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REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Steve Awodey (Managing Editor), John Baldwin, John Burgess, Mark Colyvan, Anuj Dawar, Mirna Džamonja, Marcelo Fiore, Hannes Leitgeb, Roger Maddux, André Nies, Carsten Schürmann, Kai Wehmeier, and Matthias Wille. Authors and publishers are requested to send, for review, copies of books to *ASL, Box 742, Vassar College, 124 Raymond Avenue, Poughkeepsie, NY 12604, USA*.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

MATTHIAS WILLE. ***Die Mathematik und das synthetische Apriori. Erkenntnistheoretische Untersuchungen über den Geltungsstatus mathematischer Axiome.*** Mentis, Paderborn, 2007, 234 pp.

The book has three chapters (in this review, I have translated German into English where appropriate): 1. Possible ways of defining mathematics, 2. Acting in mathematics, 3. Are mathematical propositions synthetic or analytic? Chapter 3 may be divided in two parts: sections 1–3 defends the view that mathematical truths are synthetic a priori, and section 4 argues against one particular rival conception according to which they are analytic, neo-Fregeanism (in the context of arithmetic).

The central question the book seeks to answer is: 'What is mathematics?'; it is raised in order to serve investigations into the relations between mathematics and proof theory. Central questions in those investigations are: 1. Does it make sense to speak, with regard to mathematics and proof theory, of synthetic a priori judgements?, 2. Can results from proof theory be considered to be mathematical results?, and 3. To what extent do proof theoretical practices require philosophical reflection on mathematics? (p. 11). Indeed, shortly after the book under review appeared, the author published a companion volume on proof theory: ***Beweis und Reflexion. Philosophische Untersuchungen über die Grundlagen beweistheoretischer Praxen***, Mentis, Paderborn, 2008. That second book is well worth a separate review by a specialist in proof theory.

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To his central question in the first book, Wille proposes the following answer:

> [We] here understand mathematics as the science of the generation, by means of acts immanent to a calculus, of formally-synthetically true statements that are a priori valid. (p. 69)

This answer is proposed in the context of justification, not of discovery (pp. 106–107): that is, Wille does not claim that his answer is descriptively adequate of factual mathematical practice, but he does claim that any activity, if it is to count as doing mathematics, should allow recasting in the proposed form.

Wille calls his view and approach 'neo-Kantian': Kantian to the extent that it supports Kant's claim that mathematical theorems are examples of synthetic a priori truths, and neo-Kantian in the sense that it modifies Kant's original approach by taking the linguistic and pragmatic turns. The linguistic turn leads Wille to concentrate on the language of mathematics (p. 13), the pragmatic turn to analyzing how mathematicians deal with statements in that language (p. 15). He emphasizes that his definitions of the key terms 'analytic' and 'synthetic' are not Kant's, and so his approach is Kantian more in letter than in spirit, all the more so as he does not accept a (foundational) notion of mathematical intuition. For Wille, a proposition is not either analytic or synthetic, but, in general, functions analytically and synthetically at the same time (p. 124). It functions analytically to the extent that it constrains how an undefined non-logical relation or function symbol can be used; it functions synthetically to the extent that it constrains the domain of the potential models of the theory. No axiom functions only analytically (for then it would be a truth not of mathematics but of logic); every mathematical axiom functions synthetically, and some only so.

Wille claims that his characterization of mathematics captures 'substantially all currently established mathematical theories' (p. 69). (It is not made explicit what is meant by 'established', nor what more is perhaps captured.) In particular, Wille claims that he can show that the axioms of ZFC are synthetic in such a way that their a priori validity follows (p. 19). That is not the strong claim it may seem, as on Wille's definitions of the terms, it does not imply that a model exists:

> Axioms of a mathematical theory T extend our knowledge [i.e., are synthetic], yet cannot be defeated by experience [i.e., are a priori], because
> - validity of the axioms of T is claimed only with respect to the potential models of T,
> - with their successive addition, an extension of our knowledge of the nature of the potential domains of T is generated. (p. 133)

In Wille's scheme, therefore, an axiom of a mathematical theory is a priori by definition (p. 141). While this has the desired consequence that mathematical axioms cannot be refuted by (sense) experience, it also means that this notion of the a priori lacks any connotation of mathematical absoluteness or universality, and incompatible mathematical propositions may both be qualified a priori truths. Although this goes against traditional ideas about (the unity of) mathematics, one may note that the phenomenon is familiar from Carnap's linguistic frameworks, where no overarching notion of truth is assumed to exist.

