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REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

Quintessence: Basic readings from the philosophy of W. V. Quine, edited by Roger Gibson, The Belknap Press of Harvard University, 2004, 416 pp.

There are many reasons to be grateful to Roger Gibson for his work on Quine. The present collection adds to that list. Quine considered Gibson one of the foremost interpreters of his work and the present collection testifies to that fact. ***Quintessence*** arranges and reprints some of Quine's most important philosophical work. It will be useful both to old Quine hands and to those wanting a firm foundation.

There are Five Sections; "Analyticity and Reduction", "Indeterminacy and Inscrutability", "Ontology", "Epistemology and Philosophy of Mind" and "Extensionalism". Many of the thoughtfully chosen pieces are papers which can be found elsewhere in the Quine corpus, e.g., ***From a Logical Point of View***, ***Ways of Paradox***, ***Word and Object***, ***Theories and Things*** etc.

While most of the papers might well be familiar to Quine enthusiasts, there are two which I was not familiar with: "Confessions of a Confirmed Extensionalist" and "Naturalism: Or Living Within One's Means". The former essay nicely sets the stage in the section entitled "Extensionalism". Later pieces in that section concern non-extensional, (i.e., intensional) subjects such as his rejection of modal logic and intensional objects, properties vs. sets, and his treatment of belief contexts (some speak of these in terms of 'intentionality'). I could not believe my eyes when I spotted a very, very rare occurrence in Quine publications, a terminological confusion. In the "Confessions" essay p. 333 the word 'intentionality' is used

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twice. Since Quine is here contrasting extensionality with its opposite, the correct term would appear to be 'intensionality'. It was Quine who taught many of us that the two are different though related in special ways. One is left wondering about the origins of the confusion. I checked with Dagfinn Follesdal. He confirmed my suspicion that neither Quine nor Gibson were responsible. It may have had its origin in the editing or production process for the collection in which the essay first appeared. Follesdal informed me that Quine died before the contributors proofread their pieces.

The other essay I was not familiar with, "Naturalism: Or Living Within One's Means", is a gem. In it Quine presents a shortened and lively version of his vision of naturalism (nature know thyself) and how it provides the framework for what we know.

The full title of this book has the closing phrase, 'Basic Readings From the Philosophy of W. V. Quine'. This title does not lie.

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Richard Kaye. ***The mathematics of logic: A guide to completeness theorems and their applications.*** Cambridge University Press, Cambridge, 2007, xi + 204 pp.

Kaye's book is devoted to extracting interesting mathematical content of Completeness Theorems and showing some of their more sophisticated applications.

Kaye starts his book with introducing the notion of a tree, explaining and proving König's Lemma and discussing its extensions and applications.

The second chapter introduces the notion of a poset and some related notions, and explains how trees can be viewed as posets. It also contains a very clear discussion of Zorn's lemma and its relation to the Axiom of Choice.

Chapter three explains what a formal system is, from a tree-theoretic perspective. Kaye gives a slightly unusual but fairly simple example of a formal system, uses it to introduce a few related notions, and produces a few examples of proof-theoretic facts about this system. He provides first Soundness and Completeness Theorems.

Chapter four describes a formal proof system for posets. Proofs of Soundness and Completeness Theorems are given. The completeness proof is didactically nicely chosen, for it illustrates the idea that to prove completeness it is enough to prove that consistent sets are satisfiable, and yet it doesn't involve any complicated model construction on the scale needed for completeness proofs for classical logic (neither the Axiom of Choice, nor Lindenbaum's nor Zorn's lemmas are used).

Chapter five discusses Boolean algebras. The author introduces some lattice-theoretic notions and discusses properties of lattices. The chapter contains also an interesting historical remark explaining why Boole himself wasn't strictly speaking working with boolean algebras.

Chapters six and seven contain a presentation of propositional logic in terms of algebraic semantics. Kaye proves the decidability of provability for propositional logic and soundness, completeness and compactness of the proof system under discussion. Quite interesting is additional material in chapter seven, treating about the complexity of the satisfiability problem for propositional logic and the relation between **P** and **NP** problems.

Chapter eight is devoted to applications of Soundness and Completeness Theorems to algebraic theories of boolean algebras. Kaye explains how the Soundness Theorem can be expressed in terms of filters of free algebras and uses the Completeness Theorem to prove Boolean Prime Ideal Theorem, Tychonov's Theorem and Stone Representation Theorem.

In chapter nine Kaye describes a proof system for first-order logic. The explanation is clear and concise, enriched with a nice selection of examples. Somewhat briefly, semantics for first-order languages is introduced and Soundness Theorem is proved. In fact, Kaye does something slightly surprising. He proves the Soundness Theorem by giving a three-liner:

> Formally, the proof is by induction on the (finite) length of the derivation. We simply need to inspect each proof role to check that it preserves the property [that if a set proves a formula, the set also semantically entails this formula]. [p. 134]

However trivial the Soundness Theorem for first-order logic may be, one might think, a book that's all about soundness and completeness and their applications might contain a more detailed treatment of the proof (Kaye did post the full proof at `http://web.mat.bham.ac.uk/R.W.Kaye/logic/contents`, however).

Kaye also explains in this chapter how second-order logic is different from first-order logic and points out that it's devoid of a complete proof system. He does seem to be quite dismissive about categoricity and the role played by second-order logic in obtaining it. This treatment, however, is understandable, given that the book is not concerned with second-order logic.

Chapter ten contains proofs of Completeness and Compactness Theorems. The proof uses Henkinisation and uses Zorn's lemma. The Compactness Theorem is a fairly simple consequence of Soundess and Completeness theorems.

The first mathematical example of an application of the Completeness Theorem for first-order logic is a proof that cyclic groups aren't first-order definable. Another interesting application of the Compactness Theorem is related to topology. We can consider the set of all theories of all models as a topological space. Compactness Theorem is used to prove that this topological space is compact. Yet another interesting example in topology is the Omitting Types Theorem. First, the Compactness Theorem is used to prove Baire's Theorem about Hausdorff topological spaces, and then Baire's Theorem is used to prove the Omitting Types Theorem.

