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THE COMPLEXITY OF CLASSIFICATION PROBLEMS FOR MODELS OF ARITHMETIC

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Abstract. We observe that the classification problem for countable models of arithmetic is Borel complete. On the other hand, the classification problems for finitely generated models of arithmetic and for recursively saturated models of arithmetic are Borel; we investigate the precise complexity of each of these. Finally, we show that the classification problem for pairs of recursively saturated models and for automorphisms of a fixed recursively saturated model are Borel complete.

§1. Introduction. It is well-known that models of Peano Arithmetic (PA) are highly unclassifiable. In this note, we aim to make this statement more precise by showing that many natural classification problems related to countable nonstandard models are of high complexity according to the descriptive set theory of equivalence relations. Our main tool will be Gaifman's minimal types of [4], which provide a method of constructing models of PA "along" linear orders. The book [9] provides all of the necessary details of this method, as well as the background concerning recursively saturated models. The model-theoretic arguments that we shall use are standard. We will try to give enough details in our arguments so that readers unfamiliar with models of arithmetic can understand the most important special cases.

In order to rigorously discuss the complexity of classification problems, we must use the language of *Borel equivalence relations*, an area of descriptive set theory. This subject was initiated in [3] and [6], and a good introduction can be found in [7]. To explain how it applies, we shall demonstrate how each of the classification problems which we shall consider (along with a great many others) can be identified with an equivalence relation on some standard Borel space. Recall that a *standard Borel space* is complete separable metric space equipped just with its σ-algebra of Borel sets. The most important example for us is the following. If $\mathcal{L}$ is a countable relational language and Θ is an $\mathcal{L}$-theory (or more generally, a sentence of the infinitary language $\mathcal{L}_{\omega_1,\omega}$ in which infinite conjunctions and disjunctions are allowed), then the set

$$X_\Theta := \{M : \text{the domain of } M \text{ is } \omega \text{ and } M \models \Theta\}$$

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is called the *space of countable models of* Θ.[1] Studying the classification problem for countable Θ-models now amounts to studying the *isomorphism equivalence relation* $\cong_\Theta$ on X_Θ.

Now, if E, F are (not necessarily Borel) equivalence relations on the standard Borel spaces X, Y, then we say that E is *Borel reducible* to F (written $E \leq_B F$) iff there exists a Borel function $f\colon X \to Y$ such that

$$x \; E \; x' \iff f(x) \; F \; f(x').$$

The function f is said to be a *Borel reduction* from E to F. Informally, we take $E \leq_B F$ to imply that the classification problem for elements of Y up to F is at least as hard as the classification problem for elements of X up to E.

DEFINITION 1.1. Let $\mathcal{L}$ be a countable language and Θ a sentence of $\mathcal{L}_{\omega_1,\omega}$. The class of Θ-models is said to be *Borel complete* iff for any $\mathcal{L}'$ and any sentence Θ' of $\mathcal{L}'_{\omega_1,\omega}$, we have $\cong_{\Theta'} \leq_B \cong_\Theta$.

We remark that the terminology is unfortunately misleading, since if $\cong_\Theta$ is the isomorphism relation for a Borel complete class, then $\cong_\Theta$ is a properly analytic set pairs. The Borel complete equivalence relations form a single bireducibility class which is of course quite high in the $\leq_B$ hierarchy. Many familiar classes are known to be Borel complete. For some examples, it is shown in [3] that the class of countable groups, of countable connected graphs, and of countable linear orders are all Borel complete.

In the next section, we shall show that the classification problem for countable models of arithmetic is also Borel complete. Afterwards, we turn our attention to the classification problems for various important collections of countable models of PA. In the third section, we consider the class of finitely generated models, and in the fourth the recursively saturated models. The classification problem for each of these classes of models is Borel.

In the final two sections, we consider the isomorphism problem for particular expansions of models of PA. In the fifth section, we shall show that the classification problem for elementary pairs of recursively saturated models is Borel complete. As an application, we show in the last section that the conjugacy problem for automorphisms of a recursively saturated model is also Borel complete.

We would like to thank Jim Schmerl for his careful reading of the preliminary version of this paper. Jim caught a serious error and replaced it with an interesting result (contradicting our erroneous claim) which is Theorem 3.4 below. The theorem and its proof are presented here with his kind permission.

[1]More precisely, if $a(R)$ denotes the arity of $R \in \mathcal{L}$, then X_Θ can be regarded as a Borel subset of the space $\prod_{R\in\mathcal{L}} \mathcal{P}(\omega^{a(R)})$ of all $\mathcal{L}$-structures with domain ω. It follows from the general theory that X_Θ is a standard Borel space in its own right.

§**2. Canonical I-models.** In this section, we will outline how Gaifman used minimal types to build canonical models of PA along a given linear order (refer to Section 3.3 of [9] for the full details). From the details of this construction, we shall see that the isomorphism relation for countable linear orders is Borel reducible to the that for countable models of PA and hence that the class of countable models of PA is Borel complete. Lastly, we will give some additional facts concerning these canonical models that will be useful in later sections.

For us, a type $p(x)$ is *minimal* iff it satisfies the following two properties:

(1) *unbounded*: $(t < x) \in p(x)$ for each closed Skolem term t, and
(2) *indiscernible*: for every model M, and all increasing finite sequences $\bar{a}, \bar{b}$ of elements realizing $p(x)$ in M, if $\bar{a}$ and $\bar{b}$ are of the same length, then $(M, \bar{a}) \equiv (M, \bar{b})$.

