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BOREL REDUCIBILITY AND CLASSIFICATION OF VON NEUMANN ALGEBRAS

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Abstract. We announce some new results regarding the classification problem for separable von Neumann algebras. Our results are obtained by applying the notion of Borel reducibility and Hjorth's theory of turbulence to the isomorphism relation for separable von Neumann algebras.

§1. Introduction. Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$, which we give the weak topology. A separable von Neumann algebra is a weakly closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. A von Neumann algebra is called a *factor* if its center consists of the scalar multiples of the identity. The factors make up the building blocks of von Neumann algebra theory: Any von Neumann algebra can be represented as a direct integral of factors (see [3, III.1.6]). A central problem in the theory of von Neumann algebras is to classify factors up to isomorphism (see [7]). The first steps towards a classification were obtained by Murray and von Neumann, [25], when they introduced the notion of the *type* of a von Neumann algebra and gave examples of factors in each of the classes. Another major advance towards classifying factors was A. Connes' thesis [5] where he further extended the notion of type to split the type III case in the subtypes III_λ, $0 \leq \lambda \leq 1$.

Denote by $\mathrm{vN}(\mathcal{H})$ the set of von Neumann algebras on $\mathcal{H}$. E. Effros [11], [12] defined a Borel structure on $\mathrm{vN}(\mathcal{H})$ and proved that this structure is standard and that the set of factors $\mathcal{F}(\mathcal{H})$ is Borel. One of Effros's goals was to show that the set of isomorphism classes of factors endowed with the quotient Borel structure was not standard, which would imply the existence

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of uncountably many non-isomorphic von Neumann factors. At that time only few examples were known of non-isomorphic infinite dimensional factors, thus a solution of this problem was of major importance. The existence of a continuum of non-isomorphic factors was finally shown by Powers in 1967 for the type III case and by McDuff in 1969 for the type II_1 case. In 1971 J. Woods solved Effros's problem: The isomorphism relation for factors is not smooth. In modern terminology, what Woods proved was that E_0, the equivalence relation on $2^{\mathbb{N}}$ of eventual equality, is Borel reducible to isomorphism of so-called ITPFI factors (see §4.2 or [3, III.3.1] for a definition).

Until recently, Woods's result was one of the few theorems of its kind in the study of von Neumann algebras. For example, it remained an open problem to show that isomorphism of factors of type II_1 is not smooth. In a forthcoming paper [33] we apply the notion of Borel reducibility from descriptive set theory to obtain information about the classification problem for separable von Neumann factors.

Let us briefly recall the key notions surrounding Borel reducibility, but otherwise refer to the excellent introduction in [20], or the survey [23]. If E and F are equivalence relations on standard Borel spaces X and Y, respectively, then we say that E is *Borel reducible* to F, written $E \leq_B F$, if there is a Borel $f\colon X \to Y$ such that

$$x\,E\,y \iff f(x)\,F\,f(y).$$

Thus if $E \leq_B F$ then the points of X can be classified up to E equivalence by a Borel assignment of invariants that we may think of as F-equivalence classes. E is *smooth* if it is Borel reducible to the equality relation on $\mathbb{R}$. While smoothness is desirable, it is most often too much to ask for: As mentioned above, E_0 is not smooth. A more generous class of invariants which seem natural to consider are countable groups, graphs, fields, or other countable structures, considered up to isomorphism. Thus, following [20], we will say that an equivalence relation E is *classifiable by countable structures* if there is a countable language $\mathcal{L}$ such that $E \leq_B \simeq^{\mathrm{Mod}(\mathcal{L})}$, where $\simeq^{\mathrm{Mod}(\mathcal{L})}$ denotes isomorphism in $\mathrm{Mod}(\mathcal{L})$, the Polish space of countable models of $\mathcal{L}$ with universe $\mathbb{N}$. We note that E_0 may be seen to be classifiable by countable structures.

It turns out that even allowing this more generous class of invariants is not enough: There are many natural classification problems in mathematics where countable structures do not suffice as invariants. Hjorth conceived of his theory of *turbulence* (see again [20]) as a general tool to prove that various equivalence relations are not classifiable by countable structures. One of the early applications of this theory was due to Foreman and Weiss [14], who showed that the measure preserving ergodic transformations on the unit interval are not classifiable, up to conjugacy, by countable structures.

Subsequently, similar results have been achieved for the weaker notion of orbit equivalence (see §2) of measure preserving actions of non-amenable groups, see [35], [21].

Our main results are

THEOREM 1. *The isomorphism relation for factors of type* II_1, II_∞ *and* III_λ, $0 \leq \lambda \leq 1$ *is not classifiable by countable structures.*

THEOREM 2. *If* $\mathcal{L}$ *is a countable language, then* $\simeq^{\mathrm{Mod}(\mathcal{L})} <_B \simeq^{\mathcal{F}_{\mathrm{II}_1}}$, *where* $\simeq^{\mathcal{F}_{\mathrm{II}_1}}$ *denotes the isomorphism relation for factors of type* II_1.

