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THE SENSES OF FUNCTIONS IN THE LOGIC OF SENSE AND DENOTATION

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Abstract. This paper discusses certain problems arising within the treatment of the senses of functions in Alonzo Church's Logic of Sense and Denotation. Church understands such senses themselves to be "sense-functions," functions from sense to sense. However, the conditions he lays out under which a sense-function is to be regarded as a sense presenting another function as denotation allow for certain undesirable results given certain unusual or "deviant" sense-functions. Certain absurdities result, e.g., an argument can be found for equating any two senses of the same type. An alternative treatment of the senses of functions is discussed, and is thought to do better justice to Frege's original theory.

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§1. Introduction. Alonzo Church's Logic of Sense and Denotation, has had a many-storied history, initially taking on three alternative forms, and

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undergoing a number of revisions at Church's own hands and those of others [2, 3, 4, 8, 9, 10, 11, 13, 36, 37]. In this paper, I focus on the treatment of the senses of function expressions within these systems. Hitherto these senses have almost always been treated as functions from sense to sense. While greatly simplifying their syntax, this approach has generated significant problems. My first aim in this paper is to make plain the extent of the difficulties, especially those concerning the specification of the conditions under which a function from sense to sense is itself to be regarded as a sense having another function as denotation. My second aim is to describe a rival understanding of the senses of functions, and sketch how the systems could be restructured accordingly. It is argued that the rival understanding solves some of the more difficult problems with the previous approach, is in some ways philosophically superior, and perhaps closer to Frege's own understanding of such senses.

§**2. Church's Logic of Sense and Denotation: a recap.** The Logic of Sense and Denotation represents an attempt to translate Frege's [18, 20] theory of sense (*Sinn*) and denotation (*Bedeutung*) into a full-fledged and general purpose intensional logic. While Church knowingly deviates from Frege's own views at a number of points, the logical theory underlying the systems is still largely inspired by Frege. According to the functional logics favored by both Frege and Church, "formulas" are understood as terms for truth-values, of which there are two, the True and the False. Predicates are understood as standing for functions in the strictest sense, mapping individuals to truth-values, and propositional connectives are taken as functions mapping truth-values to truth-values. In addition to denoting an individual, truth-value or function, each expression is also thought to be correlated with another entity: a "sense" in Frege's terminology or a "concept" in Church's terminology. (In what follows, I use Frege's terminology, to avoid the confusion of Church's "concept" with Frege's term "*Begriff*" which differs in meaning.) In certain intensional contexts (propositional attitudes, modal contexts), these entities, typically only expressed, are instead thought to be denoted. According to the "direct discourse" method favored by Church (see [28, pp. 23–9]), the same ordinary language expressions are transcribed using different signs in the logical language when they appear in intensional contexts, making clear that they there denote senses.

It is useful to introduce officially the vocabulary that will be used when discussing the relationships between expressions, their senses, and their denotations. I shall use the word "express" for the relation between a part of language, e.g., the name "Socrates," and its sense or intensional meaning: the name *expresses* the sense. I shall use the word "denote" for the relation between the linguistic expression and the entity denoted: here the name "Socrates" *denotes* the person Socrates. This terminology comports with common English translations of Frege's own. I shall use the word "presents" for the relation between the sense and the denotation; the sense of the name "Socrates" *presents* Socrates. Frege had no official word for this relation, but "presents" is used by at least some later commentators.

Church builds his systems upon a functional calculus of lambda conversion employing the simple theory of types. (In [13], ramified type-theory is introduced to block certain intensional and semantic paradoxes; more on this later.) The simple types are further divided according to the sense hierarchy. The types of non-functions include two hierarchies, $\iota_0, \iota_1, \iota_2, \ldots$ and $o_0, o_1, o_2, \ldots$. Type ι_0 consists of individuals that are not senses, and type ι_{n+1} consists of senses that (potentially) present entities of type ι_n. Type o_0 consists of the two truth-values, the True and the False, and type o_{n+1} consists of senses that (potentially) present entities of type o_n. Frege's "thoughts" (*Gedanken*), as senses that present truth-values, fall into type o_1. For any types α and β, there exists a type $(\alpha \mapsto \beta)$ consisting of functions that take arguments of type α and yield values of type β.[1] Thus, predicates would have the type $(\iota_0 \mapsto o_0)$, taking individuals as argument and yielding truth-values as value. Relations and functions of multiple arguments are treated with the method suggested by Schönfinkel [42] as functions with one argument that yield functions as value. For example, the conditional function has the type $(o_0 \mapsto (o_0 \mapsto o_0))$: it is understood as a function that takes one truth-value as argument and yields as value a function with one remaining argument place and having a truth-value as result.

We adopt the conventions that a subscript on an index can be left off when it is 0, and that parentheses can be dropped from compound types with the convention of association to the right. Thus, the type symbol "$(o_0 \mapsto (o_0 \mapsto o_0))$" can be abbreviated simply "$o \mapsto o \mapsto o$". Moreover, unless another type symbol is explicitly included at their first occurrence, the letters x, y, and z (without asterisks) are used as variables of type ι, the letters p, q, and r as variables of type o, and the letters f, g, and h as variables or constants of type $\iota \mapsto o$. (Apostrophes can be added to ensure an infinite supply of such variables.) When written with asterisks, the letters x^*, y^*, and z^* should be taken as variables of type ι_1, the letters p^*, q^*, and r^* as variables or constants of type o_1, and the letters f^*, g^*, and h^* as variables or constants of type $\iota_1 \mapsto o_1$. When giving informal examples, the letters a, b, and c are used as *constants* of type ι, and a^*, b^*, and c^* as constants of type ι_1. When a schematic letter such as α is used for some arbitrary type symbol, α_{+1} is to be understood as the type symbol obtained from α by adding one to each numerical subscript on the primitive type symbols making up α. Boldface letters such as **M** and **N** are

[1] In Church's own notation, no arrow is used, and the type symbol for the value is written first, and the type symbol of the argument is written second, thus the reverse of that used here.

used in the metalanguage schematically for arbitrary expressions. Following Church, the notation $|\mathrm{S}^{\mathbf{A}}_{\mathbf{B}}\,\mathbf{M}|$ is used in the metalanguage for the expression that results when $\mathbf{B}$ is substituted for $\mathbf{A}$ throughout $\mathbf{M}$.

The extensional portion of the system is built upon the following function constants: $C_{o\mapsto o\mapsto o}$ for material implication, $\sim_{o\mapsto o}$ for negation, $\Pi_{(\alpha\mapsto o)\mapsto o}$ for the universal quantifiers, and $\imath_{(\alpha\mapsto o)\mapsto\alpha}$ for description functions. (The description functions take a function into truth-values as argument and yield the sole argument for which that function yields the True as value if there is such an argument, or yield a chosen member of the appropriate type if not.) More usual notation for quantification, $\ulcorner(\forall\mathbf{x}_\alpha)\,\mathbf{M}_o\urcorner$, is defined as $\ulcorner\Pi_{(\alpha\mapsto o)\mapsto o}(\lambda\mathbf{x}_\alpha\,\mathbf{M}_o)\urcorner$. The notation $\ulcorner\mathbf{A}_o \supset \mathbf{B}_o\urcorner$ is used to abbreviate $\ulcorner C_{o\mapsto o\mapsto o}\mathbf{A}_o\mathbf{B}_o\urcorner$. In some formulations, Church did not take negation as primitive, but defined $\ulcorner\sim\mathbf{A}_o\urcorner$ as $\ulcorner\mathbf{A}_o \supset (\forall p_o)\,p_o\urcorner$; little turns on this, however, and for the remainder of the paper, negation is taken as primitive solely for the sake of ease of comparison with other systems. Other logical operators (&, $\vee$, $\equiv$, $\exists$) are defined as usual from negation, material implication and universal quantification. The notation $\ulcorner\mathbf{A}_\alpha = \mathbf{B}_\alpha\urcorner$ is used to abbreviate $\ulcorner(\forall\mathbf{f}_{\alpha\mapsto o})(\mathbf{f}_{\alpha\mapsto o}\mathbf{A}_\alpha \supset \mathbf{f}_{\alpha\mapsto o}\mathbf{B}_\alpha)\urcorner$, where $\mathbf{f}_{\alpha\mapsto o}$ is the first variable of the appropriate type that does not occur in either $\mathbf{A}_\alpha$ or $\mathbf{B}_\alpha$. In addition to standard inference rules (*modus ponens*, etc.), we also have three rules for λ-conversion:

I. *Innocuous change of bound variable*: if $\mathbf{A}_o$ and $\mathbf{B}_o$ are well-formed expressions of type o, and $\mathbf{A}_o$ has a well-formed part $\mathbf{M}_\alpha$, then if $\mathbf{x}_\beta$ is a variable of type β which has no free occurrence in $\mathbf{M}_\alpha$, and $\mathbf{y}_\beta$ is a variable of type β which does not occur in $\mathbf{M}_\alpha$, and $\mathbf{B}_o$ results from $\mathbf{A}_o$ by replacing a particular occurrence of $\mathbf{M}_\alpha$ by $|\mathrm{S}^{\mathbf{x}_\beta}_{\mathbf{y}_\beta}\,\mathbf{M}_\alpha|$, then from $\mathbf{A}_o$, one may infer $\mathbf{B}_o$.

II. *Reduction*: If $\mathbf{A}_o$ and $\mathbf{B}_o$ are well-formed expressions of type o and $\mathbf{A}_o$ differs from $\mathbf{B}_o$ only by containing a well-formed part $\ulcorner((\lambda\mathbf{x}_\beta\,\mathbf{M}_\alpha)\mathbf{N}_\beta)\urcorner$ where $\mathbf{B}_o$ contains $|\mathrm{S}^{\mathbf{x}_\beta}_{\mathbf{N}_\beta}\,\mathbf{M}_\alpha|$ and the bound variables of $\mathbf{M}_\alpha$ are distinct from both $\mathbf{x}_\beta$ and the free variables of $\mathbf{N}_\beta$, then from $\mathbf{A}_o$, one may infer $\mathbf{B}_o$.

III. *Expansion*: If $\mathbf{A}_o$ and $\mathbf{B}_o$ are well-formed expressions of type o and $\mathbf{A}_o$ differs from $\mathbf{B}_o$ only by containing a well-formed part $\ulcorner((\lambda\mathbf{x}_\beta\,\mathbf{M}_\alpha)\mathbf{N}_\beta)\urcorner$ where $\mathbf{B}_o$ contains $|\mathrm{S}^{\mathbf{x}_\beta}_{\mathbf{N}_\beta}\,\mathbf{M}_\alpha|$ and the bound variables of $\mathbf{M}_\alpha$ are distinct from both $\mathbf{x}_\beta$ and the free variables of $\mathbf{N}_\beta$, then from $\mathbf{B}_o$, one may infer $\mathbf{A}_o$.

The axioms of the various formulations divide into three groups. First, there are the schemata needed for the core, extensional, portion of the logic. The second group deals with the relationship between senses and their denotations generally. The third group involves the identity conditions of senses, and differs between rival "alternatives," i.e., different theories about under what conditions senses are identical. Church develops three alternatives, dubbed "Alternative (0)," "Alternative (1)," and "Alternative (2)."

Alternative (2) has the least stringent conditions for sense-identity. Therein, two sentences are held to express the same thought if and only if they are logically equivalent, and generally, two expressions are thought to express the same sense if and only if the identity statement formed from them is a logical truth. These conditions are suitable for developing a direct discourse form of modal logic, but not for a logic dealing with other intensional contexts such as propositional attitudes. My primary interest in what follows shall be Alternatives (0) and (1). These two alternatives adopt criteria for the identity conditions of senses based on a revision of Carnap's [6] notion of *intensional isomorphism* that Church [12] calls *synonymous isomorphism*. According to this view, two complex expressions are thought to be synonymous (or to express the same thought) if and only if one can be derived from the other by a series of synonym replacements among the parts. Consequently, two sentences can be synonymous only if they share a common form, and each expression in one sentence is synonymous with (i.e., has the same sense as) the corresponding expression in the other. Unless the language in question has redundant primitives or contains constants introduced as synonymous with pre-existing complex expressions, this means that distinct sentences are nearly always regarded as having distinct senses. In Alternative (0), distinct sentences are thought synonymous only if they can be obtained from one another by inference rule I, i.e., differ from each other only by innocuous change of bound variable. In Alternative (1), inference rules II and III (the λ-reduction/expansion rules) are also thought to preserve the sense expressed.

