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BERNAYS AND SET THEORY

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Abstract. We discuss the work of Paul Bernays in set theory, mainly his axiomatization and his use of classes but also his higher-order reflection principles.

Paul Isaak Bernays (1888–1977) is an important figure in the development of mathematical logic, being the main bridge between Hilbert and Gödel in the intermediate generation and making contributions in proof theory, set theory, and the philosophy of mathematics. Bernays is best known for the two-volume 1934,1939 ***Grundlagen der Mathematik*** [39, 40], written solely by him though Hilbert was retained as first author. Going into many reprintings and an eventual second edition thirty years later, this monumental work provided a magisterial exposition of the work of the Hilbert school in the formalization of first-order logic and in proof theory and the work of Gödel on incompleteness and its surround, including the first complete proof of the Second Incompleteness Theorem.[1] Recent re-evaluation of Bernays' role actually places him at the center of the development of mathematical logic and Hilbert's program.[2]

But starting in his forties, Bernays did his most individuated, distinctive mathematical work in set theory, providing a timely axiomatization and later applying higher-order reflection principles, and produced a stream of wide-ranging essays in the philosophy of mathematics. Bernays' axiomatization, built on one of von Neumann's, would become important for the

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[1]There is no extant translation of the ***Grundlagen*** into English, but a translation project is underway; see `http://www.ags.uni-sb.de/~cp/p/hilbertbernays/goal.htm`. The second edition was translated into Russian as ***Osnovaniya mathematiki*** by N.M. Nagonyi (Nauka, Moscow, 1979) and into French as ***Fondments de mathématiques*** by Francǫis Gaillard, Eugéne Guillaume and Marcel Guillaume (l'Harmattan, Paris, 2001). The second volume of the second edition has been translated into Japanese by Sakaé Fuchino (Springer-Verlag Tokyo, 1993).

[2]See [18].

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development of set theory through its adoption, with mainly cosmetic simplifications, by Gödel in his work on the constructible universe. We refer to Mancosu [49] for Bernays' early philosophy; Parsons [59] for his later philosophy; and Sieg–Tait [65] for a comprehensive selection of essays and extended account of the philosophy. Here, summarizing his involvements with proof theory when incumbent to round out the picture, we set out Bernays' work in set theory against the backdrop of historical circumstance and mathematical interaction, assessing its role in the development of modern set theory.

§**1. To the axiomatization.** Bernays was born in London on 17 October 1888, officially a Bürger of the Swiss city of Zürich. After a time in Paris he grew up in Berlin and attended university there for two years. In 1909 he continued at Göttingen, becoming involved in mathematics with Hilbert's circle and in philosophy with the "neo-Friesian school" of Leonard Nelson.[3]

To frame the time, we quickly recall some pivotal junctures. Hilbert [35] at the Heidelberg 1904 International Congress of Mathematicians advocated a simultaneous, axiomatic development of the laws of logic and of the arithmetic of the real numbers, particularly to avoid the recent paradoxes of logic and set theory. Stimulated by events at the congress Ernst Zermelo, at Göttingen and in Hilbert's circle, soon formulated [77] the Axiom of Choice and with it established the Well-Ordering Theorem. Then he in 1908 provided [78] the first full-fledged axiomatization of set theory, partly to establish set theory as a discipline free of paradoxes and particularly to put his Well-Ordering Theorem on a firm footing. In 1910 Zermelo left Göttingen to become a professor at the University of Zürich.[4]

In 1912 Bernays received a doctorate under Edmund Landau with a thesis on binary quadratic forms. He then took the opportunity to follow Zermelo to Zürich and completed his *Habilitation* there at the end of 1912 with a thesis on modular elliptic functions. Bernays served as an *Assistent* to Zermelo and was a *Privatdozent* at the university.

Bernays' first involvement with set theory, though elliptical, occurred in this period, and this in connection with the development of the ordinals. As is well-known, for Cantor the ordinal numbers were the ordertypes of well-orderings, autonomous and separate from sets. In the early 1920s John von Neumann [71] would formulate the set concept of ordinal, with the basic idea of taking precedence in a well-ordering to be membership. Using the Axiom of Replacement, von Neumann established the key instrumental property of Cantor's ordinal numbers for ordinals, that every well-ordering is order-isomorphic to exactly one ordinal with membership. Von Neumann forthwith ascribed to the ordinals the role of Cantor's ordinal numbers,

[3]We refer to Moore [55] and Ebbinghaus [17] pp. 125–7 for many of the biographical details about Bernays given here and below.

[4]We refer to Ebbinghaus [17] for biographical details about Zermelo given here and below.

drawing them into set theory. But in point of fact, Zermelo was most probably the first chronologically to have formulated the concept of ordinal, and this by 1915 in Zürich. The rudiments of the theory appear in items in his *Nachlass*,[5] and indications are there of collaboration with Bernays.[6] With Zermelo never to publish this work, the first published comments about it would appear in a later paper by Bernays [4] pp. 6,10.[7]

During his years in Zürich Zermelo was plagued with tuberculosis, went several times on sick leave, and had to arrange for others to give his lectures. In 1916 he was finally retired from his professorship.[8] Zermelo went back to Göttingen, where notably he gave a lecture on his new theory of ordinals in November 1916, and eventually settled in Freiburg in 1921.

In the autumn of 1917, Hilbert gave his pivotal "Axiomatisches Denken" [36] lecture at Zürich, in which with renewed interest in foundations he praised the ***Principia*** work of Russell and newly advocated the axiomatization of logic itself and the reduction of number theory and set theory to logic in the quest for consistency. Hilbert thereupon invited Bernays back to Göttingen as his *Assistent* in the investigations of the foundations of mathematics, an invitation that Bernays readily accepted.

The collaboration of Hilbert and Bernays led to a remarkable sequence of formative lectures through the years 1917–1922 in which one sees the emergence of first-order logic and proof theory.[9] In 1918 Bernays completed a second *Habilitation* with a thesis that established the completeness of propositional logic.[10] In the 1920 summer semester lectures Hilbert presented Zermelo's conceptualization of the ordinals, and through it the Burali-Forti paradox, and a version of Zermelo's 1908 axioms in a first-order context. The lectures were written up by Bernays and Moses Schönfinkel,[11] and this

[5] See Hallett [31] pp. 277ff for an analysis.

[6] See Ebbinghaus [17] 3.4.3.