In Chapter 11 Kaye proves that there are countably many formulas of a language built over a countable alphabet and that the complete Henkinisation of such a language also will be countable. This gives a countable version of the Completeness Theorem. Next, he proves in general that for any first-order language with an infinite alphabet the cardinality of the set of finite strings over that alphabet is equal to the cardinality of the alphabet (this applies to uncountable languages as well). Similarly: the cardinality of the complete Henkinisation of a language equals to the cardinality of that language. This gives us a completeness theorem for arbitrary cardinalities. Another useful theorem is that any consistent set of sentences of a first-order language is such that if that language is of an infinite cardinality, the set has models for each of the cardinalities equal or greater to the cardinality of that language. The author discusses negation-completeness and categoricity and proves that every set of formulas that's k-categorical for some infinite cardinal k is such that, if it has no finite models, it is negation-complete. An application of this result is an argument for the negation-completeness of an axiom system for F-vector spaces over a field F. Another interesting proof is that of Morley's theorem: any set of sentences that has no finite models and is k-categorical for some cardinality k greater than the cardinality of that language is l-categorical for any cardinal l greater than the cardinality of the language. Among the applications of these developments we find the fact that the theory of dense linear orders is $\aleph_0$-categorical and hence negation-complete. Another example is Schröder–Bernstein Theorem.

Chapter twelve is concerned with non-standard analysis. The basic idea is that the Completeness and Compactness Theorems can be used (as discovered by Robinson) in a construction of the hyperreal number line and in relating reals and hyperreals.

The author's intention is to present "the material usually treated in a first course in logic, but in a way that should appeal to a suspicious mathematician wanting to see some genuine mathematical applications" without the "off-putting and didactically unnecessary logical precision insisted on by some of the authors of the standard undergraduate textbooks." (p. vii) In this task he succeeds.

There are some trade-offs resulting from selecting such a task. Even the less-advanced sections of the book require a more mathematically sophisticated reader than usual presentations of the same material in standard logic textbooks. Kaye assumes that the reader is used to proofs by induction, and that they are ready to go into more set theory than is usually needed for the proofs of metalogical facts.

The author does a good job explaining the Completeness Theorems and indicating their mathematically interesting application (although, one wonders why Kaye didn't bring up non-standard models of Peano Arithmetic in this context). Writing from the perspective of a philosopher, even though I wouldn't recommend this book as a first logic textbook, I'm inclined to say that despite the fact that some parts of the book are not really directed to a philosophically-minded reader, most of the material presented (perhaps excluding some applications of the Completeness Theorem and chapter 12) is a must for any philosopher who wants to work with philosophy of logic or philosophy of mathematics.

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Peter Smith. ***An introduction to Gödel's theorems.*** Cambridge Introductions to Philosophy, Cambridge University Press, 2007, xiv + 362 pp.

This is a comprehensive introduction to Gödel's incompleteness theorems and their implications, intended to be accessible and useful for both advanced philosophy students and mathematics (or computer science) students. Some of its main features are:

- The pace is slow, and (as the author says in the preface) the style is relatively relaxed. The author is doing his best to achieve first an understanding of the ideas. Thus a proof is often first sketched, frequently in an informal way, and only later a full version of it is given. In some cases *only* a sketch is provided (or even just a "sketch of a sketch").
- The main methodological principle that guides the presentation of the material is to go from the particular to the general. Thus a theorem may first be formulated and proved for a particular system like *PA*, and then generalized to some class of theories. Another example: the diagonalization lemma is formulated and proved only after the construction of Gödel sentences is explained in detail.

  Another important methodological guiding principle of the book is to introduce basic notions and ideas only before they are really needed. While this is usually a good policy, because of it general computability theory and the exact formulation of Church's Thesis (and so the formal definitions of "an axiomatic theory" etc) are left here to the end of the book—and I am not sure that *this* is a good idea.
- In order to make the book "reader friendly", the book stops from time to time to reflect on what has been achieved up to that point and what has not, and to give "road-maps" to what is coming next.
- The book tries (and succeeds) to provide a broader perspective of the subject by including relevant philosophical and historical discussions. One of its strongest points is the inclusion (sometimes in a sketchy way) of important related subjects that usually cannot be found in other books on the subject (examples are given below). However, I believe the author should make greater efforts to separate between his personal opinions and objective material. Thus the presentation might make it difficult for a non-professional reader to see the difference between the "proof" given in Chapter 35 to Church's thesis, and other informal proofs given in the book.
- From the technical side, the presentation of the material is rather conventional, essentially following Gödel's original presentation. This is in sharp contrast to Smullyan's "Gödel's Incompleteness Theorems" or Fitting's recent book on "Incompleteness in the Land of Sets".

Now I turn to a detailed description of the content of the book.

Chapter 1 contains a nice, concise overview of the content of the theorems, the central ideas of their proofs, and the possible implications (e.g., the Lucas–Penrose thesis).

Chapters 2–7 contain proofs of intuitive versions of Gödel's first incompleteness theorem, Tarski's Theorem, and Church's Theorem. The proofs are based on certain (not fully specified) expected properties of the notions of "decidable", "effectively computable", and "effectively enumerable", which are explained in an informal way in Chapter 2. This is done there first for numerical functions, and then, via the possibility of coding, for other types of data structures. The chapter contains also a first, again informal, acquaintance with the Church–Turing Thesis and the evidence for it.

On the basis of the above informal notions, an informal concept of formal axiomatic theory is introduced in Chapter 3. Then two basic theorems are informally proved: that the set of theorems of formal axiomatic theories is effectively enumerable, and that (negation-)complete formal axiomatic theories are necessarily decidable.

$\mathcal{L}_A$, the language of *PA*, is defined in Chapter 4 together with its standard semantics. The fundamental concepts of expressing a relation in a language, and of representing (or "capturing", in the terminology of the present book) a relation in a theory, are introduced, and the difference between the two notions is explained and emphasized.

In Chapter 5 an intuitive notion of a "sufficiently expressive language" is introduced. Then the following two theorems are informally proved:

1. The set of true sentences of a "sufficiently expressive language" is not effectively enumerable, and so it cannot be axiomatized.
2. A sound formal axiomatic theory in a "sufficiently expressive language" cannot be complete. Moreover: it is *incompletable* in the sense that no additions of axioms would make it complete.