Since it is known that the isomorphism relation for countable graphs, say, is complete analytic, we obtain the following as a consequence of Theorem 2:

THEOREM 3. *The isomorphism relation* $\simeq^{\mathcal{F}_{\mathrm{II}_1}}$ *of factors of type* II_1 *is a complete analytic subset of* $\mathcal{F}_{\mathrm{II}_1} \times \mathcal{F}_{\mathrm{II}_1}$, *where* $\mathcal{F}_{\mathrm{II}_1}$ *denotes the Borel set of* II_1 *factors.*

The proofs of these results rely heavily on results obtained by Popa's novel "deformation rigidity techniques", in particular on the class of HT factors (discussed below) introduced in [27], as well as Hjorth's theory of turbulence. In this paper we will first in §2 give some background regarding von Neumann algebra factors, in particular regarding the *group-measure space construction*, which plays the starring role in all the proofs above. In §3 we give an outline of the proofs. In §4 we discuss some of the open problems and questions that remain.

§2. Von Neumann algebras. A separable von Neumann Algebra is a weakly closed self-adjoint algebra of operators on a separable complex Hilbert space. A von Neumann algebra is called a factor if its center only consists of the scalar multiples of the identity. A von Neumann algebra N is said to be *finite* if it admits a finite faithful normal tracial state, i.e., a linear functional $\tau \colon N \to \mathbb{C}$ such that: $\tau(x^*x) \geq 0$, $\tau(x^*x) = 0$ iff $x = 0$, $\tau(1) = 1$, $\tau(xy) = \tau(yx)$ and the unit ball of N is complete with respect to the norm given by the trace $\|x\|_\tau = \tau(x^*x)$. If such a trace exists it need not to be unique, however, a fundamental fact is that if a finite von Neumann algebra is also a factor, then it has a unique trace.

Some basic examples of finite von Neumann algebras are:

1. $L^\infty(X, \mu)$, the set of essentially bounded measurable functions on a standard Borel probability space (X, μ). They act by multiplication on $L^2(X, \mu)$. Here the trace is given by the integral. The Borel functional calculus yields that any separable Abelian finite von Neumann algebra is of this form.
2. $M_n(\mathbb{C})$, the set of $n \times n$ complex matrices with the normalized trace $\mathrm{Tr}_{M_n(\mathbb{C})}$. Any finite dimensional von Neumann factor is of this form.

3. $\bigoplus_{i=1}^{k} M_{n_i}(\mathbb{C})$ with the trace given by $\sum_{i=1}^{k} c_i \operatorname{Tr}_{M_{n_i}(\mathbb{C})}$; $c_i > 0$, $\sum_{i=1}^{k} c_i = 1$. Moreover any finite dimensional von Neumann algebra is of this form.

Von Neumann algebras are categorized into *types* according to the behavior of the lattice of projections. The types are called I, II_1, II_∞ and III_λ, $0 \leq \lambda \leq 1$, see [3, III.1.4] or [32] for an introduction. A von Neumann algebra is of type II_1 if it is finite and it doesn't have minimal projections (a projection p in a von Neumann algebra $M \subseteq \mathcal{B}(\mathcal{H})$ is said to minimal if there is no projection $q \in M$ with $0 < q < p$, where $q < p$ means $\operatorname{im}(q) \subsetneq \operatorname{im}(p)$.) If M is a II_1 factor and τ is the normalized trace on M then

$$\{\tau(p)\colon p \in M \text{ is a projection}\} = [0, 1].$$

Factors of type II_∞ are of the form

$$M \otimes \mathcal{B}(\mathcal{H})$$

where M is a factor of type II_1 and $\mathcal{H}$ is an infinite dimensional Hilbert space. A factor of type II_∞ has a semifinite trace which is unique up to scaling.

Murray and von Neumann already exhibited examples of factors of type II_1. This was done using two fundamental constructions, the *group von Neumann algebra* and the *group-measure space construction*. Since both constructions still play a preponderant role in the theory and they are at the core of some of our arguments, we give here a more or less detailed account of how they are constructed.

2.1. The group von Neumann algebra. Let G be an infinite discrete group. $\ell^2(G)$ is the infinite dimensional Hilbert space with orthonormal basis $\{\xi_g\colon g \in G\}$. The group G acts on $\ell^2(G)$ by the left regular representation $u_g(\xi_h) = \xi_{gh}$. The *group von Neumann algebra* $L(G)$ is the von Neumann algebra generated by the unitary operators $u_g \in \mathcal{B}(\ell^2(G))$, that is, $L(G) = \overline{\langle u_g\colon g \in G\rangle}^{wo}$, the closure in the weak operator topology of the algebra generated by the u_g. The trace is given by $\tau(u_g) = \langle u_g(\xi_e), \xi_e\rangle$, where e denotes the identity of G. It is easy to show that $L(G)$ is a factor iff the group G is ICC (infinite conjugacy classes) i.e., for each $g \in G\backslash\{e\}$ the conjugacy class $\{hgh^{-1}\colon h \in G\}$ is infinite.

