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AUTOMATIC CONTINUITY OF GROUP HOMOMORPHISMS

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Abstract. We survey various aspects of the problem of automatic continuity of homomorphisms between Polish groups.

CONTENTS

§1. Introduction. The questions that we consider here come out of the work on a very old question of Cauchy, namely,

PROBLEM 1.1 (A. L. Cauchy). *What are the functions* $\pi\colon \mathbb{R} \to \mathbb{R}$ *that satisfy the functional equation*

$$\pi(x + y) = \pi(x) + \pi(y)?$$

In modern terminology, this is of course just asking for a characterisation of endomorphisms of the additive group $\mathbb{R}$. The motivation of Cauchy was to know whether the only such functions are given by $\pi(x) = rx$ for

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some fixed $r \in \mathbb{R}$. He proved himself that any continuous solution is of the form $\pi(x) = rx$ and this was later extended by J. G. Darboux to functions continuous in a single point.

We shall be interested in a more general question which extends the consideration to general Polish groups, i.e., separable topological groups whose topology can be given by a complete metric. Though many of the results mentioned are not really specific to this case, we shall nevertheless stick to this setting as it encompasses what we feel are the most important examples, and since we do not want to get bogged down in excessively detailed hypotheses.

The most general form of the question considered is simply:

> *When is a homomorphism $\pi\colon G \to H$ between Polish groups continuous?*

Since this question is admittedly extremely vague, we shall specify some more concrete subproblems that we treat individually. Though some of the techniques will be similar for several of the problems, they are also very diverse and therefore deserve separate attention. Moreover, the topics touched on here represent the author's personal selection and should therefore not be considered complete in any way, but it does represent a good spectrum of the work on these problems. However, we have mostly avoided results that require specific analysis of individual groups or heavy handed combinatorial proofs and instead focused on more general techniques. Also, we shall say nothing about automatic continuity in the context of Banach algebras. This is a huge area in itself and differs a lot from our topic here, as it mixes the multiplicative and linear structure of algebras. For more information on this, one can consult the massive volume of H. G. Dales [7].

Another important topic, that is completely left out here, is that of applications of automatic continuity. While this is certainly of vital importance to motivate our study, we leave it for the reader to follow up on the references. Let us just mention that many of the individual results presented here are motivated by various specific questions concerning the algebraic structure of topological groups, phenomena of rigidity in ergodic theory, geometry and model theory and also by applications to the dynamics of large Polish groups. Moreover, we feel that the topic has an intrinsic interest as a framework for studying the tight connections between algebraic and topological structure of Polish groups.

Before delving deeper into the theory, let us note that the general form of Cauchy's question is actually non-trivial by exhibiting some discontinuous homomorphisms between Polish groups. As will be shown later, in any such example at least a certain amount of the axiom of choice is necessary, so in order to find examples one will have to consider in which way to get choice to bear on the groups in question. We essentially have five different examples, though some of these might also be considered to belong to the same category. These will also indicate the limitations of possible positive results in later sections.

EXAMPLE 1.2. *Discontinuous functionals ϕ on a separable infinite-dimensional Banach space X.*

To construct such ϕ, choose a basis $\{x_i\}_{i\in I}$ of X as an $\mathbb{R}$-vector space such that $\{x_i\}_{i\in I}$ is dense in X and define

$$\phi\Big(\sum_{i\in I} a_i x_i\Big) = a_{i_0}$$

for some fixed $i_0 \in I$. By density, we can find $j_n \in I \setminus \{i_0\}$ such that $x_{j_n} \to x_{i_0}$. Then ϕ is discontinuous since

$$\phi(x_{i_0}) = 1, \text{ while } \phi(x_{j_n}) = 0.$$

EXAMPLE 1.3. *An isomorphism of say* $(\mathbb{R}, +)$ *and* $(\mathbb{R}^2, +)$.

This can be constructed using a Hamel basis. I.e., if $\mathcal{A}$ and $\mathcal{B}$ are bases for $\mathbb{R}$ and $\mathbb{R}^2$ as $\mathbb{Q}$-vector spaces, then $|\mathcal{A}| = 2^{\aleph_0} = |\mathcal{B}|$ and thus $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic as $\mathbb{Q}$-vector spaces. Since the $\mathbb{Q}$-vector space structure encompasses the group structure, $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic as groups. But not topologically.

Before stating the next example, let us note that if G is a Polish group, any open subgroup $H \leq G$ is also closed. For if H is open, so are all its cosets gH, and so the complement $\sim\!H$ is a union of open cosets and hence open too. Also, if $H \leq G$ is a closed subgroup of countable index, then H is open. For by Baire's Theorem, H cannot be meagre and hence, by the Banach–Pettis Theorem (see Lemma 2.1 below),

$$1 \in \text{Int}\,(H^{-1}H) = \text{Int}\,H.$$

So, as H is homogeneous, H is open.

Let also S_∞ be the *infinite symmetric* group, i.e., the group of all permutations of $\mathbb{N}$ (not just finitely supported), where the topology has as subbasis the sets of the form

$$\{g \in S_\infty \mid g(n) = m\}$$

for all $n, m \in \mathbb{N}$. S_∞ is a Polish group.

Now, if $H \leq G$ is a non-open/closed subgroup of *denumerable* (i.e., countably infinite) *index*, then we can define a discontinuous homomorphism

$$\pi\colon G \to S_\infty,$$

as follows. First, since the set G/H of left cosets is countable, we can see S_∞ as the group $\text{Sym}(G/H)$ of all permutations of G/H. Now set $\pi(f) = L_f \in \text{Sym}(G/H)$, where

$$L_f(gH) = fgH.$$

But, $L_f(1H) = 1H$ if and only if $f \in H$, and hence

$$\pi^{-1}(\{\alpha \in \mathrm{Sym}(G/H) \mid \alpha(1H) = 1H\}) = H,$$

which is not open. So π is discontinuous.

Of course, if $H \leq G$ is a non-open subgroup of *finite* index, it is even easier to construct discontinuous homomorphisms defined on G. For by a theorem of Poincaré, there is a finite index normal subgroup $N \trianglelefteq G$ contained in H. Now, N cannot be open as otherwise H would be open too, being the union of finitely many cosets of N. This implies that the quotient mapping $\pi\colon G \to G/N$ is discontinuous, where G/N is taken discrete. Therefore, in order to produce discontinuous homomorphisms on a Polish group, it suffices to find non-open subgroups of countable index. We shall now see some examples of these.

EXAMPLE 1.4. *Discontinuous homomorphisms from infinite powers of finite groups.*

Suppose that F is any finite group $\neq \{1\}$. Then we construct a non-open subgroup of finite index in $F^{\mathbb{N}}$ as follows: Fix a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and define a normal subgroup $H \trianglelefteq F^{\mathbb{N}}$ of index $|F|$ by

$$H = \{(f_n)_{n\in\mathbb{N}} \mid \{n \in \mathbb{N} \mid f_n = 1\} \in \mathcal{U}\}.$$

Since, $\mathcal{U}$ is non-principal, H is dense in $F^{\mathbb{N}}$ and thus fails to be open. Therefore, the mapping $\pi\colon F^{\mathbb{N}} \to F^{\mathbb{N}}/H$ is discontinuous. For example, the complement of a non-principal ultrafilter on $\mathbb{N}$ is itself a non-open subgroup of finite index in the Cantor group $(\mathbb{Z}_2)^{\mathbb{N}}$.

EXAMPLE 1.5 (S. Thomas [32], R. R. Kallman [15]). *Some matrix groups, e.g., $SO_3(\mathbb{R})$, embed discontinuously into S_∞.*

EXAMPLE 1.6. *Infinite compact Polish Abelian groups have non-open subgroups of countable index.*

This follows from the fact that infinite Abelian groups have subgroups of denumerable index. And, of course, if H is compact and $K \leq H$ has denumerable index, then K cannot be open, since otherwise the covering of H by left cosets of K would have no finite subcovering.

§2. Measurable homomorphisms.

To further specify our main problem, we begin by considering what happens when placing restrictions on the type of homomorphisms.

> *If $\pi\colon G \to H$ is a homomorphism between Polish groups that is definable or is assumed to have various regularity properties, is π continuous?*

We recall that if A is a subset of a Polish space X,

- A has the *Baire property* if it differs from a Borel set by a meagre set,
- A is *universally measurable* if for any Borel probability measure μ on X, A differs from a Borel set by a set of μ-measure zero.

In the second case, there is unfortunately no reason to believe that we can use the same Borel set for all measures. This fact is the root of much evil.

Similarly, a map $\pi\colon X \to Y$ between Polish spaces is

- *Baire measurable* if $\pi^{-1}(V)$ has the Baire property for every open $V \subseteq Y$,
- *universally measurable* if $\pi^{-1}(V)$ universally measurable for every open $V \subseteq Y$.

2.1. The case of category. For Baire measurable homomorphisms, the question is simple and was fully solved by B. J. Pettis [22] building on a previous result by S. Banach.

