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# ALMOST DISJOINT FAMILIES AND DIAGONALIZATIONS OF LENGTH CONTINUUM

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**Abstract.** We present a survey of some results and problems concerning constructions which require a diagonalization of length continuum to be carried out, particularly constructions of almost disjoint families of various sorts. We emphasize the role of cardinal invariants of the continuum and their combinatorial characterizations in such constructions.

**§1. Introduction.** The phrase "diagonalization of length continuum" is used in set theory to refer to recursive constructions during which continuum many "requirements" must be met. Such arguments go back to the beginnings of set theory. Although the construction of any type of object may require a diagonalization of length continuum, this obligation is most often encountered during the construction of sets of reals that satisfy certain combinatorial properties. Consider, for example, the familiar construction of a Bernstein set, a set of reals such that neither it nor its complement contains a perfect set. One enumerates the continuum many perfect sets in type $\mathfrak{c}$, say as $\langle P_\alpha \colon \alpha < \mathfrak{c}\rangle$. Then one constructs the Bernstein set $X$ in $\mathfrak{c}$ steps. At stage $\alpha$, one diagonalizes against $P_\alpha$, ensuring that neither $X$ nor its complement contains $P_\alpha$. It is possible to do this since, at stage $\alpha$, fewer than $\mathfrak{c}$ reals have been put into $X$ or its complement, while $P_\alpha$ has size $\mathfrak{c}$.

The Bernstein set is an example of a diagonalization of length $\mathfrak{c}$ that can be carried out in ZFC. However, many such diagonalizations require additional hypotheses and it is, in general, not possible to tell when such a diagonalization can be done in ZFC alone. One of the key issues involves the smaller diagonalizations that must be performed at each stage of the main diagonalization. In the case of the construction of a Bernstein set, at stage $\alpha$,

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we need to diagonalize against the reals that have already been placed into $X$ or into its complement to produce two fresh reals in $P_\alpha$ that are different from all of them. This is a diagonalization of length less than $\mathfrak{c}$, and a typical feature of diagonalizations of length $\mathfrak{c}$ is that such smaller diagonalizations must be performed at each stage of the main diagonalization. We will refer to these smaller diagonalizations as *sub-diagonalizations* in this article. For the Bernstein set construction, the sub-diagonalizations proceed at each stage because perfect sets have size $\mathfrak{c}$. In general, however, one needs to make further assumptions about the continuum in order to ensure that the sub-diagonalizations can be done. The most common such assumption is CH, and this suffices for essentially all diagonalizations of length $\mathfrak{c}$. There is a deep fact underlying this. Let us say that a statement is $\Sigma^2_1$ if it has the form $\exists X \subset \mathcal{P}(\omega)\psi(X)$, where $\psi(X)$ is an arbitrary second order formula over the structure $(\omega, <, +, \times)$. In all diagonalizations of length $\mathfrak{c}$ that interest us, each of the $\mathfrak{c}$ many "requirements" can be coded as a real number, and the statement that there is a set of reals "meeting all the requirements" rephrased as, "there is a set $X \subset \mathcal{P}(\omega)$ such that for every real $r \in \mathcal{P}(\omega) \cdots$". Here the quantifier "for every real $r \in \mathcal{P}(\omega)$" is ranging over the $\mathfrak{c}$ requirements that $X$ must fulfill. Therefore, a diagonalization of length $\mathfrak{c}$ proves a statement of the form $\exists X \subset \mathcal{P}(\omega)\forall r \in \mathcal{P}(\omega)\phi(X, r)$, which is an example of a $\Sigma^2_1$ statement. The following deep result of Woodin establishes the ability of CH, in the presence of large cardinals, to decide all "reasonable" $\Sigma^2_1$ statements. One example of an "*unreasonable*" $\Sigma^2_1$ statement is "there exists a $\Delta^1_2$ well-ordering of the reals".

THEOREM 1.1 (Woodin). *Assume that there are class many measurable Woodin cardinals. If a $\Sigma^2_1$ statement $\phi$ is true in some forcing extension, then it is true in any forcing extension satisfying* CH.

This means that if the end product of a diagonalization of length $\mathfrak{c}$ can be forced to exist, which will almost certainly be the case if its existence is at all consistent, then its existence most likely already follows from CH, since it cannot fail to exist in any forcing extension satisfying CH. More details about Woodin's theorem can be found in [13].

In fact, it turns out that CH facilitates these diagonalizations in two fundamentally different ways: one by allowing the sub-diagonalizations required at each stage to be carried out, and second, by providing a small set of reals that "captures" the continuum in some way. CH achieves both goals simultaneously by the making the entire continuum small. In this article, we classify the diagonalizations of length $\mathfrak{c}$ into two types depending on whether they need both kinds of help from CH or just the first kind. We will begin by describing a famous example of each kind of diagonalization of length $\mathfrak{c}$. Then in Section 2 we will point out that cardinal invariants of the continuum can be used to separate the two kinds of diagonalizations, and in the remainder of the article, we will focus exclusively on diagonalizations of the first kind. We will describe some constructions of almost disjoint families requiring a diagonalization of length $\mathfrak{c}$ that can be carried out in ZFC. We will first describe a well-known example due to Balcar and Vojtáš [4] and then a more recent example due to the author [28]. The unifying theme in these examples is that a diagonalization of length $\mathfrak{c}$ that appears to require additional hypotheses *a priori* may be doable in ZFC through the use of deep combinatorial characterizations of appropriate cardinal invariants. Finally, in Section 6, we will end with some open questions regarding whether certain constructions of almost disjoint families that require a diagonalization of length continuum can be carried out in ZFC.

We begin with an example of a diagonalization of length $\mathfrak{c}$ of the first kind, namely one which requires an additional hypothesis only for a sub-diagonalization at each stage. Recall that a non-empty set $\mathcal{F} \subset \mathcal{P}(\omega)$ is a *filter on $\omega$* if $\mathcal{F}$ is closed under supersets and pairwise intersections, but does contain any finite sets. Recall also that an *ultrafilter on $\omega$* is a maximal filter on $\omega$—i.e., a filter such that $\forall a \subset \omega\, [a \in \mathcal{F} \vee (\omega \setminus a) \in \mathcal{F}]$; any filter on $\omega$ can be extended to an ultrafilter using Zorn's lemma. A famous problem in the history of set theory concerns the existence of a certain kind of ultrafilter on $\omega$ whose construction requires a diagonalization of length $\mathfrak{c}$. We say that $b$ is *almost contained* in $a$, and we write $b \subset^* a$, if $b \setminus a$ is finite. An ultrafilter $\mathcal{U}$ is called a *P-point* if for every countable family $\{a_n : n \in \omega\} \subset \mathcal{U}$, there is $b \in \mathcal{U}$ such that $\forall n \in \omega\, [b \subset^* a_n]$. P-points were intensively studied in connection with the question of whether $\beta\omega$ is homogeneous, and an historically important question in set theory was whether P-points exist. To construct a P-point, we must diagonalize against all countable subsets of the ultrafilter we are building (in addition to ensuring that we do end up with an ultrafilter). As there are $\mathfrak{c}$ such subsets, we need to do a diagonalization of length continuum. At any stage $\alpha < \mathfrak{c}$, we have $\mathcal{F}_\alpha$, the filter built up so far, which we may assume is generated by fewer than $\mathfrak{c}$ sets. We also have some countable family $\{a_n : n \in \omega\} \subset \mathcal{F}_\alpha$, and a set $c_\alpha \in [\omega]^\omega$. To make sure we end up with an ultrafilter, we need to decide whether to put $c_\alpha$ or its complement into $\mathcal{F}_{\alpha+1}$. To deal with $\{a_n : n \in \omega\}$, we have to perform a sub-diagonalization to produce a $b \in [\omega]^\omega$ which is positive for the filter $\mathcal{F}_\alpha$ with the property that $\forall n \in \omega\, [b \subset^* a_n]$. $c_\alpha$ can be decided after this step. To be positive for $\mathcal{F}_\alpha$, it is enough for $b$ to have infinite intersection with all the members of some filter base for $\mathcal{F}_\alpha$, which we know can be chosen to be of size less than $\mathfrak{c}$. So our sub-diagonalization is of length less than $\mathfrak{c}$, and we are required to have constructed $\mathcal{F}_\alpha$ in such a way that this sub-diagonalization is possible. But unlike in the case of a Bernstein set, it is far from clear that there are any "right choices" that we could have made at stages before $\alpha$, either regarding the $c_\xi$ for $\xi < \alpha$, or the earlier sub-diagonalizations, to guarantee that $\mathcal{F}_\alpha$ is as required. Indeed Shelah constructed a model of set theory in which P-points don't exist (see Theorem 4.4.7 of [7]). Nevertheless, note that the sub-diagonalization can be done if $\mathcal{F}_\alpha$ is countably generated, and that this will be the case for every $\alpha$ provided $\mathfrak{c} = \omega_1$. Thus CH allows us to build a P-point by making the necessary sub-diagonalizations be only of countable length.