Let $\boldsymbol{GP}$ denote the Polish space of all countable groups with universe $\mathbb{N}$, and consider the equivalence relation $\sim_{\mathrm{vN}}$ in $\boldsymbol{GP}$ defined by

$$G \sim_{\mathrm{vN}} H \iff L(G) \text{ is isomorphic to } L(H).$$

We do not know how complex this equivalence relation is (cf. Problem 7 below.) Two outstanding (yet seemingly unrelated) open problems in the theory are concerned with this equivalence relation:

(a) Is it true that $\mathbb{F}_n \nsim_{\mathrm{vN}} \mathbb{F}_m$ when $n \neq m$, $n, m \geq 2$? That is, when is $L(\mathbb{F}_n)$ isomorphic to $L(\mathbb{F}_m)$? Here $\mathbb{F}_n$ denotes the free group on n generators.

Free probability was first envisioned by Voiculescu as an attempt to solve this problem (see [38], [39] for an overview.)

(b) If G and H are countably infinite ICC property (T) groups, does $G \sim_{\mathrm{vN}} H$ imply that G is isomorphic to H? This problem is known as Connes's conjecture.[1]

The first appearance of property (T) in the context of operator algebras is Connes result [8] stating that the fundamental group of the group von Neumann algebra of an ICC property (T) group is countable. Recall that the fundamental group of a II_1 factor M is defined as

$$F(M) = \{\tau(p)/\tau(q) \mid pMp \simeq qMq\}$$

where $p, q \in M$ are non-zero projections and τ denotes the trace on M. The fundamental group is a subgroup of $\mathbb{R}_{>0}$. As a consequence of his work on HT factors, Popa gave in [27] the first example of a type II_1 factor with trivial fundamental group, solving a longstanding problem in the theory. Going back to Connes's conjecture, in [28] Popa gave what may be seen as a partial affirmative answer to the conjecture but for actions of property (T) groups. In §3 we explain some aspects of Popa's work that are pertinent to our results while for a more thorough introduction to Popa's theory and its many applications we refer the reader to the survey papers [10], [29] and [36].

It is worth mentioning that in sharp contrast with these two problems, Connes's seminal work on injective factors [6] shows that if a group G is ICC and amenable, $L(G)$ is isomorphic to the unique hyperfinite II_1 factor R.

2.2. The group-measure space construction. Let G be a countably infinite discrete group which acts in a measure preserving way on a Borel probability space (X, μ). For each $g \in G$ and $\zeta \in L^2(X, \mu)$ the formula

$$\sigma_g(\zeta)(x) = \zeta(g^{-1} \cdot x)$$

defines a unitary operator on $L^2(X, \mu)$.

We identify the Hilbert space $\mathcal{H} = L^2(G, L^2(X, \mu))$ with the Hilbert space of formal sums $\sum_{g \in G} \zeta_g \xi_g$, where the coefficients ζ_g are in $L^2(X, \mu)$ and satisfy $\sum_g \|\zeta_g\|^2_{L^2(X,\mu)} < \infty$, and ξ_g are indeterminates indexed by the elements of G. The inner product on $\mathcal{H}$ is given by

$$\langle \sum_{g \in G} \zeta_g(x)\xi_g, \sum_{g \in G} \zeta'_g(x)\xi_g \rangle = \sum_{g \in G} \langle \zeta_g, \zeta'_g \rangle_{L^2(X,\mu)}.$$

Both $L^\infty(X, \mu)$ and G act by left multiplication on $\mathcal{H}$ by the formulas

$$f(\zeta_g(x)\xi_g) = ((f(x)\zeta_g(x))\xi_g,$$
$$u_h(\zeta_g(x)\xi_g) = \sigma_h(\zeta_g)(x)\xi_{hg},$$

[1] See [2] for more information regarding Kazhdan's property (T).

where $f \in L^\infty(X,\mu)$, $\zeta_g(x) \in L^2(X,\mu)$ and $g,h \in G$. Thus if we denote by $\mathcal{FS}$ the set of finite sums,

$$\mathcal{FS} = \{\sum_{g\in G} f_g u_g \colon\ f_g \in L^\infty(X,\mu),\ f_g = 0, \text{ except for finitely many } g\},$$

then each element in $\mathcal{FS}$ defines a bounded operator on $\mathcal{H}$. Moreover, multiplication and involution in $\mathcal{FS}$ satisfy the formulas

$$(f_g u_g)(f_h u_h) = f_g \sigma_g(f_h) u_{gh}$$

and

$$(f u_g)^* = \sigma_{g^{-1}}(f^*) u_{g^{-1}}$$

and so $\mathcal{FS}$ is a $*$-algebra. By definition, the *group-measure space von Neumann algebra* is the weak operator closure of $\mathcal{FS}$ on $\mathcal{B}(\mathcal{H})$ and it is denoted by $L^\infty(X,\mu) \rtimes_\sigma G$. The trace on $\mathcal{FS}$, defined by

