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REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

RALF KRÖMER. ***Tool and object: A history and philosophy of category theory.*** Science Networks. Historical Studies, vol. 32. Birkhäuser, Basel, 2007, xxxvi + 367 pp.

Ralf Krömer's book is the first global and comprehensive study of the history and philosophy of category theory. It is very well-informed, interesting, stimulating and should be read by historians and philosophers of mathematics. The main thesis of the book is that a form of pragmatism best describes the implicit philosophy one finds in the development of category theory (CT) from 1945 to 1970. The thesis is defended by looking at the history of CT and the latter is organized by following the interactions of CT with three fields: algebraic topology, homological algebra and algebraic geometry. This follows the actual development of the field and is thus entirely justified.

Chapter 1 presents the philosophical position adopted by the author. A form of constructivism is assumed, captured by the slogan that every mathematical object is first considered as an historical artefact.

The philosophical methods used throughout rest upon the interplay between the history of the field and a pragmatist point of view on knowledge. By looking at the various ways CT was used and by focusing on the epistemological properties of these usages, philosophical insights into the nature of CT ought to be gained. Ensue a presentation of Peirce's and Wittgenstein's positions on various issues related to mathematical knowledge.

Chapter 2 takes a careful look at the origins of category theory in the work on Eilenberg and Mac Lane and its applications in algebraic topology by Eilenberg and Steenrod, Mac Lane's

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paper on duality in groups and Kan's paper on adjoint functors. The historical analysis is detailed, interesting and to the point. In some sections, the author brings back in his philosophical outlook, for instance in his section on Eilenberg and Steenrod's book.

Chapter 3 is a fascinating, detailed and instructive analysis of the interactions between category theory and homological algebra.

The author carefully looks at the birth of the discipline, launched by the publication of Cartan and Eilenberg's book in 1956 and then proceeds to analyse systematically Grothendieck's monumental *Tôhoku* paper, published in 1957, seen by many as being one of the turning points in the development of category theory. The author claims that here, one *has* to have an historical perspective to see and understand the nature of the shift that occurred. In a nutshell, the claim is that although CT before and after Grothendieck's work is still CT, that is, from a logical point of view, the theory did not change, its status and place in the overall mathematical network of ideas changed. This can only be seen diachronically, whence the necessity of an historical standpoint according to Krömer.

Chapter 4 turns to the daunting task of analysing the role of CT in Grothendieck's formulation of the foundations of algebraic geometry. Anyone familiar with Grothendieck's work in the area knows how hard and complex such an analysis has to be. Grothendieck and his colleagues published thousands of pages of original mathematics, developing at the same time algebraic geometry *and* the required conceptual foundations as they went along. The enterprise was nothing less than monumental but bear its fruits in many ways. Not only was the test case successfully solved, namely the proof of Weil's conjectures, although perhaps not in the way envisaged by Grothendieck, but the foundations of algebraic geometry are nowadays taken from this work.

Chapter 5 shifts from the uses of category theory in the early applications to categorical concepts themselves. First, Krömer looks at mathematical concepts that pre-existed CT but were radically transformed when absorbed by CT and then moves to categorical concepts themselves. This leads to a discussion of structuralism and CT.

The author wants to show that there is no need to interpret categories as collections of structured sets and structure preserving homomorphisms between them. Various arguments are offered in favour of that claim.

The discussion unravels with an examination of the basic concepts of CT itself: objects, arrows and categories themselves. The major portion of this chapter is philosophical. It refers at various times to the previous chapters but much of it relies on a conceptual analysis of the notions involved.

In chapter 6, the author explores the interactions between sets and categories, thus the question of the foundations of category theory from a set theoretical point of view. The author discusses how the problem evolved from Eilenberg and Mac Lane to the late sixties. It is during that period that category theorists slowly came to realize that the standard solutions to the set-theoretical paradoxes were not adequate for the development of CT. The need of functor categories and functors between functor categories in various contexts and the conviction that these constructions had to be treated like any other mathematical object lead to the conviction that the usual distinction between sets and classes was insufficient. Mac Lane's various positions are carefully examined, followed by a detailed presentation of the origins and developments of Grothendieck's universes. An interesting discussion relating Grothendieck's universes to Tarski's work on inaccessible cardinals throws new light on the issue. Once again, the author exhibits his exquisite philosophical sensitivity and knowledge of the literature: fine technical points drive the discussion and the analysis of the situation.