Now, we give a well-known example of a diagonalization of length $\mathfrak{c}$ that requires extra hypotheses both to guarantee the existence of a small set of reals "capturing" the continuum in some way, and to perform a sub-diagonalization at each stage. Whether $S$ and $L$ spaces exist was a famous problem in the history of set-theoretic topology. We will not deal directly with this problem, but with another problem about sets of reals which is very closely related to it. Let us say that a family $X \subset [\omega]^\omega$ is *well founded* if the poset $\langle X, \subset \rangle$ has no infinite descending chains—i.e., if there is no sequence $\langle a_n : n \in \omega \rangle \subset X$ so that $a_{n+1} \subsetneq a_n$. Given $X \subset [\omega]^\omega$, we say a family $A \subset X$ is an antichain if $\forall a, b \in A \, [a \neq b \Longrightarrow (a \not\subset b \wedge b \not\subset a)]$. We are interested in the question of whether there is an uncountable well founded family $X \subset [\omega]^\omega$ with no uncountable antichains. First of all notice that if such a family exists, then there is one of size $\omega_1$. Notice also that a well founded family $X \subset [\omega]^\omega$ of size $\omega_1$ can be enumerated as $\langle x_\alpha : \alpha < \omega_1 \rangle$ in such a manner that $\forall \alpha, \beta < \omega_1 \left[ x_\alpha \subset x_\beta \Longrightarrow \alpha \leq \beta \right]$. Readers familiar with $S$ and $L$ spaces will immediately see that such a family $X = \langle x_\alpha : \alpha < \omega_1 \rangle$ is a right separated subspace of $[\omega]^\omega$ when endowed with the Vietoris topology. This is the topology on $[\omega]^\omega$ generated by sets of the form $[s, a] = \{ b \in [\omega]^\omega : s \subset b \subset a \}$, where $s \in [\omega]^{<\omega}$ and $a \in [\omega]^\omega$. Moreover, not containing any uncountable antichains, $X$ has no uncountable discrete subspaces. Therefore, in the Vietoris topology, $X$ is an example of an $S$ space which is moreover first countable. This is because for each $a \in [\omega]^\omega$, $\{ [s, a] : s \in [a]^{<\omega} \}$ is a local base at $a$.

Let us now return to the purely combinatorial question of whether there exists a family $X = \langle x_\alpha : \alpha < \omega_1 \rangle \subset [\omega]^\omega$ such that for all $\alpha < \beta < \omega_1$, $x_\beta \not\subset x_\alpha$, which doesn't contain any uncountable antichains. It seems that to ensure that $X$ contains no uncountable antichains, we need to diagonalize against all its uncountable subsets, which are $2^{\aleph_1}$ in number. Moreover, the sentence expressing the existence of such a set of reals $X$ is not $\Sigma^2_1$. However, Van Douwen and Kunen [36] first observed that only $2^{\aleph_0}$ requirements need to be met if we replace the condition of not containing any uncountable antichains with the following stronger requirement:

$$\begin{array}{l} \text{for each countable set } A \subset \omega_1, \text{ there is } \alpha < \omega_1 \text{ such that for} \\ \text{all } \beta \geq \alpha, \text{ if } x_\beta \text{ is in the closure (with respect to the usual} \\ \text{topology on } [\omega]^\omega) \text{ of } \{x_\gamma : \gamma \in A\}, \text{ then } \exists \gamma \in A \left[ x_\gamma \subset x_\beta \right]. \end{array} \tag{$*$}$$

Let us first see that requirement $(*)$ is indeed stronger than not containing any uncountable antichains. Let $Y \in [\omega_1]^{\omega_1}$. There is a countable set $A \subset Y$ such that $\{ x_\gamma : \gamma \in A \}$ is dense in $\{ x_\beta : \beta \in Y \}$. Applying $(*)$ to $A$, we get an $\alpha < \omega_1$ satisfying the condition given there. Now choose $\beta \in Y \setminus A$ with $\beta \geq \alpha$. Since $\{x_\gamma : \gamma \in A\}$ is dense in $\{x_\beta : \beta \in Y\}$, $x_\beta$ is in the closure of $\{x_\gamma : \gamma \in A\}$. As $\alpha$ is as in $(*)$, there is a $\gamma \in A \subset Y$ so that $x_\gamma \subset x_\beta$, whence $Y$ is not an antichain.

To fulfill condition $(*)$, we must diagonalize against all countable subsets of $\omega_1$, and there are precisely $\mathfrak{c}$ of these. Notice also that the sentence expressing the existence of a $X$ satisfying $(*)$ is $\Sigma^2_1$. But now the problem is that we must "catch" each countable subset of $\omega_1$ at some *countable ordinal* $\alpha$, and then we need to "respect" that countable set at all stages $\beta \geq \alpha$. Of course, we can do this if $\mathfrak{c} = \omega_1$. Given CH, we can let $\langle A_\alpha : \alpha < \omega_1 \rangle$ enumerate all countable subsets of $\omega_1$ in such a way that $\forall \alpha < \omega_1 \, [A_\alpha \subset \alpha]$. At a stage $\alpha < \omega_1$, we want to "respect" all the countable sets $\{A_\beta : \beta \leq \alpha\}$, which by assumption, are all subsets of $\alpha$. We must do a sub-diagonalization against $\{x_\beta : \beta < \alpha\}$ and against $\{A_\beta : \beta \leq \alpha\}$ to produce a set $x_\alpha \in [\omega]^\omega$ satisfying:

1. $\forall \beta_0, \dots, \beta_k < \alpha \left[\omega \setminus \left(x_{\beta_0} \cup \dots \cup x_{\beta_k} \cup x_\alpha\right) \text{ is infinite}\right]$
2. for each $\beta \leq \alpha$, if $x_\alpha$ is in the closure of $\{x_\gamma : \gamma \in A_\beta\}$, then $\exists \gamma \in A_\beta \left[x_\gamma \subset x_\alpha\right]$
3. $\forall \beta < \alpha \left[x_\alpha \not\subset x_\beta\right]$.

Since $\alpha$ is countable, it is easy to see that this sub-diagonalization can be performed, provided that $\forall \beta_0, \dots, \beta_k < \alpha \left[\omega \setminus \left(x_{\beta_0} \cup \dots \cup x_{\beta_k}\right) \text{ is infinite}\right]$ (which we guarantee during the construction).

We have seen that CH facilitates diagonalizations of length $\mathfrak{c}$ by allowing the necessary sub-diagonalizations to be be carried out and also by allowing some features of the continuum to be "captured" in a "small" number of steps. CH achieves both of these things by making the continuum as small as possible. But there is a more subtle approach. There are appropriate assumptions about cardinal invariants of the continuum, which we discuss in the next section, guaranteeing that sub-diagonalizations of any length less than $\mathfrak{c}$ can be performed. There are also assumptions about these cardinal invariants that provide a small set of reals capturing some combinatorial property of the entire continuum. In general, these assumptions tend to contradict those assumptions ensuring the feasibility of sub-diagonalizations of uncountable length, and hence cardinal invariants can be used to separate the two roles played by CH in diagonalizations of length $\mathfrak{c}$ that become conflated when CH itself is assumed.

**§2. Cardinal invariants of the continuum.** A cardinal invariant of the continuum marks the place where a given type of diagonalization argument that works for any countable ordinal first fails; a cardinal invariant can be associated with each type of diagonalization argument. Moreover, there is always a set of size $\mathfrak{c}$ for which these diagonalization arguments fail, so that every cardinal invariant lies between $\omega_1$ and $\mathfrak{c}$ (since the diagonalization always works for countable ordinals). We will illustrate the idea here with some examples, and we will introduce more of them as and when we need them for our discussion. A general survey of cardinal invariants may be found in [6] and [9]. We start with one of the most important types of diagonalization argument. Let us say that a family $\mathscr{A} \subset [\omega]^\omega$ has the *finite intersection property* (FIP) if for any finite $\{a_0, \ldots, a_k\} \subset \mathscr{A}$, $|a_0 \cap \cdots \cap a_k| = \omega$. Given a countable family $\mathscr{A} \subset [\omega]^\omega$ with the FIP, it is easy to diagonalize against it to produce a set $b \in [\omega]^\omega$ which is almost contained in all $a \in \mathscr{A}$. Indeed, if $\mathscr{A} = \{a_n : n \in \omega\}$, then simply let $b = \{k_0 < k_1 < \cdots\}$, where $k_n$ is the least element of $a_0 \cap \cdots \cap a_n$ that is greater than $k_{n-1}$. Note that by assumption $a_0 \cap \cdots \cap a_n$ is an infinite set. This is a diagonalization of countable length. Now, the cardinal invariant $\mathfrak{p}$, called the *pseudointersection number*, marks the first place where this diagonalization *cannot* be carried out. Notice there is always a set of size $\mathfrak{c}$ for which this diagonalization fails (e.g., an ultrafilter).