$$\tau(\sum_{g\in G} f_g u_g) = \int_X f_e \, d\mu,$$

extends to a faithful normal tracial state in $L^\infty(X) \rtimes_\sigma G$ by the formula $\tau(T) = \langle T(\xi_e), \xi_e\rangle$, where e represents the identity of G. Observe that $L^\infty(X,\mu)$ embeds into $L^\infty(X) \rtimes_\sigma G$ via the map $f \mapsto f u_e$ and has the property that its normalizer inside $L^\infty(X) \rtimes_\sigma G$,

$$\begin{aligned}&\mathcal{N}_{L^\infty(X)\rtimes_\sigma G}(L^\infty(X,\mu)) = \\ &\{u \in \mathcal{U}(L^\infty(X) \rtimes_\sigma G) \colon u\, L^\infty(X,\mu)\, u^* = L^\infty(X,\mu)\}\end{aligned}$$

generates a weakly dense subalgebra in $L^\infty(X) \rtimes_\sigma G$.

If σ is a free action, then $L^\infty(X,\mu)$ is a MASA (maximal abelian subalgebra) of M, in which case it is called a *Cartan subalgebra* of $L^\infty(X) \rtimes_\sigma G$, (i.e., a MASA with a weakly dense normalizer). If σ is free then $L^\infty(X) \rtimes_\sigma G$ is a factor (of type II_1, since G is infinite) if and only if σ is an ergodic action.

There is an important connection between the notion of *orbit equivalence* and certain isomorphisms between group-measure space von Neumann algebras. Recall that if σ and τ are measure preserving actions on standard Borel probability spaces (X,μ) and (Y,ν), respectively, of possibly different groups G and H, we say that σ and τ are *orbit equivalent* if there is a measure preserving bijection $\theta \colon X \to Y$ such that

$$x \, E_\sigma \, x' \iff \theta(x) \, E_\tau \, \theta(x') \text{ (a.e.)},$$

i.e., if σ and τ generate "isomorphic" orbit equivalence relations E_σ and E_τ. Feldman and Moore showed in [13] that two free ergodic measure preserving actions σ and τ are orbit equivalent if and only if their corresponding inclusions of Cartan subalgebras $L^\infty(X) \subset L^\infty(X) \rtimes_\sigma G$ and

$L^\infty(Y) \subset L^\infty(Y) \rtimes_\tau H$ are isomorphic. Thus the study of orbit equivalence of measure preserving group actions can be translated into a problem regarding inclusions of finite von Neumann algebras.

§3. Outline of the proofs.

3.1. Theorem 1. Let $a\colon \mathrm{SL}(2,\mathbb{Z}) \curvearrowright \mathbb{Z}^2$ be the natural linear action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{Z}^2$. Consider the natural measure preserving ergodic a.e. free action σ_0 of $\mathrm{SL}(2,\mathbb{Z})$ on $X = \mathbb{T}^2$ equipped with the Haar measure μ and given by

$$\sigma_0(g)(\chi)(h) = \chi(a(g^{-1})(h)),$$

where we identify $\mathbb{T}^2$ with the character group of $\mathbb{Z}^2$. The matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a copy of $\mathbb{F}_2$ as a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$, and so we get an action $\sigma\colon \mathbb{F}_2 \curvearrowright \mathbb{T}^2$ by letting $\sigma = \sigma_0 \mid \mathbb{F}_2$. This action is still ergodic, see for instance [34, §2]. The group measure space factor $L^\infty(X,\mu) \rtimes_\sigma \mathbb{F}_2$ was studied in detail by Sorin Popa in [27], where it was shown that $L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes_\sigma \mathbb{F}_2$ is a so-called HT_s Cartan subalgebra (see below), and that it is the unique HT_s Cartan subalgebra in $L^\infty(X,\mu) \rtimes_\sigma \mathbb{F}_2$ up to conjugation by a unitary. In effect this means that the unitary conjugacy class of $L^\infty(X,\mu)$ is definable inside of $L^\infty(X,\mu) \rtimes_\sigma \mathbb{F}_2$ and depends only on the isomorphism type of $L^\infty(X,\mu) \rtimes_\sigma \mathbb{F}_2$.

We refer to [27, Definition 6.1] for the exact definition of HT_s, but in short, the H stands for Haagerup property, meaning that $L^\infty(X,\mu)$ has the relative Haagerup property in $L^\infty(X,\mu) \rtimes_\sigma \mathbb{F}_2$, and the T means that it also has the relative property (T), that is, $L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes_\sigma \mathbb{F}_2$ is a rigid inclusion in the sense defined by Popa. These are von Neumann algebra generalizations of the corresponding properties for discrete groups. Rather than explaining the technical definition of the Haagerup property for groups, we refer the reader to the monograph [4], and to the recent survey paper [26] in this journal for applications and open questions regarding property H. Here we just mention that amenable groups and the free groups $\mathbb{F}_n$ have the Haagerup property [17], and hence by [27, Theorem 3.1] the inclusion $L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes_\sigma \mathbb{F}_2$ has the relative property H. For groups, the property H and the property (T) are mutually exclusive, in the sense that a discrete group that satisfies both properties must be finite. A remarkable fact is that the inclusion of groups $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ satisfies both the relative property (T) and the property H. It is the combination of deformation (i.e., the Haagerup property) and rigidity (i.e., property (T)), in particular the inclusion $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$, that is the engine behind Popa's results in [27], and in turn, the engine behind our results.

