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A DESCRIPTIVE VIEW OF COMBINATORIAL GROUP THEORY

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Dedicated to the memory of Greg Hjorth.

Abstract. In this paper, we will prove the inevitable non-uniformity of two constructions from combinatorial group theory related to the word problem for finitely generated groups and the Higman–Neumann–Neumann Embedding Theorem.

§1. Introduction. Since the fundamental papers of Friedman–Stanley [3] and Hjorth–Kechris [7], it has been well-known that descriptive set theory provides a framework for measuring the complexity of the possible complete invariants for many naturally occurring classification problems and hence also for measuring the relative complexity of these problems. For example, see Thomas–Velickovic [22], Hjorth–Kechris [8], Clemens–Gao–Kechris [1], Thomas [20] and Ferenczi–Louveau–Rosendal [2]. It is less well-known that descriptive set theory also provides a framework for explaining the inevitable non-uniformity of many classical constructions in mathematics. In this paper, we will illustrate this point by considering two constructions from combinatorial group theory related to the word problem for finitely generated groups and the Higman–Neumann–Neumann Embedding Theorem.

We will begin by considering the word problem for finitely generated groups. For each $n \geq 1$, let $\{w_k(x_1, \dots, x_n) \mid k \in \mathbb{N}\}$ be an effective enumeration of the (not necessarily reduced) words in $x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}$. If $G = \langle a_1, \dots, a_n \rangle$ is a finitely generated group, then

$$\operatorname{Rel}(G) = \{k \in \mathbb{N} \mid w_k(a_1, \dots, a_n) = 1\}.$$

Of course, there is a slight abuse of notation here, since the set $\operatorname{Rel}(G)$ clearly depends on the sequence of generators $a_1, \dots, a_n$. However, if $b_1, \dots, b_m$ is any other sequence of generators of G, then it is easily seen that

$$\{k \in \mathbb{N} \mid w_k(a_1, \dots, a_n) = 1\} \equiv_T \{\ell \in \mathbb{N} \mid w_\ell(b_1, \dots, b_m) = 1\}.$$

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Here $\equiv_T$ denotes the Turing equivalence relation on $2^{\mathbb{N}}$ defined by

$$A \equiv_T B \iff A \leq_T B \text{ and } B \leq_T A.$$

It is well-known that for each subset $A \subseteq \mathbb{N}$, there exists a finitely generated group G_A such that $\mathrm{Rel}(G_A) \equiv_T A$. The usual constructions of the group G_A are highly dependent on the specific subset $A \subseteq \mathbb{N}$, in the sense that if $A \neq B$ are subsets such that $A \equiv_T B$, then the groups G_A, G_B are usually nonisomorphic. Consequently, it is natural to ask whether there is a more uniform construction with the property that if $A \equiv_T B$, then $G_A \cong G_B$. However, Theorem 1.1 below implies that no such construction exists.

Throughout this paper, $\mathcal{G}$ denotes the Polish space of countably infinite groups and $\mathcal{G}_{fg}$ denotes the Polish space of finitely generated groups. (These spaces will be defined in Section 2.) As usual, the powerset $\mathcal{P}(\mathbb{N})$ will be identified with the Cantor space $2^{\mathbb{N}}$ by identifying subsets of $\mathbb{N}$ with their characteristic functions.

THEOREM 1.1. *Suppose that $A \mapsto G_A$ is a Borel map from $2^{\mathbb{N}}$ to $\mathcal{G}_{fg}$ with the property that* $\mathrm{Rel}(G_A) \equiv_T A$ *for all $A \in 2^{\mathbb{N}}$. Then there exists a Turing degree $\mathbf{d}_0$ such that for all Turing degrees $\mathbf{d}$ with $\mathbf{d}_0 \leq_T \mathbf{d}$, there exists an infinite subset $\{A_n \mid n \in \mathbb{N}\} \subseteq \mathbf{d}$ such that the groups $\{G_{A_n} \mid n \in \mathbb{N}\}$ are pairwise incomparable with respect to embeddability.*

Next recall that the Higman–Neumann–Neumann Embedding Theorem [5] states that any countable group G can be embedded into a 2-generator group K_G. In the standard proof of this classical theorem, the construction of the group K_G involves an enumeration of a set $\{g_n \mid n \in \mathbb{N}\}$ of generators of the group G; and it is clear that the isomorphism type of K_G usually depends upon both the generating set and the particular enumeration that is used. Once again, it is natural to ask whether there is a more uniform construction with the property that if $G \cong H$, then $K_G \cong K_H$. In this case, the main result of Thomas [21] shows that no such construction exists. However, it turns out that we can obtain a much more striking result if we are willing to make use of a relatively mild large cardinal assumption.

Throughout this paper, we will write (RC) to indicate that the proof of a given result makes use of the assumption that a Ramsey cardinal exists.

THEOREM 1.2 (RC). *Suppose that $G \mapsto K_G$ is a Borel map from $\mathcal{G}$ to $\mathcal{G}_{fg}$ such that $G \hookrightarrow K_G$ for all $G \in \mathcal{G}$. Then there exists a perfect family $\mathcal{F} \subseteq \mathcal{G}$ of pairwise isomorphic groups such that the groups $\{K_G \mid G \in \mathcal{F}\}$ are pairwise incomparable with respect to relative constructibility; i.e., if $G \neq H \in \mathcal{F}$, then $K_G \notin L[K_H]$ and $K_H \notin L[K_G]$.*

Here $L[K_G]$ denotes the smallest inner model of *ZFC* which contains the finitely generated group K_G.