[7] After Zermelo and before von Neumann, Dmitry Mirimanoff, a professor at Geneva, published several papers [51, 52, 53] in the French Swiss journal ***L'Enseignment Mathématique***, in which he formulated the ordinals and went a considerable distance toward shaping the set-theoretic universe as von Neumann would later do. See Hallett [31] pp. 185ff for an analysis.

[8] Abraham Fraenkel, in his autobiographical *Lebenskrise*, recounts the following (see [17] p. 113): Shortly before the World War he [Zermelo] spent a night in the Bavarian alps. He filled the column "Nationality" in the hotel's registration form with the words: "Not Swiss, thank goodness." Misfortune would have it that shortly after that the head of the Education Department of the Canton Zürich stayed at the same hotel and saw the entry. It was clear that Zermelo could not stay much longer at the University of Zürich." This tale may be apocryphal but was well-known during Zermelo's lifetime; the substantive issue should of course have been Zermelo's failing health.

[9] See Sieg [63], in which Bernays is accorded equal credit with Hilbert.

[10] See Zach [76], in which Bernays is accorded a central role in the development of propositional logic.

[11] See [18].

may have been Bernays' first serious encounter with axiomatic set theory. In 1922 Bernays became a Professor Extraordinarius of the university. In 1922 Hilbert [37] and Bernays [2] set out what is now widely known as "Hilbert's program" for establishing the consistency of ongoing mathematics through finitary reasoning. Much, of course, has been written about this and even more about Gödel's incompleteness results and their transforming effect on the program, and we only provide the main narrative impingements.

The three who would newly axiomatize set theory—von Neumann, Bernays, and Gödel—interacted first in connection with the new incompleteness results, so that the nexus at set theory would in fact be interwoven with the nexus at incompleteness. By 1923 von Neumann had come to the ordinals and in developing their theory had accorded a central place to the Axiom of Replacement. This axiom had been suggested earlier by Abraham Fraenkel [20, 21] and Thoralf Skolem [66] as an addition to the Zermelo [78] axiomatization, but to ensure that a certain collection of large cardinality resulting from a simple recursion be a set. Also by 1923, von Neumann had worked out his new axiomatization of set theory,[12] the first significant axiomatization since Zermelo's and the subject of von Neumann's 1926 Budapest doctoral thesis. He spent 1926–7 at Göttingen and was a *Privatdozent* at Berlin 1927–9 and at Hamburg 1929–30.[13] Von Neumann's [73] evidenced his engagement with Hilbert's program, and he together with Bernays and Wilhelm Ackermann came to be regarded the "Hilbert school" in proof theory.

When Gödel first spoke in September 1930 at Königsberg on his First Incompleteness Theorem, von Neumann saw not only its broad significance but its particular relevance to the work of the Hilbert school. Some weeks after his lecture Gödel established his Second Incompleteness Theorem, the unprovability of consistency, and a few days later heard from von Neumann that he too had established this result.[14] Thus, there had to be something wrong with Hilbert's epsilon-term substitution argument for formulas as carried out by Ackermann [1] to establish the consistency of number theory, and von Neumann soon provided a formula for which the argument failed.[15] Beyond the common impression that Gödel's Second Incompleteness Theorem largely precluded Hilbert's consistency program, this close interplay between Gödel and von Neumann brings out the specific *mathematical* impact that

[12]This becomes clear from an important 15 August 1923 letter of his to Zermelo (in Zermelo's *Nachlass* under signature C 129/85 and reprinted in [50] pp. 271–3).

[13]Von Neumann was again in Berlin in the winter of 1930, where Jacques Herbrand visited him; see Sieg [64] p. 175.

[14]See the initial letters from von Neumann to Gödel of 20 November and 29 November 1930, Gödel [30] pp. 337ff.

[15]The counterexample is given in the Hilbert-Bernays ***Grundlagen*** [40] pp. 123ff.

Gödel's result had on a concerted effort then being made by the Hilbert school.[16]

Bernays would maintain a long correspondence with Gödel,[17] and as with von Neumann's with Gödel, the main topic of the initial letters was incompleteness. Having received from Gödel the galleys of his epochal paper [22] and having "thoroughly digested" it in four days, Bernays in a letter of 18 January 1931 to Gödel discussed at length some mathematical implications vis-à-vis work of the Hilbert school. At the end he mentioned:[18]

> I have laid out a modified version of von Neumann's set theory which, first of all, establishes a closer relation to the ordinary logical processes of set formation, and furthermore eliminates various unnecessary deviations from Zermelo's system and makes the formulation of the axioms more easily understandable. (I lectured on it last summer before the Göttingen Mathematical Society.)

In his reply of 2 April 1931, Gödel dealt at length with the issues raised by Bernays, and wrote at the end:[19] "Your improvement of the axiom system of set theory would interest me very much, and I would be very grateful to you for sending me a manuscript of your talk (in case such a thing is available)." Gödel was already thinking ahead to set theory, for as he stated in a well-known, prescient footnote 48a to his [22], his undecidable propositions become decidable "whenever appropriate higher types are added". Fully one-third of a summary, dated 22 January 1931, of a talk given on his incompleteness results is given over to set theory:[20]

> In case we adopt a type-free construction of mathematics, as is done in the axiom system of set theory, axioms of cardinality (that is, axioms postulating the existence of sets of ever higher cardinality) take the place of type extensions, and it follows that certain arithmetic propositions that are undecidable in Z [first-order Peano arithmetic] become decidable by axioms of cardinality, for example, by the axiom that there exist sets whose cardinality is greater than every α_n, where $\alpha_0 = \aleph_0$, $\alpha_{n+1} = 2^{\alpha_n}$.

This is Gödel's first remark on set theory of substance, and significantly, his example of an "axiom of cardinality" has as the thrust the existence of the set that both Fraenkel [20, 21] and Skolem [66] had pointed to as the one to be secured by adding Replacement to Zermelo's [78] axiomatization.

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[16]Correspondence between Gödel and Herbrand also illuminates a mathematical impact of the Second Incompleteness Theorem; see Sieg [64].

[17]See [29]; Solomon Feferman's introductory note to the corrspondence, pp. 41ff, is very informative.

[18]See Gödel [23] p. 91.

[19]See Gödel [29] pp. 99ff.

[20]See Gödel [23] pp. 234ff.