Returning to the proof, we now proceed as in [34]: Let a, b be generators for $\mathbb{F}_2$, and let $T_a, T_b \in \mathrm{Aut}(X, \mu)$ be the measure preserving transformations corresponding to the action of a and b according to σ. (Here $\mathrm{Aut}(X, \mu)$ denotes the group of measure preserving transformations, equipped with Polish group topology it inherits when naturally identified with a weakly closed subgroup of the unitary group of $L^2(X, \mu)$.) It was shown in [34, §3] that the set

$$\mathrm{Ext}(\sigma) = \{S \in \mathrm{Aut}(X, \mu)\colon T_a, T_b \text{ and } S \text{ generate an a.e. free action of } \mathbb{F}_3\}$$

forms a dense G_δ in $\mathrm{Aut}(X, \mu)$ and that if we, for $S \in \mathrm{Ext}(\sigma)$, denote by $\sigma_S\colon \mathbb{F}_3 \curvearrowright (X, \mu)$ the resulting a.e. free ergodic $\mathbb{F}_3$-action, then the equivalence relation

$$S_1 \sim_{oe} S_2 \iff \sigma_{S_1} \text{ is orbit equivalent to } \sigma_{S_2}$$

has meagre classes and the set of transformations with dense $\sim_{oe}$-class is comeagre. It was pointed out by Kechris in [22, Theorem 17.1] that this equivalence relation is *generically S_∞-ergodic*, meaning that if Y is a Polish S_∞ space and $f\colon \mathrm{Aut}(X, \mu) \to Y$ is a Baire measurable map which satisfies

$$S_1 \sim_{oe} S_2 \Longrightarrow (\exists g \in S_\infty) g \cdot f(S_1) = f(S_2)$$

then f must be constant on a comeagre set. Since $\simeq^{\mathrm{Mod}(\mathcal{L})}$ is induced by a continuous S_∞ action, this shows that $\sim_{oe}$ is not classifiable by countable structures.

For $S \in \mathrm{Ext}(\sigma)$, let

$$M_S = L^\infty(X, \mu) \rtimes_{\sigma_S} \mathbb{F}_3.$$

The fact that $\mathbb{F}_3$ has the Haagerup property and that $L^\infty(X, \mu) \rtimes_\sigma \mathbb{F}_2 \subseteq L^\infty(X, \mu) \rtimes_{\sigma_S} \mathbb{F}_3$ can be seen to imply that $L^\infty(X, \mu)$ is the unique (up to perturbation by a unitary) HT_s Cartan subalgebra of $L^\infty(X, \mu) \rtimes_{\sigma_S} \mathbb{F}_3$.

One now shows that the map $S \mapsto M_S$ is Borel. Further, if $S \sim_{oe} S'$ then $M_S \simeq M_{S'}$ by Feldman and Moore's Theorem. On the other hand, if $M_S \simeq M_{S'}$ then any isomorphism $\varphi\colon M_S \to M_{S'}$ must, after possibly perturbing it with a unitary, map $L^\infty(X, \mu) \subset M_S$ to $L^\infty(X, \mu) \subset M_{S'}$. But then by Feldman–Moore, we must have that σ_S is orbit equivalent to $\sigma_{S'}$. Thus $S \mapsto M_S$ provides a Borel reduction of $\sim_{oe}$ to $\simeq^{\mathcal{F}_{\mathrm{II}_1}}$. Consequently, since $\sim_{oe}$ is not classifiable by countable structures, neither is $\simeq^{\mathcal{F}_{\mathrm{II}_1}}$.

The II_∞ and III_λ cases are consequences of the II_1 case, but this requires more sophisticated use of the rigidity properties of the factors M_S above. For the II_∞ case, one shows that the map

$$S \mapsto M_S \otimes \mathcal{B}(l^2(\mathbb{N})),$$

where $\mathcal{B}(l^2(\mathbb{N}))$ denotes the bounded operators on $l^2(\mathbb{N})$, is a Borel reduction of $\sim_{oe}$ to $\simeq^{\mathrm{II}_\infty}$. For the III_λ case, the map

$$S \mapsto M_S \otimes R_\lambda$$

provides a Borel reduction of $\sim_{oe}$ to $\simeq^{\mathrm{III}_\lambda}$, where R_λ is a (fixed) injective factor of type III_λ.

3.2. Theorem 2. Theorem 2 relies on another deformation-rigidity result of Sorin Popa. Recall that if G is a countably infinite group then the (left) Bernoulli shift $\beta\colon G \curvearrowright [0,1]^G$ is defined by

$$\beta(g)(x)(h) = x(g^{-1}h).$$

The Bernoulli shift is ergodic and preserves the product measure.