REMARK 1.3. Working in *ZFC*, we can obtain the weaker conclusion that there exists a perfect family $\mathcal{F} \subseteq \mathcal{G}$ of pairwise isomorphic groups such that the groups $\{K_G \mid G \in \mathcal{F}\}$ are pairwise incomparable with respect to embeddability. (In fact, as we will explain in Section 4, it is possible to prove significantly stronger results.)

REMARK 1.4. The existence of a Ramsey cardinal is certainly not the minimum large cardinal assumption necessary to prove Theorem 1.2. For example, Philip Welch has pointed out that Theorem 1.2 is a consequence of the weaker assumption that $\omega_1^{L[r]} < \omega_1$ for all $r \in \mathbb{R}$. (With this assumption, working with the inner models $L[r]$ for suitable reals r instead of with generic extensions, the arguments of this paper go through virtually unchanged.)

The remainder of this paper is organised as follows. In Section 2, we will discuss the Polish space $\mathcal{G}$ of countably infinite groups and the Polish space $\mathcal{G}_{fg}$ of (marked) finitely generated groups. In Section 3, we will use Borel Determinacy to prove Theorem 1.1; and in Section 4, we will use a generic absoluteness argument to prove Theorem 1.2.

Our notation is standard. For example, we write $G \hookrightarrow H$ to indicate that the group G embeds into the group H. Throughout this paper, $\boldsymbol{\Sigma}^1_n$ and $\boldsymbol{\Pi}^1_n$ will denote the classical (boldface) pointclasses. As this paper is intended to be intelligible to a general audience of logicians, it contains detailed explanations of some points which will be obvious to the experts in recursion theory and descriptive set theory.

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§**2. Spaces of groups.** In this section, we will discuss the Polish space $\mathcal{G}$ of countably infinite groups and the Polish space $\mathcal{G}_{fg}$ of (marked) finitely generated groups.

Throughout this paper, we will use the usual representation of the class of countably infinite groups by the elements of a Polish space. In more detail, let $\mathcal{G}$ be the set of countably infinite groups G with underlying set $\mathbb{N}$ and let $2^{\mathbb{N}^3}$ be the Polish space of all 3-ary functions $f\colon \mathbb{N}^3 \to \{0,1\}$ with the natural product topology. Then, identifying each group $G \in \mathcal{G}$ with the graph of its multiplication operation $m_G \in 2^{\mathbb{N}^3}$, it is easily checked that $\mathcal{G}$ is a G_δ-subset of $2^{\mathbb{N}^3}$ and hence $\mathcal{G}$ is a Polish subspace of $2^{\mathbb{N}^3}$.

Although this method can also be adapted to construct a Polish space of finitely generated groups, we will prefer to use an alternative approach due to Grigorchuk [4], which more faithfully reflects various important features of the class of finitely generated groups. The Polish space $\mathcal{G}_{fg}$ of (marked) finitely generated groups is defined as follows. A *marked group* $(G, \bar{s})$ consists of a finitely generated group with a distinguished sequence $\bar{s} = (s_1, \dots, s_m)$ of generators. (Here the sequence $\bar{s}$ is allowed to contain repetitions and we also allow the possibility that the sequence contains the identity element.) Two marked groups $(G, (s_1, \dots, s_m))$ and $(H, (t_1, \dots, t_n))$ are said to be *isomorphic* if $m = n$ and the map $s_i \mapsto t_i$ extends to a group isomorphism between G and H.

DEFINITION 2.1. For each $m \geq 2$, let $\mathcal{G}_m$ be the set of *isomorphism types* of marked groups $(G, (s_1, \dots, s_m))$ with m distinguished generators.

Let $\mathbb{F}_m$ be the free group on the generators $\{x_1, \dots, x_m\}$. Then for each marked group $(G, (s_1, \dots, s_m))$, we can define an associated epimorphism $\theta_{G,\bar{s}} \colon \mathbb{F}_m \to G$ by $\theta_{G,\bar{s}}(x_i) = s_i$. It is easily checked that two marked groups $(G, (s_1, \dots, s_m))$ and $(H, (t_1, \dots, t_m))$ are isomorphic iff $\ker \theta_{G,\bar{s}} = \ker \theta_{H,\bar{t}}$. Thus we can naturally identify $\mathcal{G}_m$ with the set $\mathcal{N}_m$ of normal subgroups of $\mathbb{F}_m$. Note that $\mathcal{N}_m$ is a closed subset of the compact space $2^{\mathbb{F}_m}$ of all subsets of $\mathbb{F}_m$ and so $\mathcal{N}_m$ is also a compact space. Hence, via the above identification, we can regard $\mathcal{G}_m$ as a compact space.