LEMMA 2.1 (The Banach–Pettis Theorem [22]). *Suppose G is a Polish group and $A, B \subseteq G$ are subsets. Let $U(A)$ and $U(B)$ be the largest open subsets of G in which A, resp. B, are comeagre. Then*

$$U(A) \cdot U(B) \subseteq AB.$$

PROOF. We note that if $x \in U(A)U(B)$, then the open set $V = xU(B)^{-1} \cap U(A) = U(xB^{-1}) \cap U(A)$ is non-empty and so xB^{-1} and A are comeagre in V. It follows that $xB^{-1} \cap A \neq \emptyset$, whereby $x \in AB$. ⊣

THEOREM 2.2 (S. Banach–B. J. Pettis [22]). *Any Baire measurable homomorphism $\pi\colon G \to H$ between Polish groups is continuous.*

PROOF. It is enough to prove that π is continuous at 1, i.e., that for any open $V \ni 1$ in H, $\pi^{-1}(V)$ is a neigbourhood of 1 in G. So suppose $1 \in V \subseteq H$ is given and find an open set $W \in 1$ such that $WW^{-1} \subseteq V$. Then $\pi^{-1}(W)$ is non-meagre, as it covers G by countably many left translates, and also has the Baire property. Thus, $U(\pi^{-1}(W))$ is non-empty open, and hence by Lemma 2.1

$$1_G \in U(\pi^{-1}(W))U(\pi^{-1}(W))^{-1} \subseteq \pi^{-1}(W)\pi^{-1}(W)^{-1} \subseteq \pi^{-1}(V),$$

whereby $1 \in \mathrm{Int}(\pi^{-1}(V))$. ⊣

In particular, this applies to Borel measurable homomorphisms, which thereby are all continuous. On another note, by results of R. M. Solovay [31] and S. Shelah [28], it is known to be consistent with ZF that all sets of reals are Baire measurable, which implies that all subsets of Polish groups are Baire measurable. Therefore, by Lemma 2.1, it is consistent with ZF that all homomorphisms between Polish groups are continuous. Thus, in order to produce discontinuous homomorphisms the axiom of choice must intervene in some fashion, e.g., via the existence of ultrafilters, Hamel bases, etc.

2.2. The case of measure. Around the beginning of the 20th century, several people were working on Cauchy's problem and on finding additional assumptions that would imply that a solution be continuous. In an article published in 1915, M. Fréchet eventually proved that any Lebesgue measurable solution is continuous, and in the very first issue of Fundamenta

Mathematicae from 1920 there were no less than three articles by respectively S. Banach, W. Sierpiński and H. Steinhaus providing alternative proofs or treating the finer aspects of this question. Steinhaus' article, which seems to have had the biggest impact, improved on an earlier result result of Sierpiński stating that for any measurable non-null sets $A, B \subseteq \mathbb{R}$, $(A - B) \cap \mathbb{Q} \neq \emptyset$, by proving that $A - B$ actually contains an open interval. And later, with the advent of abstract harmonic analysis in the 1930's, Steinhaus' result was further extended by A. Weil to all locally compact Polish groups.

Theorem 2.3 (H. Steinhaus–A. Weil). *Suppose G is a locally compact Polish group with (left) Haar measure λ. Then for any λ-measurable set of positive measure, $A \subseteq G$, we have that*

$$AA^{-1}$$

is a neighbourhood of 1.

Proof. By inner and outer regularity there are a compact set K and an open set U such that $K \subseteq A \subseteq U$ and $\lambda(U) < 2\lambda(K)$. Now pick some open neighbourhood V of 1 such that $VK \subseteq U$. Then if $g \in V$, gK is a subset of U of measure $\lambda(K)$ and so $K \cap gK \neq \emptyset$, whereby $g \in KK^{-1} \subseteq AA^{-1}$. So $V \subseteq AA^{-1}$. ⊣

By a trivial adaptation of the proof of Theorem 2.2, we get:

Corollary 2.4. *Any universally measurable homomorphism from a locally compact Polish group G into a Polish group H is continuous.*

Unfortunately, locally compact groups are about as far as this argument goes. For a simple argument due to Weil shows that locally Polish groups are the only ones that carry non-zero, (quasi-)invariant, σ-finite Borel measures. To see this, notice that if G is a Polish group with such a measure λ, then by inner regularity and σ-finiteness there is a K_σ-subset $M \subseteq G$ such that $\lambda(G \setminus M) = 0$. Therefore, if g is any element of G, we have $\lambda(gM) > 0$ and so $gM \cap M \neq \emptyset$, whereby $g \in MM^{-1}$. Thus, $G = MM^{-1}$ is a K_σ Polish group and hence, by Baire's Category Theorem, a locally compact group.

Therefore, in order to deal with universally measurable sets in arbitrary Polish groups, one will need different tools. One approach, that now seems to be universally favoured, is due to J. P. R. Christensen [4, 5, 6], who noticed that though one cannot hope for an invariant measure, at least one can find an invariant notion of null set. Apart from its uses in automatic continuity, Christensen's definition and results have proved extremely useful in the literature on dynamical systems (see, e.g., W. Ott and J. A. Yorke [21]), where it used as a measure of smallness in various infinite-dimensional function spaces.

Definition 2.5 (J. P. R. Christensen [4]). *Let G be a Polish group and $A \subseteq G$ a universally measurable subset. We say that A is* Haar null *if there is a Borel probability (or equivalently, non-zero σ-finite) measure μ on*

G such that for all $g, h \in G$,

$$\mu(gAh) = 0.$$

Also, A is left Haar null *if for all $g \in G$,*

$$\mu(gA) = 0,$$

and similarly for right Haar null.

(1) We note that A is left Haar null if and only if A^{-1} is right Haar null. For if μ witnesses that A is left Haar null, define ν by

$$\nu(B) = \mu(B^{-1}).$$

Then for any $g \in G$,

$$\nu(A^{-1}g) = \mu((A^{-1}g)^{-1}) = \mu(g^{-1}A) = 0.$$

So ν witnesses that A^{-1} is right Haar null.

(2) Let us recall how to convolve two measures μ and ν on a Polish group G: The convolution $\mu * \nu$ is the ν-average of right-sided translates of μ, or equivalently, the μ-average of left-sided translates of ν, i.e.,

$$\begin{aligned}
\mu * \nu(A) &= \int \mu(Ay^{-1})\, d\nu(y) \\
&= \int \nu(x^{-1}A)\, d\mu(x) \\
&= \int\int \chi_A(xy)\, d\mu(x)\, d\nu(y) \\
&= \mu \times \nu(\{(x,y) \in G^2 \mid xy \in A\}).
\end{aligned}$$

For example, $\mu * \delta_x = \mu(\cdot\, x^{-1})$.

(3) Using this, we show that if $A \subseteq G$ is left Haar null, as witnessed by ν, and right Haar null, as witnessed by μ, then $\mu * \nu$ simultaneously witnesses that A is left and right Haar null. For if $g \in G$, then

$$\begin{aligned}
\mu * \nu(gA) &= \int \nu(x^{-1}gA)\, d\mu(x) \\
&= 0 \\
&= \int \mu(Agy^{-1})\, d\nu(y) \\
&= \mu * \nu(Ag).
\end{aligned}$$

Nevertheless, the property of being Haar null is in general much stronger than being simultaneously left and right Haar null (we shall come back to examples of this later).

(4) Note however that if A is left Haar null, then any two-sided translate gAh is left Haar null. For if μ witnesses that A is left Haar null, then the translated measure $\mu * \delta_h = \mu(\cdot\, h^{-1})$ witnesses that gAh is left Haar null.

Fortunately, in locally compact groups, the situation is simpler as Haar null sets coincide with sets of Haar measure zero.

LEMMA 2.6. *Suppose G is a locally compact Polish group with left and right Haar measure λ and ρ. Then the following are equivalent for a universally measurable subset $A \subseteq G$.*

(1) *A is Haar null,*
(2) *A is left Haar null,*
(3) $\lambda(A) = 0$,
(4) $\rho(A) = 0$.

Let us recall that for a measure μ and a property P, we write $\forall^\mu x\, P(x)$ to denote that the set $\{x \mid P(x)\}$ has full μ-measure.

PROOF. Suppose A is left Haar null as witnessed by μ. Then, as

$$\forall^\rho x\, \mu(x^{-1}A) = 0,$$

we have by Fubini's Theorem

$$\rho \times \mu\big(\{(x, y) \in G^2 \mid xy \in A\}\big) = 0,$$

and so, by Fubini again,

$$\forall^\mu y\, \rho(Ay^{-1}) = 0.$$

By right invariance of ρ, it follows that $\rho(A) = 0$, and, as $\rho \sim \lambda$, we conclude $\lambda(A) = 0$.

On the other hand, if $\lambda(A) = 0$, then for all $g \in G$, $\lambda(gA) = 0$ and so, as $\rho \sim \lambda$, also $\rho(gA) = 0$. By right invariance of ρ, we thus have $\rho(gAf) = 0$ for all $g, f \in G$, so A is Haar null. ⊣

So the question that immediately poses itself is which type of Haar null set to work with, Haar null or left Haar null. Of course in locally compact or Abelian groups, the two notions are the same, so only in the context of more complicated groups does this problem appear. As we shall see below, the class of Haar null sets is in general more well behaved than the class of left Haar null sets, as the former forms a σ-ideal, while the latter fails, in general, even to be closed under finite unions. Of course, for the applications to automatic continuity, the main issue is rather whether we have an analogue of the Steinhaus–Weil Theorem. I.e., if A is universally measurable, but not left Haar null, does $A^{-1}A$ contain a neighbourhood of 1? In this generality the answer turns out to be negative and instead we shall present some alternative approaches to the question of automatic continuity for specific groups due to Christensen, S. Solecki and the author.

Our first result is a slight variation of a result by Christensen.

THEOREM 2.7 (J. P. R. Christensen [4]). *Suppose G is a Polish group and $A \subseteq G$ is a universally measurable subset which is not right Haar null. Then* *for any neighbourhood W of 1 there are finitely many $h_1, \ldots, h_n \in W$ such that*

$$h_1 AA^{-1} h_1^{-1} \cup \cdots \cup h_n AA^{-1} h_n^{-1}$$

is a neighbourhood of the identity.