Instead of sending a manuscript, Bernays in an extended letter of 3 May 1931 wrote out a careful account of the new axiomatization, the first starting with both set and class as primitive notions.[21] Years later Gödel would use a version of Bernays' axiomatization in his formalized proof with the constructible universe L of the relative consistency of the Axiom of Choice and the Generalized Continuum Hypothesis.[22] In his 1940 monograph [25], Gödel routinely acknowledged Bernays by citing his published account [3] of 1937, and this lends itself to a surface misleading of causal connections. Gödel had actually assimilated Bernays' axiomatization through that May 1931 letter,[23] and Bernays [3] p. 65 himself stated that he had lectured on his axiomatization already in 1929–30.

§**2. The axiomatization.** The following is Bernays' axiomatization with sets and classes essentially as in his May 1931 letter to Gödel, but organized according to his first published account [3, 4]. In his letter Bernays had written:

> ... I mention at the outset that I will here adopt the *contentual* [*inhaltlichen*] standpoint, i.e., I will employ the logical symbolism only *as a means of communication* and also only insofar as it appears advantageous to me for facilitating the overview.
>
> A *complete* formalization, and in fact in a *first-order* [*ersten Stufe*] framework, can be carried out without difficulty.

In his published account he wrote out the axioms in prose. We rely on logical symbolism, at least at first, to approach what a first-order formalization would be. There are two sorts of variables, lower case for sets and upper case for classes, and two membership relations, $a \in b$ for sets and $x \,\eta\, A$ for sets in classes. Bernays evidently adopted this use of η from von Neumann's last paper [74] p. 231 on axiomatization, a paper from which Bernays draws significantly as we shall see. As to "first-order", that the axiomatization was historically and technically seen to be first-order is significant. There would be axiomatizations of set theory in second-order terms, starting with Zermelo's contemporaneous one [79] discussed below, but this amounts to the submergence of classes *qua* arbitrary collections into the logic, whereas Bernays was axiomatizing class as a specific, delineated concept. More broadly speaking, first-order logic had been isolated in 1917 lectures of Hilbert, in the first year that Bernays was his *Assistent*,[24] and Bernays'

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[21] See Gödel [29] pp. 105ff.

[22] For Gödel's work in set theory, see Kanamori [42].

[23] This becomes clear from Gödel's letter of 19 June 1939 to Bernays (Gödel [29] p. 115ff).

[24] The transcription of those lectures was done by Bernays and was actually an Ausarbeitung, so that one can speculate that it was actually Bernays who first isolated first-order logic.

axiomatization can be seen as another conduit from Hilbert to Gödel in connection with the later ascendancy of first-order logic in set theory.

I. Axioms of Extensionality

(1) $\forall x(x \in a \longleftrightarrow x \in b) \longrightarrow a = b$.
(2) $\forall x(x \,\eta\, A \longleftrightarrow x \,\eta\, B) \longrightarrow A = B$.

II. Axioms of Direct Construction of Sets

(1) $\exists a \forall x(x \notin a)$.
(2) $\forall a \forall b \exists c \forall x(x \in c \longleftrightarrow x \in a \vee x = b)$.

This last, asserting the existence of $a \cup \{b\}$ for sets a and b, had appeared in the aforementioned 1920 Hilbert lectures. One now defines an ordered pair $\langle a, b\rangle$[25] and continues:

III. Axioms for Construction of Classes

a(1) $\forall a \exists A \forall x(x \,\eta\, A \longleftrightarrow x = a)$.
a(2) $\forall A \exists B \forall x(x \,\eta\, B \longleftrightarrow \neg x \,\eta\, A)$.
a(3) $\forall A \forall B \exists C \forall x(x \,\eta\, C \longleftrightarrow x \,\eta\, A \;\;\&\;\; x \,\eta\, B)$.
b(1) $\exists A \forall x(x \,\eta\, A \longleftrightarrow \exists a \forall y(y \in x \longleftrightarrow y = a))$.[26]
b(2) $\exists A \forall x(x \,\eta\, A \longleftrightarrow \exists a \exists b(a \in b \;\;\&\;\; x = \langle a, b\rangle))$.
b(3) $\forall A \exists B \forall x(x \,\eta\, B \longleftrightarrow \exists a \exists b(x = \langle a, b\rangle \;\;\&\;\; a \,\eta\, A))$.
c(1) $\forall A \exists B \forall x(x \,\eta\, B \longleftrightarrow \exists y(\langle x, y\rangle \,\eta\, A))$
c(2) $\forall A \exists B \forall a \forall b(\langle a, b\rangle \in B \longleftrightarrow \langle b, a\rangle \in A)$.
c(3) $\forall A \exists B \forall a \forall b \forall c(\langle\langle a, b\rangle, c\rangle \in B \longrightarrow \langle a, \langle b, c\rangle\rangle \in A)$.

For these last three, Bernays actually had implications starting from the assumption that A is a relation, "a class of pairs". One formally defines this concept as well as the concepts of function, one-to-one correspondence, and so forth, and what it means for a set a to *represent* a class A (i.e., $\forall x(x \in a \longleftrightarrow x \,\eta\, A)$) and continues:

IV. Axiom of Choice

Every relation C has a subclass which is a function and has the same domain.[27]

[25]In the May 1931 letter Bernays defined the ordered pair as $\{\{a\}, \{\emptyset, \{b\}\}\}$ and in [3] he adopted the Kuratowski ordered pair $\{\{a\}, \{a, b\}\}$.

[26]Instead of this axiom positing the class of all singletons, Bernays in his May 1931 letter to Gödel had the axiom positing the class of all $\langle a, a\rangle$. The latter axiom is more natural in getting directly at what was wanted, the equality relation; see Bernays' letter of 21 June 1939 to Gödel (Gödel [29] p. 117) and footnote 11 of Bernays [3]. Since by [3] Bernays had adopted the Kuratowski ordered pair, b(1) suffices in the presence of the axioms to get the equality relation.

Eventually, Bernays realized that in any case b(1) is redundant; see [9] §20.

[27]This formulation, evidently drawn from von Neumann [74], is easily seen to admit a simple self-refinement in the presence of the other axioms, that there is a global choice function. See the forthcoming discussion.

V. Axioms Concerning the Representation of Classes by Sets

a. (Separation) $\forall a \forall A \exists b \forall x(x \in b \longleftrightarrow x \in a \;\;\&\;\; x \,\eta\, A)$.

b. (Replacement) If the domain of a one-to-one correspondence is represented by a set, then so is the range.

c. (Union) $\forall a \exists b \forall x(x \in b \longleftrightarrow \exists y(y \in a \;\;\&\;\; x \in y))$.

d. (Power Set) $\forall a \exists b \forall x(x \in b \longleftrightarrow x \subseteq a)$.