The topologies on $\mathcal{N}_m$ and $\mathcal{G}_m$ can be described more explicitly as follows. For each marked group $(G, \bar{s})$ and integer $\ell \geq 1$, let $B_\ell(G, \bar{s})$ be the closed ball of radius ℓ around the identity element in the (labelled directed) Cayley graph $\mathrm{Cay}(G, \bar{s})$ of G with respect to the generating sequence $\bar{s}$. Then, letting $\bar{x} = (x_1, \dots, x_m)$, a neighborhood basis in $\mathcal{N}_m$ of the normal subgroup N is given by the collection of open sets

$$U_{N,\ell} = \{M \in \mathcal{N}_m \mid M \cap B_\ell(\mathbb{F}_m, \bar{x}) = N \cap B_\ell(\mathbb{F}_m, \bar{x})\}, \quad \ell \geq 1.$$

If $(G, \bar{s}) \in \mathcal{G}_m$ corresponds to the normal subgroup $N \in \mathcal{N}_m$, then the set of relations $N \cap B_{2\ell+1}(\mathbb{F}_m, \bar{x})$ contains essentially the same information as the closed ball $B_\ell(G, \bar{s})$ in the Cayley graph of $(G, \bar{s})$. It follows that a neighborhood basis in $\mathcal{G}_m$ of the marked group $(G, \bar{s})$ is given by the collection of open sets

$$V_{(G,\bar{s}),\ell} = \{(H, \bar{t}) \in \mathcal{G}_m \mid B_\ell(H, \bar{t}) \cong B_\ell(G, \bar{s})\}, \quad \ell \geq 1.$$

Finally, for each $m \geq 2$, there is a natural embedding of $\mathcal{N}_m$ into $\mathcal{N}_{m+1}$ defined by

$$N \mapsto \text{the normal closure of } N \cup \{x_{m+1}\} \text{ in } \mathbb{F}_{m+1}.$$

This enables us to regard $\mathcal{N}_m$ as a clopen subset of $\mathcal{N}_{m+1}$ and to form the locally compact Polish space $\mathcal{N} = \bigcup \mathcal{N}_m$. Note that $\mathcal{N}$ can be identified with the space of normal subgroups N of the free group $\mathbb{F}_\infty$ on countably many generators such that N contains all but finitely many elements of the basis $X = \{x_i \mid i \in \mathbb{N}^+\}$. Similarly, we can form the locally compact Polish space $\mathcal{G}_{fg} = \bigcup \mathcal{G}_m$ of finitely generated groups via the corresponding natural embedding

$$(G, (s_1, \dots, s_m)) \mapsto (G, (s_1, \dots, s_m, 1)).$$

From now on, we will identify $\mathcal{G}_m$ and $\mathcal{N}_m$ with the corresponding clopen subsets of $\mathcal{G}_{fg}$ and $\mathcal{N}$. If $\Gamma \in \mathcal{G}_{fg}$, then we will write $\Gamma = (G, (s_1, \dots, s_m))$, where m is the least integer such that $\Gamma \in \mathcal{G}_m$. Following the usual convention, we will completely identify the Polish spaces $\mathcal{G}_{fg}$ and $\mathcal{N}$; and we will work with whichever space is most convenient in any given context. For example, if $\Gamma \in \mathcal{G}_{fg}$ is a marked group, then there is no longer any abuse of notation in writing $\mathrm{Rel}(\Gamma)$. On the other hand, if we fix an effective enumeration of $\mathbb{F}_\infty$ and then identify $\Gamma \in \mathcal{G}_{fg}$ with the corresponding normal subgroup $N \in \mathcal{N}$, then we see that Γ is essentially a subset of $\mathbb{N}$ and hence $L[\Gamma] = L(\Gamma)$ is the smallest inner model of *ZFC* which contains Γ.

In the remainder of this paper, we will only consider the usual isomorphism relation $\cong$ on the space $\mathcal{G}_{fg}$ of finitely generated groups; i.e., two marked groups are $\cong$-equivalent iff their underlying groups (obtained by forgetting about their distinguished sequences of generators) are isomorphic. And we will often abuse notation by writing $G \in \mathcal{G}_{fg}$ instead of $\Gamma = (G, (s_1, \dots, s_m)) \in \mathcal{G}_{fg}$. Notice that $\cong$ is a countable Borel equivalence relation on $\mathcal{G}_{fg}$; i.e., for each $G \in \mathcal{G}_{fg}$, the set $\{H \in \mathcal{G}_{fg} \mid H \cong G\}$ is countable. Similarly, for each $G \in \mathcal{G}_{fg}$, the set $\{H \in \mathcal{G}_{fg} \mid H \hookrightarrow G\}$ is also countable.

§3. Borel determinacy and the word problem for finitely generated groups. In this section, we will present the proof of Theorem 1.1. Our proof makes use of a remarkable consequence of Borel Determinacy which classifies the $\leq_T$-cofinal Borel subsets $X \subseteq 2^{\mathbb{N}}$. We will begin by considering the special case when X is $\equiv_T$*-invariant*; i.e., whenever $x \in X$ and $y \equiv_T x$, then $y \in X$. Here the canonical examples are the cones $C_z = \{x \in 2^{\mathbb{N}} \mid z \leq_T x\}$, where $z \in 2^{\mathbb{N}}$. (We have included the proof of Theorem 3.1 in order to give the reader a flavor of the more complicated determinacy argument that is involved in the proof of Theorem 3.4.)