PROOF. Suppose that the conclusion fails for A, i.e., for all $h_1, \ldots, h_n \in W$ and any neighbourhood $V \ni 1$, there is some

$$g \in V \setminus \left(h_1^{-1} AA^{-1} h_1 \cup \cdots \cup h_n^{-1} AA^{-1} h_n \right).$$

Then we can inductively choose $g_0, g_1, g_2, \cdots \to 1$ in W such that for all $i_0 < i_1 < i_2 < \cdots$ and n,

(a) the infinite product $g_{i_0} g_{i_1} g_{i_2} \ldots$ converges fast,
(b) $g_{i_n} \notin (g_{i_0} \ldots g_{i_{n-1}})^{-1} AA^{-1} (g_{i_0} \ldots g_{i_{n-1}})$.

Using (a), we can define a continuous map $\phi \colon 2^{\mathbb{N}} \to G$ by

$$\phi(\alpha) = g_0^{\alpha(0)} g_1^{\alpha(1)} g_2^{\alpha(2)} \ldots,$$

where $g^0 = 1$ and $g^1 = g$.

Now let λ be Haar measure on the Cantor group $(\mathbb{Z}_2)^\omega = 2^{\mathbb{N}}$ and notice that, as A is not right Haar null, there is some $f \in G$ such that

$$\lambda(\phi^{-1}(Af)) = \phi_* \lambda(Af) > 0.$$

So by Weil's Theorem,

$$\phi^{-1}(Af) \phi^{-1}(Af)^{-1}$$

is a neighbourhood of the identity 0^ω in $2^{\mathbb{N}}$. In particular, there are two elements α and β differing in exactly one coordinate, say $\alpha(n) = 1$ and $\beta(n) = 0$, such that

$$\phi(\alpha) = hg_n k, \phi(\beta) = hk \in Af,$$

where $h = g_{i_0} g_{i_1} \ldots g_{i_l}$, $i_0 < i_1 < \cdots < i_l < n$, and $k \in G$. It follows that

$$hg_n h^{-1} = hg_n k \cdot k^{-1} h^{-1} \in Aff^{-1} A^{-1} = AA^{-1}$$

and so

$$g_n \in h^{-1} AA^{-1} h = (g_{i_0} g_{i_1} \ldots g_{i_l})^{-1} AA^{-1} (g_{i_0} g_{i_1} \ldots g_{i_l}),$$

contradicting the choice of g_n. ⊣

COROLLARY 2.8 (J. P. R. Christensen [4]). *Suppose $\pi \colon G \to H$ is a universally measurable homomorphism from a Polish group G to a Polish group H, where H admits a 2-sided invariant metric compatible with its topology. Then π is continuous.*

This happens when, for example, H is Abelian, compact or a countable direct product of discrete groups.

PROOF. As always, it is enough to prove that π is continuous at 1. So suppose that $V \ni 1$ is any neighbourhood of 1 in H and find a smaller *conjugacy invariant* neighbourhood $U \ni 1$ such that $UU^{-1} \subseteq V$. Now, $\pi^{-1}(U)$ covers G by countably many right translates, so it fails to be right Haar null. Therefore, by Theorem 2.7, there are $h_1, \dots, h_n \in G$ such that

$$\bigcup_{i=1}^{n} h_i \pi^{-1}(U)\pi^{-1}(U)^{-1} h_i^{-1}$$

contains a neighbourhood W of 1 in G. It follows that for $g \in W$, there is i such that

$$\begin{aligned}\pi(g) &\in \pi(h_i)UU^{-1}\pi(h_i)^{-1}\\ &= \pi(h_i)U\pi(h_i)^{-1} \cdot \pi(h_i)U^{-1}\pi(h_i)^{-1}\\ &= UU^{-1}\\ &\subseteq V.\end{aligned}$$

Thus, $\pi(W) \subseteq V$, showing continuity at 1. ⊣

COROLLARY 2.9 (J. P. R. Christensen [4]). *Suppose $\pi\colon G \to H$ is a universally measurable homomorphism from an Abelian Polish group G to a Polish group H. Then π is continuous.*

PROOF. For, $\overline{\pi(G)}$ is an Abelian subgroup of H and hence $\pi\colon G \to \overline{\pi(G)}$ is a universally measurable homomorphism into an Abelian Polish group and thus continuous. ⊣

Obviously, the classes of left Haar null and Haar null sets are hereditary, i..e, closed under taking universally measurable subsets. What is less obvious is that the class of Haar null sets actually form a σ-ideal.

THEOREM 2.10 (J. P. R. Christensen [5]). *The class of Haar null sets is a σ-ideal.*

PROOF. Suppose $A, B \subseteq G$ are Haar null sets in a Polish group G as witnessed by probability measures μ and ν respectively. Then for all $g, f \in G$

$$\mu * \nu(gAf) = \int \mu(gAfh^{-1})\, d\nu(h) = 0$$

and

$$\mu * \nu(gBf) = \int \nu(h^{-1}gBf)\, d\mu(h) = 0.$$

So $\mu * \nu$ witnesses that *both* A and B are Haar null, and hence also witnesses that $A \cup B$ is Haar null. For the case of infinite unions, one has to consider infinite convolutions of measures. ⊣

Much of the recent work on left Haar null sets in general Polish groups is due to S. Solecki, whose approach to automatic continuity can be summarised as follows:

- There are more left Haar null sets than Haar null sets, so not being left Haar null is stronger information than not being Haar null. Therefore, replace Haar null with *left* Haar null.
- Determine the extent of the class of Polish groups in which the left Haar null sets form a reasonable class.

In [30] Solecki isolated the Polish groups that seem particularly amenable to an analysis via left Haar null sets. These are defined in analogy with Christensen's proof that Haar null sets form a σ-ideal.

DEFINITION 2.11 (S. Solecki [30]). *A Polish group G is* amenable at 1 *if for any sequence μ_n of Borel probability measures on G with $1 \in \operatorname{supp} \mu_n$, there are Borel probability measures ν_n and ν such that*

(1) $\nu_n \ll \mu_n$,
(2) *if $K \subseteq G$ is compact,*

$$\lim_n \nu * \nu_n(K) = \nu(K).$$

Examples of groups that are amenable at 1 include

(1) Abelian Polish groups,
(2) locally compact Polish groups,
(3) countable direct products of locally compact Polish groups such that all but finitely many factors are amenable.

Moreover, the class of amenable at 1 groups is closed under taking closed subgroups and quotients by closed normal subgroups. So being amenable at 1 is a weakening of being amenable.

THEOREM 2.12 (S. Solecki [30]). *Suppose G is amenable at* 1. *Then*

(1) *the left Haar null sets form a σ-ideal,*
(2) *if $A \subseteq G$ is universally measurable and not left Haar null, then $A^{-1}A$ is a neighbourhood of* 1.

PROOF. (1) We show that if A_n is left Haar null for every n, then so is $\bigcup_n A_n$. So find a sequence μ_k of probability measures such that for every n there are infinitely many k such that for every f,

$$\mu_k(fA_n) = 0.$$

Also, by translating the measures on the left, we can suppose that $1 \in \operatorname{supp} \mu_k$. Let $\nu_k \ll \mu_k$ and ν be given as in the definition of amenable at 1. Then for all compact $K \subseteq A_n$,

$$\begin{aligned} \nu(gK) &= \lim_k \nu * \nu_k(gK) \\ &= \liminf_k \nu * \nu_k(gK) \\ &= \liminf_k \int \nu_k(h^{-1}gK)\, d\nu(h) \end{aligned}$$

$$\leq \liminf_k \int \nu_k(h^{-1}gA_n)\,d\nu(h)$$
$$= 0,$$

where the last equality follows from $\nu_k \ll \mu_k$. So by inner regularity, we see that ν witnesses that A_n is left Haar null for all n simultaneously, and so $\bigcup_n A_n$ is left Haar null too.

(2) Suppose G is amenable at 1 and $A \subseteq G$ is universally measurable but not left Haar null. We shall show that $1 \in \mathrm{Int}(A^{-1}A)$. So suppose towards a contradiction that this fails and pick a sequence $g_n \notin A^{-1}A$ such that $g_n \to 1$. We can then define probability measures μ_k on G with $1 \in \mathrm{supp}\,\mu_k$ by setting

$$\mu_k = \sum_{i=k}^{\infty} 2^{-i+k-1}\delta_{g_i}.$$

Let $\nu_k \ll \mu_k$ and ν be given as in the definition of amenability at 1. As $\nu_k \ll \mu_k$, ν_k is some infinite convex combination

$$\nu_k = \sum_{i=k}^{\infty} a_{i,k}\delta_{g_i}.$$

And for compact $K \subseteq G$,

$$\nu(K) = \lim_k \nu * \nu_k(K) = \lim_k \sum_{i=k}^{\infty} a_{i,k}\nu(Kg_i^{-1}).$$

We claim that ν witnesses that A is left Haar null. For if not, there is some h such that $\nu(hA) > 0$. So pick some compact, respectively open, $K \subseteq A \subseteq U$ such that

$$\nu(hU \setminus hK) < \frac{1}{2}\nu(hA).$$

Then for sufficiently large k,

$$\sum_{i=k}^{\infty} a_{i,k}\nu(hKg_i^{-1}) > \frac{1}{2}\nu(hA)$$

and hence for such k, there is some $i_k \geq k$ such that

$$\nu(hKg_{i_k}^{-1}) > \frac{1}{2}\nu(hA).$$

But, as $g_{i_k}^{-1} \to 1$, we have for large k

$$hKg_{i_k}^{-1} \subseteq hU$$

and so, using

$$\nu(hKg_{i_k}^{-1}) + \nu(hA) > \nu(hU),$$

we have

$$hKg_{i_k}^{-1} \cap hA \neq \emptyset.$$

It follows that

$$g_{i_k} \in A^{-1}K \subseteq A^{-1}A,$$

which is a contradiction. ⊣

For good measure, let us mention the main conclusion.