VI. Axiom of Infinity

There is a set in one-to-one correspondence with a proper subclass.[28]

VII. Axiom of Foundation[29]

$\forall A(\exists x \,\eta\, A \longrightarrow \exists b(b \,\eta\, A \;\;\&\;\; \neg \exists z(z \in b \;\;\&\;\; z \,\eta\, A)))$.

After the formulation of Foundation in his May 1931 letter Bernays notably defines "transitive" (really, "vollzählig" in the German) for sets and provides a definition, with Foundation implicitly assumed, of (von Neumann) ordinal as a transitive set each of whose members is transitive. This first direct definition is in contradistinction to Neumann's involved definition through order-isomorphisms and may be attributable to Bernays' earlier interaction with Zermelo.

What are the similarities between the Bernays and von Neumann systems? Von Neumann's was the first significant axiomatization after Zermelo's, and the first, through the I-objects and II-objects, to allow sets together with proper classes, as we would now say, while avoiding the paradoxes, this being accomplished by having only sets be members. Bernays transported this basic framework.

Bernays' axiom group III for the construction of classes has a direct counterpart in von Neumann, and those axioms establish in modern terms the *Predicative Comprehension Schema*:

> For any formula $\varphi(v_1, \ldots, v_n)$ of the formalized system in the free set variables as displayed and no quantified class variables, there is a corresponding class, i.e., a class A such that $\varphi(a_1, \ldots, a_n)$ holds exactly when $\langle a_1, \ldots, a_n \rangle \,\eta\, A$.

In the recursive proof of this schema a(2) handles negation, a(3) handles conjunction, b(1) and b(2) provide for atomic formulas, b(3) handles $\forall$, c(1) handles $\exists$, and c(2) and c(3) handle the changing of the order of appearance of variables. Thus, a few instances of the Predicative Comprehension Schema

[28] Instead of this classical Dedekind formulation, Bernays in his May 1931 letter to Gödel preferred the specific $\exists x(\emptyset \in x \;\;\&\;\; \forall a(a \in x \longrightarrow a \cup \{a\} \in x))$. This now-standard version had appeared in the aforementioned 1920 Hilbert lectures.

[29] This is drawn from von Neumann [74] as we shall soon discuss. Bernays [3] had "Restriction" here instead of "Foundation"; the now standard term is from Zermelo [79]. A significant refinement, that Foundation only for sets suffices, is due to Gödel in 1939 correspondence and discussed in the next section.

lead to the full schema, and this *finite axiomatizability*, particularly the drawing out of the technical c(2) and c(3), is a notable feature of the move into classes.

In the historical development of set-theoretic axioms, Zermelo's original [78] Separation axiom, with its separating out of members of a given set according to a "definite property", was widely regarded as in need of clarification. Von Neumann [72] took as one of the crucial accomplishments of his axiomatization such a clarification, with his axioms for generating classes sufficient so that all definite properties can be correlated with classes according to his Reduction Theorem [72] §3, essentially the Predicative Comprehension Schema.[30] In his May 1931 letter to Gödel, Bernays in effect asserts the full schema, regarding it as the realization of Zermelo's definite property—so axiom Va serves indeed as the Separation Axiom.

What are the differences between the Bernays and von Neumann systems? Commentators gravitate to the most evident difference, that von Neumann had taken function as a primitive notion with his I-objects and II-objects being functions, whereas Bernays reverted to collections, sets and classes. However, between von Neumann and Bernays there is a striking conceptual, and as we shall see, strategic difference that we draw out by looking at the thrust of the main axioms.

The focal axiom of von Neumann's system, the provenance of its power, is his axiom IV 2. We state it in terms of sets and classes, with V the class of all sets:

> A class A is not (represented by) a set exactly when there is a surjection of A onto V.

Thus, von Neumann crucially transformed the negative concept of proper class, which had appeared in various guises, e.g., Cantor's "inconsistent multiplicities", into the positive concept of having a surjection onto V. IV 2 is an existence principle that plays the role of regularizing proper classes, much as the Axiom of Choice does for sets, by extending the Cantorian canopy of cardinality over them through the positing of functional correspondences, and this will become more explicit through a deduction below. Of IV 2 von Neumann [72] §3 wrote:

> Axiom IV 2, finally, deviates quite essentially from what Zermelo and Fraenkel have, and indeed it is the distinctive feature of our axiomatization. It is, to be sure, related in a certain sense to the axioms of separation and replacement, but it goes much further. On the one hand it guarantees the existence of subsets and image sets, and in general it makes possible the theory of ordinals and alephs (which can hardly be developed successfully in an axiom system

[30] Already in that letter of 15 August 1923 to Zermelo (in Zermelo's *Nachlass* under signature C 129/85 and reprinted in [50] pp. 271–3), von Neumann wrote that he does not introduce definiteness explicitly but rather through schemes.

> that lacks the axiom of replacement); yet all that could essentially be achieved by the axiom of replacement alone. But beyond this, IV 2 occupies an altogether central position in the axiom system; in several cases it enables us to prove that a set is "not too big", and finally it yields the well-ordering theorem.

Indeed, von Neumann's IV 2 entails Replacement Vb as a natural consequence: If F is a one-to-one correspondence whose range is not represented by a set, then there is a surjection G of that range onto V; but then F composed with G is a surjection of the domain of F onto V, and consequently the domain is not represented by a set.

On how VI 2 "yields the well-ordering theorem", von Neumann argued that since the class On of all ordinals cannot be a set by the Burali-Forti argument, IV 2 implies that there is a surjection of On onto V, and inverting the surjection according to least preimages induces a well-ordering of V itself. This immediately implies Zermelo's Axiom of Choice (for sets). With the proof of Zermelo's Well-Ordering Theorem, von Neumann's conclusion is equivalent to the following formulation of modern set theory, *Global Choice*:

> There is a choice function F on V, i.e., for any non-empty set a, $F(a) \in a$.

Global Choice is quickly seen to be a simpler, equivalent formulation of Bernays' Axiom of Choice IV, since the one choice function F can be applied to any relation.