Theorem (Popa, [28, 7.1]). *Suppose G_1 and G_2 are countably infinite discrete groups, β_1 and β_2 are the corresponding Bernoulli shifts on $X_1 = [0,1]^{G_1}$ and $X_2 = [0,1]^{G_2}$, respectively, and $M_1 = L^2(X_1) \rtimes_{\beta_1} G_1$ and $M_2 = L^2(X_2) \rtimes_{\beta_2} G_2$ are the corresponding group-measure space II_1 factors. Suppose further that G_1 and G_2 are ICC groups having the relative property* (T) *over an infinite normal subgroup. Then $M_1 \simeq M_2$ iff $G_1 \simeq G_2$.*

The group $\mathrm{SL}(3,\mathbb{Z})$ has property (T) outright (see [2]) and is ICC, and so any group of the form $H \times \mathrm{SL}(3,\mathbb{Z})$, where H is ICC, satisfies the hypotheses of Popa's Theorem. Thus, to prove Theorem 2, it suffices to show that if $\mathcal{L}$ is a countable language, then $\simeq^{\mathrm{Mod}(\mathcal{L})}$ is Borel reducible to isomorphism of groups of the form $H \times \mathrm{SL}(3,\mathbb{Z})$, H ICC, i.e., that isomorphism of groups of the form $H \times \mathrm{SL}(3,\mathbb{Z})$ is *Borel complete* for countable structures, in the sense of [15].

To this end, we modify a construction by Mekler, [24]. Mekler defines a notion of 'nice graph', and proves (in effect) that the isomorphism relation of countable connected nice graphs is Borel complete for countable structures. Mekler then defines from a given countable nice graph Γ (and a prime p, which we shall keep fixed here) a countable group $G(\Gamma)$, which we will call the *Mekler group* of Γ, and shows that for nice graphs, $\Gamma_1 \simeq \Gamma_2$ iff $G(\Gamma_1) \simeq G(\Gamma_2)$. The association $\Gamma \mapsto G(\Gamma)$ is Borel, and moreover, for every graph automorphism of Γ there is a corresponding group automorphism of $G(\Gamma)$. However, the groups $G(\Gamma)$ are generally *not* ICC.

To remedy this, we consider for each connected nice graph Γ the nice graph $\Gamma_{\mathbb{F}_2}$, defined by

$$(m,g)\,\Gamma_{\mathbb{F}_2}\,(n,h) \iff m\,\Gamma\,n \wedge g = h,$$

consisting of $\mathbb{F}_2$ copies of Γ. ($\Gamma_{\mathbb{F}_2}$ is not connected, but still nice.) Clearly, $\mathbb{F}_2$ acts by graph automorphisms on $\Gamma_{\mathbb{F}_2}$. Going to the corresponding Mekler group $G(\Gamma_{\mathbb{F}_2})$, we have a corresponding action of $\mathbb{F}_2$ by group automorphisms on $G(\Gamma_{\mathbb{F}_2})$. Thus we may form the semi-direct product $G(\Gamma_{\mathbb{F}_2}) \rtimes \mathbb{F}_2$. One now checks that this groups is indeed ICC. Thus the group

$$G_\Gamma = \mathrm{SL}(3,\mathbb{Z}) \times G(\Gamma_{\mathbb{F}_2}) \rtimes \mathbb{F}_2,$$

is an ICC group with the relative property (T) over $\mathrm{SL}(3,\mathbb{Z})$. The argument is finished by arguing that $\mathrm{SL}(3,\mathbb{Z})$, as a subgroup of G_Γ, consists exactly of the elements of G_Γ which commutes with all elements of

$$\{g \in G_\Gamma : (\exists \chi \in \mathrm{Char}(G_\Gamma))\ \chi(g) \neq 1\}.$$

Supposing now that G_{Γ^1} and G_{Γ^2} are isomorphic, it follows that $G(\Gamma^1_{\mathbb{F}_2}) \rtimes \mathbb{F}_2$ is isomorphic to $G(\Gamma^2_{\mathbb{F}_2}) \rtimes \mathbb{F}_2$, from which it may in turn be deduced that $G(\Gamma^1_{\mathbb{F}_2})$ is isomorphic to $G(\Gamma^2_{\mathbb{F}_2})$. Then by Mekler's construction, $\Gamma^1_{\mathbb{F}_2} \simeq \Gamma^2_{\mathbb{F}_2}$, so if Γ^1 and Γ^2 are connected nice graphs then $\Gamma^1 \simeq \Gamma^2$. Thus the isomorphism relation of connected nice graphs is Borel reducible to isomorphism of countable groups with the relative property (T) over an infinite normal subgroup, which by Popa's Theorem is all we needed to show.

§4. Some open problems. In this section we briefly discuss some open problems related to the results stated above that we find may be of interest to logicians.