Corollary 2.13 (S. Solecki [30]). *If G is a Polish group, amenable at* 1, *then any universally measurable homomorphism $\pi\colon G \to H$ into a Polish group is continuous.*

On the other hand, Solecki also identified a class of groups in which the class of left Haar null sets seems to be much less useful and certainly does not behave as wanted; these groups are at the extreme opposite of being amenable. Recall that the canonical examples of non-amenable locally compact groups are those containing a (closed) discrete copy of $\mathbb{F}_2$, i.e., the free non-Abelian group on 2 generators.

Definition 2.14 (S. Solecki [30]). *A Polish group G has a* free subgroup at 1 *if it has a non-discrete free subgroup all of whose finitely generated subgroups are discrete.*

For example, $(\mathbb{F}_2)^\omega$, or any group containing it as a closed subgroup, has a free subgroup at 1. This includes groups such as S_∞, $\mathrm{Aut}(\mathbb{Q})$, and $\mathrm{Homeo}([0,1]^n)$.

Theorem 2.15 (S. Solecki [30]). *Suppose G is a Polish group with a free subgroup at* 1. *Then there is a Borel set $B \subseteq G$ which is left Haar null, but*

$$G = B \cup Bf$$

for some $f \in G$.

Since the class of left Haar null sets is closed under two-sided translations, this shows that G is covered by two left Haar null sets, and thus, by taking inverses, $G = B^{-1} \cup f^{-1}B^{-1}$, G is also covered by two right Haar null sets. But then

$$G = (B \cap B^{-1}) \cup (B \cap f^{-1}B^{-1}) \cup (Bf \cap B^{-1}) \cup (Bf \cap f^{-1}B^{-1})$$

is a covering of G by four sets that are simultaneously left and right Haar null, though, of course, they cannot all be Haar null.

As the set B above cannot be right Haar null, this points to a fundamental asymmetry in these groups that shows that being left and right Haar null are very different properties in general.

To obtain counter-examples to the conclusion of the Steinhaus–Weil Theorem in groups with a free subgroup at 1, one needs a bit more. Obviously, if G is locally compact, then, letting U be a relatively compact neighbourhood of 1, any non-empty open set V covers U by finitely many 2- sided translates

$$gVh.$$

But does this characterise local compactness?

DEFINITION 2.16 (S. Solecki [30]). *A Polish group G is* strongly non-locally compact *if for any open $U \ni 1$, there is an open $V \ni 1$ such that U cannot be covered by finitely many 2-sided translates of V.*

We should mention that it remains an open problem if any non-locally compact Polish group is actually strongly non-locally compact. Again, the examples of strongly non-locally compact groups include S_∞, $\mathrm{Aut}(\mathbb{Q})$, and $\mathrm{Homeo}([0,1]^n)$.

THEOREM 2.17 (S. Solecki [30]). *Suppose G is strongly non-locally compact and has a free subgroup at 1. Then there is a Borel set $A \subseteq G$ which is not left Haar null, but*

$$1 \notin \mathrm{Int}\, A^{-1}A.$$

So does this mean that we can produce discontinuous homomorphisms via left Haar null sets? Well, not really. For since A is not left Haar null, it does not have a continuum of disjoint right translates. So, as $A^{-1}A$ is analytic, it must be somewhere comeagre and so $A^{-1}AA^{-1}A$ contains a neighbourhood of 1. Rather, Theorems 2.15 and 2.17 point to a fundamental asymmetry in groups having a free subgroup at 1. The sets constructed in these results are very far from being symmetric and therefore one would like to have a symmetric example in Theorem 2.17. However, it remains unknown whether such sets can exist. So Christensen's question

> *Is every universally measurable homomorphism between Polish groups continuous*?

is very much still open. However, one could hope to make piecemeal progress on this by attacking more concrete questions. One way of doing this is by considering other types of range groups than those with 2-sided invariant metrics.

THEOREM 2.18 (C. Rosendal [25]). *Any universally measurable subgroup of a Polish group is either open or has continuum index. In particular, any universally measurable homomorphism from a Polish group into S_∞ is continuous.*

Actually, using this result one can gain additional information about universally measurable subsets satisfying an algebraic largeness condition.

THEOREM 2.19 (C. Rosendal [25]). *Let A be a universally measurable, symmetric subset of a Polish group G containing 1 and covering G by a countable number of left translates. Then for some $n \geq 1$, A^n is a neighborhood of 1.*

This result falls short of solving Christensen's problem, since in general the n could depend on A. If, on the the hand, one could make n independent of A, then this would imply a positive solution to Christensen's problem.

The following questions could hopefully lead to fruitful considerations.

(1) Is there an n such that whenever G is a Polish group and $A \subseteq G$ a universally measurable, symmetric subset containing 1, which is not

left Haar null, we have

$$1 \in \operatorname{Int} A^n?$$

(2) What if, moreover, A covers G by a countable number of left translates?
(3) Let G be a closed subgroup of S_∞. Is any universally measurable homomorphism from G into a Polish group automatically continuous?

We recall that a positive answer to (1) or (2) would also imply a positive solution to Christensen's question. Despite the results of Solecki, the case $n = 2$ still seems to be open.

§**3. Dudley's Theorem.** We shall now completely discard any measurability assumptions on the homomorphisms and instead restrict the classes of groups considered. We begin by placing restrictions on the target group.

Are there Polish groups H such that all homomorphisms from any other Polish group into it is continuous?

One of the simplest and most powerful theorems in this direction is due to the probabilist R. M. Dudley [9]. His result has since been rediscovered a number of times in varying degrees of generality and complications, but the original result and proof does not seem to have been superseded in its utmost simplicity.

Definition 3.1. *A* norm *on a group G is a function* $\|\cdot\|\colon G \to \mathbb{N}$ *such that*

- $\|gf\| \leq \|g\| + \|f\|$,
- $\|1_G\| = 0$,
- $\|g\| = \|g^{-1}\|$,
- $\|g^n\| \geq \max\{n, \|g\|\}$ *for all* $g \neq 1_G$.

The class of normed groups is fairly restricted and points in the direction of discrete groups. In fact as we shall see, no continuous Polish group can be normed.

- The class of normed groups is closed under direct sums and free products.
- contains the free non-Abelian groups,
- and the free Abelian groups.

For example, the additive group of integers $\langle \mathbb{Z}, + \rangle$ is normed, where the norm $\|\cdot\|$ on $\mathbb{Z}$ is just the absolute value function $|\cdot|$.

Theorem 3.2 (R. M. Dudley [9]). *Any homomorphism from a Polish group G into a normed group H equipped with the discrete topology is continuous.*

Proof. Let us just prove the case of $H = \mathbb{Z}$. So suppose G is a Polish group and $\pi\colon G \to \mathbb{Z}$ is a homomorphism. If π fails to be continuous, then we can find $g_i \in G$ converging very fast to 1_G, and such that, on the other hand, the absolute value $|\pi(g_i)|$ grows equally fast.

Using these, we can find integers $k_m \geq 1$ and, through a limiting process,

$$y_m = g_m(g_{m+1}(g_{m+2}(\dots)^{k_{m+2}})^{k_{m+1}})^{k_m} \in G$$

satisfying

(1) $k_m < |\pi(g_{m+1})|$,
(2) $y_m = g_m y_{m+1}^{k_m}$,
(3) $k_m = m + \sum_{i=1}^{m} |\pi(g_i)|$.

Here it is important to notice that, as π is assumed discontinuous, there is no way of directly controlling $|\pi(y_i)|$, only y_i itself. On the other hand, as the g_i are chosen explicitly, this also controls $\pi(g_i)$. Now, if $\pi(y_{m+1}) = 0$, then

$$|\pi(y_m)| = |\pi(g_m y_{m+1}^{k_m})| = |\pi(g_m)| > k_{m-1}.$$

And if $\pi(y_{m+1}) \neq 0$, then

$$|\pi(y_m)| = |\pi(g_m y_{m+1}^{k_m})| \geq k_m \cdot |\pi(y_{m+1})| - |\pi(g_m)| \geq k_m - |\pi(g_m)| > k_{m-1}.$$

It follows that for all m

$$\begin{aligned} |\pi(y_1)| &= |\pi(g_1 y_2^{k_1})| \\ &\geq |\pi(y_2^{k_1})| - |\pi(g_1)| \\ &\geq |\pi(y_2)| - |\pi(g_1)| \\ &= \cdots \\ &\geq |\pi(y_{m+1})| - \sum_{i=1}^{m} |\pi(g_i)| \\ &> k_m - (k_m - m) \\ &= m. \end{aligned}$$

So the absolute value of $\pi(y_1)$ is infinite, which is impossible. $\dashv$

COROLLARY 3.3. *The free group $\mathbb{F}_\mathfrak{c}$ on a continuum of generators cannot be equipped with a Polish group topology.*

Notice however that $\mathbb{F}_\mathfrak{c}$ is often found as a K_σ subgroup of larger Polish groups.

Also, as a function $f: X \to D^\omega$, where X is a Polish space and D is a countable discrete set, is continuous if and only if the compositions with the coordinate projections $P_n: D^\omega \to D$ are all continuous, we have:

COROLLARY 3.4. *Any homomorphism π from a Polish group into $(\mathbb{F}_2)^\omega$ is continuous.*

Another corollary of Dudley's result is that any homomorphism from, e.g., $\mathbb{Z}^\omega$ into $\mathbb{Z}$ is continuous and thus only depends on a finite number of coordinates.