The striking difference with Bernays' axiomatization is that von Neumann's axiom IV 2 is missing. Instead, its consequences Choice IV and Replacement Vb appear, and moreover Foundation VII. This last warrants a discussion:

In his last paper on axiomatization von Neumann [74] defined the *cumulative hierarchy* by transfinite recursion. In modern terms, he had defined the class of well-founded sets through their stratification into cumulative ranks V_α, where

$$V_0 = \emptyset;\ V_{\alpha+1} = \mathcal{P}(V_\alpha);\ \text{and } V_\delta = \bigcup_{\alpha<\delta} V_\alpha \text{ for limit ordinals } \delta.$$

Mirimanoff [51] pp. 51ff had been first to study the well-founded sets, and the cumulative hierarchy is distinctly anticipated in his work. In the axiomatic tradition Fraenkel [21], Skolem [66], and von Neumann [72] had considered the salutary effects of restricting the universe of sets to the well-founded sets. Then von Neumann [74] formulated, in his functional terms, Foundation VII and observed that it is equivalent to the assertion that the cumulative hierarchy *is* the universe, $V = \bigcup_\alpha V_\alpha$. Moreover, by restricting the universe to the cumulative hierarchy he established the first relative consistency result via "inner models"; his argumentation, as we would now say, established the consistency of Foundation relative to Zermelo's [78] axioms plus Replacement.

Zermelo in his remarkable [79] subsequently provided his final axiomatization of set theory, incorporating for the first time both Replacement and Foundation in an axiomatization. Like Bernays, Zermelo worked with proper classes, but he took the diametrically opposite approach of proceeding with sets in what we would now say is a full second-order context and not distinguishing a concept of class. The now standard axiomatizations ZFC and ZF of set theory are recognizable, the main difference being of course that these are theories of first-order logic. Zermelo was actually presenting a dramatically different view of set theory as based on a procession of models, *domains* each having a basis of *urelements*, sets without members yet distinct from each other, and a height a *Grenzzahl*, either ω or an inaccessible cardinal. Significantly, Zermelo pointed out (p. 38) that those domains starting with one urelement satisfy von Neumann's axiom IV 2. However, in Zermelo's general approach IV 2 would not always hold, this depending on the cardinality of the starting collection of urelements.[31]

Von Neumann [74] himself had established in the presence of Foundation VII that the two consequences Choice IV and Replacement Vb of his axiom IV 2 actually imply it:

For any class A, A is the union of the layers $A \cap (V_{\alpha+1} - V_\alpha)$ by Foundation. If A is not represented by a set, then these layers are nonempty for arbitrarily large α by Replacement. But each such layer has a well-ordering by Choice, and these well-orderings can be put together to well-order all of A, again by Choice. Hence, there is a bijection between A and the class On of all ordinals.

Hence, assuming Foundation VII, von Neumann's IV 2 is *equivalent* to Choice IV and Replacement Vb in the presence of the other axioms.[32] This result has the notable thematic effect of localizing the thrust of IV 2 through Choice IV to its equivalent Global Choice, having *one* choice function on the universe. What Bernays did in his axiomatization was to adopt Foundation and replace IV 2 with the conjunction of Choice IV and Replacement Vb.

While von Neumann did not adopt Foundation as an axiom, perhaps preferring that his central axiom IV 2 have its full sway in a larger setting, both Zermelo and Bernays espoused it. With Bernays having first lectured on his axiomatization in 1929–30,[33] he and Zermelo must have arrived at the idea of incorporating Foundation almost at the same time. But Bernays, like von Neumann, worked in "pure" set theory without urelements and thus drew in von Neumann's IV 2 into set theory as a consequence of Foundation.

[31] See Kanamori [41] p. 525 and generally for Zermelo's work in set theory.

[32] Levy [45] latterly showed by a clever argument that the Union Axiom Vc also follows from von Neumann's IV 2, so that IV 2 is equivalent to Choice IV, Replacement Vb and Union Vc in the presence of the other axioms.

[33] Again, see Bernays [3] p. 65.

Foundation would henceforth be part of axiomatizations of set theory, with Zermelo [79] being the first published instance and the Bernays axiomatization filtering through Gödel's later adoption. These axiomatizations rounded out but also focused the notion of set, with Replacement providing the means for transfinite recursion and induction and Foundation making possible the application of those methods to get results about *all* sets, they now being in the cumulative hierarchy. In a notable inversion, what has come to be regarded as an underlying *iterative conception* based on the cumulative hierarchy picture has become a heuristic for motivating the axioms of set theory generally, particularly through writings of Gödel. Foundation may be the one axiom unnecessary for the recasting of mathematics in set-theoretic terms, but the axiom ascribes to membership the salient feature that distinguishes investigations specific to modern set theory as an autonomous field of mathematics.

In summary, the features of Bernays' axiomatization that would commend its further use and influence were that it recast von Neumann's work to present a viable theory starting with sets and classes as primitive notions that allowed the full sway of logical constructions, and it incorporated Replacement and Foundation into set theory, as did Zermelo's late axiomatization, but in a first-order context and without the relativism of having urelements.

§**3. Ramifications.** Soon after the National Socialists came to power in 1933, Bernays as a "non-Aryan" was no longer allowed to teach, and for six months during this period Hilbert employed Bernays privately. In his time in Göttingen, Bernays would be advisor to one student and this role he played together with Hermann Weyl for Saunders Mac Lane, who getting his doctorate in 1934 can be regarded as the last mathematician from Hilbert's Göttingen.

By the Spring of 1934 Bernays had moved to Zürich, being a Bürger of the city. He would thus no longer associate directly with Hilbert, but around then he brought out the first volume of the ***Grundlagen*** and working in Zürich he would bring out the second volume five years later. At Zürich Bernays at first held a temporary position at the Eidgenössische Technische Hochschule (ETH). He would eventually become a half-time Professor Extraordinarius in 1945, a position that he continued to hold until he became Professor Emeritus in 1959.

Bernays visited the Institute for Advanced Study at Princeton for the year 1935–6, two years after it became instituted, and had direct association with Gödel.[34] Gödel had by then come to the constructible universe L and the relative consistency of the Axiom of Choice, but because of severe depression

[34]They crossed the Atlantic together on a liner in September, and on the trip and later at the Institute Bernays absorbed the details of the Second Incompleteness Theorem from Gödel,

he did not continue to work and returned to Vienna for recuperation in November. Bernays lectured on his axiomatization at the Institute. According to an autobiographical note, Bernays [14] "had hesitated to publish it because he felt that [the] axiomatization was, to a certain extent, artificial. He expressed this feeling to Alonzo Church, who replied with a consoling smile: 'That cannot be otherwise'; this persuaded him to publish." This Bernays proceeded to do in ***The Journal of Symbolic Logic*** in several parts starting in 1937, and we discuss these publications soon below.