As for the second part of this quotation, I do not think that the argument here warrants the conclusion that any axiom A is synthetic knowledge. If we consider T as a theory in the process of being formed as axioms are being added, then adding A indeed allows us to make a true judgement that we could not have made before, namely: if T (when finished) will have a model, then A will be true in it. From that judgement neither A nor 'A is true' (in any sense of that term) can be inferred; so even though adding the axiom A occasions an extension of our knowledge, this extension does not include A. (If we define a theory (given some background logic) statically by the set of its axioms, then adding axiom A to theory T results in the different theory $T + A$; and the analytic truth that in potential models of $T + A$

A should hold extends neither our knowledge of the properties of potential models of T, nor of those of T + A. Something similar can be said about formal proofs instead of models.)

If the notion of knowledge is taken to be dependent on a notion of truth, for example truth in a model, then in order to justify the claim to know A, one has to know that models of T (as a finished theory, including A) indeed exist. For someone who shares Wille's foundational concerns, these models should of course themselves be foundationally unproblematic, and not be merely formal. But Wille's neo-Kantian program is not an alternative to foundational programs, but a complement (pp. 137–138). It is a theory about how we set up and use axiomatic theories, not a theory of how to build models for them (p. 137). Such questions are explicitly excluded and left to a foundational program. So, if the neo-Kantian program is, by itself, unable to justify the claim that mathematical theorems are synthetic a priori *truths*, this is by design.

Although Wille realizes that for a claim to knowledge one might well demand a model (p. 133), he insists on qualifying the added axiom A as knowledge even in absence of a (foundationally unproblematic) model, on the ground that this is 'how in factual mathematical practice, theories are set up and knowledge is generated [. . .] Whether models are available or not, it is indisputable that mathematical reality and thereby also the possibility for mathematical knowledge are generated by the mathematical means themselves. And as factual mathematical practice proceeds axiomatically, we are here interested only in the knowledge thus generated' (p. 134, cf. p. 138). In saying this, Wille gives priority to a descriptive conception of knowledge over a normative one. (This move should, I think, also alleviate concerns raised by the fact that 'factual mathematical practice' is not a wholly unambiguous term.) One would have liked to see an account why, if, as Wille proposes, a normative view should be taken with respect to mathematical acts, this should not also be done with respect to the notion of mathematical knowledge. Wille's answer to the question why, to take his example, certain classical statements about $\mathbb{R}$ express *synthetic* a priori truths comes down to the fact that classical mathematicians *believe* that $\mathbb{R}$ exists (see p. 134). This will do little to convince someone who doubts that there are such truths.

Admittedly, it is clear from various passages that Wille wishes to accord particular significance to 'constructive or predicative foundational programs' (p. 137, cf. p. 142). But it is taken for granted, rather than argued, that one should wish to do that; and there is no reflection on how this wish relates to the readiness to ascribe knowledge on the basis of 'factual mathematical practice', or to the idea that mathematical reality is wholly constituted by the axiomatic theories themselves. The latter idea entails that models can only be formal. That is problematic, because, as has been argued forcibly by Poincaré, Brouwer, and Gödel, in foundational dealing with formal systems, some notion of a priori intuition (in particular, of the natural numbers) cannot be dispensed with; and hence the natural numbers must be accepted as given in a sense that, however it may be construed exactly, is different from that of being described by a theory. Wille does not want to go so far as to identify mathematics with what can be constructively or predicatively justified, saying that to do so would be to throw 'general [i.e., classical] set theory' out of the philosophy of mathematics, 'per definitionem' (p. 59). Constructivists may reply that the definition in question, whatever its merits or lack thereof, would not be stipulative, but grounded on general epistemological or ontological views; and that, in any case, classical mathematics need not be considered as having been thrown out altogether, as, in a formalized setting, it may be conceived of as dealing with relative consistency proofs. Such a reply would not be new.

It is a pity that Wille does not pause to present an *Auseinandersetzung* with the views on axiomatic systems developed by Carnap, e.g. *Empiricism, semantics, and ontology*, ***Revue Internationale de Philosophie***, vol. 4 (1950), pp. 20–40, or the earlier ***Untersuchungen zur allgemeinen Axiomatik***, hrsg. von Thomas Bonk und Jesús Mosterin. *Wissenschaftliche Buchgesellschaft*, Darmstadt, 2000, neither of which is included in the bibliography. Carnap's only appearance occurs when his rejection of the synthetic a priori (in Kant's sense) is duly noted (p. 114). But Wille's program has much affinity with Carnap's idea that mathematics does not transcend linguistic frameworks and with his Principle of Tolerance; on the other hand, unlike Carnap, Wille sees a need for (in particular, constructive) foundational programs.

Of independent interest is Wille's treatment of the neo-logicist program to found arithmetic on logic alone. In particular, Wille presents an argument that a type-free logic and Hume's principle cannot be jointly epistemically justified (p. 208ff), and he favours adopting a typed logic.

In Wille's other book, mentioned above, proof theory is described as a specific form of reflection on calculus-immanent acts (p. 80). As that characterization is compatible with a wide range of answers to the question what mathematics is, it does not quite necessitate accepting Wille's answer to that question in the book reviewed here.

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