We remark that this is not Gaifman's original definition, but rather a convenient characterization.

We now show how to to build, given a linear order I and a completion T of PA, the *canonical I-model* of T, which we shall denote $M_T(I)$. First, fix a minimal type $p(x)$. There are $2^{\aleph_0}$ many such types and it is not important which one we pick, but to make our constructions parameter-free, we can always choose one which is uniformly arithmetic in T. Next, form the type

$$\Delta(x_i)_{i\in I} := \bigcup \{p(x_i) \colon i \in I\} \cup \bigcup \{x_i < x_j \colon i, j \in I \wedge i < j\},$$

and let $M_T(I)$ be the Skolem closure of a sequence realizing $\Delta(x_i)_{i\in I}$. From the results of Section 3.3 of [9], we in fact have that I can be recovered from $M_T(I)$. In other words, if $(I, <)$ and $(J, <)$ are linear orders, then

$$(I, <) \cong (J, <) \iff M_T(I) \cong M_T(J).$$

Although this works for linear orders of any cardinality, it is easy to see that for countable I, the construction of $M_T(I)$ is arithmetic in T and I. In particular, there exists a Borel function f from the space of countable linear orders to the space of countable T-models such that whenever $x = (\omega, <)$ is of ordertype I then $f(x)$ is of isomorphism type $M_T(I)$.

We have established the following result. Let $\cong_{\mathsf{LO}}$ denote the isomorphism equivalence relation on the space of countable linear orders and $\cong_T$ the isomorphism equivalence relation on the space of countable T-models.

THEOREM 2.1. *There exists a Borel reduction from $\cong_{\mathsf{LO}}$ to $\cong_T$ which sends a linear order I to a canonical I-model of T. In particular, $\cong_T$ is Borel complete.*

We remark that for I infinite, the model $M_T(I)$ is not finitely generated. It follows from the results of the next section that the use of non-finitely generated models is essential for Theorem 2.1.

In later sections, we shall require another important feature of canonical I-models. First, recall that an extension $K \prec M$ is said to be an *end extension*, written $K \prec_{\mathsf{end}} M$, iff K is an initial segment of M. Next, for a structure M, we let $\mathrm{Def}(M)$ denote collection of all subsets of M which are definable from parameters in M. An extension $K \prec M$ is said to be *conservative* iff for every $X \in \mathrm{Def}(M)$, we have that $X \cap K \in \mathrm{Def}(K)$. The MacDowell–Specker Theorem (see for instance Theorem 2.2.8 of [9]) states that any model of PA has a conservative elementary end extension. The following classical result, due to Gaifman, can again be found in more detail in Section 3.3 of [9].

THEOREM 2.2. *Let $M_T(I)$ be the canonical I model and suppose that J is a proper initial segment of I. Then $M_T(I)$ is a conservative elementary end extension of $M_T(J)$.*

For further applications, let us note that the construction of $M_T(I)$ works in a much more general context. For $\mathcal{L} \supset \{+, \times, 0, 1\}$ a countable language, we let $\mathsf{PA}(\mathcal{L})$ be the theory obtained from PA by adding instances of the induction schema for all $\mathcal{L}$-formulas. We will use the notation PA^* as a stand-in for $\mathsf{PA}(\mathcal{L})$ for any countable $\mathcal{L}$. The construction of canonical I-models can also be carried out for models of PA^*, and everything which has been said in this section holds in this general situation as well.

§3. **Finitely generated models.** While each $\cong_T$ is Borel complete, the isomorphism equivalence relation $\cong_T^{\mathsf{fg}}$ on the space of *finitely generated* models of T is Borel. In this section, we shall see that according to the $\leq_B$ hierarchy, $\cong_T^{\mathsf{fg}}$ lies among the countable Borel equivalence relations. After introducing this important class, we present a theorem of Schmerl which helps us further understand the complexity of $\cong_T^{\mathsf{fg}}$.

For each arithmetic formula $\varphi(x, \bar{y})$ there is a corresponding Skolem term $t_\varphi(\bar{y})$, which is defined to be $\min\{x : \varphi(x, \bar{y})\}$ if this set is nonempty, and 0 otherwise. If M is a model of PA, then the *Skolem closure* of some $\bar{a} \in M^n$ is the set

$$\mathrm{Scl}(\bar{a}) = \{t(\bar{a}) : t \text{ is a Skolem term}\}.$$

M is said to be *finitely generated* iff there is an $\bar{a} \in M^n$ such that $M = \mathrm{Scl}(\bar{a})$. Since it is possible to code a finite sequence of natural numbers as a single natural number, and this can be done definably in PA, we can always suppose that a finitely generated model is generated by a single element. If M and N are finitely generated, then $M \cong N$ iff there are a and b such that $M = \mathrm{Scl}(a)$, $N = \mathrm{Scl}(b)$ and $\mathrm{tp}(a) = \mathrm{tp}(b)$. One can verify that the condition on the right hand of this equivalence is Borel (in M and N).