Next, an intuitive notion of a "sufficiently strong theory" is introduced in Chapter 6, and it is proved that a formal axiomatic theory which is both consistent and "sufficiently strong" can be neither decidable nor complete. Moreover: this situation cannot be changed by adding axioms. Again the proof heavily relies on intuitions concerning the properties that the notions of "computability" and "decidability" should have.

The advantage of all these informal proofs (which Smith takes as "Proofs from the Book") are that they nicely represent the main ideas of the proofs of the above-mentioned, more precise theorems (especially diagonalization). Their main disadvantage (in my opinion) is the danger that the reader would not quite understand that these are not exact theorems, and might fail to see what is missing (despite the author's attempt in Chapter 7 to explain this).

Chapter 7 reviews what has been achieved so far. Then there is an explanation what extra knowledge the real theorems of Gödel provide that what has been done so far does not, and what will be done later in order to get Gödel's original results.

Chapters 8–10 describe the systems *Q* and *PA*. The first is introduced in Chapter 8, where it is shown that it is true but not negation complete. The chapter also explains (without a proof yet) why it is interesting: because it is "sufficiently strong". In Chapter 9 it is proved that *Q* is Σ_1-complete. Before that the notions of Δ_0-formulas, Σ_1-formulas and Π_1-formulas are introduced, and their basic properties are investigated. Next *PA* is introduced and investigated in Chapter 10. It is noted that unlike *Q*, this is a rich theory, and it is not easy to show that it is not negation complete.

Chapters 11–14 are devoted to primitive recursive (p.r.) functions and their relation with *Q* and *PA*. This class of functions is defined in Chapter 11, where it is shown that all of them are computable, but they do not include all intuitively computable functions. Chapters 12 introduces three notions of *capturing* functions in a theory: as a relation (the weakest notion), as a function, and "fully as a function" (the strongest notion). The relations between these notions are discussed. Then it is proved in Chapters 13 that Q captures as a function every p.r. function. It is also mentioned without a proof that *PA fully* captures as a function every p.r. function. In the last section it is noted that the proofs are constructive, and that the formula they provide for capturing a given p.r. function reflects the p.r. definition of that function and so "canonically captures" it (this notion is not defined in exact terms). Finally, Chapter 14 contains a short historical survey (especially about *Principia Mathematica*) and a "road-map" for the next stages.

Chapters 15–21 are devoted to the technical details of the formulation and the proof of the first incompleteness theorem.

Chapter 15 introduces the arithmetization of syntax through Gödel numbers. Then it is proved that the main syntactic relations and functions related to *PA* are p.r. relative to the encoding used. The chapter contains an important discussion on the dependence of its results on the coding system, explaining why any other "sensible" one would do just as well.

Chapter 16 constructs the Gödel sentence G for *PA*, and shows its undecidability in *PA* using either the truth in N of the theorems of *PA* ("the semantic argument") or the assumption that *PA* is ω-consistent ("the syntactic argument"). It is also shown that G is of the "Goldbach type" (i.e., Π_1-sentence). The Chapter ends with an important discussion whether it is correct to claim that G "says that G is unprovable in *PA*".

Chapter 17 observes first that *PA* is not only incomplete but actually incompletable. Then this result and the results of Chapter 16 are generalized to every consistent p.r. theory which is either arithmetically true (in the "semantic" version of the first incompleteness theorem) or an ω-consistent extension of Q (in the "syntactic" version).

Chapter 18 compares between the informal theorems of chapters 2–6 and the formal ones of chapter 17. Then it explains from a wider perspective what has been achieved, and (no less important) what has *not* been achieved (this is done by analyzing some misconceptions of the incompleteness theorem). There is also a short section which extends the reach of incompleteness to non-arithmetical languages and theories by (implicitly and informally) using translations. One corollary is that the natural numbers cannot be defined within the pure first-order theory of the real numbers.

Chapter 19 broadens the scope of the incompleteness theorems in various directions. First it outlines a proof of Craig's Theorem concerning axiomatized theories. This intuitively implies that the results of Chapter 17 apply to any such theory, not only to p.r. theories (but formal counterparts cannot be proved in this stage, since computability theory is developed later in the book). Then it outlines Rosser's proof that the condition of ω-consistency can be replaced by simple consistency. Finally it shows that actually even Gödel's arguments require assumptions which are weaker than soundness and ω-consistency: Σ_1-soundness and 1-consistency.

Chapter 20 introduces the provability predicate $Prov_T$, and shows that $G_T \leftrightarrow \neg Prov_T(\ulcorner G_T \urcorner)$ is provable in T if T is "nice" (i.e., a consistent p.r. extension of Q). This construction is generalized to get the full diagonalization lemma.

Chapter 21 uses the diagonalization lemma first to reprove the general Gödel (first) incompleteness theorem(s), and then for full proofs of the Gödel–Rosser theorem, Tarski's Theorem, and a version of Church's Theorem. It is noted that Tarski's Theorem immediately implies that truth in N and provability in some formal theory cannot be identical (which easily implies the semantic version of the incompleteness theorem). The chapter ends with two results (with sketchy proofs) concerning the length of proofs in nice theories.

Chapter 21 also contains a short (but interesting) discussion on the difference between the "canonically" constructed Gödel sentences for a nice theory T, and between "Gödel sentences" defined as fixed points in T of $\neg Prov_T$. It is noted that if a nice T is not sound (for N) than according to the broader concept there exists "Gödel sentences" for T which are *false* (unlike the "canonical" ones, which should be true). This is supposed to indicate the importance of canonicality. However, any fixed point of $\neg Prov_T$ in Q (rather than in T) is necessarily true (if T is nice), and such fixed points need not be "canonical". Hence the argument only shows that it is better to define "Gödel sentences" as fixed points *in* Q of $\neg Prov_T$ (rather than in T).

Chapters 22–23 look at the incompleteness theorem from a wider perspective. Proofs in them are either sketchy or altogether omitted. The first, Chapter -22, is devoted to a sketchy but informative discussion of PA_2 (second-order arithmetics). Its main goal is to explain the apparent conflict between the categoricity of PA_2 and the incompleteness theorem. The chapter describes the language of PA_2, its standard semantics (the term "standard" is not used here, though), and the standard corresponding formal system. It then provides a proof sketch of its categoricity. Together with the incompleteness theorem, this implies that the formal proof system is incomplete, and that this fact cannot be repaired. As a major alternative to PA_2 the chapter suggests the (predicatively justified) system ACA_0, It notes (without proofs) that it is conservative over *PA*, but on the other hand it allows to enormously shorten proofs of some of *PA*'s theorems.