DEFINITION 2.1.

$$\mathfrak{p} = \min\{|\mathscr{A}| : \mathscr{A} \subset [\omega]^\omega \text{ has FIP } \wedge \neg\exists b \in [\omega]^\omega \forall a \in \mathscr{A} \left[b \subset^* a\right]\}.$$

Another important type of diagonalization involves functions from $\omega$ to $\omega$. Given $f$ and $g$ in $\omega^\omega$, we write $f <^* g$ to mean $\forall^\infty n \in \omega \left[f(n) < g(n)\right]$. We say that $f$ *bounds* a family $\mathscr{A} \subset \omega^\omega$ if $\forall g \in \mathscr{A}\, [g <^* f]$. Again, an easy diagonalization of countable length shows that for every countable family $\mathscr{A} \subset \omega^\omega$, there is $f \in \omega^\omega$ that bounds it. Notice that the same diagonalization also produces a function (namely $f$) that is *not bounded* by any element of $\mathscr{A}$ (which is weaker than bounding $\mathscr{A}$). The *bounding number* $\mathfrak{b}$ is the first place where the diagonalization needed to produce a function bounding a given family $\mathscr{A}$ cannot be carried out, and the *dominating number* $\mathfrak{d}$ is the first place where a function not bounded by any element of $\mathscr{A}$ cannot be produced. Again notice that there is an $\mathscr{A}$ of size $\mathfrak{c}$ for which neither of these diagonalizations can be done (e.g., $\omega^\omega$).

DEFINITION 2.2.

$$\mathfrak{b} = \min\{|\mathscr{A}| : \mathscr{A} \subset \omega^\omega \wedge \mathscr{A} \text{ is unbounded}\}.$$
$$\mathfrak{d} = \min\{|\mathscr{A}| : \mathscr{A} \subset \omega^\omega \wedge \forall f \in \omega^\omega \exists g \in \mathscr{A} \left[f <^* g\right]\}.$$

We also mention here two cardinal invariants that, at least on the face of it, seem unlike the three considered so far in that their rationale lies with some topological properties of the reals. Recall that a set $A \subset X$ of some topological space $X$ is *nowhere dense* if the closure of $A$ has empty interior. $A \subset X$ is *meager* if it is a countable union of nowhere dense sets. For the Baire space $\omega^\omega$, we know from the Baire category theorem, which is a diagonalization argument of countable length, that countably many meager sets cannot cover $\omega^\omega$, and we also have, by definition, that any countable set $A \subset \omega^\omega$ must be meager. Thus, we are led to the following natural cardinal invariants.

Definition 2.3.

$\operatorname{cov}(\mathcal{M}) = \min\{|\mathcal{F}| \colon \forall A \in \mathcal{F}\,[A \subset \omega^\omega \text{ is meager in } \omega^\omega] \wedge \bigcup \mathcal{F} = \omega^\omega\}$.

$\operatorname{non}(\mathcal{M}) = \min\{|A| \colon A \subset \omega^\omega \text{ is not meager in } \omega^\omega\}$.

The failure of diagonalization processes associated with cardinal invariants implies the existence of small sets of reals which share some combinatorial properties with the whole continuum. Consider, for instance, the hypothesis $\mathfrak{d} = \omega_1$. Clearly, $\mathfrak{d} = \omega_1$ is equivalent to the statement that the poset $(\omega^\omega, <^*)$ has a cofinal subset of size $\omega_1$. So the failure of the diagonalization associated with $\mathfrak{d}$ at a small value gives a small subset of $(\omega^\omega, <^*)$ capturing essential combinatorial information about that poset.

Since cardinal invariants mark the places where a diagonalization which always works for countable sets first fails, and since the failure of such diagonalizations at small values points to the existence of small sets capturing some combinatorial essence of the continuum, CH can usually be replaced in diagonalizations of length $\mathfrak{c}$ with appropriate assumptions about cardinal invariants.

Notice that CH has two implications for cardinals invariants: it implies that every cardinal invariant is equal to $\mathfrak{c}$; it also implies that every cardinal invariant is equal to $\omega_1$. CH can usually be replaced with one or the other of these implications to separate out the two goals simultaneously achieved by CH. Observe that an assumption of the form $\mathfrak{x} = \mathfrak{c}$, where $\mathfrak{x}$ is some cardinal invariant, allows the diagonalization process associated with $\mathfrak{x}$ to proceed up to any length less than $\mathfrak{c}$, and hence can replace CH in diagonalizations of length $\mathfrak{c}$ of the first kind. On the other hand, assumptions of the form $\mathfrak{x} = \omega_1$ provide a small set of reals that captures some combinatorial feature of the continuum, and hence can be used for diagonalizations of length $\mathfrak{c}$ of the second kind. This pattern obtains for the examples given in Section 1. Ketonen proved (see Theorem 4.4.5 of [7]) that if $\mathfrak{d} = \mathfrak{c}$, then there is a P-point. The following result of Todorčević [33] shows that the object constructed in the second example from Section 1 can be constructed as long as $\mathfrak{b} = \omega_1$ (it is easy to see that the hypothesis of the theorem below holds iff $\mathfrak{b} = \omega_1$).

Theorem 2.4 (see Lemma 1.0. of [33]). *Let* $L = \langle f_\alpha \colon \alpha < \omega_1 \rangle \subset \omega^\omega$ *be such that*

1. $f_\alpha(n) \leq f_\alpha(n+1)$,
2. $\forall \beta < \alpha < \omega_1 \left[f_\beta <^* f_\alpha\right]$,
3. *$L$ is unbounded.*

*For each* $\alpha < \omega_1$, *put* $x_\alpha = \{\langle n, m \rangle \in \omega \times \omega \colon m \leq f(n)\}$. *Then the set* $X = \langle x_\alpha \colon \alpha < \omega_1 \rangle \subset [\omega \times \omega]^\omega$ *satisfies*

(a) *for each* $A \in [\omega_1]^\omega$, *there is an* $\alpha < \omega_1$ *so that for all* $\beta \geq \alpha$, *if* $x_\beta$ *is in the closure of* $\{x_\gamma \colon \gamma \in A\}$, *then* $\exists \gamma \in A \left[x_\gamma \subset x_\beta\right]$,

(b) $\forall \beta < \alpha \left[x_\alpha \not\subset x_\beta\right]$.

When CH is replaced with assumptions about cardinal invariants, the deep and fundamental differences between the two types of diagonalizations of length $\mathfrak{c}$ become apparent. For one thing, assumptions of the form $\mathfrak{x} = \mathfrak{c}$ are often inconsistent with assumptions of the form $\mathfrak{y} = \omega_1$ (unless, of course, $\mathfrak{c} = \omega_1$). But more importantly, natural forcing axioms such as Martin's Axiom (MA) and the Proper Forcing Axiom (PFA) imply that all cardinal invariants are equal to $\mathfrak{c}$. Therefore, all diagonalizations of length $\mathfrak{c}$ of the first kind can be carried out under these forcing axioms. On the other hand, MA (or more accurately $\mathrm{MA}_{\aleph_1}$) and PFA tend to *contradict* the existence of the objects that are constructed in diagonalizations of length $\mathfrak{c}$ of the second kind. For example, the following theorem of Kunen rules out all objects of the sort constructed in the second example of Section 1 under $\mathrm{MA}_{\aleph_1}$.

THEOREM 2.5 (see Lemma 1 of Baumgartner [8]). *Assume* $\mathrm{MA}_{\aleph_1}$. *Let $X \subset [\omega]^\omega$ be well founded and uncountable. Then there exists an uncountable $A \subset X$ which is an antichain.*

More generally, PFA implies that there are no $S$ spaces at all (see Todorčević [33]), and $\mathrm{MA}_{\aleph_1}$ implies that there are no first countable $L$ spaces (Szentmiklóssy [32]). Other natural combinatorial statements that are consequences of forcing axioms also imply the conclusion of Theorem 2.5. For instance, the Open Coloring Axiom of Todorčević [33], a consequence of PFA, implies that every uncountable $X \subset [\omega]^\omega$ either contains an uncountable chain or an uncountable antichain, and hence that if $X$ is well founded, it must contain an uncountable antichain. This is also implied by the P-Ideal Dichotomy of Todorčević [34] provided that $\mathfrak{p} > \omega_1$. Analogous conclusions hold for several other objects constructed using a diagonalization of the second kind, such as two c.c.c. posets whose product is not c.c.c.

On the other hand, if a diagonalization of length $\mathfrak{c}$ requires an assumption of the form $\mathfrak{x} = \mathfrak{c}$, then in order to produce a model where the object constructed fails to exist, one must typically iterate a forcing that adds a certain kind of real while not adding other kinds of reals. Section 11 of Blass [9] provides a quick introduction to such forcing notions, and Bartoszyński and Judah [7] describes many such examples in detail.