Let us consider first those that are invariant across the alternatives. Because different signs are used for sense and denotation, Church introduces methods for capturing the relation between the two. The sign $\Delta_{\alpha\mapsto\alpha_{+1}\mapsto o}$ stands for a two-place function whose value is the True if its second argument presents its first argument, and the False otherwise. (In what follows, when this operator is followed by its arguments, its type symbol is left off as it should be obvious from the context what it must be.) For example, if "a" stands for Russell, and "a^*" stands for the sense of the name "Russell," then "Δaa^*" stands for the True, whereas if "b" stands for Frege, then "Δba^*" stands for the False.[2] Intuitively, a sense presents at most one denotation,

[2]Those familiar with the notation for intensional logic adopted by Montague [34] may be accustomed to working with an operator, "$^\vee$", understood as standing for a function mapping a sense (or intension) to the (unique) denotation (or extension) it presents (so that, if "a^*" is a name of a sense, "$^\vee a^*$" would name the entity the sense presents). Such a sign can be defined by making use of the description operator, i.e., $^\vee{}_{\alpha_{+1}\mapsto\alpha} =_{\mathrm{df}} (\lambda x_{\alpha_{+1}}\, \imath(\lambda y_\alpha\, \Delta yx))$. With this definition in place, if Δaa^* then $^\vee a^* = a$. The primary danger in using this instead of the relation sign "Δ" involves the possibility of "empty senses," those that present no entity as denotation. With the above definition of "$^\vee$", if b^* is an empty sense, then $^\vee b^*$

Hence we have the following schema:

(Axiom 17^α) $\quad (\forall x_\alpha)(\forall y_\alpha)(\forall x^*_{\alpha_{+1}})[\Delta x x^* \supset (\Delta y x^* \supset x = y)].$

Most of the remaining schemata in the second group have to do with the senses of functions. As mentioned above, Church treats the senses of functions as functions from sense to sense. These functions are typically called "sense-functions." Let us first give an ordinary-language example. The name "Socrates" denotes a certain person and expresses a sense. The whole expression "Socrates is human," is thought to denote the True, and to express a certain *thought* or *proposition*. In it, the predicate " ... is human" denotes a certain function, which has the True as value for Socrates as argument (but the False as value for Boston as argument, and so on). According to Church, the sense expressed by the predicate should also be taken as a function, in this case, as a sense-function taking the sense of the name "Socrates" as argument and yielding the thought expressed by "Socrates is human" as value. This sense-function would yield the thought that *Aristotle is human* for the sense of the name "Aristotle" as argument, the thought that *Boston is human* for the sense of the name "Boston" as argument, and so on. This view leads to a nice parallel between the realms of sense and denotation; just as the denotation of the predicate can be taken as a function of type $(\iota_0 \mapsto o_0)$, its sense can be understood as a function of type $(\iota_1 \mapsto o_1)$. In the Logic of Sense and Denotation, a constant of the former type would be used to transcribe " ... is human" when it appears in a standard context, and a constant of the latter type would be used to transcribe " ... is human" in an intensional context.

In Frege's [22] notation, a function expression's argument place must always be filled somehow, so that when one function is taken as the argument to a higher-type function, the higher-type function expression must be written as a variable-binding operator, with the bound variable being used to fill the argument position of the lower-type function, e.g., "$M_\beta(f(\beta))$" for M taking f as argument. In Church's notation, however, function variables and lambda abstracts may appear in subject position, that is, as arguments to higher-type functions, without their own argument positions being filled, provided the type-restrictions are obeyed. Therefore, he can use a formula of the form $\ulcorner \mathbf{\Delta f}_\alpha \mathbf{f}^*_{\alpha_{+1}} \urcorner$ to express that a certain sense-function presents a function as its denotation. If we suppose that "h" stands for the function denoted by the ordinary language predicate " ... is human" (type $\iota \mapsto o$),

is some chosen object of type ι. Some authors have advocated a simplifying assumption whereby every sense is taken to have some denotation, in which case "$^\vee$" could be used everywhere in place of "Δ", or even taken as primitive instead. However, this assumption is philosophically undesirable. After all, Frege's [20] suggestion that expressions such as "Romulus" and "the least rapidly converging series" can be meaningful parts of sentences because they express senses, despite lacking denotation, was part of the original motivation for the sense/denotation distinction.

and "h^*" stands for the sense of the same predicate, i.e., the sense-function discussed above (type $\iota_1 \mapsto o_1$), then "Δhh^*" stands for the True. However, this does not capture the relationship between the *values* of the two functions. That relationship seems to be this: for any pair of sense, and denotation presented by that sense, the value of the sense-function h^* for the sense as argument is a thought that presents the truth-value yielded by the function h for the denotation as argument. For example, when h^* takes the sense of "Socrates" as argument, it yields the thought that *Socrates is human*, and this thought presents the True, which is the value of h for Socrates as argument. When h^* takes the sense of "Boston" as argument, it yields the thought that *Boston is human*, which presents the False, i.e., the value of h for Boston as argument. To put it formally, we have:

$$(1) \qquad (\forall x)\,(\forall x^*)[\Delta x x^* \supset \Delta(hx)(h^* x^*)].$$

Whenever it holds that $(\forall x)\,(\forall x^*)[\Delta x x^* \supset \Delta(fx)(f^* x^*)]$ for a given f and f^*, the sense-function f^* is said to *characterize* the function f.

To capture the relation between characterization and presentation, Church adds certain additional axioms to the Logic of Sense and Denotation. First, he adds the following, relatively uncontroversial, axiom schema:

(Axiom $15^{\alpha\beta}$)
$$(\forall f_{\alpha\mapsto\beta})\,(\forall f^*_{\alpha_{+1}\mapsto\beta_{+1}})\,(\forall x_\alpha)\,(\forall x^*_{\alpha_{+1}})\{\Delta f f^* \supset [\Delta x x^* \supset \Delta(fx)(f^* x^*)]\}.$$

In other words, if f^* presents f, then f^* also characterizes f. More controversially, Church also adds the converse of axioms 15:

(Axiom $16^{\alpha\beta}$)
$$(\forall f_{\alpha\mapsto\beta})\,(\forall f^*_{\alpha_{+1}\mapsto\beta_{+1}})\{(\forall x_\alpha)\,(\forall x^*_{\alpha_{+1}})[\Delta x x^* \supset \Delta(fx)(f^* x^*)] \supset \Delta f f^*\}.$$

If f^* characterizes f, then f^* presents f. These axioms have some initial plausibility, but not as much as axioms 15, and they are not clearly an essential assumption of the system. Their main use for Church seems to be in the proof of what he calls "The Sense Relationship Theorem." This theorem involves the relationship between a given closed expression $\mathbf{M}_\alpha$ and what Church calls its "first ascendant," $\mathbf{M}_\alpha$, which is another expression standing for the sense expressed by the original expression. Every primitive constant in the Logic of Sense and Denotation is introduced along with a hierarchy of constants for senses. Thus, the sign for the conditional function, "$C_{o\mapsto o\mapsto o}$", expresses a sense $C_{o_1\mapsto o_1\mapsto o_1}$, and the sign "$C_{o_1\mapsto o_1\mapsto o_1}$" itself expresses a sense, $C_{o_2\mapsto o_2\mapsto o_2}$, and so on. Similar hierarchies of constants are introduced for the other constants, including $\Delta_{\alpha\mapsto\alpha_{+1}\mapsto o}$. Axiom schemata governing these hierarchies of constants are also introduced:

(Axiom 11^n) $\Delta C_{o_n\mapsto o_n\mapsto o_n} C_{o_{n+1}\mapsto o_{n+1}\mapsto o_{n+1}}$,
(Axiom $12^{n\alpha}$) $\Delta\Pi_{(\alpha\mapsto o_n)\mapsto o_n}\Pi_{(\alpha_{+1}\mapsto o_{n+1})\mapsto o_{n+1}}$,

(Axiom $13^{n\alpha}$) $\Delta\imath_{(\alpha\mapsto o_n)\mapsto\alpha}\imath_{(\alpha_{+1}\mapsto o_{n+1})\mapsto\alpha_{+1}}$,
(Axiom $14^{n\alpha}$) $\Delta\Delta_{\alpha\mapsto\alpha_{+1}\mapsto o_n}\Delta_{\alpha_{+1}\mapsto\alpha_{+2}\mapsto o_{n+1}}$,
(Axiom 18^{n}) $\Delta\sim_{o_n\mapsto o_n}\sim_{o_{n+1}\mapsto o_{n+1}}$.

For any closed expression $\mathbf{M}_\alpha$, its first ascendant, $\mathbf{M}_{\alpha_{+1}}$, can be obtained from it simply by raising the subscripts on all primitive type symbols by one.[3] The first ascendant of the expression

$$[\lambda y_o\,\Pi_{(o\mapsto o)\mapsto o}(C_{o\mapsto(o\mapsto o)}y_o)]$$

is the expression

$$[\lambda y_{o_1}\,\Pi_{(o_1\mapsto o_1)\mapsto o_1}(C_{o_1\mapsto(o_1\mapsto o_1)}y_{o_1})].$$

While the former expression stands for a function from truth-values to truth-values, the latter stands for a sense-function from senses of truth-values to senses of truth-values.

The "Sense Relationship Theorem" is the following desired result:

(SRT) If $\mathbf{M}_{\alpha_{+1}}$ is the first ascendant of $\mathbf{M}_\alpha$ then $\vdash \Delta\mathbf{M}_\alpha\mathbf{M}_{\alpha_{+1}}$.

For our example above, we should have:

(2) $\vdash \Delta[\lambda y_o\,\Pi_{(o\mapsto o)\mapsto o}(C_{o\mapsto(o\mapsto o)}y_o)][\lambda y_{o_1}\,\Pi_{(o_1\mapsto o_1)\mapsto o_1}(C_{o_1\mapsto(o_1\mapsto o_1)}y_{o_1})]$.

The proof of this, however, makes use of axiom 16^{oo}. Given axioms 11 and 12, it easily follows from axioms 15 and λ-conversion that:

(3) $\vdash (\forall x_o)\,(\forall x^*_{o_1})\,\big(\Delta xx^* \supset \Delta\{[\lambda y_o\,\Pi_{(o\mapsto o)\mapsto o}(C_{o\mapsto(o\mapsto o)}y_o)]x\}$
$\{[\lambda y_{o_1}\,\Pi_{(o_1\mapsto o_1)\mapsto o_1}(C_{o_1\mapsto(o_1\mapsto o_1)}y_{o_1})]x^*\}\big)$.

Axiom 16^{oo} is then used to derive (2) from (3). Indeed, with axioms 11–16, we can easily prove (SRT) in its general form for all closed expressions by induction on the length of $\mathbf{M}_\alpha$.

Nevertheless, there are at least two possible broad bases for doubt regarding Church's approach to sense-functions in general and axioms 16 in particular. One set of difficulties involves the Russell–Myhill antinomy and worries regarding the cardinalities of the domains of certain types of senses. These will be considered in section 4. My main interest in this paper is another set of difficulties stemming from the core understanding of the senses of functions as themselves being functions, and the problems that then arise in the wake of axioms 16. To these I now turn.

[3] I am overlooking temporarily the complication in Alternative (0) that necessitates a hierarchy of λ-operators, and would require also that the subscripts on the operators also be raised. This is discussed later.

§3. **Problems regarding deviant sense-functions.** The notion of a function in Fregean logic, picked up by Church, is one essentially borrowed from mathematics, but expanded to include entities other than numbers as arguments and values (see, e.g., [19]). According to this approach, a function exists for every determinate mapping of entities in the argument-type to entities of the value-type. This understanding of functions, when carried over to sense-functions, can create difficulties in the Logic of Sense and Denotation given axioms 16. Earlier we considered a sense-function, h^*, which maps senses presenting individuals to thoughts in a regular fashion. For the sense of the name "Socrates" it yields the thought expressed by "Socrates is human," for the sense of the name "Aristotle" it yields the thought expressed by "Aristotle is human," and so on. In this case it seems harmless to regard this sense-function as the sense of the predicate " ... is human," a sense that presents the function h mapping all humans to the True and non-humans to the False. Here, there is no problem with axiom $16^{\iota o}$.

Axiom schema 16 makes f^* characterizing f a sufficient condition for f^* presenting f. The problem is that this is suspect in the case of more unusual or what I call "deviant" sense-functions, i.e., those mapping argument senses to values in a less than regular fashion. One case has been discussed by Terence Parsons [37].[4] There are, so it would seem, senses without denotation, such as those expressed by such names as "Odysseus," "Superman," and "Excalibur." However, whether or not a given sense-function *characterizes* something depends entirely on its behavior with regard to senses that do have denotation. Consider the sense-function $h^{*\prime}$ that has the same value as h^* for all arguments, with one exception. Instead of yielding the thought that *Excalibur is human* for the sense of "Excalibur" as argument, it yields as value the thought that *snow is white*. Using "e^*" as a constant (type ι_1) for the sense of the name "Excalibur," and "q^*" as a constant (type o_1) for the thought that *snow is white*, $h^{*\prime}$ can be defined in terms of h^* thusly:

$$h^{*\prime} =_{\mathrm{df}} \big(\lambda x^*\, \imath\{\lambda p^*[(x^* \neq e^*) \,\&\, (p^* = h^*x^*)] \vee [(x^* = e^*) \,\&\, (p^* = q^*)]\}\big).$$

Notice that $h^{*\prime}$ also characterizes h, i.e., the following formula holds:

$$(1') \qquad (\forall x)\,(\forall x^*)[\Delta x x^* \supset \Delta(hx)(h^{*\prime}x^*)].$$

The deviant sense-function $h^{*\prime}$ differs from h^* only with regard to one value of x^*, and for this value, the antecedent of the conditional is false for all values of x. It then follows by axiom $16^{\iota o}$ that $\Delta hh^{*\prime}$, i.e., that $h^{*\prime}$ is also a sense presenting h. However, $h^{*\prime}$ seems to be a very strange sense-function to regard as a sense. Indeed, as Parsons has explained, if we were to imagine there to exist a predicate expressing $h^{*\prime}$, we would arrive at an

[4]My example is slightly different from Parsons's, but the main line of his argument is preserved.

absurdity. Suppose that $h^{*\prime}$ were expressed by some predicate, " ... is such-and-such." Consider the sentence, "Excalibur is such-and-such." The thought expressed by this sentence is the value of the sense of " ... is such-and-such," i.e., $h^{*\prime}$, for the sense of "Excalibur," e^*, as argument, i.e., the thought that *snow is white*. Since this thought is true, the sentence "Excalibur is such-and-such" is a true sentence. However notice that the predicate " ... is such-and-such" expresses a sense that presents the same denotation as the sense expressed by the predicate " ... is human." By the basic Fregean principle that two expressions with the same denotation can be substituted for each other *salva veritate*, we get as a result that the sentence "Excalibur is human" is also true, which is absurd. (There may be disagreement about *why* it is absurd. Frege's own attitude towards sentences containing empty names was that they were neither true nor false, i.e., they express thoughts that do not present any truth-value. Others may want to hold this sentence to be false. The result that it is *true* is absurd either way.)