Chapter 23 discusses two subjects. The first is whether there are "interesting" (my quotation marks) arithmetical truths which cannot be decided in *PA*. The author brings Goodstein's theorem as an example of such a mathematical theorem (touching on the way induction up to ε_0, and Gentzen's proof of the consistency of *PA* which uses it). The second topic of Chapter 23 is what is called here "Isaacson's conjecture", according to which "*PA* exactly reflects what follows from our purely arithmetical knowledge". The conservativity over *PA* of ACA_0 is taken as evidence for this thesis. Another evidence brought here is the conservativity over *PA* of yet another system, called here PA^*. PA^* is a counterpart of *PA* in the framework of ancestral logic, based on Myhill's incomplete deductive system for that logic. Now I do share Smith's opinion about the importance of ancestral logic. However, I think that the argument from PA^* at best reduces the truth of "Isaacson's conjecture" (whatever it actually means) to the thesis that Myhill's system exactly reflects what follows from our understanding of the concept of ancestral. Unfortunately, the author provides no evidence for *this* Thesis.

Chapters 24–28 are devoted to Gödel's second incompleteness theorem. The presentation in them is top-down, leaving the difficult technical parts to later stages:

Chapter 24 introduces the sentence(s) Con_T for nice T, and formulates (without a proof yet) the formalized version of the first incompleteness theorem for T. Assuming that this version is provable in T, the second incompleteness theorem follows. The rest of the chapter discusses the implications and meaning of this theorem, including a short discussion of its relations with Gentzen's consistency proof for *PA*.

Chapter 25 introduces the Hilbert–Bernays–Löb derivability conditions, using the language of modal logic. Then it is shown that any nice theory T that satisfies them proves the formalized version of the first incompleteness theorem for T, and so does not prove Con_T. It is further shown that Con_T and G_T, as well as any other fixed point of $\neg Prov_T$, are equivalent in T. All these proofs are essentially proofs in the modal logic $K4$ (though this is not mentioned in the text). The chapter ends with two proofs of Löb Theorem.

Chapter 26 *sketches* a proof that the derivability conditions obtain for nice theories in which induction for Σ_1-formulas is available. This proves the second incompleteness theorem for such theories. Some stronger versions are then described, including Jeroslow's version in which one of the derivability conditions may be dropped.

Chapter 27 shows that in formulating the second incompleteness theorem, one should be careful about the way consistency statements are formalized. Then it is proved that the consistency of an extension T of Q is equivalent to the validity of Π_1-reflection for T. The chapter ends with two sections discussing the possible philosophical implications of these results.

The first part of Chapter 28 describes Hilbert's program and the implications of Gödel's theorems to it (including Detlefsen's attempts to rescue it, which are refuted). The second part discusses the relevance of the theorems to the question whether our minds can be machines, and refutes the Lucas–Penrose claim that from Gödel's theorems it follows that they are not.

The last part of the book, Chapters 29–35, is devoted to computability theory and Church's Thesis.

In Chapter 29 general recursive functions are introduced (as μ-recursive functions). It is shown (in a sketchy way) that the Ackermann–Péter function is μ-recursive but not primitive recursive. Then Church's thesis is introduced and justified, and the two different ways of its use are explained.

Chapter 30 shows that a function or relation can be captured in a given nice theory iff it is recursive. This easily entails that nice theories cannot be decidable (Church's Theorem). The undecidability of first order logic immediately follows. Church's Theorem is used also for yet another argument why a consistent nice theory cannot be complete. Finally, it is shown that all interesting results proved so far in the book for p.r. theories obtain for recursively axiomatized theories as well.

Chapter 31 introduces Turing Machines and Turing-computable functions.

Chapter 32 shows that a function on N is μ-recursive iff it is Turing-computable. Then it discusses possible generalizations of Turing Machines and explains why none of them adds anything to their power.

Chapter 33 shows that the halting problem for Turing machines is not recursively solvable. Then this result and its proof are used for some new arguments for the first incompleteness theorem.

Chapter 34 describes the history of Church's Thesis, and discusses its status and exact content (what it says and what it does *not* say).

In Chapter 35 it is argued that it might be possible to *prove* Church's thesis. Then an attempt is made to provide such a proof, by "squeezing" the computable functions between the Turing-computable ones and those which can be computed using Kolmogorov–Uspenskii algorithms. Personally, I do not find the argument convincing, since Kolmogorov–Uspenskii algorithms and Turing algorithms are based on similar assumptions, and the arguments for a "Kolmogorov–Upsenski thesis" are in my opinion not essentially different than those given by Turing for the "Turing Thesis". Still, a chapter like this is interesting, but as noted at the beginning of this review, it should be written more carefully. Such a chapter should also give references to other attempts to prove Church's thesis, like those of Sieg, and the recent one of Dershowitz and Gurevich.

One last comment: at the end of Chapter 30 there are two proof sketches of the claim that the set of truths of basic arithmetics is not r.e. (with some technical gaps left for the following chapters). Actually, this claim follows immediately from the much stronger Theorem 21.5 of Chapter 21 (Tarski's Theorem). The author of the book promises to make the corresponding necessary changes in the next revised edition of the book.

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JAMES H. SCHMERL. *Peano models with many generic classes*. ***Pacific Journal of Mathematics***, vol. 43 (1973), pp. 523–536.

JAMES H. SCHMERL. *Correction to: "Peano models with many generic classes"*. ***Pacific Journal of Mathematics***, vol. 92 (1981), no. 1, pp. 195–198.

JAMES H. SCHMERL. *Recursively saturated, rather classless models of Peano arithmetic*. ***Logic Year 1979–80 (Proceedings, Seminars, and Conferences in Mathematical Logic, University of Connecticut, Storrs, Connecticut, 1979/80).*** edited by M. Lerman, J. H. Schmerl, and R. I. Soare, Lecture Notes in Mathematics, vol. 859. Springer, Berlin, pp. 268–282.