In the positive direction, cardinal invariants can sometimes help us eliminate all extra assumptions from a diagonalization of length $\mathfrak{c}$. In the rest of this article, we illustrate this by describing two constructions of almost disjoint families that can be done without any additional hypotheses even though they appear to require assumptions of the form $\mathfrak{x} = \mathfrak{c}$ at first sight. The two problems (and their solutions) share several formal similarities. Both constructions manage to avoid all extra assumptions by making use of non-trivial combinatorial characterizations of certain cardinal invariants.

Before taking leave of diagonalizations of the second kind, we mention that it is also occasionally possible to do some of these in ZFC. The main technique here is that of minimal walks invented by Todorčević. This technique is useful for dealing with (among other things) problems that are "really about" the combinatorics of $\omega_1$. For each limit ordinal $\alpha < \omega_1$, one chooses a strictly increasing $\omega$-sequence $C_\alpha \subset \alpha$ that is cofinal in $\alpha$. One can then define various functions from $\omega_1$ to $\omega$ using the fact that for any $\beta < \alpha < \omega_1$, $C_\alpha \cap \beta$ is a finite set. With the help of these functions it is possible to build using just ZFC combinatorial structures on $\omega_1$ that may require a diagonalization of the second kind. A recent success of this technique is Moore's construction of an $L$ space on the basis of ZFC alone. For more details about the technique of minimal walks see Todorčević [35] and Moore [24].

§3. **Almost disjoint families.** In this section we introduce almost disjoint families. We say that two infinite subsets $a$ and $b$ of $\omega$ are *almost disjoint or a.d.* if $a \cap b$ is finite. We say that a family $\mathscr{A}$ of infinite subsets of $\omega$ is *almost disjoint or a.d.* if its members are pairwise almost disjoint. A trivial example of an a.d. family is $\{\{n\colon n \text{ is even}\}, \{n\colon n \text{ is odd}\}\}$. For a more interesting example, consider a one to one map $\Sigma\colon 2^{<\omega} \to \omega$ and take the family $\{\{\Sigma(f \upharpoonright n)\colon n \in \omega\}\colon f \in 2^\omega\}$. Observe that the first of these is maximal in the sense that no a.d. family can properly contain it. However, in set theory, we are mainly interested in infinite examples that have this property. So we shall reserve the term *Maximal Almost Disjoint family, or* MAD *family* for an infinite a.d. family that is not properly contained in a larger a.d. family. And we will refer to a family as in the first example—i.e., a finite a.d. $\mathscr{A} \subset [\omega]^\omega$ such that $\bigcup \mathscr{A}^* \supset \omega$—as a *finite partition of* $\omega$. Note that any infinite a.d. family, such as the second example above, can be extended to a MAD family using Zorn's Lemma. It should be clear that $\omega$ can be replaced in this entire discussion by an arbitrary countable set $X$.

Now, any family of pairwise *disjoint* infinite subsets of $\omega$ must be countable. But it is an interesting combinatorial fact that every MAD family of infinite subsets of $\omega$ must be uncountable. And, of course, the argument is a diagonalization of length $\omega$ that we leave to the reader. In the spirit of Section 2 we want to associate a cardinal number with this specific type of diagonalization, telling us the first place where it fails.

Definition 3.1. $\mathfrak{a} = \min\{|\mathscr{A}|\colon \mathscr{A} \subset [\omega]^\omega \text{ is MAD in } [\omega]^\omega\}$.

MAD families have been intensively studied in set theory. They have numerous applications in set theory as well as general topology. For example, the technique of almost disjoint coding has been used in forcing theory (see [21]) and MAD families are used in the construction of the Isbell–Mrówka space in topology (see [15]). Another connection with topology is the relation between almost disjoint refinements and $\mathfrak{c}$-points in the Stone–Čech compactification of $\omega$ (see [1] and [4]). We will consider this last topic in greater detail soon.

There are several important variations on the notion of a MAD family of subsets of $\omega$ that will interest us. Two functions $f$ and $g$ in $\omega^\omega$ are said to be *almost disjoint or a.d.* if they agree in only finitely many places. Such functions are sometimes also referred to as being eventually different. Identifying functions with their graphs—i.e., with certain subsets of $\omega \times \omega$—we see that $f$ and $g$ are a.d. iff $|f \cap g| < \omega$. Similarly, we say that a family $\mathscr{A} \subset \omega^\omega$ is a.d. if its members are pairwise a.d., and finally we say that an a.d. family $\mathscr{A} \subset \omega^\omega$ is MAD if $\forall f \in \omega^\omega \exists h \in \mathscr{A}\, [|f \cap h| = \omega]$ (we do not need to stipulate that the family be infinite because no finite a.d. family of functions in $\omega^\omega$ can be maximal). We will refer to an a.d. family of infinite subsets of a countable set $X$ as an a.d. family in $[X]^\omega$, and to an a.d. family of functions as an a.d. family in $\omega^\omega$. Notice that any a.d. family $\mathscr{A} \subset \omega^\omega$ is also an a.d. family in $[\omega \times \omega]^\omega$, although it can *never* be a MAD family in $[\omega \times \omega]^\omega$ because every function is a.d. from all the vertical columns of $\omega \times \omega$. Once again, a countable diagonalization shows that no countable a.d. family $\mathscr{A} \subset \omega^\omega$ can be maximal. And, of course, analogously to $\mathfrak{a}$, associated with MAD families in $\omega^\omega$ is yet another cardinal invariant.

Definition 3.2. $\mathfrak{a}_\mathfrak{e} = \min\{|\mathscr{A}| : \mathscr{A} \subset \omega^\omega \text{ is MAD in } \omega^\omega\}$.

One of the two constructions of a.d. families we will present concerns almost disjointness in $[\omega]^\omega$ while the other is about $\omega^\omega$.

We close this section with one more notion of almost disjointness that is related to the second example of an a.d. family given above. We say that a set $b \subset \omega^{<\omega}$ is a *branch* if it has the form $b = \{f \restriction n : n \in \omega\}$ for some $f \in \omega^\omega$. A set $a \subset \omega^{<\omega}$ is called *off branch* if it is a.d. from each branch. For example, every antichain $a \subset \omega^{<\omega}$ is off branch. An a.d. family $\mathscr{A} \subset \left[\omega^{<\omega}\right]^\omega$ is said to be *off branch* if each $a \in \mathscr{A}$ is off branch. An off branch $\mathscr{A} \subset \left[\omega^{<\omega}\right]^\omega$ is called a *maximal off branch* family or MOB family if $\mathscr{A} \cup \{b \subset \omega^{<\omega} : b \text{ is a branch}\}$ is a MAD family in $\left[\omega^{<\omega}\right]^\omega$. The cardinal invariant $\mathfrak{o}$ denotes the size of the smallest MOB family. These notions were introduced by Leathrum [22], who showed, among other things, that $\mathfrak{a} \leq \mathfrak{o}$. They are further studied in Brendle [10].

§4. **Almost disjoint refinements.** We deal here with the first of the two constructions of almost disjoint families we have alluded to above. The problem, or rather, group of related problems concerns when we can find an almost disjoint refinement of a family $\mathscr{C} \subset [\omega]^\omega$. Given $\mathscr{C} \subset [\omega]^\omega$, we say that a family $\mathscr{A} = \{a_c : c \in \mathscr{C}\} \subset [\omega]^\omega$ is an *almost disjoint refinement* (ADR) of $\mathscr{C}$ if

1. $\forall c \in \mathscr{C}\, [a_c \subset c]$,
2. $\forall c_0, c_1 \in \mathscr{C}\, \left[c_0 \neq c_1 \implies |a_{c_0} \cap a_{c_1}| < \omega\right]$.