Parsons's diagnosis of this problem is that there is a problem with axioms 16 because they do not adequately consider what sense-functions do with senses that lack denotation. He suggests a patch in which we first revise our understanding of the sense/denotation relation so that all senses are to be understood technically as presenting a denotation, but in the case of so-called empty senses, they are all to be understood as, technically speaking, presenting some chosen object of the appropriate type, called a *zip*. Hence, the sense of "Excalibur" is to be regarded as presenting the zip of type ι. The zip of type o would be some third truth-value, and the function h must be regarded as having this zip of type o as value for the zip of type ι as argument. On this new understanding, while the argument that h^* characterizes h goes through, the argument that $h^{*\prime}$ characterizes h does not, since the value of $h^{*\prime}$ for the sense of "Excalibur" as argument would have to present the zip of type o, which it does not.[5]

I think Parsons has misdiagnosed the problem. Deviant sense-functions cause trouble for axioms 16 due to aberrant behavior with regard to senses that do present a denotation as well. Consider another deviant sense-function, $h^{*\prime\prime}$, having the same value as h^* for every argument, except that instead of yielding the thought that *Russell is human* for the sense of the name "Russell" as argument, it yields the thought that *snow is white*. Again using "a^*" as a constant for the sense of the name "Russell," this function could be defined as follows:

$$h^{*\prime\prime} =_{\mathrm{df}} \big(\lambda x^*\ \imath\{\lambda p^*[(x^* \neq a^*) \,\&\, (p^* = h^* x^*)] \vee [(x^* = a^*) \,\&\, (p^* = q^*)]\}\big).$$

[5]Parsons says quite a bit more on this topic than I can discuss here. It should be noted that in an "afterwords" to [37, p. 537], he claims that after composing it that he became aware of problems with his position that need patching, and it may very well be that he no longer holds the views outlined here.

Again we get the result that $h^{*\prime\prime}$ characterizes h:

$$(1'') \qquad (\forall x)\,(\forall x^*)[\Delta xx^* \supset \Delta(hx)(h^{*\prime\prime}x^*)].$$

In the problematic case, that in which x is Russell and x^* is the sense of the name "Russell," the consequent of $(1'')$, $\Delta(hx)(h^{*\prime\prime}x^*)$, still holds. Here, hx is the True (since Russell is human), and $h^{*\prime\prime}x'$ is the thought that *snow is white*. This thought *does* present the True, and so the appropriate relation holds. Again, by axiom $16^{\prime o}$, we have the unexpected result that $\Delta hh^{*\prime\prime}$, i.e., the deviant sense-function $h^{*\prime\prime}$, is still to be regarded as a sense presenting h. No finagling with zips would provide help here, since the sense of "Russell" presents Russell, not a zip.

However, it might be thought that, in this case, no absurdity results. Certainly we do not face precisely the same problem as we did with the Excalibur case. Suppose that the $h^{*\prime\prime}$ is expressed by some predicate " ... is thus-and-so." By an argument parallel to the previous example, we again get that " ... is thus-and-so" is codenotative with " ... is human." Hence, "Russell is thus-and-so" must denote the same truth-value as "Russell is human." The former sentence expresses the thought that *snow is white*, and the latter the thought that *Russell is human*; these have the same truth-value, so there is no violation here of the Fregean substitutivity principles for codenotative expressions.

This response, however, misses the heart of the matter. It simply *accepts* that the sentence "Russell is thus-and-so," where the name "Russell" expresses the same sense it normally does, could express the thought that *snow is white*, a thought that has nothing to do with Russell. This is intuitively suspect. It does not seem that there could be any such predicate as " ... is thus-and-so," which, when predicated of *Russell*, expresses the thought that *snow is white*. One must take care not to confuse the strange predicate " ... is thus-and-so" with the unobjectionable predicate " ... is an entity such that, there is some sense presenting it, which, when taken as argument to the sense-function $h^{*\prime\prime}$, the result is a thought presenting the True." The latter predicate does not express $h^{*\prime\prime}$, but instead expresses a much more complex sense involving some sense even higher in the sense-hierarchy that has $h^{*\prime\prime}$ as *denotation*. If we were to attribute this latter predicate to Russell, the result would not be the thought that *snow is white*, but another more complex thought that is true *because* the thought that *snow is white* is true. The predicate " ... is thus-and-so" is much more strange. To claim that "Russell is thus-and-so" somehow *is* to claim that snow is white (and only that).

Consider an even more deviant sense-function g^* whose values are different from the values of h^* for *all* arguments, and have no common form or similarities except that each has the same truth-value as the corresponding value of h^*. The value of g^* might be the thought that *Barack Obama is the*

44*th President of the United States* for the sense of "Russell," the thought that *Stockholm is in Sweden* for the sense of "Socrates," and the thought that $2 + 2 = 5$ for the sense of "Boston," etc. It would still hold that g^* characterizes h, and hence, by axiom 16^{to} that g^* is a sense presenting h. Suppose there were some predicate "… is thingamajiggy" expressing g^*. This predicate would be codenotative with the predicate "… is human," and yet, to predicate it of Socrates would be to claim that Stockholm is in Sweden, to predicate it of Boston would be to claim that $2 + 2 = 5$, and so on. Again, this seems bizarre. The connection between g^* and h seems much more remote than the relation between sense and denotation, and g^* simply doesn't seem like it could be a sense at all.

This issue cannot be fully addressed without delving further into the nature of senses generally, their relationship with language, and their identity conditions. However, if two sentences express the same thought if and only if they are synonymously isomorphic (as explained earlier), obviously "Russell is thus-and-so" and "snow is white" cannot plausibly be taken to express the same thought, for that would suggest that "Russell" is synonymous with "snow" and "… is thus-and-so" is synonymous with "… is white." Hence, it might be argued that the result that $h^{*\prime\prime}$ and g^* are senses is inconsistent with the guiding principles of Alternatives (0) and (1).[6] One might respond to these concerns by noting that just because $h^{*\prime\prime}$ and g^* are senses does not necessarily mean that there could be predicates expressing them. However, this would leave us in the dark about exactly what senses are, or at least, about what sort of "senses" our deviant sense-functions $h^{*\prime\prime}$ and g^* are supposed to be.

Worse, for present purposes, the results of axioms 16 regarding deviant sense-functions threaten to make it impossible to add principles in line with certain otherwise plausible conditions regarding the identity conditions of senses to the Logic of Sense and Denotation without disaster.

To see this, it is worth noting that on any fine-grained understanding of the identity conditions of senses, such as those involved in Alternatives (0) and (1), those sense-functions which are most plausibly regarded as the senses of primitive function signs would be one-one functions. Two sentences

[6]Axioms 16 in their stronger form are inconsistent with the basic principles of Church's other alternative understanding of the identity conditions of senses, Alternative (2), as well. On this alternative, senses are identified when they are necessarily codenotative. However, David Kaplan and A. F. Bausch, apparently independently, discovered that axioms 16 lead to a proof in Alternative (2) that there are is only one true thought/proposition, and one false thought/proposition (see [37, p. 515f]). Church attempts to rectify this in [10] by modifying Alternative (2) using a device meant for Alternative (0), where it is clearly out of place and somewhat *ad hoc*. I do not discuss Alternative (2) at length in this paper. However, although I have not explored this option myself, I believe it possible that my alternative approach, outlined in sec. 6, could also be made to provide a different solution to the Kaplan/Bausch problem.

of the form ⌜**n** is human⌝ and ⌜**m** is human⌝ express different thoughts if the names **n** and **m** have different senses. Consequently, the sense function h^*, considered earlier, would appear to yield distinct values for distinct senses as argument. Indeed, in Alternative (0), it is plausible to suppose that all sense-functions that are themselves senses having functions as denotation would be one-one. In Alternative (1), for reasons explained later, this cannot be held for all types, but it is at least plausible for the simplest types. Consider then the following axiom for given types α and β:

$$\text{(Axiom } 64^{\alpha\beta}) \quad (\forall f_{\alpha\mapsto\beta})\,(\forall f^*_{\alpha_{+1}\mapsto\beta_{+1}})\,(\forall x_\alpha)\,(\forall x^*_{\alpha_{+1}})\,(\forall y_\alpha)\,(\forall y^*_{\alpha_{+1}})\,\big[\Delta f f^* \supset \big(\Delta x x^* \supset \{\Delta y y^* \supset [(f^* x^* = f^* y^*) \supset (x^* = y^*)]\}\big)\big].$$

In other words, those sense-functions that are senses yield different senses as value for different senses as argument. Church [11, pp. 151–52] adds this principle and other axioms capturing the identity conditions of senses to his formulation of Alternative (0).[7] While he did not add the principle in a general form in his first formulation of Alternative (1) [13, pp. 23–24], he included a similar principle for each primitive sense-function constant.

Something like axiom $64^{\iota o}$ is arguably the lesson to be learned from the Fregean solution to the belief puzzles. In "oblique" contexts, words shift from having their usual sense and denotation to having their customary sense as denotation. Hence in the sentences "Elizabeth believes that Hesperus is a planet" and "Elizabeth believes that Phosphorus is a planet," the embedded clauses "Hesperus is a planet" and "Phosphorus is a planet" denote thoughts rather than truth-values. For this to solve the belief puzzle, they must be *different* thoughts. The *reason* they are different thoughts is that the customary sense of " … is a planet"—the denotation of " … is a planet" in these contexts—yields different thoughts for different argument senses. Indeed, Frege [21, pp. 255–56] explicitly makes such claims as that when one name in a sentence is replaced with another having a different sense, the sentence as a whole changes sense as well, and this provides rationale for axioms 64.

However, such principles cannot be added to a system containing axioms 16 in unmodified form without catastrophe. To see this, consider now our previous example of h and h^*, and a deviant sense-function $h^{*\prime\prime\prime}$, similar to h^* and the others except that its value for the sense of "Russell" as argument is the thought expressed by "Frege is human." If we use "b^*" to stand for the sense of the name "Frege," we can define $h^{*\prime\prime\prime}$ as follows:

$$h^{*\prime\prime\prime} =_{\text{df}} \big(\lambda x^*\, \imath\{\lambda p^*[(x^* \neq a^*)\ \&\ (p^* = h^* x^*)] \vee [(x^* = a^*)\ \&\ (p^* = h^* b^*)]\}\big).$$

[7] I ignore for present purposes the superscripts on the Δ-relation Church placed on this axiom schema in line with his 1974 tack for solving the semantical paradoxes.

For reasons that should already be clear from the above discussion, we still have:

$$(1''') \qquad (\forall x)\,(\forall x^*)[\Delta xx^* \supset \Delta(hx)(h^{*'''}x^*)]$$

and hence, by axiom $16^{\iota o}$:

$$(4) \qquad \Delta hh^{*'''}.$$

In conjunction with axiom $64^{\iota o}$, this leads to:

$$(5) \quad (\forall x_\alpha)\,(\forall x^*_{\alpha_{+1}})\,(\forall y_\alpha)\,(\forall y^*_{\alpha_{+1}})\,\big(\Delta xx^* \supset \{\Delta yy^* \supset [(h^{*'''}x^* = h^{*'''}y^*) \supset (x^* = y^*)]\}\big).$$

According to our definition, the value of $h^{*'''}$ for the sense of "Russell" as argument (a^*) is the thought that "Frege is human" (h^*b^*); the same thought results for the sense of "Frege" as argument (b^*). If now, in the above, we take x as a (Russell), x^* as a^* (the sense of "Russell"), y as b (Frege), and y^* as b^* (the sense of "Frege"), we arrive at the absurd result that the sense of "Russell" is the same as the sense of "Frege" $(a^* = b^*)$, because these yield the same thought as value when taken as arguments to the same deviant sense-function. Innumerable other difficulties follow, such as that the thought expressed by "Frege is German" is the same as the thought expressed by "Russell is German," and hence, that these have the same truth-value, etc. Worse, given the determinacy of senses, captured in Church's system by axioms 17 listed earlier, the above results entail that Russell *is* Frege. The problem, in sum, is that axioms 64 entail that all those sense-functions that are senses must be one-one, but axioms 16 entail otherwise.

The point can be stated in general terms as follows. Suppose $a^*_{\alpha_{+1}}$ and $b^*_{\alpha_{+1}}$ are different senses of the same type presenting different denotations, a_α and b_α, respectively. Further suppose that $f_{\alpha\mapsto o}$ is a function such that $f_{\alpha\mapsto o}a_\alpha$ and $f_{\alpha\mapsto o}b_\alpha$ are the same truth-value. Let $f^*_{\alpha_{+1}\mapsto o_1}$ be a non-deviant sense-function that is a sense presenting $f_{\alpha\mapsto o}$. Then, $f^*_{\alpha_{+1}\mapsto o_1}a^*_{\alpha_{+1}}$ and $f^*_{\alpha_{+1}\mapsto o_1}b^*_{\alpha_{+1}}$ are senses presenting the same truth value, $f_{\alpha\mapsto o}a_\alpha$. However, one can define a sense-function $f^{*\prime}_{\alpha_{+1}\mapsto o_1}$, whose value is the same as $f^*_{\alpha_{+1}\mapsto o_1}$ for all arguments except $a^*_{\alpha_{+1}}$, for which its value is $f^*_{\alpha_{+1}\mapsto o_1}b^*_{\alpha_{+1}}$. Axiom $16^{\alpha o}$ entails that $f^{*\prime}_{\alpha_{+1}\mapsto o_1}$ is also a sense of $f_{\alpha\mapsto o}$, despite that $f^{*\prime}_{\alpha_{+1}\mapsto o_1}a^*_{\alpha_{+1}}$ is a thought that, intuitively, has nothing to do with the value of $f_{\alpha\mapsto o}$ for a as argument. Outright absurdities follow when certain principles governing the identity conditions of thoughts are adopted. Note that $f^{*\prime}_{\alpha_{+1}\mapsto o_1}b^*_{\alpha_{+1}}$ is the same thought as $f^{*\prime}_{\alpha_{+1}\mapsto o_1}a^*_{\alpha_{+1}}$, because both are defined as $f^*_{\alpha_{+1}\mapsto o_1}b^*_{\alpha_{+1}}$. If we accept axiom $64^{\alpha o}$, we get that $a^*_{\alpha_{+1}}$ and $b^*_{\alpha_{+1}}$ are the same sense, and that a_α and b_α are the same entity, contrary to what was assumed. We have just given an argument for equating any two senses of the type α_{+1}, *regardless of their denotations*, and with axiom 17^α, this leads to an argument equating any two entities of the type α.