The observation that $\cong_T^{\mathsf{fg}}$ is Borel, combined with Theorem 2.1, already yields an interesting corollary. By Theorem 2.1.12 of [9], every countable model M of T has a finitely generated minimal elementary end extension. The construction used in the proof Theorem 2.1.12 of [9] is not canonical, it depends on the choice of enumeration of the model M. The following result shows that in fact there is no canonical construction.

Corollary 3.1. *Let T be a completion of* PA. *Then there is no Borel map taking each countable model M of T to a finitely generated minimal elementary end extension of M.*

Proof. Suppose that f is such a map. If both extensions $M \prec_{\mathsf{end}} M'$ and $N \prec_{\mathsf{end}} N'$ are minimal, then any isomorphism between M' and N' must map M onto N. Hence if M and N are nonisomorphic, then M' and N' are nonisomorphic. It follows that f is in fact a Borel *reduction* from $\cong_T$ to $\cong_T^{\mathsf{fg}}$. Hence, the composition of f with the Borel reduction $(I, <) \mapsto M_T(I)$ given by Theorem 2.1 would yield a Borel reduction from $\cong_{\mathsf{LO}}$ to $\cong_T^{\mathsf{fg}}$. But this is impossible, since $\cong_T^{\mathsf{fg}}$ is Borel complete, and a Borel complete equivalence relation cannot be Borel. ⊣

We next observe that $\cong_T^{\mathsf{fg}}$ has the stronger property that it is essentially countable. Here, a Borel equivalence relation E is called *countable* iff every E-class is countable, and E is called *essentially countable* iff it is Borel bireducible with a countable Borel equivalence relation. Let us also say that a class $\mathcal{C}$ of countable models is essentially countable iff the isomorphism equivalence relation $\cong_C$ on $\mathcal{C}$ is essentially countable. We will need the following characterization from [6] of the essentially countable classes.

Theorem 3.2 (Hjorth–Kechris). *Let Θ be a sentence of $\mathcal{L}_{\omega_1,\omega}$. Then the class of models of Θ is essentially countable iff there is a countable fragment F of $\mathcal{L}_{\omega_1,\omega}$ with $\Theta \in F$ such that for every countable $M \models \Theta$ there exists $n \in \omega$ and $\bar{a} \in M^n$ such that $\mathrm{Th}_F(M, \bar{a})$ is $\aleph_0$-categorical.*

Many classes of models which are finitely generated in some sense turn out to be essentially countable. For instance, the class of finitely generated groups is essentially countable, as is the class of fields of finite transcendence degree.

Proposition 3.3. *$\cong_T^{\mathsf{fg}}$ is essentially countable.*

Proof. Let Θ be the conjunction of the axioms of PA together with the sentence

$$\exists x \, \forall y \bigvee \{y = t(x) \colon t \text{ is a Skolem term}\}.$$

If F is any countable fragment of $\mathcal{L}_{\omega_1,\omega}$ containing Θ, then the sentence

$$\forall y \bigvee \{y = t(a) \colon t \text{ is a Skolem term}\}$$

is in $\mathrm{Th}_F(M, a)$, and the result follows from Theorem 3.2. ⊣

We now briefly discuss the structure of the countable Borel equivalence relations. Here, we will work only on *uncountable* standard Borel spaces; it is a classical result that there is a unique such space up to Borel bijections. By a theorem of Silver, the *equality* equivalence relation $=_{2^\omega}$ on 2^ω is the least complex countable Borel equivalence relation. An equivalence relation E which is Borel reducible to $=_{2^\omega}$ is called *smooth*, or *completely classifiable* because the Borel reduction gives a system of complete invariants for the classification problem up to E. The next least complex equivalence relation is the *almost equality* relation E_0 on 2^ω defined by $x \mathrel{E_0} x'$ iff $x(n) = x'(n)$ for all but finitely many n. By Harrington–Kechris–Louveau [5], a Borel equivalence relation E is nonsmooth iff $E_0 \le_B E$.

It also turns out that there exists a *universal* countable Borel equivalence relation, which we denote by E_∞. For instance, the class of finitely generated groups lies at the level of E_∞, as does the class of connected locally finite graphs. It seems likely that $\cong_T^{\mathsf{fg}}$ is also bireducible with E_∞, but we don't know this yet for sure. We now present an argument of Schmerl which at least eliminates the possibility that $\cong_T^{\mathsf{fg}}$ is smooth.

Theorem 3.4. *If T is any completion of* PA, *then E_0 is Borel reducible to $\cong_T^{\mathsf{fg}}$.*

For the proof, let M be a prime model of T and let G be the group of definable permutations of M. Then G acts on the space $S(T)$ of complete 1-types over T by setting $gp(x) =$ the unique complete type in $S(T)$ containing $\{\varphi(g^{-1}(x)) \colon \varphi(x) \in p(x)\}$. (Here, each $g \in G$ is identified with a Skolem term for g.) Notice that if $p(x)$ is the type of a, then $gp(x)$ is the type of $g(a)$. Let $\sim_T$ denote the orbit equivalence relation on $S(T)$ induced by the action of G. Theorem 3.4 follows immediately from the following two lemmas.