§**4. Subgroups of small index.** In Dudley's Theorem, the automatic continuity relies heavily on very specific algebraic features of the target group; namely, that it is constructed from normed groups, i.e., essentially free groups or free Abelian groups. We shall now look at a property related to the topological structure of S_∞. We begin by recalling a basic result.

PROPOSITION 4.1 (see, e.g., [3], (1.5.1)). *A Polish group G is topologically isomorphic to a closed subgroup of S_∞ if and only if G has a neighbourhood basis at* 1 *consisting of (necessarily open) subgroups of countable index.*

This includes , for example, automorphism groups, $\mathrm{Aut}(\mathcal{M})$, of countable first order structures $\mathcal{M}$ and, by a result of D. van Dantzig, any totally disconnected, locally compact, second countable group.

DEFINITION 4.2. *A Polish group G is said to have the* small index property *if any subgroup $H \leq G$ of countable index is open.*

We should note that one often requires this to hold for any subgroup of index $< 2^{\aleph_0}$. But in the context of this paper, we shall stick to the weaker condition.

PROPOSITION 4.3. *Suppose G is a Polish group with the small index property. Then any homomorphism*

$$\pi\colon G \to S_\infty$$

is continuous. In particular, if G acts on a countable set, it does so continuously.

The following result seems to be the first pointing in this direction.

THEOREM 4.4 (S. W. Semmes [27], J. D. Dixon, P. M. Neumann and S. Thomas [8]). *S_∞ has the small index property.*

This is just the first in a long list of automorphism groups of highly homogeneous countable first order structures that are also known to have this property. Moreover, S. Thomas [32] has classified the countable products $\prod_{n\in\mathbb{N}} F_n$ of finite, simple, non-Abelian groups F_n, that have this property. More curiously, by a result of Solecki and the author [26], $\mathrm{Homeo}_+(\mathbb{R})$ has the small index property. So, as it is connected, it simply has no proper subgroups of countable index and hence cannot act non-trivially on a countable set.

We shall now turn our attention to subgroups of even smaller index, namely, finite index subgroups.

DEFINITION 4.5. *A compact Hausdorff group G is* profinite *if it has a neighbourhood basis at* 1 *consisting of open subgroups of finite index.*

The Polish profinite groups are easy to recognise. For having a neighbourhood basis at the identity consisting of open subgroups, they embed into S_∞. Thus, the Polish profinite groups are exactly the compact subgroups of S_∞. These are alternatively the closed subgroups of countable products $\prod_{i=1}^{\infty} F_i$, where the F_i are finite.

Since the neighbourhood basis at 1 in a profinite group is given by subgroups of finite index, the following three conditions on a Polish group G are easily seen to be equivalent:

- Every subgroup of finite index is open,
- any homomorphism $\pi\colon G \to F$ into a finite group F is continuous,
- any homomorphism $\pi\colon G \to H$ into a profinite group is continuous.

A first result in this direction is due to J.-P. Serre sometime in the 1960's. His result deals with *pro-p groups*, i.e., profinite groups G such that for all open normal subgroups $N \trianglelefteq G$, the quotient G/N is a (discrete) p-group.

THEOREM 4.6 (J.-P. Serre). *Any finite index subgroup of a pro-p group is open.*

However, it remained open for a long time how much this result generalises. Very recently, N. Nikolov and D. Segal [19, 20] extended this to all topologically finitely generated profinite groups. Here, a topological group G is *topologically n-generated* if it has a dense subgroup with n generators, or, equivalently, if there is a homomorphism

$$\pi\colon \mathbb{F}_n \to G$$

with dense image $\overline{\langle \pi(\mathbb{F}_n)\rangle} = G$. And G is *topologically finitely generated* if it is topologically n-generated for some finite n.

THEOREM 4.7 (Nikolov and Segal [19, 20]). *If G is a topologically finitely generated profinite group, then any finite index subgroup of G is open.*

We shall now give a brief introduction to a part of their proof. Notice first that if F is a finite group, then $F^{\mathbb{N}}$ is *locally finite*, i.e., any finitely generated subgroup is finite. For if $f_1, \dots, f_n \in F^{\mathbb{N}}$, then, as F is finite, we can find a partition

$$\mathbb{N} = A_1 \cup \cdots \cup A_k$$

such that each f_i is constant on every piece A_j. It follows that $\langle f_1, \dots, f_n\rangle$ embeds into F^k and hence is finite. This argument also works if the orders $|F_n|$ are bounded. So to get infinite examples of topologically finitely generated, profinite, Polish groups, one has to consider subgroups of products

$$\prod_{n\in\mathbb{N}} F_n,$$

where $|F_n|$ is unbounded.

Now suppose G is an (abstract) group and $w(x_1, \dots, x_k)$ is a group word. The *verbal subgroup* determined by w is simply the subgroup generated by all evaluations of w in G, i.e.,

$$w(G) = \langle w(g_1, \dots, g_k) \mid g_i \in G\rangle.$$

Note that $w(G)$ is a normal subgroup of G, since its generating sets is conjugacy invariant:

$$h \cdot w(g_1, \dots, g_k) \cdot h^{-1} = w(hg_1h^{-1}, \dots, hg_kh^{-1}).$$

The main part of Nikolov and Segal's paper concerns the proof of the following result, which is proved using methods of finite group theory.

THEOREM 4.8 (N. Nikolov and D. Segal [19, 20]). *Suppose $w(x_1, \ldots, x_k)$ is a group word such that*

$$[\mathbb{F}_n \colon w(\mathbb{F}_n)] < \infty.$$

Then there is a positive integer r such that whenever G is a topologically n-generated profinite group, any element of $w(G)$ can be written as a product of r elements of the form

$$w(g_1, \ldots, g_k), \quad g_i \in G.$$

An equivalent way of stating the conclusion of this theorem is to say that $w(G)$ is a compact subgroup of G. This follows from the simple fact that, given the conclusion, $w(G)$ is a continuous image of G^{rk}. For the converse implication, note that if $w(G)$ is compact, write $A = \{w(g_1, \ldots, g_k), w(g_1, \ldots, g_k)^{-1} \mid g_i \in G\}$. Then, as $w(G) = \bigcup_{n \in \mathbb{N}} A^n$, where A^n is compact, by the Baire Category Theorem, some A^n must have non-empty interior. But $w(G) = \bigcup_{g \in A^{<\omega}} gA^n$, so by compactness there are $g_1, \ldots, g_k \in A^{<\omega}$ such that $w(G) = g_1A^n \cup \cdots \cup g_kA^n$. Therefore, if l is large enough such that $g_i \in A^l$ for all i, we have $w(G) = A^{l+n}$.

LEMMA 4.9 (Nikolov and Segal [19, 20]). *Suppose H is a finite group and $n \geq 1$. Then there is a group word $w(x_1, \ldots, x_k)$ such that $[\mathbb{F}_n \colon w(\mathbb{F}_n)] < \infty$ and $w(H) = \{1\}$.*

PROOF. To see this, let Θ be the finite set of all homomorphisms $\vartheta \colon \mathbb{F}_n \to H$ and let

$$K = \bigcap_{\vartheta \in \Theta} \ker \vartheta.$$

Then K, being the intersection of finitely many finite index subgroups, has finite index in $\mathbb{F}_n$ and so is finitely generated by some

$$w_1(a_1, \ldots, a_n), \ldots, w_m(a_1, \ldots, a_n) \in \mathbb{F}_n$$

where the a_i are the free generators of $\mathbb{F}_n$. Now, for any $f_1, \ldots, f_n$ and i, we have $w_i(f_1, \ldots, f_n) \in K$. For otherwise there would be some homomorphism $\vartheta \colon \mathbb{F}_n \to H$ such that

$$\vartheta(w_i(f_1, \ldots, f_n)) \neq 1.$$

Letting $\rho \colon \mathbb{F}_n \to H$ be defined by $\rho(a_i) = \vartheta(f_i)$, we see that also

$$\rho(w_i(a_1, \ldots, a_n)) = \vartheta(w_i(f_1, \ldots, f_n)) \neq 1,$$

contradicting that $w_i(a_1, \ldots, a_n) \in K \subseteq \ker \rho$. So $w_i(\mathbb{F}_n) \subseteq K$ for all i.

Now let $\overline{y}_1, \ldots, \overline{y}_m$ be disjoint n-tuples of variables and set

$$w(\overline{y}_1, \ldots, \overline{y}_m) = w_1(\overline{y}_1) \ldots w_m(\overline{y}_m).$$

Then $w(\mathbb{F}_n) = K$ and hence $[\mathbb{F}_n \colon w(\mathbb{F}_n)] < \infty$.

It remains to see that $w(H) = \{1\}$. But, if $h_1, \ldots, h_n \in H$, define $\vartheta \colon \mathbb{F}_n \to H$ by

$$\vartheta(a_i) = h_i.$$

Then

$$w_i(h_1, \ldots, h_n) = \vartheta(w_i(a_1, \ldots, a_n)) = 1,$$

and so $w(H) = \{1\}$. ⊣

We can now deduce Theorem 4.7 from Lemma 4.9 and Theorem 4.8.

Proof. Suppose G is topologically n-generated. By a theorem of Poincaré, any finite index subgroup contains a subgroup of finite index which moreover is normal. Thus, it is enough to show that any finite index *normal* subgroup $N \leq G$ is open. Now, by Lemma 4.9 applied to the finite group $H = G/N$, we find a group word $w(x_1, \ldots, x_k)$ such that

$$[\mathbb{F}_n \colon w(\mathbb{F}_n)] < \infty$$

and $w(G/N) = \{1\}$, i.e., $w(G) \leq N$.

