] and [44].

§5. A corollary and some objections. Let me now state a corollary to the understanding of meaning and intuitiveness that I suggest. My analyses allow for incompatible meanings of 'set' and conflicting systems of set theoretic intuitions (i.e., intuitively plausible statements about sets) in mathematics. Consider successful but mutually contradictory systems of set theory like ZFC + $V = L$ and ZFC + (large) large cardinals, and also consider the meanings of 'set' and the systems of intuitions connected to them. Indeed, some would say that this is the actual situation in set theory.[36] Others would instead point out that it is a matter of fact that the existence of (large) large cardinals, but not $V = L$ and its consequences, is part of what we now mean by 'set'. Accordingly, they regard working under systems alternative to ZFC + (large) large cardinals as nothing more than a "temporary adoption of a restrictive assumption as a convenient device for avoiding irrelevant structures."[37] Some would finally observe that the greater epistemic force of (large) large cardinal axioms over $V = L$ is a counterexample to my view that meaning and intuitiveness in set theory hinge basically on success. For nobody can deny that, although ZFC + $V = L$ and ZFC + (large) large cardinals are both successful systems, the existence of $0^{\#}$ is mostly regarded as an *axiom* today, i.e., as a proposition with which new set theoretic developments should be consistent, and that CH (the *Continuum Hypothesis*) is usually supposed to be open, no matter what the case may be under ZFC + $V = L$ or alternative systems.[38]

I am ready to accept the corollary ensuing from the view of meaning and intuitiveness that I endorse. Although they are all compatible with ZFC, there may be different meanings of 'set' in mathematics and conflicting systems of intuitively plausible statements about sets. But I am also ready to acknowledge the facts of the practice of set theory. I believe that work done under ZFC + (large) large cardinals yields a meaning of 'set' that is dominant over that produced by work done under ZFC + $V = L$. Is there a contradiction here? I think not. That a univocal meaning of 'set' and a univocal system of set theoretic intuitions seem to have emerged over the last few decades can be explained as depending both on the greater success of a

[36] See [41].

[37] An adoption which one should exclude as part of the "broadest point of view on sets" ([43], p. 422).

[38] $0^{\#}$ does not exist if $V = L$—and we know what the universe would look like if $0^{\#}$ did not exist at all (this is shown by Jensen's *Covering Lemma*). CH, instead, holds under ZFC + $V = L$. Of course there may be exceptions to the view I report here and endorsed by most set theorists. For example, Shelah in [41] appears to be cautious about the status of even (small) cardinal axioms, and of $0^{\#}$, which he regards as "semi-axioms". On the other hand he seems to regard the independence of statements like CH as an answer to the question of whether or not they hold. This is not to deny either that there are authors, like Woodin, who at some point (cf. [47]) asserted the existence of evidence for a solution of CH which differs from that implied by ZFC + $V = L$.

peculiar system of set theory over its competitors and on a preferred option for *unify* over alternative methodological maxims within the set theoretic community.

As I introduced the notion in the previous section, success is not a matter of all-or-nothing. It admits degrees. Accordingly, there is nothing odd about saying that, from an overall perspective, ZFC + (large) large cardinals has proved more successful than ZFC + $V = L$. A detailed analysis of the reasons for the greater success of the former over the latter system would fall outside the scope of this paper.[39] Here I only briefly recall the arguments of Hauser and Steel. L can be studied from within V, under (large) large cardinal hypotheses, but not vice versa. The theory of projective sets implied by (large) large cardinals (exactly *PD*, which is an implication of them) and contradicted by ZFC + $V = L$, fulfills mathematical expectations regarding projective sets of reals grounded on *successful* developments in descriptive set theory, and it makes further significant achievements in the field possible.[40] In fact, the theory one derives from *PD* extends to all the projective hierarchy 'regularity' properties implied by ZFC for its first two levels, properties which projective sets, as 'simple' sets of reals, are expected to possess.[41] Moreover, the negative results about projective sets implied by $V = L$ depend on the existence of a (Δ^1_2) well-ordering of the reals and

> [. . .] the central idea of descriptive set theory is that definable sets of reals are free from the pathologies one gets from a wellorder of the reals. Since $V = L$ implies there is a Δ^1_2 wellorder of the reals, under $V = L$ this central idea collapses low in the projective hierarchy, and after this, there is, in an important sense, *no* descriptive set theory. ([43], p. 429.)