Lemma 3.5. *$\cong_T^{\mathsf{fg}}$ is Borel bireducible with $\sim_T$.*

Proof. To see that $\sim_T \le_B \cong_T^{\mathsf{fg}}$, consider the map which sends a type $p(x)$ to a canonically defined prime model of $p(x)$. To see that $\cong_T^{\mathsf{fg}} \le_B \sim_T$, consider a map which sends a finitely generated model of T to the type of one of its generators. $\dashv$

Lemma 3.6. *E_0 is Borel reducible to $\sim_T$.*

Proof. We will construct a family $\langle X_s \colon s \in 2^{<\omega}\rangle$ of unbounded definable subsets of M with the following properties:

(1) $X_s \subset X_t$ whenever $s \supset t$;
(2) $X_s \cap X_t = \emptyset$ whenever $|s| = |t|$ and $s \neq t$;
(3) for every $b \in 2^\omega$, there exists a unique type $p_b(x)$ such that $X_{b\restriction n} \in p_b$ for all $n \in \omega$. (Here we say that a definable set X is in p iff the formula that defines it is in p.);
(4) for every $b, b' \in 2^\omega$, we have $b \mathrel{E_0} b'$ iff $p_b \sim_T p_{b'}$.

Thanks to property (4), the proof will be complete once this is done. Our construction will have the following additional property. First, for all $s, t \in 2^{<\omega}$ such that $|s| = |t|$, let $\alpha_{s,t} \colon X_s \to X_t$ denote the unique definable order-preserving bijection. Then we will have:

(5) $\alpha_{s,t}\restriction X_{sr} = \alpha_{sr,tr}$ for all s, t such that $|s| = |t|$, and for all r.

To begin the construction, let $\langle \phi_i(x, y) \colon i \in \omega\rangle$ be a fixed enumeration of the binary formulas. Let $X_\emptyset = M$, and given X_s for all $s \in 2^n$, we define

X_{s0}, X_{s1} as follows. First, repeatedly using Ramsey's Theorem (formalized inside PA) and the functions $\alpha_{s,t}$, we get unbounded, definable subsets $Y_s \subset X_s$ such that

(6) Y_s is homogeneous for $\phi_n(x, \alpha_{s,t}(y))$ for all $s, t \in 2^n$; [Here, Y is said to be *homogeneous* for $\varphi(x, y)$ iff for all $x, y, u, v \in Y$ with $x < y$ and $u < v$, we have $(\varphi(x, y) \longleftrightarrow \varphi(u, v)) \wedge (\varphi(y, x) \longleftrightarrow \varphi(v, u)) \wedge (\varphi(x, x) \longleftrightarrow \varphi(u, u))$.]

(7) $\alpha_{s,t}(Y_s) = Y_t$ for all $s, t \in 2^n$.

Next, let X_{s0}, X_{s1} be a partition of Y_s into disjoint unbounded and definable sets (you could take every other element in an enumeration in Y_s). Now, for each $b \in 2^\omega$ we define p_b by

$$\varphi(x) \in p_b \iff \exists n\, X_{b\restriction n} \subseteq \varphi(M).$$

Each of the properties (1)–(3) and (5)–(7) is easily verified; it remains only to establish (4). Suppose first that $b\ E_0\ b'$, and let $n \in \omega$ be the last index such that $b(n-1) \neq b'(n-1)$. Then by (5), $\alpha_{b\restriction n, b'\restriction n}$ maps $X_{b\restriction i}$ onto $X_{b'\restriction i}$ for all $i \geq n$. It is not difficult to extend $\alpha_{b\restriction n, b'\restriction n}$ to a definable permutation of M which also maps $X_{b\restriction i}$ onto $X_{b'\restriction i}$ for all $i < n$. It follows that $p_b \sim_T p_{b'}$.

For the converse, suppose that $p_b(x) \sim_T p_{b'}(x)$ and let $g \in G$ be a definable permutation such that $p_{b'} = g p_b$. Then we have:

$$\text{for all definable } X, \text{ if } X \in p_b \text{ then } g(X) \in p_{b'}.$$

Let n be such that $\varphi_n(x, y)$ is the formula $y = g(x)$. Let $s = b\restriction n$ and $t = b'\restriction n$. Then Y_s is homogeneous for $\varphi_n(x, \alpha_{s,t}(y))$. Since $\varphi_n(x, \alpha_{s,t}(y))$ defines the relation $g(x) = \alpha_{s,t}(y)$, by (6) one of the following holds:

(a) For all $x \in Y_s$, $g(x) = \alpha_{s,t}(x)$, or

(b) For all $x, y \in Y_s$, if $x \neq y$, then $g(x) \neq \alpha_{s,t}(y)$.

But (b) implies that g sends Y_s completely outside of Y_t, contradicting that $p_{b'} = g p_b$. Thus (a) holds, and this implies that $g \restriction Y_s = \alpha_{s,t} \restriction Y_s$. It follows that $\alpha_{s,t}$ maps $X_{b\restriction i}$ to $X_{b'\restriction i}$ for all $i > n$, and together with (5) this implies that $b(i) = b'(i)$ for all $i > n$. Thus, $b\ E_0\ b'$, and the proof is complete. $\dashv$

It is worth remarking that as a consequence of property (6) of the above construction, the types p_b are each unbounded and 2-indiscernible. Such types are indiscernible and minimal in the sense of Gaifman. Since minimal types are extremely special, this gives some evidence that $\sim_T$ is much more complex than E_0.