Recall that by Theorem 4.8, $w(G)$ is a closed subgroup of the profinite group G and hence the intersection of all finite index, open, normal subgroups $M \leq G$ containing $w(G)$. Since G is topologically n-generated and such an M is open, G/M is topologically n-generated, and, being discrete, it is outright n-generated. Also, as $w(G) \leq M$, the quotient G/M is the epimorphic image of $\mathbb{F}_n/w(\mathbb{F}_n)$. So the indices $[G \colon M]$ are bounded by $[\mathbb{F}_n \colon w(\mathbb{F}_n)]$ and hence the intersection $w(G)$ of the M also has index bounded by $[\mathbb{F}_n \colon w(\mathbb{F}_n)]$.

So $w(G)$ is a closed subgroup of finite index and hence open. Since $w(G) \leq N \leq G$, N is open too. ⊣

To produce examples of uncountable, topologically finitely generated, profinite groups, one can take, for example, a sequence (F_n) of finite groups having elements $f_n \in F_n$ whose orders $|f_n|$ tend to infinity. Let now $f = (f_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} F_n$. Clearly, f has infinite order and so the compact group $\overline{\langle f \rangle}$ is infinite and therefore uncountable. Moreover, $\overline{\langle f \rangle}$ is topologically generated by a single element, i.e., $\overline{\langle f \rangle}$ is monothetic and so, in particular, Abelian. However, any infinite Abelian group has a denumerable, i.e., countably infinite, quotient and we can therefore find a normal subgroup $H \leq \overline{\langle f \rangle}$ of denumerable index. Since $\overline{\langle f \rangle}$ is compact and the covering by cosets of H has no finite subcovering, H cannot be open.

Example 4.10. *There are monothetic profinite Polish groups having non-open subgroups of countable index, but all of whose subgroups of finite index are open.*

Such groups have discontinuous embeddings into S_∞, but any homomorphism into a profinite group is continuous.

§5. **Ample generics.** The final question we are considering will be the least restrictive of all, namely,

> *Are there conditions on a Polish group G that ensure that every homomorphism*
>
> $$\pi\colon G \to H$$
>
> *from G into a Polish group H is continuous*?

It turns out that the exact condition on H that it be Polish is somewhat inessential. Rather, the right condition on H is that its topology should not be too large. To see this, we need a couple of definitions:

Definition 5.1. *A subset A of a group G is said to be* (left) σ-syndetic *if there are $g_n \in G$ such that $G = \bigcup_{n\in\mathbb{N}} g_n A$, i.e., A covers G by countably many left translates.*

Also, a topological group G is said to be $\aleph_0$-bounded *if any non-empty open subset $V \subseteq G$ is σ-syndetic.*

Using a classical characterisation due to I. I. Guran [10] of the $\aleph_0$-bounded groups as those that are embeddable into a direct product of second countable groups and a result of V. V. Uspenskiĭ [33] stating that any Polish group is isomorphic to a closed subgroup of Homeo$([0,1]^{\mathbb{N}})$, we have the following equivalence:

Proposition 5.2 (C. Rosendal [24]). *Suppose G is a Polish group. Then the following conditions are equivalent.*

(1) *Any homomorphism $\pi\colon G \to$ Homeo$([0,1]^{\mathbb{N}})$ is continuous,*
(2) *any homomorphism $\pi\colon G \to H$ into a separable group is continuous,*
(3) *any homomorphism $\pi\colon G \to H$ into a second countable group is continuous,*
(4) *any homomorphism $\pi\colon G \to H$ into an $\aleph_0$-bounded group is continuous.*

In order to find Polish groups G that satisfy the above version of automatic continuity, one is of course tempted to simplify the problem by getting rid of the quantification over all Polish groups H. Superficially, this problem seems to be solved by condition (1) of Proposition 5.2 above, but that condition is really not more helpful, as Homeo$([0,1]^{\mathbb{N}})$ is universal for all Polish groups.

So are there any other conditions on a Polish group assuring that homomorphisms from G into separable groups are always continuous? Well, here is one:

Definition 5.3 (C. Rosendal & S. Solecki [26]). *A Polish group G is said to be* Steinhaus *if there is an integer constant $k \geq 1$ such that whenever $A \subseteq G$ is a symmetric, i.e., $A = A^{-1}$, σ-syndetic subset containing* 1, *we have*

$$1 \in \operatorname{Int} A^k.$$

We say in this case that G is Steinhaus with exponent k.

Notice that if $\pi\colon G \to H$ is any homomorphism into a separable group, then for any symmetric neighbourhood $V \subseteq H$ of 1, the inverse image $\pi^{-1}(V)$ is σ-syndetic. So by the standard proof of Steinhaus and Weil's theorem on automatic continuity, we have

PROPOSITION 5.4 (C. Rosendal & S. Solecki [26]). *If G is a Steinhaus, Polish group, then any homomorphism $\pi\colon G \to H$ into a separable group is continuous.*

The main problem then becomes to identity non-discrete, Steinhaus, Polish groups. Though we shall present some examples of Polish transformation groups that can be proved to be Steinhaus by brute force, we will be more interested in the finer techniques that were first considered in a seminal paper by W. Hodges, I. Hodkinson, D. Lascar and S. Shelah [13], namely *ample generics*. A version of ample generics in [13] was introduced in the course of a proof to the effect that automorphism groups of ω-stable, ω-categorical structures have the small index property and only made sense for automorphism group, being tied up with the permutation group structure of these. However, the concept was further developed A. S. Kechris and the author in [16], where the reference to permutation groups was dispensed with and their methods were extended. The results presented here are improvements of results from [16], which again are inspired by the results of [13].

Suppose G is a Polish group acting continuously on a Polish space X. Then for any positive integer n, we can define the diagonal action $G \curvearrowright X^n$ by

$$g \cdot (x_1, \ldots, x_n) = (g \cdot x_1, \ldots, g \cdot x_n).$$

DEFINITION 5.5 (A. S. Kechris & C. Rosendal [16]). *Suppose G is a Polish group acting continuously on a Polish space X. We say that the action has* ample generics *if for every $n \geq 1$ there is a comeagre orbit in X^n under the diagonal action of G.*

We shall refer to elements $(x_1, \ldots, x_n)$ of the comeagre orbit of dimension n as generics.

Easy examples of such actions are, for example, the natural actions of S_∞ on $X = 2^{\mathbb{N}\times\cdots\times\mathbb{N}}$ by simultaneous permutation of the coordinates:

$$(g \cdot x)(n_1, \ldots, n_k) = x(g^{-1}(n_1), \ldots, g^{-1}(n_k)).$$

For more interesting examples, consider a Polish group G acting on itself by conjugation. Then the diagonal action is given by

$$g \cdot (h_1, \ldots, h_n) = (gh_1g^{-1}, \ldots, gh_ng^{-1}).$$

If this action has ample generics, we simply say that G has ample generics itself.

At this time, the only groups known to have ample generics are closed subgroups of S_∞, i.e., automorphism groups of countable structures. It is well known that any closed subgroup of S_∞ is topologically isomorphic to the automorphism group of a countable *ultrahomogeneous* structure $\mathcal{M}$. And, by Fraïssé's theory (see, e.g., [12]), any countable ultrahomogeneous structure can be seen as the Fraïssé limit of its *age*, i.e., as being built up from the class of its finitely generated substructures, Age($\mathcal{M}$), by a generic process of amalgamation. Using the correspondence between ages of ultrahomogeneous structures and their automorphism groups, we have the following examples of Polish groups with ample generics:

- S_∞,
- $\mathrm{Aut}(\mathbb{N}^{<\mathbb{N}})$ (see [16]),
- $\mathrm{Aut}(\mathcal{R})$, where $\mathcal{R}$ is the random graph (E. Hrushovski [14], Hodges et al. [13]).
- $\mathrm{Homeo}(2^{\mathbb{N}}, \lambda)$, where λ is Haar measure on Cantor space (see [16]),
- $\mathrm{Isom}(\mathbb{U}_{\mathbb{Q}})$ (Solecki [29]), where $\mathbb{U}_{\mathbb{Q}}$ is the Urysohn metric space with rational distances.

In these specific examples, the main tool behind the proof of them having ample generics is that they all satisfy the following property:

Definition 5.6 (A. S. Kechris & C. Rosendal [16]). *A Polish group G is* approximately compact *if there is an increasing sequence of compact subgroups*

$$C_0 \leq C_1 \leq C_2 \leq \cdots \leq G$$

such that $\bigcup_n C_n$ is dense in G.