Under these circumstances, if one adopts *unify*, one is likely to see ZFC + (large) large cardinals as closer to what we mean by 'set', as more natural and clearer, than any other axiomatic system for set, with the distinguished meaning of set emerging from the 'most successful' set theory being more attractive than those emerging from alternative systems. In my view, however, there is nothing to *unify* that makes it a mandatory maxim in set theory.

Before giving arguments for this claim, I would point out that Jensen is ready to acknowledge that ZFC + large cardinals has proved more successful than ZFC + $V = L$. In fact in [22], he explicitly recognizes that "the quest for new axioms of infinity has led to very interesting mathematics" and that there are good reasons for regarding these axioms as "successive approximations

[39] Far-reaching comparative accounts have been given elsewhere: e.g., in [27], [28] and [16]. See also my [1] for a detailed analysis of the extrinsic evidence of the system ZFC + large cardinal axioms.

[40] See [34] for details.

[41] See footnote 23 above for more on descriptive set theory and *PD*.

to a final universe."[42] He does not feel compelled, however, to reject the meaning of 'set' and the system of intuitions to which $V = L$ led. In fact, the study of ZFC + $V = L$ was not abandoned by set theorists after the proof of the incompatibility of the axiom of constructibility with (large) large cardinals. Rather, adjustments were made to the meaning of 'set' and to the system of intuitions developed working under $V = L$ while the mathematical techniques developed to study L were used to unfold the content of (large) large cardinals.[43] L itself can envisage the existence of models for large cardinal axioms if there are such cardinals in V. In fact, as a consequence of *Shoenfield's Absoluteness Lemma*, L and V (under $V \neq L$) see the same transitive countable models for a (large) large cardinal hypothesis H, if the latter is assumed to hold in V.[44] "Hence we could just assume ourselves to be in a countable segment of L when we assume H."[45] The moral of the story, according to Jensen, is that the relation between ZFC + $V = L$ and ZFC + large cardinals should be more correctly understood as one of mutual *interpretability* than as one of mutual exclusion.[46] This remark seems to me connected with Jensen's non-commitment to *unify*, a maxim to which I now return.

My contention that there is no a priori ground for *unify* to be mandatory in set theory hinges on the fact that I endorse no form of Platonism or conceptual realism. A Platonist or conceptual realist might well subscribe to my view that the meaning of 'set' and our system of intuitions about sets arise from successful mathematical developments rather than being prior to them. To her/him, however, this would be a description of how we become acquainted with the meaning of 'set', not of a state of affairs concerning that meaning. For the latter is regarded as univocally fixed once and for all by what is the case in a "well-determined reality" of mathematical objects,[47] or, alternatively, by a concept of set existing independently "of our definitions

[42] [22], p. 407.

[43] This is contemporary inner/core model theory. See [20], [32] and [44] for technical details.

[44] In fact if the hypothesis H holds in V, then by reflection H has a model that is a stage of V, e.g., V_κ (note: this is an informal passage in the argument) for some cardinal κ. By *Löwenheim–Skolem*'s theorem, there is a countable elementary submodel of V_κ, call it N, in which H holds. By *Mostowski's Collapsing Theorem* there is a transitive $\overline{N}$ that is a countable transitive model of H. Let $a \in \mathbf{R}$ be a code for $\overline{N}$. The formula $\exists a(\dots)$ is Σ^1_2. By *Shoenfield's Absoluteness Lemma*, it is true in L: that is L envisages the existence of a transitive countable set model for H.