Not every $\mathscr{C} \subset [\omega]^\omega$ can have an ADR. Recall that we say that a family $\mathcal{I} \subset \mathcal{P}(\omega)$ is an *ideal* on $\omega$ if $\mathcal{I}$ is closed under subsets and pairwise unions, and also contains all finite subsets of $\omega$. We say that a family $\mathscr{X} \subset [\omega]^\omega$ is *dense* if $\forall x \in [\omega]^\omega \exists y \in [x]^\omega\, [y \in \mathscr{X}]$, and we say that $\mathscr{X}$ is *open* if $\forall x \in \mathscr{X} \forall y \in [\omega]^\omega\, [y \subset^* x \Longrightarrow y \in \mathscr{X}]$. Note that any ideal on $\omega$ is automatically open. Now suppose $\mathscr{C}$ has an ADR, $\mathscr{A} = \{a_c : c \in \mathscr{C}\}$. We claim that there is a dense ideal $\mathcal{I}$ such that $\mathcal{I} \cap \mathscr{C} = 0$. To see this, let $\mathcal{I}$ be an ideal on $\omega$ maximal with respect to the condition that $\mathcal{I} \cap \mathscr{A} = 0$. Notice that if an ideal $\mathcal{I}$ fails to intersect $\mathscr{A}$, then it doesn't intersect $\mathscr{C}$ either. Now, we will argue that this $\mathcal{I}$ is dense. If not, then there is a set $x \in [\omega]^\omega$ such that no $y \in [x]^\omega$ is in $\mathcal{I}$. Moreover, by the maximality of $\mathcal{I}$, no such $y$ can be added to $\mathcal{I}$ without violating the requirement that $\mathcal{I} \cap \mathscr{A} = 0$. Therefore, for every $y \in [x]^\omega$, there is an $a \in \mathcal{I}$ and $c \in \mathscr{C}$ such that $a_c \subset y \cup a$. In particular, there is $a_0 \in \mathcal{I}$ and $c_0 \in \mathscr{C}$ such that $a_{c_0} \subset x \cup a_0$. Since $a_{c_0} \notin \mathcal{I}$, $a_{c_0} \setminus a_0$ is an infinite subset of $x$. Choose $z \in \left[a_{c_0} \setminus a_0\right]^\omega$ with $\bar{z} = (a_{c_0} \setminus a_0) \setminus z$ also infinite. Note that since $\bar{z}$ is an infinite subset of $x$, $\bar{z} \notin \mathcal{I}$. Next, since $z \in [x]^\omega$, we can find $a_1 \in \mathcal{I}$ and $c_1 \in \mathscr{C}$ such that $a_{c_1} \subset z \cup a_1$. Once again, $a_{c_1} \setminus a_1$ is an infinite subset of $z \subset a_{c_0}$. So $a_{c_0}$ and $a_{c_1}$ have infinite intersection, whence $c_0 = c_1$. Now, we have $a_{c_0} \subset z \cup a_1$. But now, $\bar{z} \subset a_1$, contradicting our observation that $\bar{z} \notin \mathcal{I}$.

How about the converse? Suppose $\mathscr{C} \subset [\omega]^\omega$ is a family so that there is a dense ideal $\mathcal{I}$ satisfying $\mathscr{C} \cap \mathcal{I} = 0$. Does $\mathscr{C}$ have an ADR? Firstly, note that this is the same as asking whether $\mathcal{P}(\omega) \setminus \mathcal{I}$ has an ADR whenever $\mathcal{I} \subset \mathcal{P}(\omega)$ is a dense ideal. Let us consider what such a construction involves. Fix a dense ideal $\mathcal{I}$, and put $\mathscr{C} = \mathcal{P}(\omega) \setminus \mathcal{I}$. Write $\mathscr{C} = \{c_\alpha : \alpha < \mathfrak{c}\}$. At each stage $\alpha < \mathfrak{c}$, we are given $\{a_\beta : \beta < \alpha\}$, an ADR for $\{c_\beta : \beta < \alpha\}$. We need to do a sub-diagonalization against the set $\{a_\beta : \beta < \alpha\}$ to find an $a_\alpha \subset c_\alpha$ that is almost disjoint from all $a_\beta$. We must have chosen the $a_\beta$ carefully for this to be possible. Given an a.d. family $\mathscr{A} \subset [\omega]^\omega$, let $\mathcal{I}(\mathscr{A})$ denote the ideal on $\omega$ generated by $\mathscr{A}$. For the sub-diagonalization to have any hope of succeeding, $c_\alpha$ must not be in $\mathcal{I}(\{a_\beta : \beta < \alpha\})$, for otherwise any set a.d. from $\{a_\beta : \beta < \alpha\}$ will also be a.d. from $c_\alpha$.

Of course this will not be a problem if we choose each $a_\beta \in \mathcal{I}$, because then $\mathcal{I}(\{a_\beta : \beta < \alpha\}) \subset \mathcal{I}$, which we know is disjoint from $\mathscr{C}$. It is here that the density of $\mathcal{I}$ comes into play. So assume that at stage $\alpha$ we have chosen $\{a_\beta : \beta < \alpha\} \subset \mathcal{I}$. Now, put $\mathscr{A}_\alpha = \{a_\beta : \beta < \alpha\}$ and $\mathscr{A}_\alpha \cap c_\alpha = \{a_\beta \cap c_\alpha : \beta < \alpha \wedge \left|a_\beta \cap c_\alpha\right| = \omega\}$. Notice that $\mathscr{A}_\alpha \cap c_\alpha$ is an almost disjoint family in $[c_\alpha]^\omega$, and that it is sufficient to find an $a_\alpha \in [c_\alpha]^\omega$ that is almost disjoint from all the members of $\mathscr{A}_\alpha \cap c_\alpha$. This requires $\mathscr{A}_\alpha \cap c_\alpha$ to be neither a MAD family in $[c_\alpha]^\omega$ nor a finite partition of $c_\alpha$. Since $c_\alpha \notin \mathcal{I}(\mathscr{A}_\alpha)$, we know that $\mathscr{A}_\alpha \cap c_\alpha$ is not a finite partition of $c_\alpha$. But, *a priori* there is no reason why it could not be a MAD family in $[c_\alpha]^\omega$. Notice however that $\mathscr{A}_\alpha \cap c_\alpha$ has fewer then $\mathfrak{c}$ members. Therefore, if there are no "small" MAD families in $[\omega]^\omega$—i.e., if $\mathfrak{a} = \mathfrak{c}$—then $\mathscr{A}_\alpha \cap c_\alpha$ cannot be a MAD family in $[c_\alpha]^\omega$. So if $\mathfrak{a} = \mathfrak{c}$, there is a $b \in [c_\alpha]^\omega$ which is a.d. from the things in $\mathscr{A}_\alpha \cap c_\alpha$. Now, since $\mathcal{I}$ is a dense ideal there is an $a_\alpha \in [b]^\omega$ which is in $\mathcal{I}$. Clearly, this $a_\alpha$ is also a.d. from the things in $\mathscr{A}_\alpha \cap c_\alpha$, and now the construction can proceed. Thus we have shown that if $\mathfrak{a} = \mathfrak{c}$, then every set of the form $\mathcal{P}(\omega) \setminus \mathcal{I}$, where $\mathcal{I}$ is dense ideal, has an ADR.

Can this construction be done in ZFC? This question remains open in full generality, even though significant progress has recently been made by Shelah [30]. It is known to be equivalent to the problem of whether there is a completely separable MAD family $\mathscr{A} \subset \mathcal{I}$ for every dense $\mathcal{I}$.

DEFINITION 4.1. A MAD family $\mathscr{A} \subset [\omega]^\omega$ is said to be *completely separable* if for any $b \in \mathcal{P}(\omega) \setminus \mathcal{I}(\mathscr{A})$, there is an $a \in \mathscr{A}$ with $a \subset b$.

It is possible to show that $\mathcal{P}(\omega) \setminus \mathcal{I}$ has an ADR for every dense ideal $\mathcal{I}$ iff for every dense ideal $\mathcal{I}$, there is a completely separable MAD family $\mathscr{A} \subset \mathcal{I}$ (we are *not* asserting that these are equivalent for a given $\mathcal{I}$). The proof may be found in [3].

Even though the ZFC existence of ADRs is an open question in general, Balcar and Vojtáš established it in 1980 for several specific cases by using a deep combinatorial property of the cardinal invariant $\mathfrak{h}$. Suppose $\{\mathscr{X}_n : n \in \omega\} \subset [\omega]^\omega$ is a countable collection of dense open sets. Suppose $b \in [\omega]^\omega$. As $\mathscr{X}_0$ is dense we can find $a_0 \in [b]^\omega \cap \mathscr{X}_0$, and as $\mathscr{X}_1$ is dense there is $a_1 \in [a_0]^\omega \cap \mathscr{X}_1$. Continuing in this way we get a sequence $b \supset a_0 \supset a_1 \supset \cdots$, where $a_i \in \mathscr{X}_i$. We can find a $c \in [b]^\omega$ such that $\forall i \in \omega\, [c \subset^* a_i]$. As $\mathscr{X}_i$ is open, we have $c \in \mathscr{X}_i$, and we conclude that the intersection of countably many dense open sets is dense. $\mathfrak{h}$ marks the first place where this diagonalization cannot be carried out.

DEFINITION 4.2.

$$\mathfrak{h} = \min\{|\mathcal{F}| : \forall \mathscr{X} \in \mathcal{F}\, [\mathscr{X} \subset [\omega]^\omega \text{ is dense open}] \wedge \bigcap \mathcal{F} \text{ is not dense}\}.$$

The definition would not change if we replaced "$\bigcap \mathcal{F}$ is not dense" by "$\bigcap \mathcal{F} = 0$". An alternative way to define $\mathfrak{h}$ is as the least $\kappa$ such that the forcing notion $\mathcal{P}(\omega)/[\omega]^{<\omega}$ adds a new function from $\kappa$ into the ground model $\mathbf{V}$.