To see what this means in the context of automorphism groups, suppose that $\mathcal{M}$ is a countable ultrahomogeneous structure and that $\mathrm{Aut}(\mathcal{M})$ is approximately compact, as witnessed by some approximating sequence $C_0 \leq C_1 \leq \cdots \leq \mathrm{Aut}(\mathcal{M})$. Then if $f_1, \ldots, f_n \in \mathrm{Aut}(\mathcal{M})$ and $A \subseteq \mathcal{M}$ is a finite subset, we can find some k and $g_1, \ldots, g_n \in C_k$ such that

$$g_i|_A = f_i|_A$$

for every i. But $\langle g_1, \ldots, g_n \rangle \leq C_k$ and, as C_k is compact, so is the discrete set $C_k \cdot A$, i.e., $C_k \cdot A$ is finite. So $C_k \cdot A \subseteq \mathcal{M}$ is a finite subset of $\mathcal{M}$ containing A and invariant under $g_1, \ldots, g_n$. It follows that the f_i can be approximated arbitrarily well by elements setwise fixing a finite set containing A. Under additional assumptions, one can prove the converse of this implication leading to:

Proposition 5.7 (A. S. Kechris & C. Rosendal [16]). *Let $\mathcal{M}$ be a locally finite (i.e., any finitely generated substructure is finite), countable, ultrahomogeneous structure. Then* $\mathrm{Aut}(\mathcal{M})$ *is approximately compact if and only if for any finite substructure $\mathcal{A} \subseteq \mathcal{M}$ and all* partial *automorphisms $\phi_1, \ldots, \phi_n$ of $\mathcal{A}$, there is a larger finite substructure $\mathcal{B}$,*

$$\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{M}$$

and full *automorphisms $\psi_1, \ldots, \psi_n$ of $\mathcal{B}$ extending $\phi_1, \ldots, \phi_n$ respectively.*

Any structure $\mathcal{M}$ satisfying the conditions of the above proposition is said to have the *Hrushovski* property (Hrushovski [14] originally proved the conclusion of the proposition for the structure $\mathcal{R}$). The deepest results in this direction are probably due to B. Herwig and D. Lascar [11], who proved a very general result to the effect that the Fraïssé limits of a large class of relational structures have the Hrushovski property. Solecki [29] subsequently used this to prove that also $\mathbb{U}_{\mathbb{Q}}$ has the Hrushovski property.

We shall now present some of the main implications of the existence of ample generics.

Lemma 5.8 (The extension lemma; A. S. Kechris & C. Rosendal [16]). *Let $G \curvearrowright X$ be a Polish group action with ample generics and suppose $A, B \subseteq X$ are arbitrary subsets such that*

- *A is non-meagre,*
- *B is nowhere meagre.*

Then, if $\overline{x} = (x_1, \dots, x_n) \in X^n$ is generic and $V \ni 1$ is open, there are $g \in V$, $y \in A$ and $z \in B$ such that $(x_1, \dots, x_n, y)$ and $(x_1, \dots, x_n, z)$ are generic, while

$$g \cdot (x_1, \dots, x_n, y) = (x_1, \dots, x_n, z).$$

Notice that the last condition implies that $g \in G_{x_i}$ for $i \leq n$.

Proof. Let $\mathcal{O} \subseteq X^{n+1}$ be the comeagre orbit of dimension $n+1$. Then, by Kuratowski–Ulam,

$$\forall^* \overline{u} \in X^n \, \forall^* y \in X \, (\overline{u}, y) \in \mathcal{O}.$$

Also, for any $\overline{u} \in X^n$ and $g \in G$,

$$\begin{aligned} \forall^* y \in X \, (\overline{u}, y) \in \mathcal{O} &\Rightarrow \forall^* y \in X \, (g \cdot \overline{u}, g \cdot y) \in \mathcal{O} \\ &\Rightarrow \forall^* z \in X \, (g \cdot \overline{u}, z) \in \mathcal{O}. \end{aligned}$$

So

$$D = \{\overline{u} \in X^n \mid \forall^* y \in X \, (\overline{u}, y) \in \mathcal{O}\}$$

is comeagre and G-invariant. Thus, D contains the comeagre orbit in X^n and hence $\overline{x} \in D$. It follows that

$$(*) \qquad \forall^* y \in X \, (x_1, \dots, x_n, y) \in \mathcal{O}.$$

So, as A is non-meagre, we can find $y \in A$ such that $(x_1, \dots, x_n, y) \in \mathcal{O}$. Notice now that

$$G_{\overline{x}} \cdot y = \{z \in X \mid (x_1, \dots, x_n, z) \in \mathcal{O}\},$$

which is comeagre in X by $(*)$.

Now, since $(G_{\overline{x}} \cap V) \cdot y$ covers $G_{\overline{x}} \cdot y$ by countably many translates, the set $(G_{\overline{x}} \cap V) \cdot y$ is somewhere comeagre in X and hence intersects B. So letting $z \in B \cap (G_{\overline{x}} \cap V) \cdot y$, we can find $g \in G_{\overline{x}} \cap V$, such that

$$g \cdot y = z,$$

whereby

$$g \cdot (\overline{x}, y) = (\overline{x}, z). \qquad \dashv$$

Lemma 5.9 (Fundamental lemma for ample generics; A. S. Kechris & C. Rosendal [16]). *Suppose $G \curvearrowright X$ is a Polish group action with ample generics and that $A_n, B_n \subseteq X$ are respectively non-meagre and nowhere meagre. Then there is a continuous map*

$$\alpha \in 2^{\mathbb{N}} \mapsto h_\alpha \in G$$

such that if $\alpha|_n = \beta|_n$ but $\alpha(n) = 0$ and $\beta(n) = 1$, then

$$h_\alpha \cdot A_n \cap h_\beta \cdot B_n \neq \emptyset.$$

Proof. Using the extension lemma, we define by induction on the length of $s \in 2^{<\mathbb{N}} \setminus \{\emptyset\}$, points $x_s \in X$ and group elements $f_s \in G$ such that for all s,

(1) $(x_{s|1}, x_{s|2}, \dots, x_s)$ is generic,
(2) $x_{s0} \in A_{|s|}$ and $x_{s1} \in B_{|s|}$,
(3) for all $\alpha \in 2^{\mathbb{N}}$, the infinite product $f_{\alpha|1} f_{\alpha|2} f_{\alpha|3} \dots$ converges,
(4) $f_{s0} = 1$,
(5) $f_{s1} \cdot (x_{s|1}, x_{s|2}, \dots, x_s, x_{s1}) = (x_{s|1}, x_{s|2}, \dots, x_s, x_{s0})$.

Set $h_\alpha = f_{\alpha|1} f_{\alpha|2} f_{\alpha|3} \dots$. And notice also that by (5), if $t \sqsubset s$, then $f_s \cdot x_t = x_t$. It follows that for all $\alpha, \beta \in 2^{\mathbb{N}}$, if $\alpha|_n = \beta|_n = s$, $\alpha(n) = 0$ and $\beta(n) = 1$, then

$$\begin{aligned}
h_\alpha \cdot x_{\alpha|n+1} &= f_{\alpha|1} f_{\alpha|2} f_{\alpha|3} \cdots \cdot x_{\alpha|n+1} \\
&= f_{\alpha|1} f_{\alpha|2} f_{\alpha|3} \dots f_{\alpha|n+1} \cdot x_{\alpha|n+1} \\
&= f_{s|1} f_{s|2} \dots f_s f_{s0} \cdot x_{s0} \\
&= f_{s|1} f_{s|2} \dots f_s \cdot x_{s0} \\
&= f_{s|1} f_{s|2} \dots f_s f_{s1} \cdot x_{s1} \\
&= f_{\beta|1} f_{\beta|2} \dots f_{\beta|n} f_{\beta|n+1} \cdot x_{\beta|n+1} \\
&= f_{\beta|1} f_{\beta|2} \cdots \cdot x_{\beta|n+1} \\
&= h_\beta \cdot x_{\beta|n+1}.
\end{aligned}$$

Since $x_{\alpha|n+1} \in A_n$ and $x_{\beta|n+1} \in B_n$, we have

$$h_\alpha \cdot A_n \cap h_\beta \cdot B_n \neq \emptyset. \qquad \dashv$$

Though we do not have any interesting applications of this lemma in the context of general actions with ample generics, when applied to Polish groups with ample generics, the consequences are quite intriguing.

THEOREM 5.10 (A. S. Kechris & C. Rosendal [16]). *Let G be a Polish group with ample generics and $\{k_iA_if_i\}_{i\in\mathbb{N}}$ a covering of G, where $k_i, f_i \in G$ and $A_i \subseteq G$ are arbitrary subsets of G. Then there is an i such that*

$$A_i^{-1}A_iA_i^{-1}A_i^{-1}A_iA_i^{-1}A_iA_iA_i^{-1}A_i$$

is a neighbourhood of 1.

PROOF. By leaving out all terms $k_iA_if_i$, such that A_i is meagre, and reenumerating, we can suppose that

(1) for every i there are infinitely many n such that

$$k_iA_if_i = k_nA_nf_n,$$

(2) $\bigcup_i k_iA_if_i$ is comeagre,

(3) each $f_i^{-1}A_if_i$ is non-meagre.