§4. Recursively saturated models. Let $\mathcal{L}$ be a finite first-order language. An $\mathcal{L}$-structure is *recursively saturated* iff for any finite $\bar{a} \in M^n$, and any recursive set of $\mathcal{L}$-formulas $p(x, \bar{y})$, if $p(v, \bar{a})$ is consistent with $\mathrm{Th}(M, \bar{a})$, then $p(v, \bar{a})$ is realizable in M. Countable recursively saturated models of PA form a robust class which has been intensively studied over the last 30 years.

In this section we shall show that, in contrast with the class of all countable models of PA, the classification problem for the countable recursively saturated models is Borel. We shall even isolate its precise complexity.

To see that the classification problem for countable recursively saturated models is Borel, we need only the most basic property of recursively saturated models. Recall that the *standard* system of a nonstandard model $M \models \mathsf{PA}$ is the collection

$$\mathrm{SSy}(M) := \{X \cap \mathbb{N} \colon X \in \mathrm{Def}(M)\}.$$

The following result is standard, see for instance Proposition 1.8.1 of [9] for a proof.

Proposition 4.1. *If M and N are recursively saturated models of a completion T of* PA, *then $M \cong N$ iff* $\mathrm{SSy}(M) = \mathrm{SSy}(N)$.

When M is countable, $\mathrm{SSy}(M)$ is a countable set of reals, and hence $\mathrm{SSy}(M)$ is coded by a real. We must now be more precise about how we code countable sets of reals. Unfortunately, the space $[\mathcal{P}(\omega)]^\omega$ of countable sets of reals does not carry a natural standard Borel structure. We work instead with the space $\mathcal{P}(\omega)^\omega$ of countable *sequences* of reals, and let E_{set} denote the equivalence relation defined on $\mathcal{P}(\omega)^\omega$ by

$$x \mathrel{E_{\mathsf{set}}} y \iff \{x(n) \colon n \in \omega\} = \{y(n) \colon n \in \omega\}.$$

(The relation E_{set} has also assumed the names $=^+$, E_{ctble} and F_2.) It is easy to see that E_{set} is a Borel equivalence relation.

Theorem 4.2. *The isomorphism equivalence relation $\cong_T^{\mathsf{rec}}$ on the space of recursively saturated models of T is Borel bireducible with E_{set}. In particular, $\cong_T^{\mathsf{rec}}$ is a Borel equivalence relation.*[2]

Proof. Let T be a completion of PA. By Proposition 4.1, the function which assigns to each model M of T a code for $\mathrm{SSy}(M)$ is a Borel reduction from $\cong_T^{\mathsf{rec}}$ to E_{set}. For the reverse direction, we shall need a notion of genericity. A subset $X \subseteq \mathbb{N}$ is said to be *Cohen generic* iff the set of restrictions of its characteristic function to the intervals $[0, n]$, for $n < \omega$, meets all arithmetically definable dense subsets of $2^{<\omega}$. Cohen generics exist over every countable expansion of $\mathbb{N}$. We will work over $(\mathbb{N}, T)$, where we have identified T with the set of Gödel numbers of the sentences in T.

Now, by Lemma 6.3.6 of [9], there exists a perfect set $\mathfrak{S}$ of subsets of $\mathbb{N}$ which are *mutually Cohen generic* over $(\mathbb{N}, T)$ in the sense that for any distinct $X_1, \dots, X_n \in \mathfrak{S}$, X_n is Cohen generic over $(\mathbb{N}, T, X_1, \dots, X_{n-1})$. Identifying $\mathcal{P}(\omega)$ with the perfect set $\mathfrak{S}$, each $C \in \mathcal{P}(\omega)^\omega$ naturally corresponds to an element $\mathfrak{S}_C \in \mathfrak{S}^\omega$. Let $\mathfrak{X}_C$ be the collection of subsets of $\mathbb{N}$ which are definable from T together with the sets enumerated in $\mathfrak{S}_C$. By mutual genericity,

[2] The referee has pointed out that Theorem 4.2 is essentially the same as the main result of Marker's [11]. Marker proved that for any first order theory in a countable language where the type space $S(T)$ is uncountable, $E_{\mathsf{set}} \leq \cong_T$; hence $\cong_T$ is not essentially countable.

if $C \neq C'$, then $\mathfrak{X}_C \neq \mathfrak{X}_{C'}$. Since $\mathfrak{X}_C$ is a Scott set and $T \in \mathfrak{X}_C$, there exists a countable recursively saturated model M_C of T such that $\mathrm{SSy}(M_C) = \mathfrak{X}_C$ (see for instance Theorem 3.5 of [12]). it follows that the map $C \mapsto M_C$ is a Borel reduction from E_{set} to $\cong_T^{\mathsf{rec}}$, which completes the proof. ⊣

The equivalence relation E_{set} is an important benchmark in the Borel reducibility hierarchy; many natural equivalence relations lie at this complexity level. E_{set} is not essentially countable, but rather lies "just above" the countable Borel equivalence relations (indeed, $E_\infty <_B E_{\mathsf{set}}$ but there is no known equivalence relation E such that $E_\infty <_B E <_B E_{\mathsf{set}}$). In particular, Theorem 4.2 implies that the class of recursively saturated models is also not essentially countable. There is, however, a simple argument of Jim Schmerl which already implies this fact, and moreover implies that many related classes of models are not essentially countable.