The following deep combinatorial characterization of $\mathfrak{h}$ was proved by Balcar, Pelant and Simon [2]. The tree $\mathcal{T}$ whose existence is proved in the following theorem is often called a *base tree* or a *base matrix*. Its proof was the original motivation for the introduction of the cardinal $\mathfrak{h}$, which stands for the height of the base tree.

THEOREM 4.3. *There is a family $\mathcal{T} \subset [\omega]^\omega$ satisfying the following properties*:

1. $\langle \mathcal{T}, {}^*\supset \rangle$ *is a tree of height* $\mathfrak{h}$,
2. *each level of $\mathcal{T}$ is a* MAD *family in* $[\omega]^\omega$ *except the root, which is* $\omega$,
3. *each node has* $\mathfrak{c}$ *immediate successors,*
4. $\mathcal{T}$ *is a dense family in* $[\omega]^\omega$.

*Moreover,* $\mathfrak{h}$ *is the least height of any such tree.*

As an aside, we remark that since $\mathcal{T}$ is dense in $[\omega]^{\omega}$, forcing with $\mathcal{P}(\omega)/[\omega]^{<\omega}$ is equivalent to forcing with $\mathcal{T}$. Because of (3), forcing with $\mathcal{T}$ adjoins an onto map from $\mathfrak{h}$ to $\mathfrak{c}$. Moreover, forcing with $\mathcal{P}(\omega)/[\omega]^{<\omega}$ does not add a new function from any $\alpha < \mathfrak{h}$ into the ground model $\mathbf{V}$. It follows that forcing with $\mathcal{P}(\omega)/[\omega]^{<\omega}$ always makes $\mathfrak{c}^{\mathbf{V}[G]} = \mathfrak{h}^{\mathbf{V}}$.

A notable feature of this characterization of $\mathfrak{h}$ is that although $\mathcal{T}$ is a dense family in $[\omega]^{\omega}$, it is quite "short" as a tree because $\mathfrak{h}$ is small compared to many other cardinal invariants. In particular, the inequalities $\mathfrak{h} \leq \mathfrak{b} \leq \mathfrak{a}$ are easily verified.

Balcar and Vojtáš [4] used this characterization of $\mathfrak{h}$ to solve a problem of Pierce and Hindman that had been open for a long time. Notice that one example of a dense ideal is a maximal one. If $\mathcal{I}$ is a maximal ideal, then $\mathcal{P}(\omega) \setminus \mathcal{I}$ is simply an ultrafilter. So we can ask does every ultrafilter have an ADR? This question was posed by Pierce [25] and Hindman [17] in the context of $\mathfrak{c}$-points in $\beta\omega \setminus \omega$, the Stone–Čech remainder of $\omega$. A point $p$ in a topological space is called a *$\mathfrak{c}$-point* if there is a family of $\mathfrak{c}$ pairwise disjoint open sets each of which has $p$ in its closure. Pierce and Hindman asked if each point $\mathcal{U} \in \beta\omega \setminus \omega$ is a $\mathfrak{c}$-point. It is easy to see that this is the same as asking if every $\mathcal{U} \in \beta\omega \setminus \omega$ has an ADR. Balcar and Vojtáš [4] provided a positive answer in ZFC.

THEOREM 4.4. *Every non-principal ultrafilter has an* ADR.

We end this section by mentioning some recent work by the author on a question of Shelah and Steprāns [31] that is connected to the notion of a completely separable MAD family. Given an ideal $\mathcal{I} \subset \mathcal{P}(\omega)$, let us say that a set $A \subset [\omega]^{<\omega}$ is *$\mathcal{I}$-positive* if for every $a \in \mathcal{I}$, $\exists s \in A\, [s \cap a = 0]$. Note that a set $a \in \mathcal{P}(\omega)$ is in $\mathcal{P}(\omega) \setminus \mathcal{I}$ iff $\{\{n\} \colon n \in a\}$ is $\mathcal{I}$-positive. Now, we may rephrase the definition of a completely separable MAD family as follows: $\mathscr{A} \subset [\omega]^{\omega}$ is completely separable iff for every $\mathcal{I}(\mathscr{A})$-positive $A \subset [\omega]^{<\omega}$ such that every $s \in A$ is a singleton, there is an infinite $B \subset A$ with $\bigcup B \in \mathscr{A}$. The notion of a strongly separable MAD family is gotten by dropping the requirement that $A$ consist of singletons. More formally, $\mathscr{A} \subset [\omega]^{\omega}$ is *strongly separable* if for every $A \subset [\omega]^{<\omega}$ that is $\mathcal{I}(\mathscr{A})$-positive, there is an infinite $B \subset A$ such that $\bigcup B \in \mathscr{A}$. Shelah and Steprāns [31] related this concept with questions about MASAs in the Calkin algebra, and asked whether their existence can be proved in ZFC. This notion is also related to the metrization problem for Fréchet groups, which we discuss in Section 6. Building upon earlier work by Brendle and Hrušák [11], we provide a negative answer in [27] to the question of Shelah and Steprāns.

THEOREM 4.5. *There is a model where there are no strongly separable* MAD *families and* $\mathfrak{c} = \aleph_2$.

§**5. Van Douwen families.** In this section we discuss the second of the two ZFC constructions of almost disjoint families we have been alluding to. This one concerns MAD families in $\omega^\omega$. It answers an old question of Van Douwen. Van Douwen asked whether there is a MAD family of functions $\mathscr{A} \subset \omega^\omega$ that is also maximal with respect to infinite partial functions. Recall that $p \in [\omega \times \omega]^\omega$ is an *infinite partial function* if for each $n$, $p(n) = \{m \in \omega : \langle n, m\rangle \in p\}$ has at most one element.

Definition 5.1. An a.d. family $\mathscr{A} \subset \omega^\omega$ is said to be *Van Douwen* if for any infinite partial function $p$ there is $h \in \mathscr{A}$ such that $|h \cap p| = \omega$.

There are several equivalent formulations of Van Douwen's question and it is instructive to consider some of them. Recall from Section 3 that any a.d. family of functions $\mathscr{A} \subset \omega^\omega$ is an a.d. family of sets in $[\omega \times \omega]^\omega$, but is never a MAD family in $[\omega \times \omega]^\omega$ because the vertical columns are almost disjoint from any function. Van Douwen MAD families are those a.d. families of functions that have nothing else preventing them from being MAD in $[\omega \times \omega]^\omega$. In other words an a.d. $\mathscr{A} \subset \omega^\omega$ is Van Douwen iff $\mathscr{A} \cup \{c_n : n \in \omega\}$ is MAD in $[\omega \times \omega]^\omega$, where $c_n$ is the $n^{\text{th}}$ vertical column of $\omega \times \omega$—that is, $c_n = \{\langle n, m\rangle : m \in \omega\}$.

Another formulation is to ask whether there is an a.d. family $\mathscr{A} \subset \omega^\omega$ which is "everywhere maximal" in the following sense. Given an a.d. family $\mathscr{A} \subset \omega^\omega$ and a set $X \in [\omega]^\omega$, we can consider the restriction of $\mathscr{A}$ to $X$, $\mathscr{A} \restriction X = \{h \restriction X : h \in \mathscr{A}\}$. This is an a.d. family of functions in $\omega^X$. It is easily seen that $\mathscr{A}$ is Van Douwen MAD iff all its restrictions are maximal—that is, $\mathscr{A} \restriction X$ is MAD in $\omega^X$ for all $X \in [\omega]^\omega$.

There are several motivations for considering such families. One natural property of interest of maximal objects (of any sort) is preservation of their maximality in forcing extensions.

Definition 5.2. Let $\mathbb{P}$ be a notion of forcing and let $\mathscr{A}$ be a MAD family (either of sets or of functions). We will say that $\mathscr{A}$ *is* $\mathbb{P}$*-indestructible* if $\Vdash_{\mathbb{P}} \mathscr{A}$ is MAD.

Obviously, if $\mathbb{P}$ is a forcing notion that does not add a real, then every MAD $\mathscr{A}$ is $\mathbb{P}$-indestructible. The most basic example of a $\mathbb{P}$ that adds a real is Cohen forcing (i.e., $\mathrm{Fn}(\omega, 2)$). When is a MAD $\mathscr{A}$ Cohen indestructible? It turns out Cohen indestructibility is closely related to the combinatorial notion of a strongly MAD family. Brendle and Yatabe [12] have provided combinatorial characterizations of the property of being a $\mathbb{P}$-indestructible MAD family of sets in $[\omega]^\omega$ for some standard posets $\mathbb{P}$.