4.1. The Effros–Maréchal topology. The space $\mathrm{vN}(\mathcal{H})$ has a natural Polish topology, called the *Effros–Maréchal* topology. It is most easily defined as follows: Let $L_1(\mathcal{H})$ denote the unit ball in $\mathcal{B}(\mathcal{H})$, which is compact in the weak topology. Then the map

$$M \mapsto M \cap L_1(\mathcal{H})$$

is 1-1, and so we may identify $\mathrm{vN}(\mathcal{H})$ with a subset of $K(L_1(\mathcal{H}))$, the space of compact subsets of $L_1(\mathcal{H})$. The Effros–Maréchal topology is the topology $\mathrm{vN}(\mathcal{H})$ inherits under this identification. It may be shown that $\mathrm{vN}(\mathcal{H})$ is Polish in this topology, see [18, Theorem 2.8]. The set of factors $\mathcal{F}$ forms a dense G_δ set in $\mathrm{vN}(\mathcal{H})$, see [19, p. 402]. The most fundamental open problem seems to be:

PROBLEM 1. *Are the isomorphism classes in* $\mathrm{vN}(\mathcal{H})$ (*equivalently,* $\mathcal{F}$) *meagre*? *Are the unitary conjugacy classes meagre*?

If either part of problem 1 is answered in the affirmative, the next natural question to ask is:

PROBLEM 2. *Does the unitary group act turbulently on* $\mathrm{vN}(\mathcal{H})$?

Even though the subsets of II_1, II_∞ or III factors are not Polish in the Effros–Maréchal topology, they all form Borel sets, and it is tempting to ask if one can find 'natural' topologies on these spaces in which Problem 1 and 2 would make sense. We remark that by [1, 5.2.1], it is possible to find a Polish topology (with the same Borel structure) on the subsets II_1, II_∞ and III such that the conjugation action of the unitary group becomes a continuous action, but by the same token, [1, 5.1.6], applying this too crudely might make a conjugacy class clopen. Thus what we are really asking is if these sets can be given Polish topologies where Problem 1 and 2 have affirmative answers.

It should be noted that Problem 1 is strongly related to the so-called Connes embedding conjecture (see for instance [26]) for separable von Neumann algebras, which states that every separable type II_1 factor can be embedded into the ultrapower $R^{\mathbb{N}}/\mathcal{U}$, where R is the injective type II_1 factor, and $\mathcal{U}$ is an ultrafilter in $\mathbb{N}$. ($R^{\mathbb{N}}/\mathcal{U}$ is usually denoted R^{ω} in the von Neumann algebra literature, since it is convention there to use ω to denote the ultrafilter.) Indeed, an affirmative answer to Problem 1 is tantamount to refuting this conjecture, by the work of Haagerup and Winsløw in [19]. Namely, Haagerup and Winsløw have shown that the Connes' embedding conjecture is equivalent to the statement that the injective factors are dense in $\mathcal{F}$. Since the set of injective factors is G_δ, Connes' embedding conjecture is equivalent to that the generic element in $\mathcal{F}$ is injective. On the other hand, Haagerup and Winsløw have also shown that the type III_1 factors form a dense G_δ subset of $\mathcal{F}$. Hence the Connes embedding conjecture is equivalent to the assertion that the isomorphism class of the (unique) injective type III_1 factor forms a dense G_δ set.

4.2. ITPFI **factors and** T**-sets.** A factor M is called an ITPFI factor (short for Infinite Tensor Product of Factors of type I, also called an Araki–Woods factor), if it has the form

$$M = \bigotimes_{k=1}^{\infty} (M_{n_k}(\mathbb{C}), \phi_k)$$

where $M_{n_k}(\mathbb{C})$ denotes the algebra of $n_k \times n_k$ matrices and the ϕ_k are faithful normal states. (We refer the reader to [3, III.3.1] for the necessary basics regarding infinite tensor products.). Among the ITPFI factors, the *Powers factors* R_λ, $0 < \lambda < 1$, are defined by taking $n_k = 2$ for all k and $\phi_k(x) = Tr(\rho_\lambda x)$ where

$$\rho_\lambda = \begin{pmatrix} \frac{1}{1+\lambda} & 0 \\ 0 & \frac{\lambda}{1+\lambda} \end{pmatrix}.$$

Historically, the importance of the Powers factors is twofold: they provided the first example of uncountably many non isomorphic von Neumann factors (all of type III) [31]. They were also the starting point of the asymptotic analysis of factors carried on in the late sixties and early seventies that culminated with Connes classification of type III factors [5] and of injective factors [6]. Since ITPFI factors are in particular injective factors, a corollary of Connes work is that up to isomorphism there is only one ITPFI factor of type III_λ, for each $\lambda \neq 0$. At the same time, Woods proved [40] that the classification problem for ITPFI factors is not smooth by showing that E_0 is Borel reducible to isomorphism of ITPFI factors. (see [41] for an historical overview of ITPFI factors and chapter §5 of [9] for an overview of Connes work.) Of course, the factors analyzed by Woods in [40] are of type III_0 and injective. In §4 of our forthcoming paper [33] we show

THEOREM 4. *The isomorphism relation for injective factors of type* III_0 *is not classifiable by countable structures.*

However the following remains open:

PROBLEM 3. *Are* ITPFI *factors classifiable by countable structures*?