Notice also that if there is some n such that

$$A_n^{-1}A_nA_nA_n^{-1}A_n$$

is somewhere comeagre, then by Lemma 2.1 we would be done. So assume towards a contradiction that this fails. Then

$$B_n = G \setminus (f_nA_n^{-1}A_nA_nA_n^{-1}A_nf_n^{-1})$$

is nowhere meagre. So by our previous lemma there is a continuous mapping

$$\alpha \in 2^{\mathbb{N}} \mapsto h_\alpha \in G$$

so that if $\alpha|_n = \beta|_n$ but $\alpha(n) = 0$ and $\beta(n) = 1$, then

$$h_\alpha f_n^{-1}A_nf_nh_\alpha^{-1} \cap h_\beta B_nh_\beta^{-1} \neq \emptyset,$$

i.e.,

$$h_\alpha f_n^{-1}A_nf_nh_\alpha^{-1} \not\subseteq h_\beta f_nA_n^{-1}A_nA_nA_n^{-1}A_nf_n^{-1}h_\beta^{-1}.$$

Now, the mapping

$$(g, \alpha) \in G \times 2^{\mathbb{N}} \mapsto g^{-1}h_\alpha \in G$$

is continuous and open, and therefore inverse images of comeagre sets are comeagre. So, as $\bigcup_{i\in\mathbb{N}} k_iA_if_i$ is comeagre in G, we have by the Kuratowski–Ulam Theorem that

$$\forall^* g \in G\, \forall^* \alpha \in 2^{\mathbb{N}} \quad g^{-1}h_\alpha \in \bigcup_{i\in\mathbb{N}} k_iA_if_i.$$

So pick some $g \in G$ with

$$\forall^* \alpha \in 2^{\mathbb{N}} \quad g^{-1}h_\alpha \in \bigcup_{i\in\mathbb{N}} k_iA_if_i,$$

and find some i such that

$$\{\alpha \in 2^{\mathbb{N}} \mid g^{-1}h_\alpha \in k_iA_if_i\}$$

is dense in some basic open set

$$N_t = \{\alpha \in 2^{\mathbb{N}} \mid t \sqsubseteq \alpha\}.$$

Let now $n > |t|$ be such that $k_i A_i f_i = k_n A_n f_n$ and find $\alpha, \beta \in N_t$ such that

$$h_\alpha, h_\beta \in g k_n A_n f_n$$

and $\alpha|_n = \beta|_n$ while $\alpha(n) = 0$ and $\beta(n) = 1$. Then, if $h_\alpha = g k_n a f_n$ and $h_\beta = g k_n b f_n$, where $a, b \in A_n$, we have

$$\begin{aligned} h_\beta^{-1} h_\alpha f_n^{-1} A_n f_n h_\alpha^{-1} h_\beta &= f_n^{-1} b^{-1} a A_n a^{-1} b f_n^{-1} \\ &\subseteq f_n A_n^{-1} A_n A_n A_n^{-1} A_n f_n^{-1}. \end{aligned}$$

But this clearly contradicts

$$h_\alpha f_n^{-1} A_n f_n h_\alpha^{-1} \nsubseteq h_\beta f_n A_n^{-1} A_n A_n A_n^{-1} A_n f_n^{-1} h_\beta^{-1}. \qquad \dashv$$

Corollary 5.11 (A. S. Kechris & C. Rosendal [16]). *If G is a Polish group with ample generics, then G is Steinhaus with exponent* 10. *In particular, any homomorphism $\pi \colon G \to H$ from G into a Polish group H is continuous.*

Corollary 5.12 (A. S. Kechris & C. Rosendal [16]). *Suppose G has ample generics. If $\{k_i H_i f_i\}_{i \in \mathbb{N}}$ is a covering of G by two-sided translates of subgroups, then some H_i is open.*

Theorem 5.13. *Suppose G is a Polish group with ample generics and $A \subseteq G$ is a symmetric subset containing* 1. *Then either A admits a continuum of disjoint left translates in G or A^{12} is a neighbourhood of* 1.

Proof. Suppose A does not admit a continuum of disjoint left translates. Note first that if A^2 is meagre, then the binary relation R on G given by

$$xRy \Leftrightarrow x^{-1} y \in A^2$$

is meagre too, since the mapping $(x, y) \in G^2 \mapsto x^{-1} y \in G$ is surjective, continuous and open. But, by Mycielski's Theorem on independent sets for category, if R is meagre, then there is a Cantor set $C \subseteq G$ such that for distinct $x, y \in C$, $(x, y) \notin R$, i.e., $xA \cap yA = \emptyset$, contradicting the assumption on A. So A^2 must be non-meagre.

We claim that A^6 must be somewhere comeagre. For if not, let $A_n = A^2$ and set $B_n = G \setminus A^6$, which then is nowhere meagre. Applying the fundamental lemma for ample generics to this pair, we find an injective mapping $\alpha \in 2^{\mathbb{N}} \mapsto h_\alpha \in G$ such that if $\alpha|_n = \beta|_n$ but $\alpha(n) = 0$ and $\beta(n) = 1$, then

$$h_\alpha A_n h_\alpha^{-1} \cap h_\beta B_n h_\beta^{-1} \neq \emptyset.$$

It follows that for distinct $\alpha, \beta \in 2^{\mathbb{N}}$, we have

$$h_\beta^{-1} h_\alpha A^2 h_\alpha^{-1} h_\beta \cap G \setminus A^6 \neq \emptyset,$$

and so, as A^2 is symmetric, $h_\beta^{-1} h_\alpha \notin A^2$, whereby $h_\alpha A \cap h_\beta A = \emptyset$, again contradicting the assumption on A.

Thus, A^6 is somewhere comeagre and therefore, by Lemma 2.1, A^{12} is a neighbourhood of 1. ⊣

We notice that the above result gives rise to a notion of smallness, namely admitting a continuum of disjoint translates, in Polish groups, which is not closed under unions and hence does not correspond to an ideal.

Of course, ample generics is not something you are likely to find in many groups, and, in fact, most bigger Polish transformation groups even have meagre conjugacy classes. To see this, we can state a fairly general condition that implies that all conjugacy classes in a non-discrete Polish group are meagre. Namely,

Proposition 5.14 (C. Rosendal [23]). *Suppose $G \neq \{1\}$ is Polish such that for all infinite $S \subseteq \mathbb{N}$ and all open $V \ni 1$, the set*

$$A(S, V) = \{g \in G \mid \exists n \in S\, g^n \in V\}$$

is dense in G. Then all conjugacy classes in G are meagre.

Proof. Let $V_0 \supseteq V_1 \supseteq \cdots \ni 1$ be a neighbourhood basis at the identity and note that for every infinite $S \subseteq \mathbb{N}$

$$\begin{aligned} C(S) &= \{g \in G \mid \exists (s_n) \subseteq S\, g^{s_n} \to 1\} \\ &= \{g \in G \mid \forall k\, \exists s \in S \setminus [1, k]\, g^s \in V_k\} \\ &= \bigcap_k A(S, V_k). \end{aligned}$$

Then $C(S)$ is comeagre and conjugacy invariant. So if $\mathcal{O} \subseteq G$ were some comeagre conjugacy class, we would have

$$\mathcal{O} \subseteq \bigcap_{\substack{S \subseteq \mathbb{N} \\ \text{infinite}}} C(S).$$

But then if $g \in \mathcal{O}$, any sequence g^{n_i}, $n_i < n_{i+1}$, would have a subsequence converging to 1, and so $g = 1$ and $\mathcal{O} = \{1\}$. This contradicts that $\mathcal{O}$ is comeagre in $G \neq \{1\}$. ⊣

Among the groups that satisfy this condition are, for example,

- $\mathrm{Aut}([0,1], \lambda)$,
- $\mathrm{Iso}(\mathbb{U})$,
- $\mathcal{U}(\ell_2)$,

(see [23] and the references therein).

On another note, the following problems concerning comeagre conjugacy classes are still open:

(1) Find a Polish group with ample generics not embedding into $\mathcal{S}_\infty$,
(2) can a locally compact Polish group have a comeagre conjugacy class or even ample generics?,
(3) what about a compact metric group with a dense set of non-meagre conjugacy classes?

Here, by an observation by K. H. Hofmann, if there is a (non-trivial) locally compact Polish group with a comeagre conjugacy class, then there is one which is isomorphic to a closed subgroup of S_∞, which also would give a positive answer to (3). In connection with this, let us mention that E. Akin, E. Glasner and B. Weiss [1] have constructed a locally compact Polish group with a dense conjugacy class. Also, the author has constructed an example of an uncountable profinite group with a non-meagre conjugacy class.

Luckily, there are other techniques available for proving automatic continuity. For example, the techniques of J. D. Dixon, P. M. Neumann and S. Thomas [8] readily adapts to a more general context. The basic notion that is being used here and elsewhere (going back to at least R. D. Anderson [2]) is that of a *moiety*. To illustrate this, a moiety in $\mathbb{N}$ is just an infinite-coinfinite subset and a moiety in a 2-dimensional manifold is a proper closed subset homeomorphic to the unit disk $D \subseteq \mathbb{R}^2$, etc. I.e., moieties are small subsets that resemble the whole space.

Using moieties, one can analyse a number of concrete transformation groups of various structures.

THEOREM 5.15 (C. Rosendal & S. Solecki [26]). *The following groups are Steinhaus*

$$\mathrm{Homeo}(2^{\mathbb{N}}), \mathrm{Homeo}(2^{\mathbb{N}})^{\mathbb{N}}, \mathrm{Aut}(\mathbb{Q}, <), \mathrm{Homeo}(\mathbb{R})$$

with exponents 28, 108, 52, *and* 194 *respectively.*

Also, using methods of geometric topology of dimension 2, we have

THEOREM 5.16 (C. Rosendal [24]). *Suppose M is a compact 2-manifold, then* $\mathrm{Homeo}(M)$ *is Steinhaus.*

One would expect this result to generalise to higher dimensions at least for triangulable manifolds. However, the geometric topology in higher dimensions becomes significantly more complicated and thus the following conjecture remains open.

CONJECTURE 5.17. *Let M be a compact triangulable manifold. Then* $\mathrm{Homeo}(M)$ *is Steinhaus and hence satisfies automatic continuity.*

Lately, J. Kittrell and T. Tsankov have applied similar techniques elsewhere:

THEOREM 5.18 (J. Kittrell & T. Tsankov [17]). *Let E be a countable Borel equivalence relation on a Polish space preserving a E-ergodic Borel probability measure. Then the full group $[E]$ is Steinhaus.*

Here, if E is a countable Borel equivalence relation on a standard Borel space X preserving an E-ergodic Borel probability measure μ, the full group $[E]$ is defined by

$$[E] = \{T \in \mathrm{Aut}(X, \mu) \mid \forall^\mu x\ T(x)Ex\}.$$

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