JAMES H. SCHMERL. *Recursively saturated models generated by indiscernibles*. ***Notre Dame Journal of Formal Logic***, vol. 26 (1985), no. 1, pp. 99–105.

JAMES H. SCHMERL. *Large resplendent models generated by indiscernibles*. ***The Journal of Symbolic Logic***, vol. 54 (1989), no. 4, pp. 1382–1388.

JAMES H. SCHMERL. *Automorphism groups of models of Peano arithmetic*. ***The Journal of Symbolic Logic***, vol. 67 (2002), no. 4, pp. 1249–1264.

JAMES H. SCHMERL. *Diversity in substructures*. ***Nonstandard models of arithmetic and set theory.*** edited by A. Enayat and R. Kossak, Contemporary Mathematics, vol. 361, American Mathematical Societey (2004), pp. 45–161.

JAMES H. SCHMERL. *Generic automorphisms and graph coloring*. ***Discrete Mathematics***, vol. 291 (2005), no. 1–3, pp. 235–242.

JAMES H. SCHMERL. *Nondiversity in substructures*. ***The Journal of Symbolic Logic***, vol. 73 (2008), no. 1, pp. 193–211.

Since 1971 Jim Schmerl has authored and co-authored 32 papers on various topics in model theory of arithmetic. The list includes several important contributions and a number of smaller gems. It is impossible to cover it all in a single review. Instead I will describe and comment on the content of selected papers listed above.

Early results from the 1950's and 60's showed difficulties in constructing nonstandard models of PA and similar formal systems. The chief obstacle is the so called Tennenbaum phenomenon: there are no effectively presented nonstandard models of sufficiently strong arithmetic theories. This rules out standard algebraic constructions. To get a nonstandard model, one can use compactness applied either to (any consistent extension of) the theory $\mathsf{TA} = \mathrm{Th}(\mathbb{N}, +, \times) \cup \{c > n \colon n < \omega\}$, where $\mathbb{N}$ is the standard model and c is a new constant symbol, or to any consistent theory extending $\mathsf{PA} + {\sim}\varphi$, where φ is an arithmetic statement true in the standard model. To get more control, one can proceed as follows. Start with a completion T of PA. Take an effective enumeration $\varphi_n(x)$, $n < \omega$ of all formulas of the language of PA in one free variable. Define a complete type $p(x) = \{\varphi_n^{\epsilon_n}(x) \colon n < \omega\}$, where $\epsilon_n \in \{0, 1\}$ and $\varphi^1 = \varphi$ and $\varphi^0 = {\sim}\varphi$. Proceed by induction to make make sure that for each n, T proves that the set defined by $\bigwedge_{i \leq n} \varphi^{\epsilon_n}(x)$ satisfies a certain condition of largeness chosen in advance. Let M_T be the prime model of T. The type $p(x)$ determines a proper elementary extension of M_T defined on the set of Skolem terms $t(c)$, where c is a new constant, and $t(c) = s(c)$ iff the formula $t(x) = s(x)$ is in $p(x)$. There are many notions of largeness for a definable set X one can choose, but typically it is one of the two: X is unbounded, or X is bounded but of the (internal) size $> 2^{c-n}$ for a fixed nonstandard c and some standard n. In the first case one usually aims at an end extension of M_T and in the second at a cofinal extension. This routine procedure becomes an art once one builds in additional conditions into the notion of largeness. To carry on the construction of $p(x)$ one usually needs a combinatorial lemma of the form: if X is a definable large set, then every definable partition of $[X]^n$ into a small number of pieces has a large definable homogeneous set $Y \subseteq X$. One could say that most of model theory of PA and its extensions follows the pattern just described. There are many variations on this main theme, resulting in several more specialized approaches requiring advanced techniques

Model theory of arithmetic begins with the MacDowell–Specker Theorem. It states that every model of PA has an elementary end extension. Gaifman's technique of definable types provides a general method for constructing such extensions, and for iterating them. Gaifman noticed that all his results remain true for PA^*, which is Peano Arithmetic formulated for an arbitrary language (containing $+$ and $\times$), and as long as the language is countable, and he asked if the results remain true for uncountable languages. They do not, as was proved in 1978 by George Mills in *A model of Peano arithmetic with no elementary end extension*, ***The Journal of Symbolic Logic***, vol. 43 (1978), pp. 563–567. Schmerl's first paper on models of PA gives a strong result in the positive direction. It is also a classic paper, that carries all characteristic traits of Schmerl's future work. In a nutshell, the strength of the results rests on a skillful blend of standard model theoretic techniques with powerful combinatorics. In the first paper under review, Schmerl proves that if $M \models \mathsf{PA}$ and $\mathfrak{X}$ is a family of mutually Cohen-generic subsets of M, then $(M, \mathfrak{X}) \models \mathsf{PA}^*$ and $(M, \mathfrak{X})$ has a conservative elementary end extension. There are no restrictions on the cardinality of $\mathfrak{X}$. By abuse of notation, we consider $(M, \mathfrak{X})$ as a model of PA^* in the language augmented with predicate symbols for all sets in $\mathfrak{X}$. Cohen-generics in models of PA are defined analogously to the usual set-theoretic definition, using 0–1 sequence coded in M, and definable dense subsets of the set of all such sequences. In the proof, Schmerl defines a multi-variable type and shows its consistency by a combinatorial argument. However, the proof given in the paper is faulty (which is not a characteristic trait). The combinatorial theorem used is too weak. Everything is mended in the correction published in 1981 with the help of the Halpern–Laüchli–Laver–Pincus partition theorem for trees. Another typical aspect of the paper is the set-theoretic component. We need a couple definitions. A class of a model $M \models \mathsf{PA}$ is a subset A of M such that for every $a \in M$, $M \cap \{x \in M : x < a\}$ is definable in M. The generic dimension of a model M is defined as a unique[1] cardinal λ such that M has a set $\mathfrak{X}$ of mutual Cohen-generics of cardinality λ such that every class of M is definable in $(M, \mathfrak{X})$. Schmerl proves that if κ is a regular cardinal, λ is an infinite cardinal or $\lambda = 1$, and there is a κ-tree with exactly λ branches of length κ, then for each model $M \models \mathsf{PA}$ of cardinality less than κ there is a κ-like elementary end extension of M with generic dimension λ. The paper carries another characteristic feature: terseness. Schmerl's papers are not easy to read. They are very carefully written, elegant, and witty, but scarcity of details often makes them impenetrable. Here is one example. In most presentations in the literature the proof of the MacDowell–Specker Theorem takes at least two good pages. Here is Schmerl's proof:

> Let $\langle \varphi_i(x, y) : i < \omega \rangle$ be a list of all formulas of two variables. Then, using the fact that induction holds, we can find formulas $\psi_i(x)$ such that for each $n < \omega$ the following is a theorem of PA:
>
> $$(\exists x)(\forall x < w) \bigwedge_{i<n} (\varphi_i(x, y) \longleftrightarrow \psi_i(x)).$$
>
> Let M be a model of PA and let
>
> $$U = \{\{b \in M : M \models \varphi_i(a, b) \longleftrightarrow \psi_i(a)\} : a \in M \text{ and } i < \omega\}.$$
>
> Then U is an ultrafilter over the Boolean algebra of definable subsets of M. Clearly M^U is a proper elementary end extension of M.

A model is rather classless if all of its classes are definable. It is easy to see that there are no countable rather classless models of PA, and hence there are no rather classless resplendent models of PA. On the other hand, for every ordinal λ every model of PA has a canonical λ-extension $M(\lambda)$ generated over M by a set of indiscernibles order isomorphic to λ. Since models of PA are linearly ordered, by indiscernibles we mean order indiscernibles, and we no not distinguish between sets and sequences of indiscernibles. Schmerl shows in *Recursively saturated, rather classless models of Peano arithmetic* that if the cofinality of λ is uncountable, then $M(\lambda)$ is rather classless. The model $M(\lambda)$ is the union of a continuous chain of elementary submodels M_α, $\alpha < \lambda$ such that for $\alpha < \beta < \lambda$, $M_\alpha \prec_{\mathsf{end}} M_\beta$ and M_β is a conservative extension of M_α i.e., for each class X of M_β, $M_\alpha \cap X$ is definable in M_α. If $M \prec_{\mathsf{end}} N$ and N is recursively saturated then the extension is not conservative. Hence, the construction outlined above cannot yield a recursively saturated model. An ω_1-like, recursively saturated model of PA was first constructed by Matt Kaufmann in *A rather classless model*, ***Proceedings of American Mathematical Society***, vol. 62 (1977), pp. 330–333, with the use if $\Diamond$, which was later eliminated by Shelah with his $\Diamond$-eliminating technique.

[1]For finite λ the uniqueness can't be guaranteed. However, if $\lambda \in \{0, 1\}$, then it is unique.

Schmerl gives a more straightforward proof which also uses $\diamondsuit$, but instead of arguments from the admissible set theory (Barwise Compactness Theorem), he applies a construction similar to the one using canonical $\aleph_1$-extension described above, but not in the language of PA. Schmerl constructs an $\aleph_1$-chain of elementary extensions of structures of the form (M, S) where M is a countable recursively saturated model of PA and S is an inductive partial satisfaction class for M. This technique and its variations turned out to be very useful in other constructions, in particular it is the main tool in proving completeness theorems for PA with Ramsey quantifiers (Schmerl, Simpson) and with the stationary quantifier (Schmerl). In the Logic Year 79-80 paper, Schmerl goes further and considers recursively saturated rather classless models of higher cardinalities. He proves, among other results, that if $\kappa > \mathrm{cf}(\kappa) > \aleph_0$, then every consistent extension of PA* has a κ-like recursively saturated, rather classless model, and if κ is a regular cardinal, and PA has a κ-like, recursively saturated, rather classless model, then there is an Aronszajn κ-tree.

It is mildly surprising, but not that difficult to prove, that there are recursively saturated models of PA which are generated by indiscernibles. Answering questions of Julia Knight and Angus Macintyre, Schmerl proves in *Recursively saturated models generated by indiscernibles* that *every* countable model of PA is generated by a set of indiscernibles. Moreover the set of indiscernibles can be of any countable order type without a last element.[2] The proof is rather involved. The elements of the generating set of indiscernibles cannot realize an indiscernible type.[3] Instead, for a given countable recursively saturated model M, Schmerl defines an ω-type $p(x_0, x_1, \dots)$ such that in any model elements realizing the type form an indiscernible set generating a model isomorphic to M. The proof of consitency of p is based on an iterated construction using chronic resplendency and a combinatorial Ramsey-like argument. In the follow-up paper *Large resplendent models generated by indiscernibles* Schmerl uses a similar, somewhat simplified argument, to show that for every countable recursively saturated model M and every infinite cardinal κ, there is a resplendent model N such that $M \equiv_{\infty,\omega} N$ and $|N| = \kappa$. The proofs in both papers are rather sketchy. Richard Kaye wrote a modified, much more detailed version unifying both results in *Indiscernibles*, ***Automorphisms of first-order structures***, Oxford University Press, (1994), pp. 257–279.

Since every countable recursively saturated model of PA is generated by a densely ordered set of indiscernibles, it follows that the automorphism group of any countable linearly ordered set can be embedded into the automorphism group of any countable recursively saturated model of PA. In *Automorphism groups of models of Peano arithmetic* Schmerl gives the ultimate answer to the question: Which groups can be represented as automorphism group s of models of PA? The main theorem of the paper says that if G is a automorphism group of a linearly ordered first-order structure $(\mathfrak{A}, <, \dots)$, then there is a model $M \models$ PA such that $G \cong \mathrm{Aut}(M)$. There is also a variant of this result which gives an interesting insight into the structure of countable recursively saturated models of PA. If $(\mathfrak{A}, <, \dots)$ and M is countable recursively saturated model of PA, then M has an elementary cut K such that the group of those automorphism s of K which have extensions in $\mathrm{Aut}(M)$ is isomorphic to G. Both theorems rely on a combinatorial result proved independently by Abramson–Harrington and Nešetřil–Rödl: Suppose $\mathfrak{A}$ is a linearly ordered finite relational $\mathcal{L}$-structure. Then there is a finite linearly ordered $\mathcal{L}$-structure $\mathfrak{B}$ such that whenever $f\colon [B]^{<\omega} \longrightarrow \{0, 1\}$, then there is a $\mathfrak{A}'$ such that $\mathfrak{A}' \cong \mathfrak{A}$ and f is homogeneous on $\mathfrak{A}'$. A function $f\colon [A]^{<\omega} \longrightarrow \{0, 1\}$ is homogeneous if whenever $X, Y \subseteq A$ and $\mathfrak{A} \restriction X \cong \mathfrak{A} \restriction Y$, then $f(X) = f(Y)$.