Let T be a completion of PA and let M be a countable model of T. If $A \subseteq \omega$ is not in $\mathrm{SSy}(M)$, then by compactness, M has an elementary extension N such that $A \in \mathrm{SSy}(N)$. In particular, N realizes a type which is not realized in M. Moreover, if M is recursively saturated, then we can make N recursively saturated as well. The following result shows that a class of models with this property cannot be essentially countable.

THEOREM 4.3. *Suppose that $\mathcal{C}$ is a class of countable models such that every $K \in \mathcal{C}$ has an elementary extension in $\mathcal{C}$ realizing a type which is not realized in K. Further suppose that $\mathcal{C}$ is closed under unions of countable elementary chains. Then $\mathcal{C}$ is not essentially countable.*

PROOF. We shall use the characterization of essential countability provided by Theorem 3.2. Let F be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and let M be a model which is a union of a continuous elementary chain in $\mathcal{C}$, and which realizes uncountably many types. By a Skolem–Löwenheim argument, for every finite (or even countable) $\bar{a} \in M^n$, we have $M = \bigcup_{\alpha<\omega_1} K_\alpha$, where $K_\alpha \in \mathcal{C}$ and $(K_\alpha, \bar{a}) \prec_F (K_\beta, \bar{a})$ for all $\alpha < \beta < \omega_1$. Hence, there must be α and β, such that $(K_\alpha, \bar{a}) \prec_F (K_\beta, \bar{a})$ and $(K_\alpha, \bar{a}) \not\cong (K_\beta, \bar{a})$. ⊣

The paragraph preceding Theorem 4.3 also applies to countable recursively saturated models of Presburger Arithmetic, which is the theory $\mathrm{Th}(\mathbb{N}, +)$. In fact, it applies to the class of countable recursively saturated models of any rich[3] theory. Hence, Schmerl's argument shows that none of these classes is essentially countable.

§5. Pairs of recursively saturated models.

We have seen that the classification problem for countable recursively saturated models is Borel. However, each such model displays a rich second-order structure which itself is a subject of further classification attempts. Much work has been done towards

[3] T is said to be *rich* iff there exists a computable sequence of formulas $\langle \varphi_n(x) : n \in \omega \rangle$ such that for all disjoint finite $A, B \subset \mathbb{N}$, $T \vdash \exists x\, [\bigwedge_{i\in A} \varphi_i(x) \wedge \bigwedge_{j\in B} \neg\varphi_j(x)]$.

classifying elementary submodels, elementary cuts, and automorphisms of recursively saturated models of PA. None of these attempts have been completed, and there are many open problems. In this section we shall treat elementary cuts, and in the next section automorphisms.

If K is an elementary cut in a countable recursively saturated model M and K itself is recursively saturated, then K and M will have the same standard system and hence $K \cong M$. Still, there are $2^{\aleph_0}$ many isomorphism types of structures of the form (M, K), where M and K are recursively saturated and $K \prec_{\mathsf{end}} M$. We shall establish the following result.

THEOREM 5.1. *Let M be a recursively saturated model of* PA. *Then the classification problem for pairs (M, K), where $K \prec_{\mathsf{end}} M$ is recursively saturated, is Borel complete.*

For the proof, we shall initially give a *single* model M satisfying the conclusion of Theorem 5.1. Afterwards, we will indicate how to modify the construction to obtain the full result.

Let $S_{\mathbb{N}}$ be the set $\{\langle \ulcorner\varphi\urcorner, n\rangle : \mathbb{N} \models \varphi(n)\}$. If (M, S) is is an elementary extension of $(\mathbb{N}, S_{\mathbb{N}})$, then S is and example of a nonstandard *full inductive satisfaction class* for M, *i.e.*, $(M, S) \models \mathsf{PA}^*$ and S satisfies Tarski's inductive definition of satisfaction for all formulas in the sense of M. The existence of a full inductive satisfaction class for a model M entails strong restrictions on $\mathrm{Th}(M)$, but M does not have to be an elementary extension of $\mathbb{N}$ (see [10]). Although we present the following two lemmas only for elementary extensions of $(\mathbb{N}, S_{\mathbb{N}})$, there are more general formulations with almost identical proofs.

LEMMA 5.2. *If $(\mathbb{N}, S_{\mathbb{N}}) \prec (M, S)$ and the extension is proper, then M is recursively saturated.*

SKETCH OF PROOF. First let us notice that for each $\varphi(v, \bar{x})$, we have

$$(\mathbb{N}, S_{\mathbb{N}}) \models \forall v \, \forall \bar{x} \, \left[\varphi(v, \bar{x}) \longleftrightarrow \langle \ulcorner\varphi\urcorner, (v, \bar{x})\rangle \in S_{\mathbb{N}}\right].$$

It follows that the same holds in (M, S). Let $p(v, \bar{x})$ be a recursive type. Let $P(x)$ be a formula which defines the set of Gödel numbers for the formulas in $p(v, \bar{x})$. Suppose that for some $\bar{b} \in M$, $p(v, \bar{b})$ is consistent. Then for each $n < \omega$,

$$(M, S) \models \exists v \, \forall \ulcorner\varphi\urcorner < n \, \left[P(\ulcorner\varphi\urcorner) \longrightarrow \left\langle \ulcorner\varphi\urcorner, (v, \bar{b})\right\rangle \in S\right].$$

By overspill, this must be true in M for all $n < c$, for some nonstandard c, and this shows that $p(v, \bar{b})$ is realized in M. ⊣

LEMMA 5.3. *Suppose that (M, S_0) and (M, S_1) are each elementary extensions of $(\mathbb{N}, S_{\mathbb{N}})$. If $(M, S_0, S_1) \models \mathsf{PA}^*$ (recall that this means M satisfies the induction schema even for formulas that refer to S_0, S_1), then $S_0 = S_1$.*

SKETCH OF PROOF. Tarski's inductive definition of satisfaction is first-order over $(\mathbb{N}, S_{\mathbb{N}})$. By elementarity, S_0 and S_1 obey the same definition in M.