The above general proof scheme works even for type o_1, senses presenting truth-values, which threatens the consistency of the system, because one could thereby demonstrate that the True is the False. For a simple formulation of this last problem, let $j^*_{o_1 \mapsto o_1}$ be a constant sense-function that maps every thought to the same tautological thought, $t^*_{o_1}$. By axiom 16^{oo}, this sense-function presents the truth-function $(\lambda p\, p \supset p)$ that maps both the True and the False to the True. Now let s^* be some self-contradictory thought. By axiom 64^{oo}, since $j^*t^* = j^*s^*$, we get $t^* = s^*$, and thus that their denotations, the True and the False, respectively, are identical. This demonstrates that if care is not taken with the set-up of the system, the result would be outright inconsistency. It also shows us that the problems arise not only for what I have called "deviant" sense-functions, but also for *constant* sense-functions.

No easy fix can be found for the problems by a modal restriction on axioms 16. If we focus our attention too much on examples similar to the one involving h and $h^{*\prime\prime\prime}$, it might seem that the difficulties could be blocked by weakening axioms 16 to claim that a sense-function f^* is a sense of a function f whenever f^* characterizes f *necessarily* (rather than whenever f^* characterizes f simply contingently). The example of $h^{*\prime\prime\prime}$ considered earlier only leads to the conclusion that the sense of "Frege" is the same as the sense of "Russell," because the thoughts that "Russell is human" and "Frege is human" happen to have the same truth-value. It is not clear that these thoughts *necessarily* have the same truth-value. However, this suggestion is insufficient to block the problems. It is not difficult to give a different example in which the characterization is necessary. For example, let a^* be the sense of "2", b^* the sense of "3", f the function denoted by " ... is prime," and f^* the sense of " ... is prime." Define $f^{*\prime}$ so that its value is the same as f^* for all arguments save a^*, for which its value is f^*b^*. It then holds that $f^{*\prime}$ characterizes f necessarily. Similar comments apply to the problem just sketched for type o_1, which was given entirely in terms of tautological and self-contradictory thoughts.[8]

[8]Moreover, a modal restriction on axiom 16 would be difficult to formulate for other reasons. Modal operators, as intensional operators, in a direct discourse system such as the Logic of Sense and Denotation, would be understood as function signs applying to expressions of type o_1 rather than of type o_0. So rather than applying such an operator directly to (1), one would make use of different expressions, and so, one would need otherwise to capture the relation between those new expressions and those appearing in the consequent, which might not be altogether easy without presupposing something like axioms 16 in the original form.

Additionally, the modal restriction would likely not be satisfied in the *desired* cases for the application of axioms 16. After all, most senses present different individuals as denotation in different possible worlds, and so, the same sense-function presents different functions in different possible worlds.

§4. **The Russell–Myhill antinomy and related problems.** The above problems are not the only difficulties plaguing axioms 16, especially when conjoined with something like axioms 64. Another set of potential problems involve difficulties regarding the cardinalities of certain sense types. In [35], John Myhill showed that the original formulation of one of Church's alternative systems for the Logic of Sense and Denotation was inconsistent, due to a violation of Cantor's theorem and a resulting antinomy similar to the paradox discussed by Russell in §500 of [41]. Consider functions of type $o_1 \mapsto o$; crudely these can be thought of as properties of thoughts or propositions. By Cantorian reasoning, there must be more such functions than there are thoughts. However, assuming that every function of this type is presented by a sense (of type $o_2 \mapsto o_1$), it would seem possible to generate a different thought for each such function, e.g., by making use of the sense-function $\Pi_{(o_2 \mapsto o_1) \mapsto o_1}$, which has as values universally quantified thoughts. The constant "$\Pi_{(o_2 \mapsto o_1) \mapsto o_1}$" represents the sense of the quantifier "$\Pi_{(o_1 \mapsto o) \mapsto o}$", and for reasons already considered, it is plausible to regard this sense-function as yielding distinct thoughts for distinct senses as argument. This corresponds to the natural language fact that sentences of the form ⌜every thought is $\mathbf{f}$⌝ and ⌜every thought is $\mathbf{g}$⌝ are synonymously isomorphic only if $\mathbf{f}$ and $\mathbf{g}$ are synonymously isomorphic. This supports the following principle:

$$(6) \quad (\forall k^*_{o_2 \mapsto o_1})\,(\forall l^*_{o_2 \mapsto o_1})[(\Pi_{(o_2 \mapsto o_1) \mapsto o_1} k^* = \Pi_{(o_2 \mapsto o_1) \mapsto o_1} l^*) \supset (k^* = l^*)].$$

Church included this as an axiom in his first formulation of Alternative (1), and something very similar results from axiom $64^{(o_1 \mapsto o)o}$ in Alternative (0). However, this leads to positing as many thoughts as senses of type $o_2 \mapsto o_1$, and, therefore, if such a sense is posited for each function of type $o_1 \mapsto o$, one gets a violation of Cantor's theorem.

This leads to outright inconsistency due to the following paradox. Some universal thoughts about thoughts fall under the property they generalize upon, others do not. The thought that *all thoughts are true* is not itself true, whereas the thought that *all thoughts are self-identical* is itself self-identical. Consider the property a thought has iff it is a universal thought that does not fall under the property it generalizes upon, e.g., consider the function $w_{o_1 \mapsto o}$:

$$\begin{aligned} w_{o_1 \mapsto o} =_{\mathrm{df}} \big(\lambda p^*\,(\exists k_{o_1 \mapsto o})\,(\exists k^*_{o_2 \mapsto o_1})\{[\Delta k k^* \\ \& \,(p^* = \Pi_{(o_2 \mapsto o_1) \mapsto o_1} k^*)] \,\&\, {\sim} k p^*\}\big). \end{aligned}$$

Abbreviating the ascendant of the above expression as $w^*_{o_2 \mapsto o_1}$, by (SRT) we have the result that $\vdash\ \Delta w w^*$. We can then consider the thought $\Pi_{(o_2 \mapsto o_1) \mapsto o_1} w^*$ (i.e., the thought that *all thoughts are w*), which we can abbreviate as r^*. We then have the impossible result, however, that $wr^* \equiv {\sim} wr^*$. This contradiction (and its ilk) has come to be known as the Russell–Myhill antinomy.

In response to this problem and other semantic antinomies, in 1974, Church suggested that the Δ-constants be affixed with superscripts to form the hierarchy $\Delta^{l}, \Delta^{l+1}, \Delta^{l+2}, \ldots$, with $l, l+1, l+2$, etc., indicating a Tarskian hierarchy of languages. The system was modified in such a way that if $\mathbf{M}_{\alpha}$ is an expression containing a Δ-constant with superscript m, and $\mathbf{M}_{\alpha_{+1}}$ is its ascendant, it would hold only that $\vdash \Delta^{m+1}\mathbf{M}_{\alpha}\mathbf{M}_{\alpha_{+1}}$ and not that $\vdash \Delta^{m}\mathbf{M}_{\alpha}\mathbf{M}_{\alpha_{+1}}$. Applying this to the above case, if m is the superscript that appears on the "Δ" in the definition of w, we get only that $\vdash \Delta^{m+1}ww^{*}$ and not $\vdash \Delta^{m}ww^{*}$, and from this no contradiction is forthcoming when we consider wr^{*}. However, Anderson [2] soon discovered that the antinomy remained unsolved, due to the consequence of axioms 16 that every object and every function has at least one sense presenting it, a result that holds even within every language when a language-hierarchy is imposed. Hence, while we may not have $\vdash \Delta^{m}ww^{*}$, we do have $\vdash (\exists k^{*})\,\Delta^{m}wk^{*}$. The antinomy can be formulated using the sense-function posited by this theorem without making use of w^{*}.

Indeed, Anderson has generalized upon the lesson of the Russell–Myhill. By Cantor's theorem, if the domain of type β has at least two members and the domain of type α_{+1} is non-empty, the cardinality of the domain $(\alpha_{+1} \mapsto \beta)$ must be greater than that of type α_{+1}. However, given axioms 16, we get the result that every function in the domain of $(\alpha_{+1} \mapsto \beta)$ is presented by a sense of type $(\alpha_{+2} \mapsto \beta_{+1})$. Because senses have unique denotations, this means that the cardinality of type $(\alpha_{+2} \mapsto \beta_{+1})$ is at least as great as that of type $(\alpha_{+1} \mapsto \beta)$. The presence, then, of any one-one sense-function of type $(\alpha_{+2} \mapsto \beta_{+1}) \mapsto \alpha_{+1}$ violates Cantor's theorem. This poses a definite problem, since the most plausible candidates for the senses of primitive function signs, when construed as sense-functions, would appear to be one-one in Alternatives (0) and (1), and some would fall into a type of the form $(\alpha_{+2} \mapsto \beta_{+1}) \mapsto \alpha_{+1}$. In his own work, Anderson has advocated dropping axioms 16, thereby blocking the result that every individual and every function is presented by at least one sense.

However, it does not seem to me that the source of the difficulties regarding the cardinalities of certain types and the failure of Church's initial proposed solution to the Russell–Myhill antinomy needs to be located with axioms 16. The result that every object and every function is presented by at least one sense is not in and of itself absurd, and indeed, has some intuitive plausibility of its own, as I have argued elsewhere [31, pp. 309–312]. Indeed, Church himself was at times attracted to taking that result as axiomatic (see, e.g., [26, 13]). If making good on Church's quasi-Tarskian solution to the Russell–Myhill antinomy necessitates blocking this result, this seems if anything to show that this is not the best way to go. Indeed, eventually, this seems to have been Church's own conclusion, and in [13] we find him advocating a richer form of ramification as a solution to semantic paradoxes instead of his earlier quasi-Tarskian approach. I agree that a richer form of ramification is the most plausible route for solving the Russell–Myhill antinomy and related cardinality problems, and hence I do not blame such difficulties on axioms 16. More is said about ramification in sec. 8, but for the most part my focus is on the other set of problems.

§5. Dropping/modifiying axiom schema 16. Even if full fledged ramification is invoked to solve the Russell–Myhill antinomy and related problems, it provides no help in solving the problems regarding many-one sense-functions turning out to be senses according to axioms 16. However we solve the cardinality problems, denying or somehow weakening these axioms is still the most *initially* tempting route for escaping these other difficulties. What makes this route tempting is that it would allow us to admit that the sense-functions $h^{*\prime}$, $h^{*\prime\prime}$, and $h^{*\prime\prime\prime}$, etc., discussed in sec. 3 characterize the function h while denying that they are therefore to be regarded as senses of h. Similarly, the sense-function $j^*_{o_1 \mapsto o_1}$ discussed at the end of sec. 3 can be admitted to characterize the function $(\lambda p\, p \supset p)$ without being taken as a sense of this function.

The danger is that if axiom schema 16 is dropped outright, we make it impossible to infer that a sense-function presents another function as denotation in virtue of some correlation of the values of the two functions. While we retain axioms 15, and preserve the relation in the other direction, we still seem to be losing certain very important results, such as (SRT). Hence, we must either replace axiom 16 with a weakened version that allows the proof of (SRT) to go through, or introduce some alternative method of getting this result. Anderson [2, 5], who, as we have seen, finds axioms 16 implausible for other reasons, has taken the latter approach in some of his work.

It is not clear to what extent Church himself appreciated the difficulties with deviant sense-functions and axioms 16 or took them sufficiently seriously. In his published works, they are scantly mentioned, and usually only in association with "empty" or "vacuous" senses (see [9, p. 4n] and [11, p. 152]), which perhaps explains in part Parsons's focus on such senses. Yet Church surely had some awareness of the more general problem, evidence of which can be found in Leon Henkin's unpublished notes on lectures Church gave on the Logic of Sense and Denotation as early as 1946. Moreover, in his 1974 formulation of Alternative (0), although Church does not discuss the relevant issues in any detail, he presents weakened versions of axioms 16 that provide some help.

For Alternative (0), Church introduces a hierarchy of operators, $\lambda_0, \lambda_1, \lambda_2$, ..., rather than a single lambda operator. The reason is that, in Alternative (0), λ-converts are not taken as synonymous. Hence, "$(\lambda x_\alpha\, f_{\alpha \mapsto \beta} x_\alpha) a_\alpha$" and "$f_{\alpha \mapsto \beta} a_\alpha$" would be thought to express different senses. The first ascendants of these expressions, therefore, must *denote* different senses. For this reason, the ascendant of "$(\lambda x_\alpha\, f_{\alpha\mapsto\beta} x_\alpha) a_\alpha$" cannot simply be "$(\lambda x_{\alpha_{+1}}\, f_{\alpha_{+1}\mapsto\beta_{+1}} x_{\alpha_{+1}}) a_{\alpha_{+1}}$" since this collapses to "$f_{\alpha_{+1}\mapsto\beta_{+1}} a_{\alpha_{+1}}$" by λ-conversion, and the latter is the ascendant of "$f_{\alpha\mapsto\beta} a_\alpha$". Hence, in Alternative (0), the first ascendant of an expression containing a given operator λ_n is formed using the operator λ_{n+1}, and λ-conversion is allowed only for the base lambda operator λ_0. In his formulation of Alternative (0), Church modified axiom schema 16 to the following:[9]

$$\text{(Axiom } 16'^{\alpha\beta}\text{)}\ \ (\forall f_{\alpha\mapsto\beta})\,(\forall f^*_{\alpha_{+1}\mapsto\beta_{+1}})\{(\forall x_\alpha)\,(\forall x^*_{\alpha_{+1}})\\ [\Delta x x^* \supset \Delta(f x)(f^* x^*)] \supset \Delta f(\lambda_1 y^*\, f^* y^*)\}.$$

Given the more particular understanding of the ascendant of an expression at work in Alternative (0), the above is all that is needed to obtain (SRT).