Woods uses an invariant $\rho(M)$, defined as

$$\rho(M) = \{\lambda \in (0,1)\colon M \otimes R_\lambda \simeq R_\lambda\}$$

to distinguish the factors constructed there up to isomorphism. (Here R_λ, $\lambda \in (0,1)$, denotes the Powers factors, see [3, III.3.1.7]). The invariant ρ has been replaced by the Connes invariant $T(M)$, called the T-*set* of M, the general definition of which is rather intricate. In the context of ITPFI factors, $T(M)$ is given by

$$T(M) = \{t \in \mathbb{R}\colon \sum_{i=1}^{\infty} \big(1 - |\sum_k (\alpha_k^{(i)})^{1+it}|\big) < \infty\},$$

where $\alpha_k^{(i)}$ denotes kth eigenvalue of ϕ_i, see [3, III.4.6.9]. From this it can be deduced that the T-set is a K_σ subgroup of $\mathbb{R}$. It has been shown that all countable subgroups of $\mathbb{R}$ and many uncountable subgroups are realizable as T-sets of an ITPFI factor (see [16]), but the following seems to be open:

PROBLEM 4. *Is every* K_σ *subgroup of* $\mathbb{R}$ *the* T*-set of some* ITPFI *factor*?

The most natural approach to this problem would be to try to construct from a given K_σ subgroup $G \leq \mathbb{R}$ a corresponding ITPFI factor M with $T(M) = G$. Can such a construction be natural? More precisely, let

$$\mathcal{S}_\sigma(\mathbb{R}) = \{(K_n) \in K(\mathbb{R})^{\mathbb{N}}\colon (\forall n) K_n = -K_n \wedge K_n \subseteq K_{n+1} \\ \wedge K_n + K_n \subseteq K_{n+1}\}$$

and let

$$(K_n) \sim (K_n') \iff \bigcup K_n = \bigcup K_n'.$$

Then we can identify a K_σ subgroup of $\mathbb{R}$ with an equivalence class in $\mathcal{S}_\sigma(\mathbb{R})/\sim$.

PROBLEM 5. *Is there a Borel* $f\colon \mathcal{S}_\sigma(\mathbb{R}) \to$ ITPFI *such that*

$$T(f(K_n)) = \bigcup K_n$$

and if $(K_n) \sim (K_n')$ *then* $f(K_n) \simeq f(K_n')$?

4.3. Group von Neumann algebras vs. group-measure space von Neumann algebras. It is clear from the outline of the proof of Theorem 1 that what we have really shown is that II_1 factors that arise from the group measure space construction are not classifiable by countable structures. Even more specifically, we are dealing with those that arise from an $\mathbb{F}_3$-action. The proof may be adapted to show that for any $n \geq 2$, the group measure space von Neumann algebras arising from an $\mathbb{F}_n$ action are not classifiable by countable structures. However, in the light of the results of [21], it is natural to ask:

PROBLEM 6. *If G is a countably infinite non-amenable group, is it true that the group measure space* II_1 *factors arising from probability measure preserving ergodic G-actions are not classifiable up to isomorphism by countable structures?*

Our last problem is about the contrasting situation for group von Neumann algebras. It is generally known that group von Neumann algebras and group measure spaces von Neumann algebras can be rather different. Indeed, one of the most striking applications of free probability theory is Voiculescu's Theorem [37] stating that the group von Neumann algebras $L(\mathbb{F}_n)$ of free groups on n generators $n \geq 2$, don't have Cartan subalgebras, thus they are not group measure space von Neumann algebras. This result was generalized recently on [30] without using free probability theory.

Recall the equivalence $\sim_{\mathrm{vN}}$ from our discussion of the group von Neumann algebra (§2.1):

$$G \sim_{\mathrm{vN}} H \iff L(G) \text{ is isomorphic to } L(H).$$

In light of Theorem 1, it is natural to ask:

PROBLEM 7. *Is the isomorphism relation for group von Neumann algebras of countable groups classifiable by countable structures? That is, is $\sim_{\mathrm{vN}}$ classifiable by countable structures?*

One could consider Problem 7 more narrowly and ask if a "weak" version of Connes' conjecture, discussed in §2.1, is true: Is the relation $\sim_{\mathrm{vN}}$ restricted to the class of ICC property (T) groups classifiable by countable structures? A negative answer to this would of course refute Connes' conjecture in a very strong way.

On the other hand one could, more broadly, ask if the classification problem for group von Neumann algebras is as difficult as the one for group-measure space von Neumann algebras. More precisely, what is the relationship between $\sim_{\mathrm{vN}}$ and the isomorphism relation for group-measure space factors in the Borel reducibility hierarchy?

All of these questions are, to our knowledge, wide open and quite interesting, since their solution may shed some light on Connes' conjecture and the general relationship between a group and its group von Neumann algebra.

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