THEOREM 3.1 (Martin [14, 15]). *If $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant $\leq_T$-cofinal Borel subset, then there exists a cone C such that $C \subseteq X$.*

PROOF. Consider the two player game $G(X)$

$$\begin{array}{lcccccccc} I & s(0) & & s(2) & & s(4) & & \dots \\ II & & s(1) & & s(3) & & s(5) & \dots \end{array}$$

where each $s(i) \in \{0, 1\}$ and I wins iff $s = (s(0)s(1)s(2)\dots) \in X$. Then, by Borel Determinacy [15], the game $G(X)$ is determined. First suppose that $\sigma \colon 2^{<\mathbb{N}} \to 2$ is a winning strategy for I. Let $\sigma \leq_T t \in 2^{\mathbb{N}}$ and consider the play of $G(X)$ where

- II plays $t = (s(1)s(3)s(5)\dots)$.
- I responds with σ and plays $(s(0)s(2)s(4)\dots)$.

Then $s \in X$ and $s \equiv_T t$. Hence $t \in X$ and so $C_\sigma = \{t \in 2^{\mathbb{N}} \mid \sigma \leq_T t\} \subseteq X$.

On the other hand, if I does not have a winning strategy, then II must have a winning strategy $\tau \colon 2^{<\mathbb{N}} \to 2$. But then, arguing as above, we see that $C_\tau \subseteq 2^{\mathbb{N}} \smallsetminus X$, which contradicts the fact that X is $\leq_T$-cofinal. ⊣

As we mentioned earlier, we will require a strengthening of Theorem 3.1 which classifies *arbitary* $\leq_T$-cofinal Borel subsets $X \subseteq 2^{\mathbb{N}}$. In this case, the canonical examples are the sets $[S]$ of infinite branches of pointed trees $S \subseteq 2^{<\mathbb{N}}$.

DEFINITION 3.2 (Sacks [18]). The tree $S \subseteq 2^{<\mathbb{N}}$ is said to be *pointed* if S is perfect and $S \leq_T x$ for all $x \in [S]$.

The following lemma summarizes some of the basic properties of pointed trees.

LEMMA 3.3 (Sacks [18]). *Suppose that $S \subseteq 2^{<\mathbb{N}}$ is a pointed tree.*

(i) *If $S \leq_T z \in 2^{\mathbb{N}}$, then there exists a branch $x \in [S]$ such that $x \equiv_T z$.*
(ii) *If $S_0 \subseteq S$ is a subtree such that $S_0 \leq_T S$, then S_0 is also a pointed tree and $S_0 \equiv_T S$.*
(iii) *If $S \leq_T z \in 2^{\mathbb{N}}$, then there exists a pointed tree $S_z \subseteq S$ such that $S_z \equiv_T z$.*

As we will soon see, Theorem 1.1 is a straightforward consequence of the following result. (The proof of Theorem 3.4 can be found in Kechris [12].)

THEOREM 3.4 (Martin). *If X is a $\leq_T$-cofinal Borel subset of $2^{\mathbb{N}}$, then there exists a pointed tree $S \subseteq 2^{<\mathbb{N}}$ such that $[S] \subseteq X$.*

Next recall that if $A, B \subseteq \mathbb{N}$, then A is *one-one reducible* to B, written $A \leq_1 B$, if there exists an injective recursive function $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$n \in A \iff f(n) \in B.$$

The proof of Theorem 1.1 also makes use of the following well-known result.

LEMMA 3.5. *If $G, H \in \mathcal{G}_{fg}$ and $G \hookrightarrow H$, then* $\mathrm{Rel}(G) \leq_1 \mathrm{Rel}(H)$.

PROOF. Suppose that $G = \langle a_1, \dots, a_n \rangle$ and $H = \langle b_1, \dots, b_m \rangle$ are marked finitely generated groups. Let $\varphi\colon G \to H$ be an embedding and let $\varphi(a_i) = t_i(\bar{b})$. Then

$$w_k(a_1, \dots, a_n) = 1 \iff w_k(t_1(\bar{b}), \dots, t_n(\bar{b})) = 1. \qquad \dashv$$

PROOF OF THEOREM 1.1. Suppose that $A \mapsto G_A$ is a Borel map from $2^{\mathbb{N}}$ to $\mathcal{G}_{fg}$ such that $\mathrm{Rel}(G_A) \equiv_T A$ for all $A \in 2^{\mathbb{N}}$ and let $\theta\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the Borel map defined by $A \mapsto \mathrm{Rel}(G_A)$. Since θ is a countable-to-one Borel map, it follows that $\theta(2^{\mathbb{N}})$ is a Borel subset of $2^{\mathbb{N}}$. Clearly $\theta(2^{\mathbb{N}})$ is $\leq_T$-cofinal and hence there exists a pointed tree S such that $[S] \subseteq \theta(2^{\mathbb{N}})$. Let $(\pi_n \mid n \in \mathbb{N})$ be an enumeration of the injective recursive maps $\pi\colon \mathbb{N} \to \mathbb{N}$. Then after replacing S by a suitable pointed subtree if necessary, we can suppose that $(\pi_n \mid n \in \mathbb{N}) \leq_T S$. Notice that if $n \in \mathbb{N}$ and $s_1, s_2 \in S$, then there exist $t_1, t_2 \in S$ such that

- $s_1 \subset t_1$ and $s_2 \subset t_2$; and
- there exists ℓ such that $t_1(\ell) \neq t_2(\pi_n(\ell))$.

Using this observation, it is routine to construct a perfect subtree $S_0 \subseteq S$ such that:

(i) $S_0 \leq_T S$; and
(ii) if $x \neq y \in [S_0]$, then x and y are incomparable with respect to the one-one reducibility relation $\leq_1$.