[45] Quoted with permission from the handout for a talk given by Jensen in Krakow in 1999. This means that in ZFC + $V = L$ + "There is a transitive countable model of H" one obtains all Σ^1_2 consequences of H. (See [43] on what he calls the 'instrumentalist dodge'.)

[46] Jensen is wont to stress that the interpretation of (large) large cardinal axioms in L is given via a countable *transitive well founded* model, which is a 'natural' set theoretic model (Jensen, personal communication).

[47] Taken to exist "independently of our constructions and of our having intuition of them individually"([13], p. 258).

and constructions".[48] As a result, for the Platonist and the conceptual realist there cannot be incompatible but equally legitimate meanings of 'set' and systems of set theoretic intuitions. Indeed, this seems to have been Gödel's position.[49] Gödel did not regard mathematical intuition, meant as the faculty both to 'perceive' mathematical objects and to grasp the concept of set, as giving us immediate knowledge of its objects. Rather, what sets exist or, alternatively, what is true for the concept of set, has become clear as set theory has developed.[50] However, both Gödel's Platonist account of mathematical objects and his conceptual realism imply that there is but one unique true theory of sets. By contrast, if one suggests, as I do, that set theory should be analyzed regardless of whether there is a "well-determined reality" of mathematical objects or a concept of set independent "of our definitions and constructions" to which set theory should adhere, one sees set theorists as no longer committed to *unify* in their practice. Accordingly, one takes the existence of incompatible meanings of set and conflicting systems of set theoretic intuitions to be a perfectly tolerable state of affairs. Under these circumstances some practitioners might still choose *unify* as a methodological option. However, nothing would force them to do so. Not even the goal that Maddy (see above, section 2) ascribes to set theory of providing a foundation for the rest of mathematics seems to do so. For set theory may be (and mostly is) pursued as an autonomous mathematical discipline with no foundational concerns. Hence one may even legitimately hold the view that set theory should be developed in as many directions as possible, even incompatible ones.

I do not argue here for my choice of endorsing no form of Platonism or conceptual realism. This would take me too far afield. I merely point out that if this means renouncing, from the outset, a characterization of set theoretic contents as true or false I would be ready to do so.[51] Indeed, this seems to be the choice made by Jensen, who recommends that $V = L$ should not be regarded as—to use his words—an axiom if by 'axiom' one means something different from a suitable hypothesis for proving theorems, e.g., a true assertion about sets.[52] Similarly, I am inclined to consider the circumstance that

[48][11], p. 128. In [46], instead, it is the iterative concept that implies that the theory of sets is univocally determined. On the other hand it is acknowledged that the practice of set theory shows us that "[. . .] our intuition of set develops" along with the advances of set theory. For "sometimes we accept an axiom once stated, sometimes it takes a fairly long time before an axiom is accepted", "it is not implied that we know all properties in advance" and "it is not excluded that we may in the process of studying the subject further come upon and accept new axioms"([46], pp. 181–222).

[49][36] provides a detailed analysis of Gödel's mathematical realism. [18] examines Gödel's philosophical position in the light of Husserl's phenomenology.

[50]See [13], p. 268.

[51]Of course one might appeal to notions of truth different from the realist one based on correspondence (see [30] on this point). I shall not enter into this issue here, however.

[52]Jensen, personal communication.

equally successful though incompatible theories of sets will develop.[53] On the other hand, the contention that discarding the idea that set theory should be faithful to a univocally determined realm of mathematical objects, or to a concept of set independent of our definitions and constructions, entails relinquishing any guarantee of rigour, non-arbitrariness and consistency in set theory, is not as a compelling argument as it might seem at the outset.[54] If issues of meaning and intuitiveness in set theory are supposed to rest on mathematical success, and if the criteria for success are as I briefly introduced them in section 3, no place seems left for arbitrariness and lack of rigour in a mathematical practice driven by the search for successful developments. Analogously, notions and principles which, owing to their success, become intuitive to the practitioners' eyes and lead in time to no contradiction may well produce confidence in their consistency, which, in its turn, will make the mathematical work under them a rationally justified enterprise even though one has no suitable consistency proof.[55]