Definition 5.3. An a.d. family of sets $\mathscr{A} \subset [\omega]^\omega$ is *strongly* MAD if for each countable collection of sets $\{a_n : n \in \omega\} \subset \mathcal{P}(\omega) \setminus \mathcal{I}(\mathscr{A})$, where $\mathcal{I}(\mathscr{A})$ is the ideal on $\omega$ generated by $\mathscr{A}$, $\exists c \in \mathscr{A} \forall n \in \omega\, [|c \cap a_n| = \omega]$. An a.d. family of functions $\mathscr{A} \subset \omega^\omega$ is *strongly* MAD if for each countable collection of functions $\{f_n : n \in \omega\} \subset \omega^\omega \cap (\mathcal{P}(\omega \times \omega) \setminus \mathcal{I}(\mathscr{A}))$, where $\mathcal{I}(\mathscr{A})$ is the ideal on $\omega \times \omega$ generated by $\mathscr{A}$, $\exists h \in \mathscr{A} \forall n \in \omega\, [|h \cap f_n| = \omega]$.

In this definition, the requirement that the collection of the $a_n$, and the collection of the $f_n$, miss $\mathcal{I}(\mathscr{A})$ is essential, for no element of $\mathscr{A}$ can have infinite intersection with two distinct members of $\mathscr{A}$.

For MAD families in $[\omega]^\omega$, strong MADness is "almost equivalent" to Cohen indestructibility. More precisely, there is a strongly MAD family of sets $A \subset [\omega]^\omega$ iff a Cohen indestructible MAD family $A \subset [\omega]^\omega$ exists. For a.d. families of functions in $\omega^\omega$, strong MADness is slightly different. An a.d. family $\mathscr{A} \subset \omega^\omega$ is strongly MAD iff $\mathscr{A} \cup \{c_n : n \in \omega\}$ is strongly MAD in $[\omega \times \omega]^\omega$. Therefore, a strongly MAD $\mathscr{A} \subset \omega^\omega$ is a Van Douwen family that is Cohen indestructibly Van Douwen.

To see that a strongly MAD $\mathscr{A} \subset \omega^\omega$ must be Van Douwen, suppose for a contradiction that $p$ is an infinite partial function a.d. from all $h \in \mathscr{A}$. Choose $h_0, h_1 \in \mathscr{A}$ so that $h_0 \neq h_1$. Put $a = \omega \setminus \operatorname{dom}(p)$, and for each $i \in 2$, set $f_i = p \cup (h_i \upharpoonright a)$. Now, applying the defining requirement of strong maximality to the collection $\{f_0, f_1\}$ gives a contradiction.

It is not known whether it is possible to construct a strongly MAD family (of either sort) just in ZFC. They can be constructed if $\mathfrak{b} = \mathfrak{c}$. However, the following natural strengthening of the notion of a strongly MAD family of functions was introduced by Kastermans [20].

Definition 5.4. Let $\mathscr{A}$ be an a.d. family and put $\kappa = |\mathscr{A}|$. We say that $\mathscr{A}$ is *very* MAD if for every cardinal $\lambda < \kappa$ and for every collection $\{f_\alpha : \alpha < \lambda\} \subset \omega^\omega \cap (\mathcal{P}(\omega \times \omega) \setminus \mathcal{I}(\mathscr{A}))$, $\exists h \in \mathscr{A}$ such that $\forall \alpha < \lambda\, [|h \cap f_\alpha| = \omega]$.

Kastermans [20] showed that very MAD families exist if $\mathfrak{p} = \mathfrak{c}$, and he asked if their existence be proved in ZFC. Regarding this it was conjectured by Brendle that if $\operatorname{cov}(\mathcal{M}) < \mathfrak{a}_\mathfrak{e}$, then there are no very MAD families, and moreover, he verified that there are no very MAD families in several natural models of the statement $\operatorname{cov}(\mathcal{M}) < \mathfrak{a}_\mathfrak{e}$, such as the Laver model, and the random real model. In [26], we proved Brendle's conjecture:

Theorem 5.5. *If $\mathscr{A}$ is a very* MAD *family, then $|\mathscr{A}| \leq \operatorname{cov}(\mathcal{M})$. In particular, if $\operatorname{cov}(\mathcal{M}) < \mathfrak{a}_\mathfrak{e}$, then there are no very* MAD *families.*

Thus the notion of a Van Douwen family is the weakest in a natural progression of notions of MADness of increasing strength. Our solution to Van Douwen's question in [28] shows that the families at the bottom of this hierarchy can be constructed in ZFC, while Theorem 5.5 shows that MAD families of the strongest variety cannot; and it is unknown if those of the middle variety can be built.

Van Douwen's question dates to the 1980s. It occurs as problem 4.2 in Miller's problem list [23]. Zhang [37] obtained some partial results on this problem. He proved that Van Douwen families of various sizes exist in certain forcing extensions.

The naive approach to constructing such a family is as follows. Let $\langle p_\alpha : \alpha < \mathfrak{c} \rangle$ enumerate all infinite partial functions. At a stage $\alpha < \mathfrak{c}$, we are given an almost disjoint family of functions $\mathscr{A}_\alpha \subset \omega^\omega$ with $|\mathscr{A}_\alpha| < \mathfrak{c}$ and an infinite partial function $p_\alpha$. If $p_\alpha$ is almost disjoint from $\mathscr{A}_\alpha$, then we must produce a $h \in \omega^\omega$ such that $\forall f \in \mathscr{A}_\alpha\, [|h \cap f| < \omega]$ and $|h \cap p_\alpha| = \omega$. If there are no "small" MAD families in $\omega^\omega$—i.e., if $\mathfrak{a}_\mathfrak{e} = \mathfrak{c}$, then there is a $g \in \omega^\omega$ such that $\forall f \in \mathscr{A}_\alpha\, [|g \cap f| < \omega]$. Now, put $a = \omega \setminus \operatorname{dom}(p_\alpha)$, and simply take $h$ to be $p_\alpha \cup (g \restriction a)$. Thus we conclude that if $\mathfrak{a}_\mathfrak{e} = \mathfrak{c}$, then there is a Van Douwen MAD family. This situation is formally analogous to the one for ADRs: both diagonalizations succeed *provided* that we do not end up with a MAD family "too soon".

Continuing the analogy with ADRs, a Van Douwen MAD family can be constructed in ZFC using a deep combinatorial characterization of an appropriate cardinal invariant. Here the right invariant turns out to be $\operatorname{non}(\mathcal{M})$ introduced in Definition 2.3. The analogue of Theorem 4.3 here is the following combinatorial characterization of $\operatorname{non}(\mathcal{M})$ proved by Bartoszyński [5].

Definition 5.6. Let $h \in \omega^\omega$. An *h-slalom* is a function $S : \omega \to [\omega]^{<\omega}$ such that for all $n \in \omega$, $|S(n)| \le h(n)$.

Theorem 5.7 (Bartoszyński). *Let $\kappa$ be an infinite cardinal. The following are equivalent*:

1. *Every set of reals of size less than $\kappa$ is meager.*
2. *For every family $F \subset \omega^\omega$ with $|F| < \kappa$, there is an infinite partial function $p$ such that $\forall f \in F\, [|p \cap f| < \omega]$.*
3. *For every $h$ and for every family of $h$-slaloms $F$ with $|F| < \kappa$, there is a $g \in \omega^\omega$ such that $\forall S \in F\, \forall^\infty n \in \omega\, [g(n) \notin S(n)]$.*

We start with a simple observation. It is easy to see that if $p$ be any infinite partial function, then $\{f \in \omega^\omega : |p \cap f| < \omega\}$ is meager in $\omega^\omega$. Therefore, any non-meager set $F = \{f_\alpha : \alpha < \operatorname{non}(\mathcal{M})\} \subset \omega^\omega$ has the property that for any infinite partial function $p$, $\exists \alpha < \operatorname{non}(\mathcal{M})\, [|p \cap f_\alpha| = \omega]$. Recall that if $\mathscr{A} \subset \omega^\omega$ is an a.d. family, and if $f \in \omega^\omega$, then $\mathscr{A} \cap f = \{h \cap f : h \in \mathscr{A} \wedge |h \cap f| = \omega\}$ is an a.d. family of sets in $[f]^\omega$. It is clear that $\mathscr{A} \subset \omega^\omega$ is Van Douwen MAD iff for every $\alpha < \operatorname{non}(\mathcal{M})$, $\mathscr{A} \cap f_\alpha$ is either a finite partition of $f_\alpha$ or is MAD in $[f_\alpha]^\omega$.

So instead of building a family of size $\mathfrak{c}$ in $\mathfrak{c}$ steps, we construct it in $\operatorname{non}(\mathcal{M})$ many steps, and at stage $\alpha < \operatorname{non}(\mathcal{M})$ we try to ensure that $\mathscr{A}_{\alpha+1} \cap f_\alpha$ is either a finite partition of $f_\alpha$ or is MAD in $[f_\alpha]^\omega$. We seem to have reduced the number of "requirements" we need to meet from $\mathfrak{c}$ to $\operatorname{non}(\mathcal{M})$. Of course, $\operatorname{non}(\mathcal{M})$ could equal $\mathfrak{c}$; but since $\operatorname{non}(\mathcal{M}) \le \mathfrak{a}_\mathfrak{e}$, we already know what to do if $\operatorname{non}(\mathcal{M}) = \mathfrak{c}$. However, the *real* point here is that we can now use Theorem 5.7 because at each stage, we would have "used up fewer than $\operatorname{non}(\mathcal{M})$ things".