Returning in 1938 to the Institute, Gödel lectured on L, having a year earlier achieved the crucial breakthrough to the relative consistency of the Generalized Continuum Hypothesis. These lectures were worked into the monograph [25], which at the beginning set out set theory with Gödel's version of the Bernays axiomatization. After a lapse of several years Gödel and Bernays would have extensive correspondence in 1939 about the monograph and L, beginning with a letter from Gödel concerning his use of the Bernays axiomatization.

Gödel in a brief account [24] of L informally presented it much as is done today: For any set x let $\mathrm{def}(x)$ denote the collection of subsets of x definable over $\langle x, \in \rangle$ via a first-order formula allowing parameters from x. Then define

$$L_0 = \emptyset;\ L_{\alpha+1} = \mathrm{def}(L_\alpha),\ L_\delta = \bigcup_{\alpha<\delta} L_\alpha \text{ for limit ordinals } \delta;$$

and the *constructible universe*

$$L = \bigcup_\alpha L_\alpha.$$

Toward the end Gödel [24] pointed out that L "can be defined and its theory developed in the formal systems of set theory themselves." This is a remarkable understatement of arguably the central feature of the construction of L. L is a class definable in set theory via a transfinite recursion that could be based on the formalizability of $\mathrm{def}(x)$, the definability of definability.

Gödel's [24] arguments for relative consistency amount to establishing that various φ hold in the sense of L, starting with the axioms of set theory themselves, and could only work if $\mathrm{def}(x)$ remains unaltered when applied in L with quantifiers restricted to L. This *absoluteness of first-order definability* is crucial if L is to be formally defined via the $\mathrm{def}(x)$ operation, but Gödel himself would never establish this absoluteness explicitly. Later in the 1960s model-theoretic methods would become infused into set theory, but at this time Gödel was presumably averse to having to formalize satisfaction-in-a-structure and in his monograph [25] proceeded without $\mathrm{def}(x)$ and by relying on the Bernays axiomatization.

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which eventually appeared in the second, 1939 volume of the ***Grundlagen***. See Dawson [16] p. 109, and generally for the biographical details on Gödel below.

There, Gödel channeled the von Neumann–Bernays class construction axioms III into eight corresponding binary operations, producing new classes from old, to generate L set by set via a transfinite recursion. In effect, Gödel relied on the aforementioned finite axiomatizability of the Predicative Comprehension Lemma to get at def(x) through evidently absolute operations. Thus the von Neumann idea of casting the logical operations through class construction to get at the "definite properties" became a specific transfinite construction in Gödel's hands, and the Bernays axiomatization served a specific *mathematical* purpose.

What are the differences between the Gödel [25] and Bernays systems? In fact, the differences have only to do with economy of presentation. The most conspicuous change is that Gödel only has one sort, class, and one membership relation and introduces a predicate for set-hood. Gödel does not bother with Separation Va, as it was well-known to follow from Replacement Vb. For Infinity Gödel reverted to the von Neumann [72] formulation: $\exists a(\exists x(x \in a) \ \& \ \forall x(x \in a \longrightarrow \exists y(y \in a \ \& \ x \subset y)))$. For Choice Gödel simply took Global Choice. Finally, several bi-conditionals in Bernays' formulation Gödel reduced to implications when sufficient.

In view of Gödel's attempt at a parsimonious presentation, there is one notable logical point arising from the 1939 correspondence. In the initial, 19 June letter from Gödel,[35] he pointed to a minor discrepancy between Bernays' May 1931 letter axiomatization and the Bernays [3] 1937 one. Gödel himself did not use Bernays' axiom b(1), the existence of the class of all singletons, and for making all order changes of variables relied on two "axioms of inversion" for 3-place relations:

$$\text{(B7)} \quad \forall A \exists B \forall x \forall y \forall z (\langle x, \langle y, z \rangle\rangle \in B \longleftrightarrow \langle y, \langle z, x \rangle\rangle \in A).$$

$$\text{(B8)} \quad \forall A \exists B \forall x \forall y \forall z (\langle x, \langle y, z \rangle\rangle \in B \longleftrightarrow \langle x, \langle z, y \rangle\rangle \in A).$$

(B7) is equivalent to Bernays' one inversion axiom c(3), as in both systems one can construct converse relations. In a letter of response of 21 June 1939, Bernays in effect pointed out to Gödel that (B8) is not necessary if one has axiom b(1), the existence of the class of all singletons. But Bernays eventually showed that b(1) is redundant in his axiomatization and noted that thus (B8) must be redundant in Gödel's axiomatization.[36] But this is simple to see: To get B as in (B8), one projects to the second coordinate, takes the converse relation, and then adjoins a coordinate x—all possible through his other axioms.

There is a more significant point of axiomatic parsimony raised by the 1939 correspondence. In the initial letter of 19 June Gödel stated Foundation *for sets* and inquired whether this goes back to von Neumann. In his response of 21 June Bernays wrote that he took over the conception wholly from von

[35]See Gödel [29] p. 115ff.

[36]See Bernays [9], especially footnote 95.

Neumann and believed that Foundation VII, *for classes*, is necessary. In a letter of 20 July Gödel insisted that the set form is sufficient and from it derived the class form, astutely pointing out the need for Infinity and Replacement:

Suppose that T is a non-empty class, say with $x \in T$. Let t be a transitive set such that $\{x\} \subseteq t$ and consider the set $t \cap T$, given by Separation. By (set) Foundation there is a $t_0 \in t \cap T$ such that $t_0 \cap t \cap T = \emptyset$. But then, since t is transitive, $t_0 \cap T = \emptyset$.[37]

Bernays in a letter of 28 September 1939 to Gödel expressed great interest in this reduction of Foundation to sets and observed that with Choice one can also make the reduction via infinite descending $\in$-sequences. It is a notable point of modern ZF that the very simply stated Foundation (for sets) is equivalent to the formalizable assertion that the universal class is exactly the recursively defined cumulative hierarchy.