It follows that S_0 is also a pointed tree. Finally let $B \in 2^{\mathbb{N}}$ be any set such that $S_0 \leq_T B$ and let $S_B \subseteq S_0$ be a pointed subtree such that $S_B \equiv_T B$.

Let $\{x_n \mid n \in \mathbb{N}\} \subseteq [S_B]$ be an infinite set of branches chosen such that each $x_n \leq_T S_B$ and hence $x_n \equiv_T S_B \equiv_T B$. For each $n \in \mathbb{N}$, choose $A_n \in 2^{\mathbb{N}}$ such that $\mathrm{Rel}(G_{A_n}) = x_n$. Then each $A_n \equiv_T B$; and if $n \neq m$, then $\mathrm{Rel}(G_{A_n})$ and $\mathrm{Rel}(G_{A_m})$ are incomparable with respect to $\leq_1$. Hence, by Lemma 3.5, the groups $\{G_{A_n} \mid n \in \mathbb{N}\}$ are incomparable with respect to embeddability. This completes the proof of Theorem 1.1. $\dashv$

§4. Collapsing cardinals and the Higman–Neumann–Neumann embedding theorem. Let $\mathrm{I}(\mathbb{N}, 2^{\mathbb{N}})$ be the Polish space of all injective maps $z\colon \mathbb{N} \to 2^{\mathbb{N}}$. In this section, we will derive Theorem 1.2 as an easy consequence of a slightly technical lemma concerning arbitrary Borel maps $\theta\colon \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}}) \to X$, where X is any Polish space. But before we can state this lemma, we need to introduce two definitions.

DEFINITION 4.1. E_{cntble} is the Borel equivalence relation on $\mathrm{I}(\mathbb{N}, 2^{\mathbb{N}})$ defined by

$$z \; E_{cntble} \; z' \iff \{z(n) \mid n \in \mathbb{N}\} = \{z'(n) \mid n \in \mathbb{N}\}.$$

DEFINITION 4.2. Let $\preceq$ be a quasi-order on the set X. Then $\preceq$ is said to be *countable* if $\{y \in X \mid y \preceq x\}$ is countable for all $x \in X$.

For example, the Turing reducibility relation $\leq_T$ on $2^{\mathbb{N}}$ and the embeddability relation $\hookrightarrow$ on $\mathcal{G}_{fg}$ are both countable Borel quasi-orders. The relative constructibility relation $\leq_c$ defined on $2^{\mathbb{N}}$ by

$$x \leq_c y \iff x \in L[y]$$

is a $\mathbf{\Sigma}^1_2$ quasi-order; and if (RC) holds, then $\leq_c$ is countable. (For example, see Jech [9, Exercise 17.28].)

LEMMA 4.3. *If X is a standard Borel space and $\theta\colon \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}}) \to X$ is any Borel map, then at least one of the following two conditions must hold*:

(a) *There exists $x \in X$ such that for all $r \in 2^{\mathbb{N}}$, there exists $z \in \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}})$ with $r \in \mathrm{range}(z)$ such that $\theta(z) = x$.*
(b) *For each countable Borel quasi-order $\preccurlyeq$ on X, there exists a perfect subset $P \subseteq \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}})$ such that*
   (i) *$y \; E_{cntble} \; z$ for all $y, z \in P$; and*
   (ii) *$\theta(y), \theta(z)$ are incomparable with respect to $\preccurlyeq$ for all $y \neq z \in P$.*

Furthermore, if (RC) holds, then the conclusion also holds with respect to the quasi-order $\leq_c$ of relative constructibility.

The proof of Theorem 1.2 also makes use of the following result.

LEMMA 4.4 (B. H. Neumann [17]). *There exists a Borel family*

$$\{H_r \mid r \in 2^{\mathbb{N}}\} \subseteq \mathcal{G}$$

of pairwise nonisomorphic infinite 2*-generator groups.*

SKETCH PROOF. For each strictly increasing sequence $\mathbf{d} = \langle d_n \mid n \in \omega \rangle$ of odd integers with $d_0 \geq 5$, let $X^n_{\mathbf{d}} = \{x^n_1, x^n_2, \dots, x^n_{d_n}\}$ and let $\Gamma_{\mathbf{d}}$ be the subgroup of $\prod_{n\in\omega} \mathrm{Alt}(X^n_{\mathbf{d}})$ generated by the two permutations

$$\alpha_{\mathbf{d}} = \prod_{n\in\omega} (x^n_1\ x^n_2\ x^n_3 \dots x^n_{d_n}),$$

$$\beta_{\mathbf{d}} = \prod_{n\in\omega} (x^n_1\ x^n_2\ x^n_3).$$

Then by B. H. Neumann [17], up to isomorphism, the finite simple normal subgroups of $\Gamma_{\mathbf{d}}$ are precisely $\{\mathrm{Alt}(d_n) \mid n \in \omega\}$. In particular, the groups $\Gamma_{\mathbf{d}}$ are infinite and pairwise nonisomorphic. The result follows easily. ⊣