This weakening of axiom schema 16 does provide some limited help in blocking the outright absurdities involving deviant sense-functions and axioms 64. Let us return to the example functions h, h^* and $h^{*\prime\prime\prime}$, from sec. 3 used to generate the argument that the sense of the name "Russell," a^*, is identical to the sense of the name "Frege," b^*. With the unmodified version of axiom $16^{\iota o}$, we got the result that $\Delta h h^{*\prime\prime\prime}$, and since it follows from the definition of $h^{*\prime\prime\prime}$ that $h^{*\prime\prime\prime} a^* = h^{*\prime\prime\prime} b^*$, axiom $64^{\iota o}$ leads to the result that $a^* = b^*$. With the weaker axiom $16'^{\iota o}$, we do not get the result that $\Delta h h^{*\prime\prime\prime}$, only the result $\Delta h(\lambda_1 y^*\, h^{*\prime\prime\prime} y^*)$. This blocks applying axiom 64 to get the absurdity because while it is still provable that $h^{*\prime\prime\prime} a^* = h^{*\prime\prime\prime} b^*$, it is impossible to prove that $(\lambda_1 y^*\, h^{*\prime\prime\prime} y^*) a^* = (\lambda_1 y^*\, h^{*\prime\prime\prime} y^*) b^*$ without conversion on λ_1. For similar reasons, the argument to the effect that the True is the False sketched at the end of sec. 3, is blocked.

While it avoids outright disaster, on the whole, I find this "solution" to the difficulties unsatisfying. We still get the result that $(\lambda_1 y^*\, h^{*\prime\prime\prime} y^*)$ is a sense presenting h as denotation. Unpacking the definition of $h^{*\prime\prime\prime}$, "$(\lambda_1 y^*\, h^{*\prime\prime\prime} y^*)$" is an abbreviation of:

$$\big[\lambda_1 y^*\, \big(\lambda x^*\, \imath\{\lambda p^*[(x^* \neq a^*)\ \&\ (p^* = h^* x^*)] \vee \\ [(x^* = a^*)\ \&\ (p^* = h^* b^*)]\}\big) y^*\big]$$

which λ-converts to:

$$\big(\lambda_1 y^*\, \imath\{\lambda p^*[(y^* \neq a^*)\ \&\ (p^* = h^* y^*)] \vee [(y^* = a^*)\ \&\ (p^* = h^* b^*)]\}\big).$$

Given this, how unpalatable is the result that the function $(\lambda_1 y^*\, h^{*\prime\prime\prime} y^*)$ is a sense of h? This is a very difficult point to assess. The intended interpretation of function abstracts making use of the hierarchy of λ-operators is really only discussed in the context of forming ascendants. However, "$(\lambda_1 y^*\, h^{*\prime\prime\prime} y^*)$" is not the ascendant of any formula, involving, as it does, functions whose

[9]Here again I omit the superscripts on the Δ-relation Church used at this time.

arguments and values are not senses such as the description function, disjunction and identity (note that the "$\imath$", "$=$" and "$\vee$" in the above are the normal extensional ones; they do not stand for sense-functions). While "$(\lambda_1 y^* \, h^{*\prime\prime\prime} y^*)$" is still thought to stand for a function of type $\iota_1 \mapsto o_1$, it is almost impossible to understand what sort of function it is supposed to be, how it differs from $h^{*\prime\prime\prime}$, and how its values relate to its arguments. Nevertheless it follows in Church's system that it is a sense that presents h as denotation. Since "$(\lambda_1 y^* \, h^{*\prime\prime\prime} y^*)$" is not the ascendant of any formula, it is unclear what sort of predicate, if any, could express it. A more elegant solution is in order, even for Alternative (0).

In any case, no similar solution is possible for Alternative (1), as it has only one λ-operator, and if a hierarchy were imposed, conversion would have to be allowed with operators other than λ_0. Oddly, Church retains axiom schema 16 in its unweakened form in his 1993 reformulation of Alternative (1). Presumably, he took it to be too central to the system, especially with regard to obtaining (SRT), to abandon. The ramification adopted to avoid semantic paradoxes provides no help with the difficulties under discussion. This formulation of Alternative (1) seems to avoid the more serious problems sketched above simply by wholly neglecting to include axioms governing the identity conditions of senses, whereas such principles are clearly called for.[10]

§**6. A more radical approach.** According to the understanding of functions adopted by Frege and Church, expanded from the common understanding in mathematics, the value of a function for a certain argument is in no way constituted by that function and its argument. The value of the function *positive square root of* for the number four as argument is the number two, but the number two is not *composed* of the number four and the square root function. The same number is the result of a strictly infinite number of different functions for different arguments. The problems that arise for deviant sense-functions for the Logic of Sense and Denotation exist precisely because there need be no direct link between the argument and value of a sense-function. The value of a sense-function for the sense of the name "Russell" as argument may be a thought that has nothing to do with Russell or the sense of "Russell." The problems do not arise for those sense-functions in which there is always a direct link between the argument and value, i.e., those in which the argument-sense can be regarded in some sense as a *constituent* or *part* of the value-sense.

[10] For reasons discussed in sec. 8, axiom schema 64 would not hold for all types under the guiding principles of Alternative (1). However, to arrive at inconsistency, we would need only the instance of axiom 64 where α and β are both type o, which, is not only consistent with, but demanded by, these principles, for reasons discussed in the appendix. Similar considerations apply for the instance of axiom 64 where α and β are types ι and o respectively, which is the instance giving rise to the more serious difficulties regarding function $h^{*\prime\prime\prime}$.

It is not surprising then that certain Frege scholars, principally Dummett ([16, pp. 291–94]; [17, pp. 249–53, 265–70]; see also [30, pp. 65–76]), have resisted attributing to Frege himself the view that the sense of a function expression is a sense-function. Frege claims repeatedly that the thought expressed by a complete sentence is a whole consisting of the senses of the parts, which is difficult to reconcile with the sense-function view of the senses of predicate expressions, especially as Frege [21, p. 255] denies that in the realm of denotation the value of a function is composed of function and argument. However, putting aside the question of what Frege's own view was, these considerations do seem to recommend an alternative understanding of the nature of the senses of function expressions. Rather than understanding them as functions in the strict sense, one could instead understand them as more literally "incomplete" or "unsaturated" entities (to borrow some Fregean phrases). These incomplete senses would, like functions, come together with an "argument," and would, like functions, thereby yield a "value." However, unlike functions, they would be understood as binding together with their argument-senses to form wholes, and these wholes would contain the arguments as constituents. Because this is somewhat different from the way that functions proper operate, I prefer to speak of *s*-arguments and *s*-values rather than arguments and values in the traditional (functional) sense.[11]

In what follows, I suggest that we modify Church's approach by introducing constants and variables for such incomplete senses, and regard these as falling into distinct types from any function types proper. In [29], David Kaplan too resists equating the senses of functions of type $\alpha \mapsto \beta$ with functions of type $\alpha_1 \mapsto \beta_1$ and instead posits a distinct type $(\alpha \mapsto \beta)_1$ for this purpose. Kaplan, however, does not develop this suggestion in detail. In what follows, I make use of a revised type system in line with Kaplan's remarks. Type symbols are defined recursively as follows: (i) ι_0 and o_0 are type symbols, (ii) if α and β are type symbols, then $(\alpha \mapsto \beta)_0$ is a type symbol, (iii) if α_n is a type symbol with n as its outermost subscript, then α_{n+1} is a type symbol. As before, type ι_0 is the type of individuals, o_0 is type of truth-values, and generally, α_{n+1} is the type of senses with denotations of type α_n. We abandon Church's assumption that the type symbol appropriate to senses of type α is to be obtained from α by raising all primitive type symbols in α by one, and, instead, simply raise the outermost subscript by one. Those type symbols with outermost subscripts greater than zero are the types for senses, and generally, we shall call such types *sense types*. Types

[11]The difference between my incomplete senses and sense-functions in Church's sense is somewhat analogous to the distinction between traditional Russellian propositional functions and Ramsey's "propositional functions in extension" in [38]. Incomplete senses are more like the former, though there are of course differences due to the differing natures of Fregean thoughts and Russellian propositions.

of the form $(\alpha \mapsto \beta)_n$ are functions in the usual mathematical sense only when $n = 0$ and are understood as types for incomplete senses otherwise. Again, the subscript on a type symbol can be omitted when it is zero, and parentheses are left off with the convention of association to the right, but only when the subscript directly *outside* the parentheses is zero. If f is a function of type $(\alpha_i \mapsto \beta_j)_0$, and a an entity of type α_i, then fa is the entity of type β_j that is the value of f with a as argument. However, if F is an incomplete sense of type $(\alpha_i \mapsto \beta_j)_{n+1}$, and a^* is a sense of type α_{i+n+1} then Fa^* is to be understood as the complex sense of type β_{j+n+1} that arises from the *composition* of F with a^*, or the *s*-value of F with a^* as *s*-argument. The term "composition" also comes from Kaplan. The terminology is appropriate as the sense a^* is to be understood as in some way a constituent or part of the complex sense Fa^*.

It should be noted that this approach does not eschew sense-functions. We still posit functions of type $(\iota_1 \mapsto o_1)_0$, i.e., functions with senses of individuals as arguments and thoughts/propositions as values. However, functions of this type are no longer regarded as the senses of functions of type $(\iota \mapsto o)_0$; instead, that role is given to entities of type $(\iota \mapsto o)_1$. Let $H_{(\iota\mapsto o)_1}$ be the sense of the predicate " ... is human," understood as an incomplete sense of type $(\iota \mapsto o)_1$. When it takes an *s*-argument of type ι_1, it yields an *s*-value of type o_1. The composition of this sense with the sense of "Russell" yields the thought that *Russell is human*, and the composition of this sense with the sense of "Boston" results in the thought that *Boston is human*. There is, to be sure, a function of type $(\iota_1 \mapsto o_1)_0$, viz., h^*, whose value for any argument x^* is the same as the *s*-value of H for x^* as *s*-argument. However, it is not the case that for every function of type $(\iota_1 \mapsto o_1)_0$ there corresponds an incomplete sense of type $(\iota \mapsto o)_1$. We can allow our deviant sense-functions $h^{*\prime}$, $h^{*\prime\prime}$, $h^{*\prime\prime\prime}$, etc., as entities of type $(\iota_1 \mapsto o_1)_0$ without positing anything corresponding to them in type $(\iota \mapsto o)_1$.

Indeed, one of the chief advantages of placing the senses of functions in a distinct logical type is that it allows us more easily to curtail how many and what sort are posited to exist. For functions proper, we can continue to posit one for every determinate mapping from entities in the argument-type to entities in the value-type. Such mappings may be many-one or one-one. This underwrites defining functions by cases, resulting in "deviant" functions. Such definition is made possible in the λ-calculus using the lambda and description operators, as seen above in the definitions of $h^{*\prime}$ and $h^{*\prime\prime\prime}$. The case must be different with incomplete senses. They must be posited only for those mappings in which the *s*-value is precisely the same for each *s*-argument except differing in containing that *s*-argument in one or more spots in which the other *s*-values contain their corresponding *s*-arguments instead. For incomplete senses with *s*-arguments of types ι_n or o_n, at least, the mapping generated of *s*-arguments to *s*-values would be one-one.

In the λ-calculus, functional "comprehension" is effected by allowing λ-abstracts as valid substituends of function variables. λ-abstraction is the typical means for defining functions other than those taken as primitive. The deviant sense-function $h^{*'''}$, as we have seen, could be defined using an abstract such as the following:

$$\big(\lambda x^*\, \imath\{\lambda p^*[(x^* \neq a^*)\ \&\ (p^* = Hx^*)] \vee [(x^* = a^*)\ \&\ (p^* = Hb^*)]\}\big).$$

Clearly, λ-abstraction using the base λ-operator should be understood as creating complex expressions of functional type, and in this case, an expression of type $(\imath_1 \mapsto o_1)_0$. The above must not be understood as a complex expression of type $(\imath \mapsto o)_1$. Nevertheless, there is a need for an abstraction notation for incomplete senses. Among other things, such notation would be necessary for forming ascendants of complex function expressions. Consider, e.g., the function expression "$(\lambda x_\imath \sim_{o\mapsto o} h_{\imath\mapsto o} x_\imath)$", while its denotation is a function mapping all humans to the False, and other individuals to the True, its sense would seem to be an incomplete sense of type $(\imath \mapsto o)_1$, yielding different thoughts when completed by different senses for individuals.

Following the notation used by Church in Alternative (0), it is convenient to use the operator λ_1 in forming the ascendants of complex function expressions using λ proper. However, this operator is to be understood as part of a notation forming expressions of type $(\alpha \mapsto \beta)_1$, not for forming sense-function expressions. (The base λ-operator can continue to be used for that purpose.) To form the ascendant of "$(\lambda x_\imath \sim_{o\mapsto o} h_{\imath\mapsto o} x_\imath)$," we could then utilize the expression "$(\lambda_1 x_{\imath_1} \sim_{(o\mapsto o)_1} h_{(\imath\mapsto o)_1} x_{\imath_1})$", which would be understood as having type $(\imath \mapsto o)_1$. To avoid misunderstanding, we should note that this use of "λ_1" is motivated primarily to distinguish incomplete-sense abstracts from function abstracts, and not, like Church's use of the sign "λ_1", as a way of blocking conversion. Indeed, we shall find it possible to allow conversion with regard to such abstracts.