In order to meet the necessary almost disjointness requirements, it turns out that we need to strengthen condition (3) of Bartoszyński's theorem so that whenever we have fewer than non $(\mathcal{M})$ many slaloms, there is an object somewhat "fatter" than a function from $\omega$ to $\omega$ that is eventually disjoint from all the slaloms.

DEFINITION 5.8. We say that a function $S: \omega \to [\omega]^{<\omega}$ is a *fat slalom* if for each $n \in \omega$, $|S(n)| = 2^n$.

The choice of $2^n$ here is arbitrary; all we need is $\lim_{n\to\infty}|S(n)| = \infty$. The strengthening of Bartoszyński's theorem required to carry out the construction is given in the next lemma.

LEMMA 5.9 (See [28]). *Let $\langle S_\xi : \xi < \lambda\rangle$ be a family of slaloms with $\lambda <$ non $(\mathcal{M})$. There is a fat slalom $S$ such that*

$$\forall \xi < \lambda \forall^\infty n \in \omega \left[S(n) \cap S_\xi(n) = 0\right].$$

As a final remark, we mention that there are other ways in which cardinal invariants aid diagonalizations of length $\mathfrak{c}$. Some diagonalizations of length $\mathfrak{c}$ may be performed by treating the various possible inequalities that may obtain between certain cardinal invariants are separate cases. For example, it may be possible to prove the existence of a certain kind of object in ZFC by giving separate, different constructions in the three cases $\mathfrak{x} < \mathfrak{y}$, $\mathfrak{x} = \mathfrak{y}$, and $\mathfrak{x} > \mathfrak{y}$, where $\mathfrak{x}$ and $\mathfrak{y}$ are appropriately chosen cardinal invariants. In arguments of this sort, even though the existence of an object of the required sort is established, some of its properties, such as, perhaps, its size, are left undetermined, as they depend on which of the three cases holds. A recent example of a construction of this sort is Shelah [30].

**§6. Some open questions.** We conclude with a discussion of some of the main open problems involving the construction of almost disjoint families of different sorts. A positive solution to any of these problems requires a diagonalization of length $\mathfrak{c}$ to be carried out in ZFC.

An outstanding question that is still open concerns the existence of ADRs (cf. Section 4)

QUESTION 6.1. *Suppose $\mathcal{I}$ is dense ideal in $\mathcal{P}(\omega)$. Is there a completely separable* MAD *family $\mathscr{A} \subset \mathcal{I}$?*

Shelah [30] has recently shown that the answer is yes if $\mathfrak{c} < \aleph_\omega$. The argument there proceeds in three cases depending on whether $\mathfrak{s} < \mathfrak{a}$, $\mathfrak{s} = \mathfrak{a}$, or $\mathfrak{s} > \mathfrak{a}$, where $\mathfrak{s}$ is the splitting number.

Another open problem concerns the existence of indestructible MAD families (cf. Section 5). There is no forcing notion $\mathbb{P}$ adding a new real for which we know how to construct a $\mathbb{P}$-indestructible MAD family (either of sets or of functions) in ZFC. A Sacks indestructible MAD family is provably the weakest such object in the following sense.

Lemma 6.2 (See [12] and [18]). *Suppose $\mathbb{P}$ is a forcing notion that adds a new real, and suppose $\mathscr{A}$ is a* MAD *family (either in $[\omega]^\omega$ or in $\omega^\omega$). If $\mathscr{A}$ is $\mathbb{P}$-indestructible, then $\mathscr{A}$ is also Sacks indestructible.*

So the weakest sort of indestructibility we can ask for is:

Question 6.3. *Is there a Sacks indestructible* MAD *family (either in $[\omega]^\omega$ or in $\omega^\omega$)? Is there one of size $\mathfrak{c}$?*

It is not too hard to see that if $\mathfrak{a} < \mathfrak{c}$, then any MAD family of size $\mathfrak{a}$ is Sacks indestructible. Therefore, for the first question, we may assume without loss that $\mathfrak{a} = \mathfrak{c}$. For the second question, it is known that the answer in yes if either $\mathfrak{b} = \mathfrak{c}$ or $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ holds.

For the case of MAD families in $[\omega]^\omega$ asking for a Cohen indestructible MAD family is the same as asking for a strongly MAD family, while for the case of $\omega^\omega$, it entails asking for a Van Douwen MAD $\mathscr{A} \subset \omega^\omega$ that is Cohen indestructibly Van Douwen.

Question 6.4. *Is there a strongly* MAD *family (either in $[\omega]^\omega$ or in $\omega^\omega$)?*

It is known that the answer is yes if $\mathfrak{b} = \mathfrak{c}$. Unlike for a Sacks indestructible MAD family, it is shown in [26] that the weak Freese–Nation property of $\mathcal{P}(\omega)$ ($\mathrm{wFN}(\mathcal{P}(\omega))$) implies that all strongly MAD families have size at most $\aleph_1$. It is shown in [14] that $\mathrm{wFN}(\mathcal{P}(\omega))$ holds in any model gotten by adding fewer than $\aleph_\omega$ Cohen reals to a ground model satisfying CH. So, in particular, it is consistent to have no strongly MAD families of size $\mathfrak{c}$. So if there is a ZFC construction of one, then it must leave the cardinality of the family indeterminate. More information about indestructible MAD families of sets can be found in [12].

The definition of strong maximality suggests the following natural weakening. An a.d. family $\mathscr{A} \subset [\omega]^\omega$ is *weakly tight* if for each countable collection of sets $\{a_n : n \in \omega\} \subset \mathcal{P}(\omega) \setminus \mathcal{I}(\mathscr{A})$, there exists $c \in \mathscr{A}$ such that for infinitely many $n \in \omega$, $|c \cap a_n| = \omega$. This notion was introduced by Hrušák, and García Ferreira [19], where they present applications of it to the Katétov ordering on MAD families. By modifying the techniques introduced by Shelah [30] I and Steprāns have recently shown in [29] that

Theorem 6.5. *If $\mathfrak{s}_\omega \leq \mathfrak{b}$, then there is a weakly tight* MAD *family.*

Here $\mathfrak{s}_\omega$ is a cardinal invariant closely related to the splitting number. In particular, such families exist if $\mathfrak{s}_\omega = \omega_1$, or if $\mathfrak{b} = \mathfrak{d}$. The ZFC existence of these families remains open:

Question 6.6. *Is there a weakly tight* MAD *family $\mathscr{A} \subset [\omega]^\omega$?*

We conclude with a discussion of a problem regarding metrizability of Fréchet groups. Recall that a topological space $X$ is *Fréchet* if whenever a point $p \in X$ is in the closure of a set $A \subset X$, there is a sequence of points in $A$ converging to $p$. A well-known question of Malykhin asks whether every countable Fréchet topological group is metrizable. If $\mathfrak{p} > \omega_1$, Malykhin's question has a negative answer. Indeed, in this situation, one can put a non-metrizable Fréchet topology on the group $\langle [\omega]^{<\omega}, \triangle \rangle$, where $\triangle$ denotes symmetric difference. Let us say that an ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ is *Fréchet* if for every $\mathcal{I}$-positive $A \subset [\omega]^{<\omega}$ (cf. Section 4), there is an infinite $B \in [A]^{\omega}$ so that $\forall a \in \mathcal{I} \forall^{\infty} s \in B\, [a \cap s = 0]$. If $\mathcal{I}$ is a Fréchet ideal which is not countably generated, then we can define a non-metrizable Fréchet topology on $\langle [\omega]^{<\omega}, \triangle \rangle$ by stipulating that

$$\{A \subset [\omega]^{<\omega} : \exists a \in \mathcal{I} \forall s \in [\omega]^{<\omega}\, [s \cap a = 0 \Longrightarrow s \in A]\}$$

is a neighborhood base at 0. If $\mathfrak{p} > \omega_1$, then *any* ideal that is $\aleph_1$ generated is Fréchet. The following question of Gruenhage and Szeptycki [16] asks if a Fréchet ideal that is not countably generated can be constructed in ZFC, and more specifically, whether there are any that are generated by almost disjoint families.

QUESTION 6.7. *Is there an uncountable a.d. family $\mathscr{A} \subset [\omega]^{\omega}$ such that $\mathcal{I}(\mathscr{A})$ is Fréchet? Is there a Fréchet ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ that is not countably generated?*

Brendle and Hrušák [11] have recently shown that it is consistent that every uncountably generated Fréchet ideal has no fewer than $\mathfrak{c}$ generators. The paper [27] also has some partial independence results on the first question.

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