PROOF OF THEOREM 1.2 (RC). Suppose that $\varphi\colon \mathcal{G} \to \mathcal{G}_{fg}$ is a Borel map such that G embeds into $\varphi(G)$ for all $G \in \mathcal{G}$. Let $\{H_r \mid r \in 2^{\mathbb{N}}\} \subseteq \mathcal{G}$ be the Borel family of pairwise nonisomorphic infinite 2-generator groups given by Lemma 4.4 and let $\psi\colon \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}}) \to \mathcal{G}$ be the injective Borel map defined by

$$\psi(z) = H_{z(0)} \times H_{z(1)} \times \cdots \times H_{z(n)} \times \cdots$$

i.e., $\psi(z)$ is the *restricted* direct product of the sequence $\langle H_{z(n)} \mid n \in \mathbb{N}\rangle$. Of course, if $y, z \in \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}})$ satisfy $y\ E_{cntble}\ z$, then $\psi(y) \cong \psi(z)$. Let $\theta\colon \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}}) \to \mathcal{G}_{fg}$ be the Borel map defined by $\theta = \varphi \circ \psi$. Applying Lemma 4.3, first suppose that there exists a group $G \in \mathcal{G}_{fg}$ such that for all $r \in 2^{\mathbb{N}}$, there exists $z \in \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}})$ with $r \in \mathrm{range}(z)$ such that $\theta(z) = G$. Then H_r embeds into G for all $r \in 2^{\mathbb{N}}$, which is impossible since G has only countably many finitely generated subgroups. Hence there a perfect set $P \subseteq \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}})$ such that

- $y\ E_{cntble}\ z$ for all $y, z \in P$; and
- the groups $\theta(y), \theta(z)$ are incomparable with respect to the quasi-order $\leq_c$ of relative constructibility for all $y \neq z \in P$.

Since ψ is an injective Borel map, it follows that $\psi[P]$ is an uncountable Borel subset of $\mathcal{G}$; and since $y\ E_{cntble}\ z$ for all $y, z \in P$, it follows that the groups in $\psi[P]$ are pairwise isomorphic. Consequently, if $\mathcal{F} \subseteq \psi[P]$ is any perfect subset, then $\mathcal{F}$ is a perfect family of pairwise isomorphic groups such that the groups $\{K_G \mid G \in \mathcal{F}\}$ are pairwise incomparable with respect to relative constructibility. ⊣

REMARK 4.5. Working in *ZFC*, we obtain the weaker conclusion that there exists a perfect family $\mathcal{F} \subseteq \mathcal{G}$ of pairwise isomorphic groups such the groups

$\{K_G \mid G \in \mathcal{F}\}$ are pairwise incomparable with respect to embeddability. Of course, we can also obtain significantly stronger results by considering other Borel quasi-orders. For example, there exists a perfect family $\mathcal{F} \subseteq \mathcal{G}$ of pairwise isomorphic groups such the sets $\{\mathrm{Rel}(K_G) \mid G \in \mathcal{F}\}$ are pairwise incomparable with respect to Turing reducibility. In fact, Philip Welch has pointed out that, working with suitable inner models instead of with generic extensions, the arguments of this paper prove the existence a perfect family $\mathcal{F} \subseteq \mathcal{G}$ of pairwise isomorphic groups such the sets $\{\mathrm{Rel}(K_G) \mid G \in \mathcal{F}\}$ are pairwise incomparable with respect to hyperarithmetic reducibility.

The remainder of this section is devoted to the proof of Lemma 4.3. From now on, we will work within a fixed base universe V of set theory and consider extensions of various projective relations in suitable generic extensions $V^{\mathbb{P}}$. If R is a projective relation on the Polish space X and $V^{\mathbb{P}}$ is a generic extension, then $X^{V^{\mathbb{P}}}$, $R^{V^{\mathbb{P}}}$ will denote the sets obtained by applying the definitions of X, R within $V^{\mathbb{P}}$. The projective relation R is said to be *absolute* for $V^{\mathbb{P}}$ if $R^{V^{\mathbb{P}}} \cap V = R$. Our proof of Lemma 4.3 is based upon the following two absoluteness theorems.

THEOREM 4.6 (Shoenfield [19]). *If $R \in V$ is a $\mathbf{\Sigma}^1_2$ relation, then R is absolute for every generic extension $V^{\mathbb{P}}$.*

THEOREM 4.7 (Martin–Solovay [16]). *Suppose that κ is a Ramsey cardinal. If $R \in V$ is a $\mathbf{\Sigma}^1_3$ relation and $|\mathbb{P}| < \kappa$, then R is absolute for $V^{\mathbb{P}}$.*

For example, suppose that $\preceq$ is a countable Borel quasi-order on the Polish space X. At first glance, it appears that the countability of $\preceq$ is a $\mathbf{\Pi}^1_3$ property. However, recall that if B is an uncountable Borel subset of a Polish space X, then B contains a nonempty perfect subset of X. Thus, letting $\mathrm{Perf}(X)$ denote the Polish space of nonempty perfect subsets of X, we see that the countability of $\preceq$ can be expressed by the following $\mathbf{\Pi}^1_2$ statement:

$$\forall x \in X \, \forall P \in \mathrm{Perf}(X) \, \exists y \in X \, [y \in P \wedge y \npreceq x].$$

Hence, by the Shoenfield Absoluteness Theorem, if $\mathbb{P}$ is any notion of forcing, then $\preceq^{V^{\mathbb{P}}}$ is a countable Borel quasi-order on the Polish space $X^{V^{\mathbb{P}}}$. (In fact, the Lusin–Novikov Uniformization Theorem [13, 18.10] allows us to express the countability of $\preceq$ by a $\mathbf{\Pi}^1_1$ statement.) Of course, if κ is a Ramsey cardinal and $|\mathbb{P}| < \kappa$, then κ remains a Ramsey cardinal in $V^{\mathbb{P}}$ and hence $\leq_c$ remains a countable quasi-order in $V^{\mathbb{P}}$. (For example, see Jech [9, Theorem 21.2].)