After having presented set theoretical foundations to CT, chapter 7 focuses on categorical foundations to set theory. The main point of the chapter is not that CT could provide alternative *mathematical* or *logical* foundations of mathematics, it is rather that the very notion of foundations of mathematics is shattered by CT. This claim is subtantiated by looking at Lawvere's work and claims about foundations, Bénabou's proposal and one of the standard objections to CT as a foundational framework. The chapter is voluntarily short, since the topics covered extend outside the historical period covered by the book.

The book ends with a general discussion of pragmatism and CT.

There are numerous regrettable grammatical constructions throughout. At times, reading the book becomes somewhat arduous. It would have greatly benefitted from a scrupulous editing process.

Open problems are scattered along. Thus any motivated researcher will find avenues for new research and possibilities of contributions to the field. In this respect, the author has to be commended for his generosity.

Overall, I really enjoyed the book. It is well-structured, challenging and instructive. One does not need to be an expert in any of the fields discussed to understand the main theses defended and everyone will learn a great deal about the history of 20th century mathematics by reading it. And although I am generally sympathetic with the author's philosophical position, I remain somewhat unconvinced by its capacity to reveal the unique features of CT as a mathematical and as a foundational framework.

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J. T. MOORE. *Set mapping reflection*. ***Journal of Mathematical Logic***, vol. 5 (2005), pp. 87–97.

J. T. MOORE. *A five element basis for the uncountable linear orders*. ***Annals of Mathematics***, vol. 163 (2006), pp. 669–688.

In[1] the last twenty years an impressive amount of results in combinatorial set theory ranging from cardinal arithmetics to general topology and model theory has been obtained by means of forcing axioms. These axioms are technical hypothesis which can be formulated as a strengthening of the Baire category theorem but require a great familiarity with forcing and set theory in order to be grasped. Nonetheless almost all of these results can be proved by interpolation using a short list of *Black box*-principles each of which is known to follow from the appropriate forcing axiom and some of which will imply the desired combinatorial result. One advantage of this approach is that most of these *"Black boxes"* can be understood and used by anyone with a modest familiarity with set theory and with no knowledge of the forcing method. For this reason also mathematicians which are not specialists in set theory may be interested in these type of hypotheses. Much in the same way as principles which holds in L like *diamond* or CH have been used to obtain the consistency of a certain solution to problems and conjectures coming from various branches of mathematics, among which general topology (the failure of Suslin's hypothesis using *diamond*, a classical result by Jensen), group theory (positive answer to Whitehead problem on free groups using *diamond*, a result of Shelah) and functional analysis (existence of outer automorphisms of the Calkin algebra using CH, a recent result by Phillips and Weaver *There are outer automorphisms of the Calkin algebra*, ***Duke Mathematical Journal***, vol. 139 (2007), pp. 185–202), there are already several applications of these "Black boxes" which provide the consistency of the opposite solutions to the same problems (in our list of examples the classical result of Solovay and Tennebaum for Suslin's hypothesis, Shelah's surprising negative solution to Whitehead's problem and Farah (unpublished) for the latter problem on the automorphisms of the Calkin algebra.

[1]A premise is in order, I decided to give precise references for results which are recent or which I consider essential for the discussion. The lack of a reference to a journal does not mean that the result I mention is unpublished, if this is the case I will explicitly state it.

Last but not least in this list of "Black Boxes" is the *mapping reflection principle* MRP introduced by Moore in the first reviewed item as a consequence of the *proper forcing axiom* PFA. There is a long list of combinatorial problems which have been solved by Moore and others by means of this principle. We mention, among others, the proof by means of MRP that PFA implies the *five elements basis*-conjecture[2] presented in the second reviewed item, the proof by means of MRP that[3] the *bounded proper forcing axiom* BPFA decides $2^{\aleph_0} = \aleph_2$ presented in the first reviewed item and the proof that MRP implies the *singular cardinal hypothesis*[4] SCH. It has also to be noted that MRP has a large consistency strength as it implies the failure of the *square principle* as it is shown in the first reviewed item and thus, for example, it implies the existence of inner models for Woodin cardinals and that the *axiom of determinacy* holds in $L(\mathbb{R})$.