To avoid a collapse back to the sorts of difficulties discussed earlier, we must adopt certain conventions regarding the proper formulation of an abstract of the form $\ulcorner(\lambda_1 \mathbf{x}_{\alpha_{i+1}}\, \mathbf{M}_{\beta_{m+1}})\urcorner$: (i) the bound variable $\mathbf{x}_{\alpha_{i+1}}$ must be of a sense type, (ii) the expression occurring within the abstract, $\mathbf{M}_{\beta_{m+1}}$, not only must itself be of a sense type, but also (a) *must* contain $\mathbf{x}_{\alpha_{i+1}}$ free, and (b) must consist entirely of constants and variables falling into sense types, and, generally, all its well-formed parts must be of sense types. Hence, even if the whole expression $\mathbf{M}_{\beta_{m+1}}$ is of a sense type, it is not allowed in the context $\ulcorner(\lambda_1 \mathbf{x}_{\alpha_{i+1}}\, \mathbf{M}_{\beta_{m+1}})\urcorner$ if it contains any variables, constants or abstracts standing for functions, individuals or truth-values (entities whose type is α_0). Consider, for example, the expression:

$$\imath\{\lambda p^*[(x^* \neq a^*)\ \&\ (p^* = Hx^*)] \vee [(x^* = a^*)\ \&\ (p^* = Hb^*)]\}.$$

This whole expression has the sense type o_1, but clearly we do not want to allow the following:

$$\big(\lambda_1 x^* \,\imath\{\lambda p^*[(x^* \neq a^*) \,\&\, (p^* = Hx^*)] \vee [(x^* = a^*) \,\&\, (p^* = Hb^*)]\}\big)$$

as a well-formed expression of type $(\imath \mapsto o)_1$. The ban on constituent function signs rules this out. The operator λ_2 is used for forming ascendants of expressions using the operator λ_1, and generally, as a way of forming complex expressions for entities of types of the form $(\alpha \mapsto \beta)_2$, and so on, creating a hierarchy. Full syntactic rules are given in the next section.

Generally, the restrictions placed on the formation of incomplete sense abstracts ensure that their s-arguments can always be understood as *constituents* of their s-values. The stricture that $\mathbf{M}_{\beta_{j+1}}$ in an abstract $\ulcorner(\lambda_1 \mathbf{x}_{\alpha_{i+1}}\, \mathbf{M}_{\beta_{m+1}})\urcorner$ must contain $\mathbf{x}_{\alpha_i}$ free rules out an abstract such as "$(\lambda_1 x^*\, a^*)$", where "a^*" is a constant. There is no incomplete sense that yields the same thought as s-value for every s-argument. Similarly, the constant sense-function j^* discussed at the end of sec. 3 does not correspond to any incomplete sense.

§7. A new formal system: the core. We begin our sketch of a new formal system by describing a syntax. We begin with primitive constants: (i) a constant $C_{(o\mapsto o\mapsto o)_n}$ for every $n \geqslant 0$, (ii) a constant $\sim_{(o\mapsto o)_n}$ for every $n \geqslant 0$, (iii) a constant $\Pi_{((\alpha_m\mapsto o)\mapsto o)_n}$ for every $n \geqslant 0$ and type α_m, (iv) a constant $\imath_{((\alpha_m\mapsto o)\mapsto \alpha_m)_n}$ for every $n \geqslant 0$ and type α_m, (v) a constant $\Delta_{(\alpha_m\mapsto\alpha_{m+1}\mapsto o)_n}$ for every $n \geqslant 0$ and type α_m. For each type α_m, we also posit an infinite number of variables:

$$a_{\alpha_m}, b_{\alpha_m}, d_{\alpha_m}, \ldots, x_{\alpha_m}, y_{\alpha_m}, z_{\alpha_m}, A_{\alpha_m}, B_{\alpha_m}, D_{\alpha_m}, \ldots, X_{\alpha_m}, Y_{\alpha_m}, Z_{\alpha_m},$$
$$a'_{\alpha_m}, b'_{\alpha_m}, d'_{\alpha_m}, \ldots, A'_{\alpha_m}, B'_{\alpha_m}, D'_{\alpha_m}, \ldots, a''_{\alpha_m}, b''_{\alpha_m}, d''_{\alpha_m}, \ldots, \text{etc.}$$

We then define a *well-formed expression* (*wfe*) recursively as follows: (i) the constants are wfes of the type given by their subscripts, (ii) variables are wfes of the type given by their subscripts, (iii) if $\mathbf{M}_{\beta_{m+n}}$ is a wfe of type β_{m+n} and every well-formed part of $\mathbf{M}_{\beta_{m+n}}$ has a type whose outermost subscript is at least n, and $\mathbf{M}_{\beta_{m+n}}$ contains the variable $\mathbf{x}_{\alpha_{i+n}}$ occurring free, then $\ulcorner(\lambda_n \mathbf{x}_{\alpha_{i+n}}\, \mathbf{M}_{\beta_{m+n}})\urcorner$ is a wfe of type $(\alpha_i \mapsto \beta_m)_n$, (iv) if $\mathbf{M}_{(\alpha_i\mapsto\beta_m)_n}$ is a wfe of type $(\alpha_i \mapsto \beta_m)_n$, and $\mathbf{A}_{\alpha_{i+n}}$ is a wfe of type α_{i+n}, then $\ulcorner(\mathbf{M}_{(\alpha_i\mapsto\beta_m)_n}\mathbf{A}_{\alpha_{i+n}})\urcorner$ is a wfe of type β_{m+n}; nothing else is a wfe. A *formula* is defined as a wfe having type o. Parentheses are dropped when no ambiguity results, with the convention of association to the left.

One consequence of the above definitions worth discussing is that vacuous variable binding is disallowed in abstracts $\ulcorner(\lambda_n \mathbf{x}_{\alpha_{i+n}}\, \mathbf{M}_{\beta_{m+n}})\urcorner$ even when $n = 0$. The reason for this change is that such abstracts would not have ascendants unless vacuous variable binding were also allowed with those abstracts where $n > 0$. However, earlier we considered reasons for avoiding such abstracts. This too perhaps brings us closer in line with the approach of

Frege himself, who disallowed all forms of vacuous variable binding. Certain of Church's axioms for the extensional portion of the system utilize vacuous variable-binding, and so it will become necessary to revise them below, but otherwise, the change has no deleterious effects on the system. A function abstract formed with a vacuously bound variable always represents a constant function, and the change does not deprive us of the ability to define constant functions. For instance, rather than using the abstract "$(\lambda x\ a)$" for the constant function whose value is always a, we can use an abstract such as "$\{\lambda x\ \imath[\lambda y(x = x\ \&\ y = a)]\}$".

What inference rules would be adopted for the system depend in part on the desired criteria for identity of senses. If we wish to conform to Church's Alternative (0), for reasons that should be clear from the discussion above, the conversion rules II and III must be restricted to cases in which the abstraction operator λ_n has the subscript 0. For Alternative (1), however, it is possible and preferable to allow conversion regardless of the subscript. On that conception, "*ha*" and "$(\lambda x\ hx)a$" are regarded as synonymous, and this is easily captured by allowing the ascendant of the latter, "$(\lambda_1 x^*\ Hx^*)a^*$" to convert to the ascendant of the former, "Ha^*". There is of course room for exploring both alternatives. However, the Fregean notion of an "incomplete sense" seems to cohere best with an account allowing such conversions, and, indeed, Frege's own logical work does not have distinct notations for "*ha*" and "$(\lambda x\ hx)a$". I have elaborated elsewhere [30, pp. 101–05] on reasons for taking this to be the most Fregean of Church's three alternatives, and indeed, I do not share the attitude expressed by Anderson [5, p. 163] that Alternative (1) was a "false start."

Allowing such conversions, the conversion rules I–III are unchanged from those considered in sec. 2 above, noting only that they are allowed regardless of the subscript on the λ-operator. It is worth noting, however, that in the revised syntax it is not always possible to perform a conversion on an expression of the form $\ulcorner(\lambda_n \mathbf{x}_{\alpha_{i+n}}\mathbf{M}_{\beta_{m+n}})\mathbf{N}_{\alpha_{i+n}}\urcorner$. For example, a formula containing the well-formed expression:

$$[\lambda_1 p^*(\lambda_1 q^*\ C_{(o\mapsto o\mapsto o)_1} p^* q^*)]\ \imath(\lambda r^*\ r^* \neq r^*)$$

cannot be converted to one containing the expression:

$$\{\lambda_1 q^*[C_{(o\mapsto o\mapsto o)_1}\ \imath(\lambda r^*\ r^* \neq r^*)]q^*\}$$

because the latter expression is not well-formed. The above expression is to be regarded as its most fully reduced form. Similarly, certain applications of the expansion rule cannot be applied to expressions found with abstracts using the λ_n operator where $n > 0$.

My formulation of the extensional portion of the system also makes use of the following additional inference rules:

IV. *Modus ponens*: if $\mathbf{A}_o$ and $\mathbf{B}_o$ are formulas, from $C_{o\mapsto o\mapsto o}\mathbf{A}_o\mathbf{B}_o$ and $\mathbf{A}_o$, infer $\mathbf{B}_o$.

V. *Generalization*: if $\mathbf{M}_{\alpha\mapsto o}$ is a wfe of type $(\alpha \mapsto o)$, and $\mathbf{x}_\alpha$ is a variable not occurring free in $\mathbf{M}_{\alpha\mapsto o}$, from $\mathbf{M}_{\alpha\mapsto o}\mathbf{x}_\alpha$ infer $\Pi_{(\alpha\mapsto o)\mapsto o}\mathbf{M}_{\alpha\mapsto o}$.

VI. *Replacement*: if $\mathbf{M}_{\alpha\mapsto o}$ is a wfe of type $(\alpha \mapsto o)$, and $\mathbf{x}_\alpha$ is a variable not occurring free in $\mathbf{M}_{\alpha\mapsto o}$, and $\mathbf{A}_\alpha$ is a wfe of type α, then from $\mathbf{M}_{\alpha\mapsto o}\mathbf{x}_\alpha$ infer $\mathbf{M}_{\alpha\mapsto o}\mathbf{A}_\alpha$.

For the extensional portion, I suggest the following axioms and schemata:

(Axiom $*1$) $\quad p \supset (q \supset p)$,
(Axiom $*2$) $\quad [p \supset (q \supset r)] \supset [(p \supset q) \supset (p \supset r)]$,
(Axiom $*3$) $\quad (\sim p \supset \sim q) \supset [(\sim p \supset q) \supset p]$,
(Axiom $*4^\alpha$) $\quad \Pi_{(\alpha\mapsto o)\mapsto o} f_{\alpha\mapsto o} \supset f_{\alpha\mapsto o} x_\alpha$,
(Axiom $*5^\alpha$) $\quad (\forall x_\alpha)(p \supset f_{\alpha\mapsto o} x_\alpha) \supset (p \supset \Pi_{(\alpha\mapsto o)\mapsto o} f_{\alpha\mapsto o})$,
(Axiom $*6^\alpha$) $\quad (\forall x_\alpha)[f_{\alpha\mapsto o} x_\alpha \equiv (x_\alpha = y_\alpha)] \supset (\imath_{(\alpha\mapsto o)\mapsto\alpha} f_{\alpha\mapsto o} = y_\alpha)$,
(Axiom $*7$) $\quad (p \equiv q) \supset (p = q)$,
(Axiom $*8^{n\alpha_i\beta_m}$) $\quad (\forall x_{\alpha_{i+n}})(f_{(\alpha_i\mapsto\beta_m)_n} x_{\alpha_{i+n}} = g_{(\alpha_i\mapsto\beta_m)_n} x_{\alpha_{i+n}}) \supset (f = g)$.

Here, I preface the axiom numbers with asterisks to differentiate them from the axioms of Church's formulation. I have not tried to preserve compatibility of the system with the possibility of the type $\imath$ being empty, as Church himself did, though this may be possible with some reformulation. I also use free variables, which Church avoided. This too could likely be fixed, although the ban on vacuous variable-binding makes the use of free variables highly convenient. Together with rule IV, axioms $*1$–3 suffice for capturing all tautologies of propositional logic. These, together with axioms $*4$–5 and the other rules, suffice for Henkin-style completeness proofs for quantification theory. Axioms $*6$ govern the description operator; if desired, "$\imath$" can be used as a choice operator instead, and axioms $*6$ replaced with:

(Axiom $*6'^{\alpha_m}$) $\quad f_{\alpha_m\mapsto o} x_{\alpha_m} \supset f_{\alpha_m\mapsto o}(\imath_{(\alpha_m\mapsto o)\mapsto\alpha_m} f_{\alpha_m\mapsto o})$.

However, it may be best to avoid this assumption. Axioms $*7$–8 capture the desired extensionality principles for the types not populated by senses, and even incomplete senses are identified when they have the same *s*-value for every *s*-argument. If this last assumption is deemed worth avoiding, this axiom can be restricted to cases in which $n = 0$, though it should be noted that the instances of axioms $*8$ when $n \neq 0$ are considerably weaker than other plausible assumptions that might be made about their identity conditions. To facilitate comparison with the numbering of the axioms in Church's systems, no axioms are listed as "Axiom $*9$" or "Axiom $*10$."