§6. $V = L$ and intrinsic evidence. To restate my thesis I maintain that there is a mode of being intuitive in set theory that originates from successful mathematics. Hence I believe it perfectly in order to characterize both ZFC + (large) large cardinal axioms and ZFC + $V = L$ as intuitively plausible although they are mutually exclusive systems and although the latter is incompatible with the maximum iterative concept. On the other hand, I do not deny that intrinsically necessary set theoretic contents, i.e., contents that are explained in terms of the maximum iterative concept, may have a privileged status compared to contents that are extrinsically evident alone. My argument is that the maximum iterative concept is a metaphorical reformulation of set theoretic notions and principles capable of strengthening their epistemic status. By giving a metaphorical account of mathematical

[53]Contrary to Maddy who, in [30], rules out this circumstance by appealing to the fact that the maxim *unify* is dictated by the foundational role of set theory.

[54]Note that neither model theoretically (without methods that are not available in ZFC) nor proof theoretically can one prove the consistency of ZFC and of theories that extend it. Nor can one prove in ZFC the relative consistency of large cardinals extensions of ZFC with regard to ZFC (the relative consistency of ZFC + $V = L$ with regard to ZFC, instead, can be proved in ZF). As far as consistency is concerned, that its assumption can only be grounded on some sort of Platonism is suggested by e.g., [46], p. 199, where the author argues: "It is most natural [. . .] to deal with objects and models rather than formulas and formal systems. One might wish to claim this is just a shorthand way of doing things even though set theory is a formal game based on analogy and hasty generalizations. The central weakness of this position is of course its apparent inability to explain how the purely formal set theory can hang together so well."

[55]Note that, as far as large cardinals are concerned, the belief in their consistency is partly grounded on mathematical results connected with the constructions of inner models for them. See [15] for a discussion of the role of the so-called inner model program in supporting the belief in the consistency of large cardinals.

contents one embeds them in a network of meanings more comprehensive than if one gave an explanation of them formulated in mathematical terms alone. As a result one's understanding of these contents increases. However, I also contend that the maximum iterative concept is intelligible qua metaphor only owing to the availability of an already existing mathematical understanding of set theoretic notions and operations. This rules out that intrinsic necessity can be an alternative to extrinsic evidence in set theory, the latter turning out to be necessary, though not sufficient, for the former.[56]

The reason why explanations of set theoretical notions and principles in terms of the maximum iterative concept are usually introduced as metaphorical is that, on being given such explanations, one is asked to regard set theoretic notions and principles 'as if' (infinite) sets and operations on (infinite) sets were produced by a mind different from ours performing acts precluded to us (e.g., running through an infinite multitude up to the point where its entire range is overviewed, completing the process of selecting different multitudes of integers from the set of all integers, etc).[57] 'Idealized' acts like these recall acts of which we are capable (running through the elements of finite multitudes), or which we conceive as possible (running through the members of an infinite multitude as a process that can continue endlessly). However, these acts are ultimately unintelligible to us as possibilities of any mind.[58] Running through an infinite multitude up to the point where its entire range is overviewed seems to contradict a basic understanding of the infinity of whatever multitude, which rules out that any mind can exhaust its elements while running them through. Analogously, discourses about "all possible ways of leaving out integers" in the process of forming subsets of ω (cf. [46]) are doomed to obscurity given the unavailability to us of an ordered manner of producing all possibilities for omitting elements of ω in forming

[56]Much has been written on the *maximum iterative concept* as a metaphor. Consider, e.g., [4], [37], [38], [6], [7], [19]. Most of these contributions criticize the traditional understanding of the maximum iterative concept, endorsed, e.g., by Wang in [46] (see section 2). In none of them, though, is the connection between intrinsic necessity (interpretability in terms of the maximum iterative concept) and extrinsic evidence (intuitiveness due to success) made explicit in the way that it is in this section.

[57]See [46]. The appeal to 'idealized' mental acts is less explicit in other accounts of the maximum iterative concept, like, e.g., [5] and [42]. Yet a constructive vocabulary is used by these authors as well, although without making explicit how exactly it should be understood.