Now, by induction on complexity of formulas, one can show that for all formulas φ (in the sense of M) and all $\bar{a} \in M^n$, $\langle \varphi, \bar{a} \rangle \in S_0 \longleftrightarrow \langle \varphi, \bar{a} \rangle \in S_1$. (Here, we used the assumption that $(M, S_0, S_1) \models \mathsf{PA}^*$; in fact it is enough to assume that (M, S_0, S_1) satisfies the Δ_0-induction schema.) $\dashv$

Now, let (M, S_0) be a fixed countable conservative elementary end extension of $(\mathbb{N}, S_{\mathbb{N}})$. Then M is recursively saturated, and since $\mathrm{SSy}(M) = \mathrm{Def}(\mathbb{N}, S_{\mathbb{N}})$ there is only one such M up to isomorphism. We shall show that this M satisfies the conclusion of Theorem 5.1.

For a countable linear order $(I, <)$, let $(M(I+1), S)$ be the canonical $(I+1)$-model of $\mathrm{Th}(\mathbb{N}, S_{\mathbb{N}})$. This model is generated by an ordered set of indiscernibles $\{a_i : i \in I+1\}$. Let $M(I)$ be the elementary submodel generated (in $(M(I+1), S)$) by the set $\{a_i : i \in I\}$ (if I is empty, then put $M(I) = \mathbb{N}$). Now, let $f : M(I+1) \longrightarrow M$ be a back-and-forth isomorphism, and let $K_I := f(M(I))$. This K_I is the 'canonical' I-cut of M. It is easy to verify that the map $I \mapsto (M, K_I)$ is Borel.

We must show that this construction yields a Borel reduction from linear orders to pairs of models. To see that the isomorphism type of (M, K_I) depends only on the isomorphism type of $(I, <)$, first observe that by the construction of canonical I-models, $(M(I+1), S)$ is a conservative elementary end extension of $(M(I), M(I) \cap S)$. Thus, it follows from Lemma 5.3 that $M(I) \cap S$ is the only full inductive satisfaction class of $M(I)$ which is coded[4] in $M(I+1)$. Moreover $(M(I), M(I) \cap S)$ is an isomorphic copy of the canonical I-extension of $(\mathbb{N}, S_{\mathbb{N}})$. It follows that K_I has a unique full inductive satisfaction class S_I which is coded in M, and such that (K_I, S_I) is an isomorphic copy of the canonical I-extension of $(\mathbb{N}, S_{\mathbb{N}})$.

To conclude the proof in this case, we must show that if $(J, <)$ is another linear order and $g : (M, K_I) \longrightarrow (M, K_J)$ is an isomorphism, then $(I, <) \cong (J, <)$. Again using Lemma 5.3, we have that $g(S_I) = S_J$, and hence that $(K_I, S_I) \cong (K_J, S_J)$. Now, since the results discussed in Section 2 regarding canonical I-models also hold for models of PA^*, we can conclude that $(I, <) \cong (J, <)$.

For the proof of Theorem 5.1 for arbitrary M, we shall require an additional fact. A set $S \subseteq M$ is *partial inductive satisfaction class* for a model $M \models \mathsf{PA}$ iff $\langle \ulcorner \varphi \urcorner, a \rangle$ is in S iff $M \models \varphi(a)$, for all formulas $\varphi(x)$ and all $a \in M$, and $(M, S) \models \mathsf{PA}^*$.

THEOREM 5.4 (Theorem 10.5.2 of [9]). *Every countable recursively saturated model $N \models \mathsf{PA}$ has a partial inductive satisfaction class S such that (N, S) is the prime model of* $\mathrm{Th}(N, S)$.

To obtain the full version of Theorem 5.1, we now modify the above proof as follows. Instead of using $(\mathbb{N}, S_{\mathbb{N}})$ and its canonical I-extensions, we fix a

[4]If $K \subseteq M \models \mathsf{PA}$, then we say that a set $A \subseteq K$ is *coded* in M, if $A = B \cap K$, for some $B \in \mathrm{Def}(M)$.

countable recursively saturated $M \models \mathsf{PA}$ and select a prime partial inductive satisfaction class S for M given by Theorem 5.4. There are $2^{\aleph_0}$ many such classes, but this is not a problem since in the construction S will serve just as an additional parameter. For a linear order $(I, <)$, we now take (M', S') to the $I+1$-canonical model of $\mathrm{Th}(M, S)$, and as before, we take $K(I)$ to be the corresponding cut in M (via an isomorphism $f \colon M' \longrightarrow M$). The rest of the argument is now similar, but one has to be more careful. In Lemma 5.3, S_0 and S_1 are *full* inductive satisfaction classes, *i.e.*, they decide the "truth" of all formulas in the sense of the model, hence the conclusion $S_0 = S_1$ is easy to get. In the present setting we cannot assume that S is full. The task can still be accomplished with the aid of a more subtle Lemma 10.5.3 of [9] and its corollary which says that every countable recursively saturated model $M \models \mathsf{PA}$ has a countable recursively saturated elementary end extension N such that for every end extension N' of N and every embedding $f \colon N' \longrightarrow N'$ such that $f(M)$ is cofinal in M, $f \restriction M$ is the identity function.