Of course, the primary difference between the system presented here and Church's is that types such as $(\imath \mapsto o)_1$ are differentiated from function types such as $\imath_1 \mapsto o_1$. Nevertheless, for any expression $\mathbf{F}_{(\imath\mapsto o)_1}$ of type $(\imath \mapsto o)_1$, there is an expression of type $(\imath_1 \mapsto o_1)$, standing for a function whose value for any sense of type $\imath_1$ as argument is identical to the *s*-value of the incomplete sense represented by $\mathbf{F}_{(\imath\mapsto o)_1}$ for that sense as *s*-argument. In particular, for each such expression $\mathbf{F}_{(\iota\mapsto o)_1}$, if $\mathbf{x}_{\iota_1}$ and $\mathbf{y}_{\iota_1}$ are variables not occurring in $\mathbf{F}_{(\iota\mapsto o)_1}$, we get:

$$\vdash (\forall \mathbf{x}_{\iota_1})[(\lambda \mathbf{y}_{\iota_1}\, \mathbf{F}_{(\iota\mapsto o)_1}\mathbf{y}_{\iota_1})\mathbf{x}_{\iota_1} = \mathbf{F}_{(\iota\mapsto o)_1}\mathbf{x}_{\iota_1}].$$

This makes it possible in many contexts to substitute expressions of type $\iota_1 \mapsto o_1$ for those of type $(\iota \mapsto o)_1$ or vice-versa. To return to the examples used in previous sections, if H is the incomplete sense of type $(\iota \mapsto o)_1$ expressed by " ... is human," there is a sense-function, h^* or $(\lambda y^*\, Hy^*)$, of type $\iota_1 \mapsto o_1$ for which it holds that $(\forall y^*)(h^*y^* = Hy^*)$. However, even given the extensionality principles above, this does not eliminate the differences between these types, and an identity statement such as "$h^* = H$" is not even syntactically well-formed. Moreover, it is not the case that for every expression of type $\iota_1 \mapsto o_1$ there is a corresponding expression of type $(\iota \mapsto o)_1$. In particular, the deviant sense-functions $h^{*\prime}$, $h^{*\prime\prime}$, $h^{*\prime\prime\prime}$ considered in sec. 3, while definable by well-formed abstracts of type $(\iota_1 \mapsto o_1)$, have nothing equivalent in type $(\iota \mapsto o)_1$.

We next turn to the intensional portion of the system. Church's axioms 11–14 and 18 are modified only slightly in line with the changes to the type system.

(Axiom $*11^n$) $\Delta C_{(o\mapsto o\mapsto o)_n} C_{(o\mapsto o\mapsto o)_{n+1}}$,
(Axiom $*12^{n\alpha}$) $\Delta \Pi_{((\alpha\mapsto o)\mapsto o)_n} \Pi_{((\alpha\mapsto o)\mapsto o)_{n+1}}$,
(Axiom $*13^{n\alpha}$) $\Delta \iota_{((\alpha\mapsto o)\mapsto \alpha)_n} \iota_{((\alpha\mapsto o)\mapsto \alpha)_{n+1}}$,
(Axiom $*14^{n\alpha_m}$) $\Delta \Delta_{(\alpha_m\mapsto \alpha_{m+1}\mapsto o)_n} \Delta_{(\alpha_m\mapsto \alpha_{m+1}\mapsto o)_{n+1}}$,
(Axiom $*18^n$) $\Delta {\sim}_{(o\mapsto o)_n} {\sim}_{(o\mapsto o)_{n+1}}$.

Versions of Church's axioms 15–17, discussed earlier, are similarly modified in line with the discussion above:[12]

(Axiom $*15^{n\alpha_i\beta_m}$) $\Delta f_{(\alpha_i\mapsto\beta_m)_n} F_{(\alpha_i\mapsto\beta_m)_{n+1}} \supset [\Delta x_{\alpha_{i+n}} y_{\alpha_{i+n+1}} \supset \Delta(fx)(Fy)]$,
(Axiom $*16^{n\alpha_i\beta_m}$) $(\forall x_{\alpha_{i+n}})(\forall y_{\alpha_{i+n+1}})[\Delta xy \supset \Delta(f_{(\alpha_i\mapsto\beta_m)_n} x)(F_{(\alpha_i\mapsto\beta_m)_{n+1}} y)] \supset \Delta f F$,
(Axiom $*17^{\alpha_m}$) $\Delta x_{\alpha_m} z_{\alpha_{m+1}} \supset (\Delta y_{\alpha_m} z_{\alpha_{m+1}} \supset x = y)$.

While we here retain a schema superficially similar to Church's axiom schema 16, analogues of the difficulties discussed in sec. 3 are not present. The variable F in $*16$ ranges over incomplete senses, not sense-functions.

[12] Alternatively, if we wished to take seriously the Fregean doctrine of the "incomplete" or "unsaturated" nature of incomplete senses, and avoid constructions such as "ΔfF" that place names of incomplete entities alone in subject position, we could avoid axioms such as $*15$ and $*16$ altogether and simply use formulas such as "$(\forall x)\,(\forall x^*)[\Delta xx^* \supset \Delta(fx)(Fx^*)]$" *in the place of* constructions such as "ΔfF". Such an approach would naturally fit best with a more Fregean-style function calculus, one that displaces the need for λ-abstracts by effecting functional comprehension directly through a strong replacement rule for function and incomplete sense variables. For the development of a system along these lines, see [30, chap. 5].

The problematic sense-functions $h^{*'''}$, j^* and their ilk are not valid substituends. The syntactic rules governing abstracts of type $(\iota \mapsto o)_1$ and $(o \mapsto o)_1$ bar defining anything yielding similar unpalatable results. This allows us to obtain (SRT) by more or less the procedure by which it is gotten in Church's systems. Of course, the definition of *first ascendant* must be modified so that the ascendant $\mathbf{M}_{\alpha_{m+1}}$ of a closed expression $\mathbf{M}_{\alpha_m}$ is gotten more simply by increasing the *outermost* subscripts on the type symbols for each variable, constant and λ-operator making up $\mathbf{M}_{\alpha_m}$ by one. (SRT) can then be proven for all closed expressions by induction on the length of the expression $\mathbf{M}_{\alpha_m}$. A full proof is left to the reader.

§8. Surrogate models, remnants of axioms 64 and the need for ramification. What remains is to develop axioms governing the identity conditions of senses. We shall not attempt here a full exploration of this topic, which raises a number of philosophical issues. Since senses are usually regarded as the meanings of linguistic phrases, an exploration of the identity conditions of senses goes hand in hand with the exploration of synonymy in language. It is still a matter of debate to what extent there is one privileged or "correct" definition of synonymy; many would allege that there are different equally legitimate notions of meaning and with them different equally legitimate conceptions of synonymy: which it is appropriate to invoke on which occasions depends on one's purposes. In some ways, a realism about intensional entities such as senses mitigates against this, for it suggests that there are facts of the matter about the identity conditions of such entities. Still, even a realist can countenance different kinds of intensions with identity conditions of differing stringencies. At any rate, different theories about the identity conditions of senses are worth exploring.

For a realist equating senses with abstract intensional entities, the *intended* interpretation of the Logic of Sense and Denotation would be such as to take these abstracta as populating the domains of quantification for sense-types. However, since the nature and existence of these abstracta is a matter of controversy, this provides little help in securing uncontroversial models for a certain set of principles regarding sense-identity. However, for those conceptions of the identity conditions of senses that can be translated into a definition of synonymy for a well-defined language, surrogate models for principles of sense-identity can be sought by taking the domains of quantification for sense-types to consist in equivalence classes of synonymous closed expressions, i.e., those expressions which are regarded as having the same sense. This general line of model construction is outlined in [1]. In particular, for the sort of system sketched above, the domain of type α_{m+1} can be thought to consist in equivalence classes of synonymous closed wfes of type α_m. The syntactic rules of the system sketched above are such that every closed wfe of an *s*-type is the ascendant of some wfe; the models would be constructed in such a way that the denotation of each such wfe would be the equivalence class of synonymous expressions containing the wfe of which it is the ascendant. Which principles of sense-identity such models would support depends on the operant notion of synonymy.

As we have seen, under the guiding principles of Alternative (0), in a formal language such as that used in the Logic of Sense and Denotation containing no redundant primitives, two closed expressions are deemed synonymous if and only if they differ from each other by at most choice of bound variable. Under Alternative (1), two sentences are synonymous if and only if they can be obtained from one another by inference rules I–III, i.e., they are λ-converts. My main interest in this context involves Alternative (1), primarily because the changes to Church's system sketched above were made to block the problems regarding deviant sense-functions, and as we saw in sec. 5 above, Church's 1993 formulation of Alternative (1) avoids outright inconsistency from these difficulties only in neglecting to include principles regarding sense identity similar to Alternative (0)'s axiom schema 64, which asserts that the senses of functions are themselves one-one. Since my main interest in this paper has been the nature of the senses of functions, I fix my attention narrowly on the issues surrounding this sort of principle.

It should be noted, however, that something akin to axioms 64 does not hold generally for all types under the guiding principles of Alternative (1). To see this, we need only consider the following counterexamples. Consider closed expressions $\mathbf{a}_{\alpha_m}$ and $\mathbf{R}_{\alpha_m \mapsto \alpha_m \mapsto o}$ of the indicated types, and let x_{α_m} and $f_{\alpha_m \mapsto o}$ be variables not occurring in them. The following expressions are all λ-converts:

(7) $\mathbf{Raa}$,

(8) $(\lambda x\, \mathbf{R}x\mathbf{a})\mathbf{a}$,

(9) $(\lambda x\, \mathbf{Ra}x)\mathbf{a}$,

(10) $(\lambda f\; f\mathbf{a})(\lambda x\, \mathbf{R}x\mathbf{a})$,

(11) $(\lambda f\; f\mathbf{a})(\lambda x\, \mathbf{Ra}x)$.

Since these expressions are interconvertible, under Alternative (1), they are regarded as expressing the same sense. Let $\mathbf{R}^*_{(\alpha_m \mapsto \alpha_m \mapsto o)_1}$ be the ascendant of $\mathbf{R}$ and let $\mathbf{a}^*_{\alpha_{m+1}}$ be the ascendant of $\mathbf{a}$. Then the ascendants of the above are:

(7*) $\mathbf{R}^*\mathbf{a}^*\mathbf{a}^*$,

(8*) $(\lambda_1 x_{\alpha_{m+1}}\, \mathbf{R}^* x \mathbf{a}^*)\mathbf{a}^*$,

(9*) $(\lambda_1 x_{\alpha_{m+1}}\, \mathbf{R}^*\mathbf{a}^* x)\mathbf{a}^*$,

(10*) $(\lambda_1 f_{(\alpha_m \mapsto o)_1}\, f\mathbf{a}^*)(\lambda_1 x_{\alpha_{m+1}}\, \mathbf{R}^* x \mathbf{a}^*)$,

(11*) $(\lambda_1 f_{(\alpha_m \mapsto o)_1}\, f\mathbf{a}^*)(\lambda_1 x_{\alpha_{m+1}}\, \mathbf{R}^*\mathbf{a}^* x)$.

These would, in the surrogate models, be assigned the same equivalence class. Indeed, the identity statements between the above expressions would follow from allowing conversion with regard to abstracts formed with λ_1, which we recommended for capturing Alternative (1). In particular, we have:

(12)
$$\vdash (\lambda_1 f_{(\alpha_m \mapsto o)_1}\, f\,\mathbf{a}^*)(\lambda_1 x_{\alpha_{m+1}}\, \mathbf{R}^* x \mathbf{a}^*) = (\lambda_1 f_{(\alpha_m \mapsto o)_1}\, f\,\mathbf{a}^*)(\lambda_1 x_{\alpha_{m+1}}\, \mathbf{R}^* \mathbf{a}^* x).$$

If something such as axiom schema 64 were allowed in full generality, then, along with various results of (SRT), one would arrive at the result that:

$$\vdash (\lambda_1 x_{\alpha_{m+1}}\, \mathbf{R}^* x \mathbf{a}^*) = (\lambda_1 x_{\alpha_{m+1}}\, \mathbf{R}^* \mathbf{a}^* x). \tag{13}$$

However, this result is obviously unpalatable in most cases, and along with the determinacy of senses (axiom schema $*17$), leads in effect to the result that all relations are symmetric, which is, of course, impossible, and contradicts many results of the system. These examples show that while the senses of the primitive function signs may yield distinct s-values for distinct s-arguments, this does not hold generally for all senses of functions under Alternative (1).

However, the following less general versions of axioms 64 remain plausible:

(Axiom $*19^{ni\alpha_m}$) $(\forall F_{(\iota_i \mapsto \alpha_m)_{n+1}})(\forall x_{\iota_{i+n+1}})(\forall y_{\iota_{i+n+1}})[(Fx = Fy) \supset (x = y)]$,

(Axiom $*20^{ni\alpha_m}$) $(\forall F_{(o_i \mapsto \alpha_m)_{n+1}})(\forall p_{o_{i+n+1}})(\forall q_{o_{i+n+1}})[(Fp = Fq) \supset (p = q)]$.

These are supported by the intended surrogate models for Alternative (1). Expressions of the form ⌜**Fa**⌝ and ⌜**Fb**⌝, where **a** and **b** are not function expressions, are interconvertible (and hence synonymous according to Alternative (1)) only if **a** and **b** are interconvertible. (For further discussion, see the appendix.) Due to the counterexamples given above, it is not always the case that ⌜**MF**⌝ and ⌜**MG**⌝ are interconvertible only if **F** and **G** are interconvertible when **F** and **G** are themselves function expressions.

In sec. 3 we showed that in Church's formulation, if axiom $16^{\iota o}$ is taken together with axiom $64^{\iota o}$, an argument making use of the many-one sense-function $h^{*\prime\prime\prime}$ results in the absurdity that the sense of "Russell" is the same as the sense of "Frege." We also saw that axiom 16^{oo}, if conjoined with axiom 64^{oo}, leads to outright inconsistency when we consider the sense-function j^* mapping all thoughts to the same tautological thought. Nevertheless, something akin to axioms $64^{\iota o}$ and 64^{oo} are demanded by the operant principles governing sense-identity in Alternative (1). The corresponding instances in the revision are the following:

(Axiom $*19^{00o}$) $(\forall F_{(\iota \mapsto o)_1})\,(\forall x_{\iota_1})\,(\forall y_{\iota_1})[(Fx = Fy) \supset (x = y)]$,

(Axiom $*20^{00o}$) $(\forall F_{(o \mapsto o)_1})\,(\forall p_{o_1})\,(\forall q_{o_1})[(Fp = Fq) \supset (p = q)]$.

However, when formulated using variables of type $(\iota \mapsto o)_1$ or $(o \mapsto o)_1$ rather than those of type $\iota_1 \mapsto o_1$ or $o_1 \mapsto o_1$, we have no instances such as $h^{*\prime\prime\prime}$ or j^*, and the problems are avoided. This is of course the primary motivation for the changes to Church's system.