The proof of Lemma 4.3 also makes use of the following notion which was abstracted by Kanovei–Reeken [11] from an argument in Hjorth [6, Section 5].

DEFINITION 4.8 (Kanovei–Reeken [11]). Working in the base universe V, suppose that E is a Borel equivalence relation on the Polish space X. If $\mathbb{P}$ is a notion of forcing, then a *virtual E-class* is a $\mathbb{P}$-name $\boldsymbol{\tau}$ such that:

(i) $\Vdash_{\mathbb{P}} \boldsymbol{\tau} \in X^{V^{\mathbb{P}}}$; and
(ii) $\Vdash_{\mathbb{P}\times\mathbb{P}} \boldsymbol{\tau}_{\text{left}}\, E^{V^{\mathbb{P}\times\mathbb{P}}}\, \boldsymbol{\tau}_{\text{right}}$.

Here $\boldsymbol{\tau}_{\text{left}}$ and $\boldsymbol{\tau}_{\text{right}}$ are the $(\mathbb{P}\times\mathbb{P})$-names such that if $G\times H$ is $(\mathbb{P}\times\mathbb{P})$-generic, then $\boldsymbol{\tau}_{\text{left}}[G \times H] = \boldsymbol{\tau}[G]$ and $\boldsymbol{\tau}_{\text{right}}[G \times H] = \boldsymbol{\tau}[H]$.

EXAMPLE 4.9. Let $E = E_{cntble}$ and let $\mathbb{P}$ consist of all finite injective partial functions $p\colon \mathbb{N} \to 2^{\mathbb{N}}$, partially ordered by $p \le q$ iff $p \supseteq q$. If $G \subseteq \mathbb{P}$ is generic, then $g = \bigcup G \in V^{\mathbb{P}}$ is a bijection between $\mathbb{N}$ and $2^{\mathbb{N}} \cap V$. Hence if $\boldsymbol{\tau}$ is the canonical $\mathbb{P}$-name such that $\boldsymbol{\tau}[G] = g$, then $\boldsymbol{\tau}$ is a virtual E_{cntble}-class.

We are finally ready to present the proof of Lemma 4.3. Let $\preceq$ be either a countable Borel quasi-order on the Polish space X or else the relative constructibility relation $\le_c$ on $2^{\mathbb{N}}$. If $x, y \in X$, then we will write $x \parallel y$ if x, y are $\preceq$-comparable and write $x \perp y$ if x, y are $\preceq$-incomparable. For the rest of this section, $\mathbb{P}$ denotes the notion of forcing consisting of all finite injective partial functions $p\colon \mathbb{N} \to 2^{\mathbb{N}}$ and τ is the virtual E_{cntble}-class defined in Example 4.9. Our analysis splits into two cases, depending on whether or not there exists an element $p_0 \in \mathbb{P}$ such that

$$\langle p_0, p_0\rangle \Vdash \theta(\boldsymbol{\tau}_{\text{left}}) \parallel \theta(\boldsymbol{\tau}_{\text{right}}).$$

We will show that if such an element exists, then Condition 4.3(a) holds; while if no such element exists, then Condition 4.3(b) holds. It should be stressed that the following argument is based very closely on the proofs in Hjorth [6, Section 5] and Kanovei–Reeken [11, Section 4]. (An account of Kanovei–Reeken [11, Section 4] can also be found in Kanovei [10, Chapter 17].)

CASE 1. Suppose that there exists an element $p_0 \in \mathbb{P}$ such that

$$\langle p_0, p_0\rangle \Vdash \theta(\boldsymbol{\tau}_{\text{left}}) \parallel \theta(\boldsymbol{\tau}_{\text{right}}).$$

CLAIM 4.10. *There exists $p_1 \le p_0$ such that $\langle p_1, p_1\rangle \Vdash \theta(\boldsymbol{\tau}_{\text{left}}) = \theta(\boldsymbol{\tau}_{\text{right}})$.*

PROOF. Suppose not and let $\mathbb{Q}$ be a notion of forcing which collapses $\mathcal{P}(\mathbb{P} \times \mathbb{P})$ to a countable set. For the remainder of this argument, we will work inside $V^{\mathbb{Q}}$. Let $\{D_n \mid n \in \mathbb{N}\}$ be an enumeration of the dense open subsets $D \subseteq \mathbb{P} \times \mathbb{P}$ such that $D \in V$. Then we can inductively define conditions $p_s \in \mathbb{P}$ for $s \in 2^{<\mathbb{N}}$ such that the following hold:

(i) $p_\emptyset = p_0$;
(ii) if $s \subset t$, then $p_s \ge p_t$;
(iii) if $s \ne t \in 2^{n+1}$, then $\langle p_s, p_t\rangle \in D_n$; and
(iv) $\langle p_{s^\frown 0}, p_{s^\frown 1}\rangle \Vdash \theta(\boldsymbol{\tau}_{\text{left}}) \ne \theta(\boldsymbol{\tau}_{\text{right}})$.