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[2]The result is more general, PA can be replaced by any theory admitting uniform coding of finite sequences.

[3]A 1-type $p(x)$ is indiscernible if it is nonprincipal and in any model M, if $I \subseteq M$ is a set of elements each realizing $p(x)$, then I is a set of indiscernibles. Minimal types of Gaifman are indiscernible and they do exist for any completion of PA* in a countable language.

One of my favorite theorems of Schmerl is from *Generic automorphisms and graph coloring*. It gives an example of a purely model theoretic statement which is, in a sense, equivalent to a combinatorial one. The question, motivated by the study of the small index property of recursively saturated models of PA, is: Suppose that $M \models \mathsf{PA}$ is countable and recursively saturated, is there an automorphism of M whose conjugacy class is dense in $\mathrm{Aut}(M)$? Density here is in the sense of the standard topology on $\mathrm{Aut}(M)$. Basic open subgroups are pointwise stabilizers of finite sets. If M is elementarily equivalent to the standard model, then, the answer is positive. However if M is not a model of True Arithmetic, then the existence of such an automorphism is equivalent to the statement: There is $k < \omega$ such that for all $D_1, D_2 \in M$

$$M \models [\chi(D_1) \geq k \wedge \chi(D_2) \geq k] \longrightarrow \chi(D_1 \times D_2) \geq 4.$$

Here D_1 and D_2 are considered as (codes of) finite directed graphs; $\chi(D)$ is the chromatic number of D; and $D_1 \times D_2$ is the directed graph whose vertices are ordered pairs (v_1, v_2), with $v_1 \in D_1$ and $v_2 \in D_2$. Two pairs form and edge in $D_1 \times D_2$ iff their first and second coordinates form edges in D_1 and D_2 respectively. The statement $\forall D_1, D_2[\chi(D_1) \geq k \wedge \chi(D_2) \geq k] \longrightarrow \chi(D_1 \times D_2) \geq 4]$ is a variant of an open problem known as the Hedetniemi's conjecture.

Elementary sumbodels of a model of PA form a lattice $\mathrm{Lt}(M)$. The lattice ordering is that of inclusion. If $M \models \mathsf{PA}$ and $M \prec N$, then $\mathrm{Lt}(N/M)$ is the sublattice of $\mathrm{Lt}(N)$ consisting of K such that $M \preccurlyeq K \preccurlyeq N$. $\mathrm{Lt}(N/M)$ is called the *interstructure lattice*. Gaifman begun a systematic study of substructure lattices for models of PA in *Models and types of Peano's arithmetic*, ***Annals of Mathematical Logic***, vol. 9 (1976), pp. 223–306. Paris showed examples of non-distributive lattices in *Models of arithmetic and the* $1-3-1$ *lattice*, ***Fundamenta Mathematicae***, vol. 95 (1977), pp. 195–199, and Wilkie, in *On models of arithmetic having non-modular substructure lattices*, ***Fundamenta Mathematicae*** vol. 95, (1977), pp. 223–237, proved that substructure lattices do not have to be modular, by constructing a model with the pentagon substructure lattice. In the following year, Schmerl in *Extending models of arithmetic*, ***Annals of Mathematicall Logic***, vol. 14 (1978), pp. 89–109, answering a question of Gaifman, solved completely the problem of representations of finite distributive lattices, by proving that if $(D, <)$ is a finite distributive lattice, then for every completion T of PA, there is a definable type $t(v)$ of T, such that whenever M is a model of T and N is a $t(v)$-extension of M, then $\mathrm{Lt}(N/M) \cong (D, <)$.

Not every infinite lattice can be represented as $\mathrm{Lt}(N/M)$. To be represented in this form the lattice must be $\aleph_1$-*algebraic*.[4] The question whether all $\aleph_1$-algebraic distributive lattices can be represented as substructure lattice of model of PA was answered affirmatively by Mills in *Substructure lattices of models of arithmetic*, ***Annals of Mathematical Logic***, vol. 16 (1979), pp. 145–180. Many questions remain open in the non-distributive case. It is not even known whether there is a finite lattice that cannot be represented as a substructure lattice. The smallest candidate for such a lattice at present is $\boldsymbol{M}_{16}$, the 18-element lattice with 16 incomparable elements.

Since the 1980's, Schmerl has been working singlehandedly on substructure representations of finite lattices. He has developed a technique based on representations of lattices as equivalence relations lattices with the so called Canonical Partition Property (CPP). Using this technique he proved several general theorems on representations of certain classes of finite lattices (including some types of non-distributive ones) as interstructure lattices in cofinal extensions. This is highly technical work using nontrivial lattice theory and combinatorics. It deserves a separate discussion. A comprehensive account is given in ***The Structure of***

[4]Let $(L, \vee, \wedge)$ be a lattice. An element $a \in L$ is *compact* if whenever $X \subseteq L$ and $a \leq \bigvee X$, then there is a finite $Y \subseteq X$ such $a \leq \bigvee Y$. Compact elements of $\mathrm{Lt}(N/M)$ are the finitely generated submodels. L is *algebraic* if it is complete, and every element of L is a supremum of a set of compact elements. L is κ-*algebraic*, if it is algebraic and for every a in L there are fewer than κ compact elements below a in L.