While these changes are sufficient to block the problems that arise from deviant or otherwise unusual sense-functions coming out as senses of functions according to Church's original axioms 16, i.e., the difficulties discussed in section 3, such as the Russell–Myhill antinomy, may continue to pose problems without the introduction of some form of ramification. It might be thought, however, that the failure of axioms 64 in their full generality may provide some help in avoiding the Russell–Myhill antinomy. Nicholas Denyer [15] recently blocked a formulation by Adam Rieger [39] of a paradox broadly similar to the Russell–Myhill in effect by drawing upon less stringent criteria for the identity of thoughts such as that in Alternative (1), though not in so many words. Indeed, given only the axioms we have considered so far the antinomy is not forthcoming. A simple formulation similar to that considered in sec. 4 would follow from the principle:

(14) $(\forall K_{(o_1 \mapsto o)_1})(\forall L_{(o_1 \mapsto o)_1})[(\Pi_{((o_1 \mapsto o)\mapsto o)_1} K = \Pi_{((o_1 \mapsto o)\mapsto o)_1} L) \supset (K = L)]$.

However, given the type restrictions on axioms 19 and 20, the above would not follow as a theorem. Indeed, the failure of axioms 64 in unrestricted form may give us reason for thinking that not all senses of type $((o_1 \mapsto o) \mapsto o)_1$ have distinct *s*-values for distinct *s*-arguments. On the current formulation, the reasoning behind the Russell–Myhill antinomy results in a theorem that there is *no* sense of type $((o_1 \mapsto o) \mapsto o)_1$ that always has distinct *s*-values for distinct *s*-arguments, i.e.:

(15) $(\forall M_{((o_1 \mapsto o)\mapsto o)_1})(\exists K_{(o_1 \mapsto o)_1})(\exists L_{(o_1 \mapsto o)_1})[(MK = ML) \,\&\, (K \neq L)]$.

One response to the Russell–Myhill paradox might be simply accepting the above result. Indeed, in a 1903 letter to Frege [24, p. 60], Russell once considered something broadly similar as a way of solving a similar Cantorian paradox of propositions.

However, I do not in the end believe that accepting (15) is a satisfactory way to respond to the Russell–Myhill antinomy, and indeed, (15) is not supported by the surrogate models we have sketched (cf. [32]). Indeed, properties of conversion make (14) extremely plausible instead. Obviously, an expression of the form $\ulcorner \Pi_{(o_1 \mapsto o)\mapsto o} \mathbf{k}_{o_1 \mapsto o} \urcorner$ is interconvertible with an expression of the form $\ulcorner \Pi_{(o_1 \mapsto o)\mapsto o} \mathbf{l}_{o_1 \mapsto o} \urcorner$ only if $\mathbf{k}_{o_1 \mapsto o}$ and $\mathbf{l}_{o_1 \mapsto o}$ are interconvertible. (Indeed, generally for any primitive constant **S** of a type of the form $(\alpha \mapsto \beta) \mapsto \delta$, for any function expression **f** and **g** of type $\alpha \mapsto \beta$, $\ulcorner \mathbf{Sf} \urcorner$ would be convertible to $\ulcorner \mathbf{Sg} \urcorner$ only if **f** is convertible with **g**.) For this reason, (14), or something more general from which (14) can be derived, is a plausible candidate for an axiom in Alternative (1). One could deny (14) while adhering to the basic principles of Alternative (1) only if one is willing to deny that the quantifiers $\Pi_{(\alpha \mapsto o)\mapsto o}$ are really primitive constants, and instead find some way of construing them as defined, but without making use of any additional primitive constants that would themselves give rise to a similar puzzle.

This makes it highly desirable to seek some other way of avoiding the paradox than accepting (15). Like Church, I find ramified type-theory to be, formally speaking, the most promising route to take. In traditional Russellian ramified type-theory, a proposition quantifying over a range of propositions must be of a higher "order" than the propositions quantified over (see, e.g., [40]). Similarly, we might suggest that the quantifier sense $\Pi_{((o_1 \mapsto o) \mapsto o)_1}$ would have as *s*-values thoughts of higher-order than those which can be taken as argument to any function presented by any of its possible *s*-arguments. It then would turn out that the thoughts of lowest-order would be less numerous than functions of type $o_1 \mapsto o$ whose arguments are of lowest-order. We could then accept that every function of this type has a sense of type $(o_1 \mapsto o)_1$, and that for every such sense it is possible to generate a distinct thought, provided that the thoughts so generated are (at least) of the next highest order. To escape the sorts of general worries about the cardinalities of sense types pointed out by Anderson, we must hold that in general, the *s*-values of any primitive function sense of type $((\alpha_{m+n+1} \mapsto \beta) \mapsto \alpha_m)_1$ is "order-raising" within the system of ramification. The full development of a system of ramification and the philosophical rationale underlying it deserves more careful and detailed scrutiny than we can give it here, and is left for another occasion. The brief sketch given here differs somewhat from the sort of ramification suggested by Church in his last paper on the topic [13], but it is possible that a suitably modified version of his suggestions could be made to work as well.[13] The system sketched in the previous section is aimed more modestly at solving those difficulties present in Church's systems stemming from unusual or deviant sense-functions for which ramification provides no help.

Appendix A. Some properties of conversion relevant to synonymy conditions under Alternative (1). To justify axiom schemata $*19$ and $*20$, I here sketch a proof that they are supported by the sorts of surrogate models suggested for Alternative (1), i.e., in which the domain of quantification for a sense type α_{m+1} consists of equivalence classes of interconvertible closed wfes of type α_m. Here, however, I limit my discussion to features of equivalence classes of interconvertible wfes of the base language, i.e., the language prior to the addition of sense-type expressions. Going beyond this would require settling the issues with regard to ramification, and studying properties of conversion with abstracts formed with λ_{n+1} rather than λ_0. These are difficult matters that require fuller exploration elsewhere.

Following Church [7], I write "**M** conv **N**" to mean that wfe **M** is λ-convertible to the wfe **N**, i.e., there is some finite number (possibly zero) of

[13]It is also possible that a suitably modified version of the alternative strategy adopted by Cocchiarella [14] for solving similar paradoxes in other systems could be adopted, though it would likely require further deviation from the core of a Fregean philosophy of language.

applications of the conversion rules by which **M** becomes **N**. The conversion rules are I–III from p. 156, though to cover wfes of different types we must expand them to apply not just when **A** and **B** have type o, but other types as well, and we must add the additional rules: (II′) where **A** is a wfe of a function type and and **x** is a variable of the appropriate type to constitute the argument to **A** but **x** does not occur free in **A** replace $\ulcorner(\lambda\mathbf{x}(\mathbf{Ax}))\urcorner$ with **A**; and (III′) where **A** is a wfe of a function type and **x** is a variable of the appropriate type to constitute the argument to **A** but **x** does not occur free in **A**, replace **A** with $\ulcorner(\lambda\mathbf{x}(\mathbf{Ax}))\urcorner$. These rules are not needed in the logical system, as they follow as derived rules from rules II and III along with axioms $*8$. Nevertheless, they need to be included as part of the conditions under which expressions are convertible when this is taken as the criterion of synonymy. Otherwise, it would not hold that "$\sim_{o\mapsto o}$" conv "$(\lambda p_o \sim_{o\mapsto o} p_o)$", yet surely these are to be regarded as synonyms under Alternative (1). The relation conv is reflexive, transitive and symmetric. Also, following Church, we say that a wfe **A** is in *normal form* iff (i) **A** is fully reduced (i.e., **A** does not contain a part of the form $\ulcorner((\lambda\mathbf{x}\,\mathbf{M})\mathbf{N})\urcorner$ to which rule II could be applied), and (ii) if **A** is a wfe of functional type, it takes the form $\ulcorner(\lambda\mathbf{x}\,\mathbf{M})\urcorner$. So "$(\lambda p_o \sim_{o\mapsto o} p_o)$", not "$\sim_{o\mapsto o}$", is in normal form. We say that a closed wfe of **A** is in *principal normal form* iff it is in normal form and its bound variables occur in strict alphabetical order, i.e., the letter a_{α_m} is used for the first occurrence of a variable of type α_m, the letter b_{α_m} used for the next, and so on, without repetition, or omission. An wfe **A** is said to have a normal form iff there is some wfe **B** in normal form such that **A** conv **B**; it has been proven for *typed* languages of λ-conversion that every wfe has a normal form [27, pp. 323–332]. Moreover, if a closed wfe **A** has a normal form, then there is a unique closed wfe **B** in *principal normal form* such that **A** conv **B** [7, p. 26]. I use the notation |pnf **A**| in the metalanguage for this unique wfe. If **A** is in normal form, then **A** differs from |pnf **A**| by at most choice of bound variable.

Axiom schemata $*19$ and $*20$ are then made plausible by the following:

Conversion Result. *If* **F** *is a closed wfe of a type with arguments of type* ι *or* o *and* **a** *and* **b** *are closed wfes of the appropriate type to constitute arguments to* **F**, *then* $\ulcorner\mathbf{Fa}\urcorner$ *conv* $\ulcorner\mathbf{Fb}\urcorner$ *only if* **a** *conv* **b**.

Proof. Assume that $\ulcorner\mathbf{Fa}\urcorner$ conv $\ulcorner\mathbf{Fb}\urcorner$. Then |pnf **Fa**| is the same as |pnf **Fb**|. Notice also that $\ulcorner\mathbf{Fa}\urcorner$ conv $\ulcorner$|pnf **F**||pnf **a**|$\urcorner$ and $\ulcorner\mathbf{Fb}\urcorner$ conv $\ulcorner$|pnf **F**||pnf **b**|$\urcorner$, and hence |pnf **Fa**| conv $\ulcorner$|pnf **F**||pnf **a**|$\urcorner$ and |pnf **Fa**| conv $\ulcorner$|pnf **F**||pnf **b**|$\urcorner$. Because |pnf **F**| is in normal form, it takes the form $\ulcorner(\lambda a_\alpha\,\mathbf{M})\urcorner$. Hence, $\ulcorner$|pnf **F**||pnf **a**|$\urcorner$ and $\ulcorner$|pnf **F**||pnf **b**|$\urcorner$ are, respectively, $\ulcorner(\lambda a_\alpha\,\mathbf{M})$|pnf **a**|$\urcorner$ and $\ulcorner(\lambda a_\alpha\,\mathbf{M})$|pnf **b**|$\urcorner$. These are not in normal form, but let **A** be the result of applying rule II to $\ulcorner(\lambda a_\alpha\,\mathbf{M})$|pnf **a**|$\urcorner$, and let **B** be the result of applying rule II to $\ulcorner(\lambda a_\alpha\,\mathbf{M})$|pnf **b**|$\urcorner$. Here, **A** results from substituting |pnf **a**| for all free occurrences of a_α in **M**. Because **a** is not a function expression, neither is $|\text{pnf}\,\mathbf{a}|$, and so $|\text{pnf}\,\mathbf{a}|$ cannot take the form $\ulcorner(\lambda\mathbf{y}\,\mathbf{N})\urcorner$. Hence $\mathbf{A}$ does not contain any portion to which rule II applies, since neither $\mathbf{M}$ nor $|\text{pnf}\,\mathbf{a}|$ contains any such portions, and substituting $|\text{pnf}\,\mathbf{a}|$ for a_α cannot create any new portions of this form. If $\mathbf{A}$ is not a function expression, it is already in normal form. If $\mathbf{A}$ is a function expression, then either $\mathbf{A}$ is already in normal form, or, where $\mathbf{x}$ is some variable not occurring in $\mathbf{A}$ or $\mathbf{B}$, $\ulcorner(\lambda\mathbf{x}\,\mathbf{Ax})\urcorner$ is in normal form, depending on whether or not $\mathbf{M}$ begins with a λ-operator. Let $\mathbf{A}^*$ be either $\mathbf{A}$ or $\ulcorner(\lambda\mathbf{x}\,\mathbf{Ax})\urcorner$, whichever is in normal form. Similar reasoning applies to $\mathbf{B}$, and let $\mathbf{B}^*$ be either $\mathbf{B}$ or $\ulcorner(\lambda\mathbf{x}\,\mathbf{Bx})\urcorner$, whichever is in normal form. (Notice that $\mathbf{A}^*$ is $\mathbf{A}$ if and only if $\mathbf{B}^*$ is $\mathbf{B}$, since they have the same type and both $\mathbf{A}$ and $\mathbf{B}$ were obtained by substitutions in $\mathbf{M}$.) Now, because $\mathbf{A}^*$ is in normal form and $\ulcorner\mathbf{Fa}\urcorner$ conv $\mathbf{A}$, and $\mathbf{A}$ conv $\mathbf{A}^*$, $\mathbf{A}^*$ differs from $|\text{pnf}\,\mathbf{Fa}|$ by at most alphabetic choice of bound variable; similarly $\mathbf{B}^*$ differs from $|\text{pnf}\,\mathbf{Fa}|$ by at most alphabetic choice of bound variable, and hence they differ from each other by at most this much. Because we have disallowed vacuous variable binding, $\mathbf{A}^*$ contains $|\text{pnf}\,\mathbf{a}|$ in certain spots where $\mathbf{B}^*$ contains $|\text{pnf}\,\mathbf{b}|$, i.e., those spots wherein $\mathbf{M}$ contains a_α free. Hence, the portion(s) $|\text{pnf}\,\mathbf{a}|$ of $\mathbf{A}^*$ must differ from the portions $|\text{pnf}\,\mathbf{b}|$ of $\mathbf{B}^*$ by at most choice of bound variable. In fact, $|\text{pnf}\,\mathbf{a}|$ and $|\text{pnf}\,\mathbf{b}|$ must be identical, because both are in principal normal form. Since $\mathbf{a}$ conv $|\text{pnf}\,\mathbf{a}|$ and $\mathbf{b}$ conv $|\text{pnf}\,\mathbf{b}|$, we can conclude that $\mathbf{a}$ conv $\mathbf{b}$. $\dashv$

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