§6. Conjugacy classes. The automorphism groups of countable saturated structures have been the subject of much study, and in many cases the conjugacy problem is known to be Borel complete. For example, the conjugacy problem for the automorphism group of the rational linear ordering $(\mathbb{Q}, <)$, the random graph, and the atomless Boolean algebra are all known to be Borel complete (for a discussion of these results, see [1]). It is shown in [8] that if M is a countable recursively saturated model of PA, then

$$\mathrm{Aut}(\mathbb{Q}, <) \leq \mathrm{Aut}(M) \leq \mathrm{Aut}(\mathbb{Q}, <)$$

but $\mathrm{Aut}(M) \ncong \mathrm{Aut}(\mathbb{Q}, <)$. The group $\mathrm{Aut}(M)$ is known to have continuum many conjugacy classes, but little is known about their classification. What is known can be summarized as follows. For every $f \in \mathrm{Aut}(M)$, let us set

$$\mathrm{fix}(f) := \{x \in M : f(x) = x\},$$

and

$$I_{\mathrm{fix}}(f) := \{x \in M : \forall y \leq x \; f(y) = y\}.$$

By a theorem of Smoryński [13], a cut I of a countable recursively saturated model of PA is of the form $I_{\mathrm{fix}}(f)$ for some $f \in \mathrm{Aut}(M)$ if and only if it is closed under exponentiation. Since each nonstandard model has continuum many pairwise nonisomorphic (or even not elementarily equivalent) cuts which are closed under exponentiation, this immediately yields continuum many conjugacy classes in recursively saturated models.

If M is arithmetically saturated,[5] then this can be refined further by considering fixed point sets of the automorphisms. It is easy to see that $\mathrm{fix}(f)$ is

[5] A recursively saturated model $M \models \mathsf{PA}$ is said to be *arithmetically saturated* iff $\mathrm{SSy}(M)$ is closed under arithmetic definability. Arithmetic saturation is stronger than recursive saturation. Every countable arithmetically saturated model has a cofinal extension which

an elementary submodel of M. Every countable recursively saturated model of PA has continuum many pairwise nonisomorphic elementary submodels, and by a theorem of Enayat [2], if M is arithmetically saturated then for every $K \prec M$ there is an $f \in \mathrm{Aut}(M)$ such that $\mathrm{fix}(f) \cong K$. However, if M is not arithmetically saturated, then as shown in [8], for every $f \in \mathrm{Aut}(M)$ we have that $\mathrm{fix}(f) \cong M$.

It is known that for a countable recursively saturated model M, a cut $I \prec_{\mathsf{end}} M$ is of the form $\mathrm{fix}(f)$ for some $f \in \mathrm{Aut}(M)$ if and only if I is *strong* in M: for each function f which is coded in M and such that $I \subseteq \mathrm{dom}(f)$, there is $c > I$ such that for all $i \in I$, $f(i) > I$ iff $f(i) > c$. However, we do not know in general which elementary pairs (M, K) are of the form $(M, \mathrm{fix}(f))$. We now establish the following consequence of Theorem 5.1.

THEOREM 6.1. *For every countable recursively saturated model $M \models$ PA the conjugacy equivalence relation on* $\mathrm{Aut}(M)$ *is Borel complete.*

Of course, the conjugacy equivalence relation on $\mathrm{Aut}(M)$ can be identified with the isomorphism equivalence relation on the class of pairs (M, f) where f is an automorphism of M. Hence, it makes sense to ask whether this relation is Borel complete.

PROOF. Let $(I, <)$ be a countable linearly ordered set. We will construct an "I-canonical" automorphism $f_I \in \mathrm{Aut}(M)$. Let $I^+ = (I, <) + (\mathbb{Z}, <)$, and let (M', S') be the canonical I^+ model of $\mathrm{Th}(M, S)$, where S is a partial inductive satisfaction class for M given by Theorem 5.4. Let $\{a_i : i \in I^+\}$ be the generators of (M', S'). Let f' be the automorphism of M' generated by $a_i \mapsto a_i$, for $i \in I$, and $a_i \mapsto a_{i+1}$, for $i \in \mathbb{Z}$. Finally, let f_I be the image of f' under a back-and-forth isomorphism $g\colon M' \longrightarrow M$. Then $\mathrm{fix}(f_I) = K(I)$ (where $K(I)$ is the 'canonical' I-cut of M defined in the previous section). If $(I, <)$ and $(J, <)$ are countable linearly ordered sets, and f_I and f_J are conjugate then $(M, \mathrm{fix}(f_I)) \cong (M, \mathrm{fix}(f_J))$. By Theorem 5.1 we must have $(I, <) \cong (J, <)$, and the result follows. ⊣

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