We tuck in here that Bernays in the subsequent discussion of L in his letter pointed out "a certain analogy to Hilbert's approach to the proof of the continuum hypothesis: the underlying theory of ordinals corresponds to the species of variables (variable types), proposition A to lemma 1, and your theorem 2 to Lemma II." Bernays was referring to Hilbert's attempted proof of CH in [38]. In a lecture of 15 December at Göttingen on L, Gödel would draw out this very analogy.[38]

The axiomatic system presented in Gödel [25] would come to be called "Gödel–Bernays" or "Bernays–Gödel", with "von Neumann" sometimes inserted. Be that as it may, the system is substantively Bernays', with the main difference the cosmetic one of making certain reductions, particularly that of contracting two membership relations into one. Still, the idea of having only one sort of object, class, was in its way a conceptual advance. Also, it was the thrust of Gödel's achievement with L, which through the details of its formal presentation propelled the new axiomatization into prominence. The impact of Gödel [25] was such that its axioms, even to their groupings, would become reverently cited in the next two decades. In particular, with first-order logic having been shown to have considerable operational efficacy in establishing striking new mathematical results, the logic as filtering from Hilbert through Bernays' axiomatization would achieve a lasting ascendancy in set theory. In sum, the Bernays' axiomatization was a substantial departure from von Neumann's in that one moves from functions to sets and classes, and one shifts from his axiom IV 2 to Replacement, Choice, and Foundation. The

[37] Where Infinity and Replacement are needed is to establish Transitive Containment, that every set is a subset of a transitive set. Notably, Zermelo [79] actually needed Foundation for classes in his framework, as he was not assuming Infinity. It is a latter-day observation that Second-Order ZF with only Foundation for sets but without Infinity does not suffice to establish Transitive Containment; see Vopěnka–Hajek [75] and Hauschild [34]. For more on Foundation for sets vs. for classes and Gödel's subvention, see Felgner [19].

[38] See Gödel [28] pp. 126–155.

second shift to the Gödel axiomatization is less logically significant, but still conceptually salutary.

Bernays published a leisurely account of his axiomatization and its ramifications in a series consisting of seven parts in ***The Journal of Symbolic Logic*** [3, 4, 5, 6, 7, 8, 9] dating from 1937 through 1954—his first and only journal publications in English. Aspects of the axiomatization itself we have already discussed, and so we focus on the development of set theory, especially those aspects that have become part of the heritage of the subject.

Part I presents the axioms I - III and is mainly devoted to establishing, in its terms, the aforementioned Predicative Comprehension Schema. A small point is that this is the first publication in which the now-standard term "transitive" for sets appears (p. 67).[39] The adoption of the Kuratowski ordered pair $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$ is from here (p. 68) as well with accreditation to Kuratowski [43]; von Neumann [72] II§5 had mentioned the possibility, and without accreditation.

Part II presents the rest of the axioms IV - VII and develops the basics of ordinals, finite recursion, and finite sets and classes. After crediting Zermelo with the "general theory", Bernays provides a now-standard definition, adopted from Robinson [61], of "ordinal" (p. 6): A transitive set a such that for any $x, y \in a$, either $x \in y$ or $x = y$ or $y \in x$.[40]

This is the source of noun use in English, which lends itself to the contraction from "ordinal number", a contraction unavailable for the German "Ordnungszahl" or "Ordinalzahl".

Part III first deals with various axioms of infinity and then proceeds to the reals as Dedekind cuts and the "foundations of analysis". To this purpose Bernays emphasizes the Countable Axiom of Choice and isolates (p. 86) the Axiom of Dependent Choices. As this principle became focal, Bernays has come to be acknowledged as the source, but it should be said that the principle had already been isolated and applied by Oswald Teichmüller [68]. Bernays in a later part does come to refer ([7] p. 93, citing a review) to this paper, which appeared in the short-lived journal ***Deutsche Mathematik*** devoted to "Aryan mathematics".

Part IV first develops the Cantorian theory of cardinality in "general set theory" as conveyed mainly by axioms I - IV and Replacement. There is a necessarily careful analysis of one-to-one correspondence distinguishing sets and classes. The rest of Part IV is devoted to well-orderability and the definition of cardinal number as initial ordinal. Part V deals with "general" recursion, provable in general set theory, and how with it one can prove Zorn's Lemma from Choice IV and formalize ordinal arithmetic. Part VI finally

[39]As mentioned earlier, the term (really "vollzählig") had appeared in Bernays' May 1931 letter to Gödel.

[40]As mentioned earlier Bernays in his May 1931 letter to Gödel had provided the first direct definition of ordinal, as a transitive set each of whose members is transitive.

draws on Foundation VII, establishing its aforementioned consequence, von Neumann's axiom IV 2. The emphasis is on how IV 2 simplifies the theory of cardinality with all the proper classes put together into one cardinality class without paradox. Bernays also defines the cumulative hierarchy and with it established von Neumann's relative consistency result about Foundation, as well as used V_ω and $V_{\omega+\omega}$, and V_{ω_1} to establish various independences among the axioms. In this Bernays can be seen as pursing axiomatic analysis in the Hilbert ***Grundlagen der Geometrie*** tradition by establishing independences through models built within the system, an approach first taken in set theory by Fraenkel [20]. In the final Part VII, Bernays continues with independences using a well known interpretation of the membership relation in arithmetic due to Ackermann and shows at the very last that the axiom III b(1) is redundant.

In that Part VII pp. 88ff Bernays establishes the independence of Foundation from the other axioms, a result that he had announced in Part II p. 10. This is a result of separate significance, both for being a counterweight to von Neumann's relative consistency result and in terms of axiomatic analysis since Bernays had newly adopted Foundation. In abstract terms one can see his approach as starting with the finite ordinals and building a cumulative hierarchy S over them, and then, with f a permutation of the finite ordinals, defining a new "membership" relation by $x \in_f y$ exactly when $f(x) \in y$. Then S together with $\in_f$ serves as a model of all the axioms except Foundation, and with a proper choice of f one can have instead $\exists a(a = \{a\})$. In his 1952 ETH Habilitationsschrift Ernst Specker (cf. [67]) similarly established the independence of Foundation by getting a's satisfying $a = \{a\}$, and moreover he coordinated such a's playing the role of urelements in his refinement of the Fraenkel-Mostowski method for deriving independence results related to the Axiom of Choice.

With the set theory ZFC coming to be taken formally as a first-order theory with Replacement as a schema, connections were drawn during this period between the two set theories ZFC and Bernays-Gödel (BG). Ilse Novak [58], in her 1948 Radcliffe thesis, constructed a model of BG within the formal syntax of ZFC to show that if the latter is consistent, then so is the former. Adapting Novak's proof Andrzej Mostowski [56] showed that BG is a conservative extension of ZFC, i.e., any sentence of ZFC (so no class variables) provable in BG is already provable in ZFC. Joseph Shoenfield [62] subsequently showed how to convert a proof in BG about sets to a proof in ZFC by finitary means, i.e., through a primitive recursive procedure. Generally speaking, if one expands a first-order theory by adding predicative (no bounded class variables) second-order logic and replacing schemas by single axioms, one gets a conservative extension. Thus, as far as propositions about sets are concerned there was formal assurance that the resources of BG do not transcend those of ZFC.