Models of Peano Arithmetic by Kossak and Schmerl, Oxford Logic Guides 50, Clarendon Press 2006, but there are also some more recent interesting developments. In the traditional substructure lattice representation results one did not care much about the isomorphism types of the substructures involved. However, motivated by the techniques involved, it makes perfect sense to ask about the spectrum of isomorphism types of elementary submodels in representations of particular lattices. In *Diversity in substructures* Schmerl defines a model M to be *diverse* if no two of its elementary substructures are isomorphic. He proves that if $M \models \mathsf{PA}$ is countable model and L is a finite lattice such that $L \cong \mathrm{Lt}(N/M)$, then M has a cofinal diverse extension N such that $\mathrm{Lt}(N/M) \cong L$. Diversity is an extreme property. One can ask similar questions about non-diverse extensions, or even more specifically, about extensions with prescribed isomorphism relation among its elementary substructures. Schmerl discusses this question in *Diversity in substructures* and *Nondiversity in substructures*. He considers the *equivalenced lattices* $\mathrm{Lt}^+(M) = \mathrm{Lt}(M, \cong_M)$, where $\cong_M$ is the isomorphism relation. $\mathrm{Lt}^+(M)$ has several properties summarized in the notion of *reasonableness*. Schmerl proves that if (L, E) is a finite reasonable equivalenced distributive lattice and M is a countable nonstandard model of PA, then M has a cofinal extension N such that $\mathrm{Lt}^+(N/M) \cong (L, E)$. It is a beautiful theorem, and its proof is quite involved. Schmerl proceeds by defining the notion of an ordered category of finite relational structures. If $(\mathbb{C}, \leq)$ is such a category and $A, C \in \mathbb{C}$, then $\left[\begin{smallmatrix} C \\ A \end{smallmatrix}\right]$ denotes the set $\{B \in \mathbb{C} \colon B \leq C \wedge B \cong A\}$. The category $(\mathbb{C}, \leq)$ has the *Ramsey Property* if for any $A, B \in \mathbb{C}$ there is $C \in \mathbb{C}$ such for every coloring $\chi \colon \left[\begin{smallmatrix} C \\ A \end{smallmatrix}\right] \longrightarrow 2$, there is $B' \in \left[\begin{smallmatrix} C \\ B \end{smallmatrix}\right]$ such that χ is constant on $\left[\begin{smallmatrix} B' \\ A \end{smallmatrix}\right]$. A refinement of this, in the form of the Canonical Induced Ramsey Property, and a Fraïssé limit construction are applied to prove the combinatorial lemmas needed in the general framework of Schmerl's CPP-representation theory, to get the desired model. The combinatorial theorem of Abramson–Harrington and Nešetřil–Rödl is used in the proof again.

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W. Kubiś and V. Uspenskij. *A compact group which is not Valdivia compact*. ***Proceedings of the American Mathematical Society***, vol. 133 (2005), no. 8, pp. 2483–2487.

W. Kubiś and H. Michalewski. *Small Valdivia compact spaces*. ***Topology and its Applications***, vol. 153 (2006), no. 14, pp. 2560–2573.

M. Burke and W. Kubiś and S. Todorčević. *Kadec norms on spaces of continuous functions*. ***Serdica. Mathematical Journal***, vol. 32 (2006), no. 2–3, pp. 227–258.

W. Kubiś. *Compact spaces generated by retractions*. ***Topology and its Applications***, vol. 153, (2006), no. 18, pp. 3383–3396.

An inverse system is a sequence S of mappings $p^{\beta}_{\alpha} \colon X_{\beta} \to X_{\alpha}, \alpha < \beta < \delta$ for some limit ordinal δ such that $\alpha < \beta < \gamma \Longrightarrow p^{\beta}_{\alpha} p^{\gamma}_{\beta} = p^{\gamma}_{\beta}$. The maps p^{β}_{α} are called the bonding mappings of S and the inverse limit of S is the subset of the product $\Pi_{\alpha<\delta} X_{\alpha}$ consisting of all x such that $p^{\beta}_{\alpha}(x(\beta)) = x(\alpha)$ for all $\alpha < \beta < \delta$. If each X_{α} is a compact space then the inverse limit is nonempty. Such systems have been studied extensively in set-theoretic topology, where most often $X_{\beta} \subseteq 2^{\beta}$ and the bonding maps are the natural projections. This concept has been around for a long time and it has been used by many people, and studied systematically by several authors. It is therefore a very pleasant surprise to have a new way of looking at this concept, allowing for the solution of a number of problems on Valdivia compacta, which is a class of compact spaces inspired by functional analysis, containing Corson compacta and dyadic spaces.

In *Small Valdivia compact spaces* it is shown that a compact space of weight $\leq \aleph_1$ is Valdivia compact iff it is the limit of an inverse sequence of metric compacta whose bonding maps are retractions. (A retraction is a continuous map which has a right inverse.) A corollary of this is that the class of Valdivia compacta of weight at most $\aleph_1$ is preserved both under retractions and under open 0-dimensional images, which answered quite a well known open question in the theory of Valdivia compacta. However, it was proved in *A compact group which is not Valdivia compact* that Valdivia compacta are generally not preserved by open continuous images—here a rather short elegant argument was based on the cohomology theory of compact connected abelian groups.

Every Valdivia compact space K has many retractions, which for instance implies that the Banach space $C(K)$ admits the so called projectional resolution of the identity (PRI, i.e., a comuting system of projections onto small subspaces). *Compact spaces generated by retractions* gives a counterexample to several questions on PRI (due to Montesinos and Kalenda) by considering a certain linear order which gives rise to a Valdivia compactum. In particular, it was shown that the property of Banach spaces having PRI is not hereditary.

A norm on a Banach space is called Kadec if the norm and the weak topology coincide on the unit sphere. It is a classical question in Banach space theory to study the class of Banach spaces that have a renorming which is Kadec. In *Kadec norms on spaces of continuous functions* the classical results about this class are greatly extended, as it is for example shown that if $C(K_1)$ has a pointwise Kadec renorming and K_2 belongs to the class of spaces obtained by closing the class of compact metrizable spaces under inverse limits of transfinite continuous sequences of retractions, then $C(K_1 \times K_2)$ has a pointwise Kadec renorming.

The new insight provided by these and other papers by Kubiś on this subject is the consideration of inverse systems in terms of category theory, on a quite abstract level. The above examples show how these abstract logical methods can then be applied to solve some very concrete problems asked outside of mathematical logic. In fact, the methods emerging in the recent close interactions of Banach space theory with mathematcal logic involve set theory, model theory and category theory, often a mixture of all. They provide an exciting new way to successfuly use foundations of mathematics to attack problems in mathematical analysis, where the classical methods of logic, such as first order logic, fail.

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