[58]The point is made extensively in [35] and in [14]. Besides the points mentioned below, they remark that the idealized infinite mind appealed to by the supporters of the maximum iterative concept should construct sets in a way that sets that appear at earlier stages of the universe could only be formed by drawing on sets constructed at later stages, which is impossible if construction means construction in time. An alternative approach not involving idealized minds is taken in [18]. It relies on ideas from Husserl's phenomenology to explicate how we gain insight into the possibilities, and the impossibilities, that are intrinsic to the concept of set and therefore identical for any mind, including minds that may be different from ours. See [18], p. 560ff. and p. 564ff., for further details.

subsets of it.[59] The only reason why these 'idealized' acts do not appear nonsensical to us,[60] seems to me to be that they can be traced back to well-understood set theoretic notions and operations (e.g., ω and the Power Set operation), which when the maximum iterative concept was first introduced were already familiar because of successful mathematical work with them.[61]

If this analysis is correct we arrive now at the conclusion, announced earlier, that already available understanding obtained through successful mathematics is necessary in order for the maximum iterative concept to be intelligible qua metaphorical reformulation of set theoretic contents. Of course, there is a circularity here, with light being shed on the maximum iterative concept by the selfsame set theoretic contents that it is supposed to explain. However, this does not prevent the former from playing the role of a paraphrase that sheds additional light on set theoretic notions and operations, thus increasing our confidence in them.[62]

Alternatively, the admonition to *maximize*—and hence to endorse a view of the universe which holds that its stages include all possible subsets of the previous stages "regardless of how or if they can be defined"([46], p. 196)—can be formulated as a purely methodological maxim, independently of the maximum iterative concept, and appealed to in order to increase the evidential status of set theoretic contents that are compatible with it.[63] As a purely methodological maxim, however, *maximize* is not uncontroversial. It entails mutually excluding conclusions concerning, e.g., the Axiom of Choice.[64] Moreover, methodological principles rivaling with *maximize* might be at work in set theory and be connected with set theoretic contents that do not conform to *maximize* but have epistemic features that resemble intrinsic necessity. Jensen's Pythagoreanism described in his 1995 study is just one example. It indeed pertains to a long tradition in the history of mathematics.[65]

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[59]For example Jané ([19], p. 379) argues that "given any list of means for specifying sets of natural numbers Cantor's diagonal method will allow us to specify, in term of the list, a set which is not specifiable by those means."

[60]In fact, in several occasions the maximum iterative concept has been explicitly characterized as a successful and powerful mathematical metaphor. See, e.g., [4], pp. 27–30.

[61]Accounts of set theoretical contents in terms of the maximum iterative concept, thought referring back to [12], first appeared in articles of the 1970's, whereas set theory was born at the end of the 19th century.

[62]This point generalizes an observation made by Jané in [19] with regard to the Axiom of Choice and its justification in terms of the maximum iterative concept. I stress that remarks similar to those reported here can be found in strong criticism against the maximum iterative concept. However, I do not mean to argue against this concept here: for it is, I believe, an a posteriori paraphrase of set theoretic contents. This neutralizes popular criticism against it.

[63]This is done, e.g., by Maddy in [27].

[64]This has been pointed out by Hauser who observed that with *maximize* one can justify both the Axiom of Choice and the principle $\exists j : V \to V$ (there is an elementary embedding of the universe into itself), which is contradictory with Choice. Cf. [17], p. 239.

[65]This is also true, however, of Newtonianism, to which the maxim of *maximize* is related.

Nobody would deny that *maximize* is more popular than other maxims at work in set theory. I suggest, however, that this may be due to the fact that it is *maximize* which is able to explain (large) large cardinals, which represent one of the most successful developments in contemporary set theory.[66] Again I assume that a circular but not vicious process is at work here, with *maximize* making successful mathematical developments even more persuasive and the success of the latter giving intellectual credibility to the former as an overall methodological principle.

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