For each $\alpha \in 2^{\mathbb{N}}$, let $G_\alpha = \{q \in \mathbb{P} \mid (\exists n) p_{\alpha\restriction n} \le q\}$ and let $x_\alpha = \bigcup G_\alpha$. Then $C = \{x_\alpha \mid \alpha \in 2^{\mathbb{N}}\}$ is a perfect subset of $\mathrm{I}(\mathbb{N}, 2^{\mathbb{N}})$ such that:

(a) $\theta \restriction C$ is injective; and
(b) if $y, z \in \theta(C)$, then $y \parallel z$.

Let $Z \subseteq \theta(C)$ be a perfect subset and consider the set

$$A = \{\langle y, z\rangle \in Z \times Z \mid y \preceq z\}.$$

If $\preceq$ is a countable Borel quasi-order, then A is a Borel subset of $Z \times Z$ and hence has the Baire property. On the other hand, if $\preceq$ is the relative constructibility relation $\leq_c$, then A is a $\mathbf{\Sigma}^1_2$ subset of $Z \times Z$; and since we are assuming (RC), it again follows that A has the Baire property. (For example, see Jech [9, 26.21].) In both cases, each section $A^z = \{y \in Z \mid y \preceq z\}$ is countable; and hence, by the Kuratowski–Ulam Theorem, it follows that A is a meager subset of $Z \times Z$. Similarly, we see that

$$B = \{\langle y, z\rangle \in Z \times Z \mid y \succeq z\}.$$

is a meager subset of $Z \times Z$. But since $Z \times Z = A \cup B$, this contradicts the Baire Category Theorem. $\dashv$

Claim 4.11. *There exists $x \in X$ such that $p_1 \Vdash \theta(\boldsymbol{\tau}) = x$.*

Proof. To simplify notation, suppose that $X = [0, 1]$. Then, working in V, we can inductively define conditions

$$p_1 \geq p_2 \geq p_3 \geq \cdots \geq p_n \geq \dots$$

and closed intervals $I_n \subseteq [0, 1]$ with rational endpoints

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset \dots$$

such that $|I_n| = 2^{-(n-1)}$ and $p_n \Vdash \theta(\boldsymbol{\tau}) \in I_n$. Let $\bigcap_{n\geq 1} I_n = \{x\}$. Then we claim that x satisfies our requirements. Otherwise, there exist $q \leq p_1$ and $n \geq 1$ such that $q \Vdash \theta(\boldsymbol{\tau}) \notin I_n$. But then $\langle q, p_n\rangle \leq \langle p_1, p_1\rangle$ satisfies

$$\langle q, p_n\rangle \Vdash \theta(\boldsymbol{\tau}_{\text{left}}) \notin I_n \text{ and } \theta(\boldsymbol{\tau}_{\text{right}}) \in I_n,$$

which is a contradiction. $\dashv$

Let $G \subseteq \mathbb{P}$ be a V-generic filter such that $p_1 \in G$. Then we have that $V[G] \vDash \theta(\boldsymbol{\tau}[G]) = x$. Recall that $\boldsymbol{\tau}[G] = \bigcup G$ is a bijection between $\mathbb{N}$ and $2^{\mathbb{N}} \cap V$. Hence for each real $r \in 2^{\mathbb{N}} \cap V$,

$$V[G] \vDash (\exists z \in \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}}))(\exists n \in \mathbb{N})[z(n) = r \text{ and } \theta(z) = x].$$

Applying the Shoenfield Absoluteness Theorem, this $\mathbf{\Sigma}^1_1$ property of the reals $r, x \in 2^{\mathbb{N}} \cap V$ must also hold in V; and so Condition 4.3(a) holds in V.

Case 2. Suppose that there does *not* exist $p \in \mathbb{P}$ such that

$$\langle p, p\rangle \Vdash \theta(\boldsymbol{\tau}_{\text{left}}) \parallel \theta(\boldsymbol{\tau}_{\text{right}}).$$

Once again, let $\mathbb{Q}$ be a notion of forcing which collapses $\mathcal{P}(\mathbb{P} \times \mathbb{P})$ to a countable set. Arguing as in the proof of Claim 4.10, we see that the following statement holds in $V^{\mathbb{Q}}$:

$$\begin{aligned}&\exists P \in \mathrm{Perf}(\mathrm{I}(\mathbb{N}, 2^{\mathbb{N}}))\, \forall x, y \in \mathrm{I}(\mathbb{N}, 2^{\mathbb{N}})\\ &\qquad [(x, y \in P \wedge x \neq y) \Longrightarrow (x\, E_{cntble}\, y \wedge \theta(x) \perp \theta(y))]. \quad (4.12)\end{aligned}$$

If $\preceq$ is a countable Borel quasi-order, then (4.12) is a $\mathbf{\Sigma}^1_2$ statement; and if $\preceq$ is the relative constructibility relation $\leq_c$, then (4.12) is a $\mathbf{\Sigma}^1_3$ statement. Hence, applying either the Shoenfield Absoluteness Theorem or the Martin–Solovay Absoluteness Theorem, it follows that statement (4.12) also holds in V. This completes the proof of Lemma 4.3.

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