We now formulate MRP.

Given a countable set A and s a finite subset of A, let $[s; A]$ be the family of all Y infinite subsets of A such that $s \subseteq Y$.

Let X be an uncountable set. The Ellentuck topology on $[X]^{\aleph_0}$, the set of countable subsets of X, is defined declaring the sets $[s; A]$ to be clopen for all $A \in [X]^{\aleph_0}$ and finite $s \subseteq A$. $\mathcal{C}$ is a club subset of $[X]^{\aleph_0}$ if there is some $f\colon X^{<\omega} \to X$ such that $A \in \mathcal{C}$ whenever $f[A] \subseteq A$.

Let $\theta > |X|$ be a regular cardinal. Let $H(\theta)$ denote the set of all sets whose transitive closure have size less than θ and M be a subset of $H(\theta)$. $\Sigma \subseteq P([X]^{\omega}) \cap M$ is M-stationary if for every $\mathcal{C} \in M$ club subset of $[X]^{\aleph_0}$, there is $Y \in \mathcal{C} \cap M \cap \Sigma$.

$$\Sigma\colon \mathcal{E} \to P([X]^{\aleph_0})$$

is an open and stationary mapping if $\mathcal{E}$ is a club subset of $[H(\theta)]^{\aleph_0}$ and $\Sigma(M)$ is open in the Ellentuck topology and M-stationary for all $M \in \mathcal{E}$.

> MRP holds if for all open and stationary maps Σ with domain $\mathcal{E}$ there is a continuous $\in$-chain[5] $\{M_\xi : \xi < \omega_1\}$ of elements of $\mathcal{E}$ such that for all non-zero limit ordinals $\xi < \omega_1$ there is $\eta < \xi$ such that $M_\gamma \cap X \in \Sigma(M_\xi)$ for $\gamma \in (\eta, \xi)$.

A sequence satisfying the conclusion of MRP for a mapping Σ is called a reflecting sequence for Σ. Here are some considerations by Moore which may help the reader to clarify the content of the above definition.

1. Let M be a countable elementary submodel of $H(\theta)$ for some large enough regular θ and c be a subset of $\delta = M \cap \omega_1$ of order type smaller than δ. The standard example of an M-stationary subset of $[\omega_1]^{\aleph_0} \cap M$ is the set $(\delta \cap M) \setminus c$.
2. Let f be a regressive map on ω_1 and Σ_f be the stationary map which sends a countable model M in $\delta \setminus f(\delta)$ where $\delta = M \cap \omega_1$. Notice that any continuous $\in$-sequence $\{M_\xi : \xi < \omega_1\}$ is a reflecting sequence for Σ_f. MRP is an attempt to maximize the set of mappings Σ for which there is a reflecting sequence.

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[2]This conjecture asserts that there are five uncountablee linear orders such that any other uncountable linear order contains an isomorphic copy of one of these five. This proof has been later refined by König, Moore, Larson and Veličković in *Bounding the consistency strength of a five element linear basis*, ***Israel Journal of Mathematics***, vol. 164, (2008), pp. 1–18, reducing the large cardinal strength of the assumptions needed to obtain the consistency of the conjecture to a very weak large cardinal hypothesis (the existence of a Mahlo cardinal suffices).

[3]Subsequently Caicedo and Veličković refined this result and showed that MRP (or BPFA) implies that there is a Δ_1-definable well ordering of $\mathbb{R}$ in the structure $H(\aleph_2)$ (see *Bounded proper forcing axiom and well orderings of the reals*, ***Mathematical Research Letters***, vol. 13 (2006), pp. 393–408).

[4]SCH holds if $\kappa^{\mathrm{cf}(\kappa)} = 2^{\mathrm{cf}(\kappa)} + \kappa^+$ for all singular cardinals κ.

[5]$\{M_\xi : \xi < \lambda\}$ is a continuous $\in$-chain if $\{M_\xi : \xi \leq \eta\} \in M_{\eta+1}$ and $\bigcup_{\xi<\eta} M_\xi = M_\eta$ for all $\eta < \lambda$.

The introduction of the first reviewed item and recent unpublished seminal works of Moore present at least two arguments to motivate the success of MRP: on one side this principle resembles closely the reflection principles which follow from *Martin's maximum* MM, a forcing axiom stronger than PFA. Thus MRP provides also in the context of PFA a type of argument which before was just peculiar to the setting of MM. This is exemplified by the main results of the first reviewed item where exactly the same pattern devised by Woodin and employed by Todorčević[6] in his proof that $2^{\aleph_0} = \aleph_2$ follows from BMM is used by Moore to obtain the same conclusion assuming the weaker BPFA. In Moore's proof MRP takes the place that the strong reflection principle SRP has in Todorčević's proof.

The second argument devised by Moore to motivate MRP is rather technical and carves deeply into the meaning of properness. Moore argues that MRP incorporates a schema of arguments to prove properness which is peculiar to a class of proper posets which are not (*weakly*) ω-*proper* and is **essential** to prove many interesting consequences of PFA.

It is outside of the scope of this review to define the notion[7] of (weak) ω-properness, it has to be noted however that any poset which has the countable chain condition or is countably closed is ω-proper, that most of the known applications of PFA are obtained by means of a weakly ω-proper poset[8] and that both properties are preserved under countable support iterations. Moore points out that there is a neat dividing line between the consequences of PFA that can be obtained by means of weakly ω-proper posets and those that cannot, MRP and the *five element basis*-conjecture being in this latter category. In particular Shelah's work shows that a (weak) club guessing sequence[9] on ω_1 is preserved by a countable support iteration of (weakly) ω-proper posets. While a standard application of MRP shows that for any $\mathcal{C}$-sequence[10] $(C_\delta : \delta < \omega_1)$, there is C club subset of ω_1 such that $C \cap C_\delta$ is finite[11] for all $\delta < \omega_1$ and thus (assuming MRP) no $\mathcal{C}$-sequence can be a weak club guessing sequence on ω_1. Now it is not hard to see that *diamond* implies that there is a club guessing sequence on ω_1 and Moore has shown in *Aronszajn lines and the club filter*, JSL LXXIII 1029, that the *five element basis*-conjecture does not hold in any model of set theory where a club guessing sequence on ω_1 exists. Thus Moore's proof that PFA implies the *five element basis*-conjecture cannot be established with the use of ω-proper posets[12] since the above discussion plus standard arguments on iterated forcing yields that the forcing axiom for ω-proper posets is consistent with the existence of a club guessing sequence on ω_1.

There are yet more arguments to motivate MRP (and in the large forcing axioms): the spectacular solution given by Moore to the *five elements basis*-conjecture follows the standard pattern for applications of BPFA: a certain poset P living in the ground model universe V is devised; an easy density argument shows that the desired conclusion holds in the generic extension V^P by P; an hard technical argument shows that P is proper; since the conclusion is a Σ_1-statement with parameters in $H(\aleph_2)$, an application of BPFA yields that the desired conclusion holds also in the ground model V. However in this proof of Moore, MRP is

[6]See *Generic absoluteness and the continuum*, ***Mathematical Research Letters***, vol. 9 (2002), pp. 465–471.

[7]Eisworth and Nyikos in *First countable, countably compact spaces and the Continuum Hypothesis*, ***Transactions of the American Mathematical Society***, vol. 357 (2005), pp. 4329–4347, introduce and analyze the class of weakly ω-proper posets.

[8]For example the *Black box*-principles PID and OCA can be forced by weakly ω-proper posets.

[9]$(C_\delta : \delta < \omega_1)$ is a weak club guessing sequence if C_δ has type ω for all limit δ and for every C club subset of ω_1 there is a δ such that $C_\delta \cap C$ is infinite. $(C_\delta : \delta < \omega_1)$ is a club guessing sequence if for all clubs C there is a δ such that $C_\delta \subseteq C$.

[10]$(C_\delta : \delta < \omega_1)$ is a $\mathcal{C}$-sequence if C_δ is cofinal in δ of type ω for all limit ordinals δ.

[11]Given a $\mathcal{C}$-sequence apply MRP to the mapping $\Sigma_{\mathcal{C}}(M) = [\delta]^{\aleph_0} \setminus C_\delta$ where $\delta = M \cap \omega_1$ and $M \prec H(\aleph_2)$ is a countable model such that $\mathcal{C} \in M$.

[12]Moore can refine this argument also to the case of weakly ω-proper posets.

crucially used in the proof that P is proper: i.e., the assumption that the ground model V satisfies MRP is necessary to establish that the required poset P is proper and it is not yet known whether the properness of P can be established rightaway in ZFC without any further assumption. This is the first known example of an application of a forcing axiom which require the forcing axiom even in the proof that the devised partial order has the required properties.

Let me close this note with a consideration linking this application of MRP to Woodin's program for the solution of the *continuum's problem*. The large number of consistency results obtained by means of forcing axioms are undoubtably due to the fact that different models of the strongest forcing axioms like MM have a strong degree of similarity. Woodin has been able to give a precise mathematical meaning to these considerations. In a few words,[13] Woodin has devised a non-constructive "proof system" $\vdash_\Omega$ and a strengthened version of BMM which we may call Woodin's maximum WM which (in the presence of large cardinals) have three remarkable properties:

1. $\vdash_\Omega$ is generically absolute i.e., $V \models$ "$T \vdash_\Omega \phi$" iff $V^{\mathbb{B}} \models$ "$T \vdash_\Omega \phi$" for any complete boolean algebra $\mathbb{B}$, thus the notion $\vdash_\Omega$ of Ω-deduction is not affected by forcing.
2. WM decides in the "proof system" $\vdash_\Omega$ the theory of $H(\aleph_2)$, i.e., for all sentences ϕ, $\mathsf{ZFC} + \mathsf{WM} \vdash_\Omega$ "$H(\aleph_2) \models \phi$" or $\mathsf{ZFC} + \mathsf{WM} \vdash_\Omega$ "$H(\aleph_2) \models \neg\phi$".
3. Any "axiom" Ψ which satisfies item 2 above denies CH.

The Ω-conjecture asserts that $\vdash_\Omega$ is complete for Π_2-sentences[14] valid in Ω-logic, where the latter is the set of sentences realized by all boolean valued model $V_\alpha^{\mathbb{B}}$ such that $V_\alpha \models$ "$\mathbb{B}$ *is a complete boolean algebra*".[15] Item 1 above is the main reason for which $\vdash_\Omega$ can be considered a proof system. Since it is a delicate matter to show that it is consistent that WM does not follow from forcing axioms,[16] item 2 above can be considered as a "proof" that virtually any concrete mathematical problem which can be formulated in the structure $H(\aleph_2)$ has a solution assuming forcing axioms. Moore's result on the *five element basis*-conjecture can be seen as another small brick added to Woodin's program as it gives another concrete and surprising example of how perspicuous can be the analysis of the structure $H(\aleph_2)$ in a universe where forcing axioms hold.

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[13]An introduction to this work of Woodin can be found in his expository articles *The continuum hypothesis, I*, ***Notices of the American Mathematical Society***, vol. 48, pp. 567–576 and *The continuum hypothesis, II*, ***Notices of the American Mathematical Society***, vol. 48, pp. 681–690.

[14]Notice that $\neg$CH is a Π_2-sentence.

[15]For α a limit ordinal, $V_\alpha^{\mathbb{B}}$ is the set of boolean valued terms in $V^{\mathbb{B}}$ of rank less than α. In order to avoid complications in the definition of $V_{\alpha+1}^{\mathbb{B}}$ only boolean valued models of the type $V_\alpha^{\mathbb{B}}$ with α a limit ordinal can be considered. This does not cause any loss of generality in the definition of Ω-truth.

[16]Nonetheless this has been shown by Larson in *Martin's Maximum and the* $\mathbb{P}_{\max}$ *axiom* $(*)$, ***Annals of Pure and Applied Logic***, vol. 106 (2000), pp. 135–149.