In 1958 Bernays brought out a full text, ***Axiomatic Set Theory*** [10], which first establishes a specific logical framework and then proceeds to develop more formally the axiomatic theory in his series of journal articles. At the beginning of his first article [3] Bernays had written that "[t]he theory is not set up as a pure formalism … " but that it "can be formalized by means of the logical calculus of first order ("Prädikatenkalkül" or "engerer Funktionenkalkül") with the addition of the formalism of equality and the ι-symbol for "descriptions" (in the sense of Whitehead and Russell)." This approach to formalization, the one that had been taken in the ***Grundlagen***, Bernays proceeds to set out for sets, but he takes a particular tack with classes. Class variables are used but no class quantification; class membership η is reduced to $\in$ à la Gödel; and he introduces *class terms* $\{x \mid \varphi(x)\}$ for formulas φ in his language and the conversion scheme $\varphi(a) \longleftrightarrow a \in \{x \mid \varphi(x)\}$. Through this means and having made his logic explicit there is no longer need for his axioms III for construction of classes, and his other axioms involving classes are newly stated schematically, precluding finite axiomatizability. Bernays thus presented a theory much closer to modern ZFC, with an elaborated underlying first-order logic that includes classes as extensions of propositions. While a modern preoccupation about Bernays-Gödel theory is its finite axiomatizability as a formal theory of classes, Bernays himself formalized a theory with classes playing a role ancillary to sets.

Bernays in this text develops set theory largely following the progression of his journal articles, but there is also axiomatic parsimony in the gradual introduction of axioms only as needed and some rearrangement to this purpose. For "general set theory", now more explicitly to correspond to Cantor's principles for generating the transfinite numbers, Bernays has his two axioms II, one giving the empty set for starting and the other giving $a \cup \{b\}$ for successors, and now a new third axiom giving $\bigcup_{x \in a} t(x)$ for class operators t (given by applying the ι-symbol to class terms). This last axiom is evidently a combination of Union and Replacement. With only these three axioms Bernays is able to go a considerable distance, in this way exhibiting the expanse in a formalized setting that the original Cantorian initiative covers. Bernays then introduces Power Set and Choice, the latter now in a set version. Remarkable is the substantial development carried out until one gets to Infinity, and with it the development of the real numbers. Bernays's final axiom is Foundation for sets. With classes in an ancillary role, von Neumann's axiom IV 2 is no longer contextually relevant, but with the historical precedent Bernays did formulate Global Choice and showed with it and Foundation that the universe V of sets and the class On of ordinals are in bijective correspondence. Significantly, even with this result, which is really one about proper classes, Bernays reverts to taking classes seriously; in modern set theory one does not assume such principles as Global Choice as part of the basic framework. Despite this, we have in ***Axiomatic Set Theory*** a remarkable development of how set theory might in several respects be presented today, with the original approaches of Cantor and Zermelo in full evidence and Foundation at the end.

Gert Müller, who was an *Assistent* to Bernays at the ETH for the years 1952–1959, wrote in the mid-1970s:[41]

> As far as I have understood from many discussions, Paul Bernays did not consider classes as real mathematical objects (in this respect his attitude differs from von Neumann's). In describing the *use* of the set concept (via some axiomatisation) in its *frame theoretic role for mathematics* (i.e., its outermost use), classes (as extensions of conditions) are considered as a *useful element* of our language with which we describe such an axiomatisation. In addition the distinction between mathematical objects as elements of something vs. classes which are not objects (in the formal sense) becomes transparent. For this purpose the axiomatic arrangement of sets and classes as given in Bernays (1958) ... seems to be the best adaptation.

This may be a fair account of Bernays' general attitude toward classes, but remarkably, a year after the appearance of ***Axiomatic Set Theory*** Bernays would be impelled by new mathematical possibilities to work with the full quantification of classes in a dramatic development.

§**4. Reflection principles.** Entering his eighth decade Bernays was newly stimulated by work of the Israeli mathematician Azriel Levy in his 1958 Jerusalem doctoral thesis. Upon receiving a copy Bernays plunged into the details in the original Hebrew, according to Müller. Levy's work on reflection principles in his dissertation and a subsequent paper [44] would lead Bernays to a new elaboration involving classes.

Reflection principles, both as heuristic and as formal propositions, have become a common feature of modern set theory due in large part to the work of Levy. To affirm terminology, φ^y denotes the relativization of the formula φ to the set y, i.e., $\forall x$ is replaced by $\forall x \in y$ and $\exists x$ by $\exists x \in y$, these latter being formalizable. The ZF *Reflection Principle*, drawn from Montague [54] p. 99 and Levy [44] p. 234, asserts that for any ZF formula $\varphi(v_1, \ldots, v_n)$ in the free variables as displayed and any ordinal β, there is a limit ordinal $\alpha > \beta$ such that for any $x_1, \ldots, x_n \in V_\alpha$,

$$\varphi[x_1, \ldots, x_n] \quad \textit{iff} \quad \varphi^{V_\alpha}[x_1, \ldots, x_n].$$

Montague showed that the principle holds in ZF, and Levy showed that it is actually equivalent to Replacement (schema) together with Infinity in

[41] See the preface p. vii of [57].

the presence of the other axioms of ZF. Through this work the ZF Reflection Principle has become well-known as making explicit how reflection is inherent to the ZF system.

Levy [44] took the ZF Reflection Principle as motivation for stronger reflection principles involving large cardinals. The first in his hierarchy asserts that for any formula $\varphi(v_1, \ldots, v_n)$, there is an inaccessible cardinal α such that for any $x_1, \ldots, x_n \in V_\alpha$,

$$\varphi[x_1, \ldots, x_n] \quad \textit{iff} \quad \varphi^{V_\alpha}[x_1, \ldots, x_n].$$

Levy showed that this principle is equivalent to the assertion that the class of inaccessible cardinals is definably stationary, i.e., every definable closed unbounded class of ordinals contains an inaccessible cardinal. At the beginnings of set theory Paul Mahlo [46, 47, 48] had studied what are now known as the *weakly Mahlo* cardinals, those cardinals κ such that the set of smaller regular cardinals is stationary in κ, i.e., every closed unbounded subset of κ contains a regular cardinal. These cardinals figured in the earliest investigation of higher fixed-point phenomena in the Cantorian transfinite, and today are at the lower end of the hierarchy of large cardinals. Levy's work established a close connection between Mahlo's cardinals and structural principles about sets. Levy recast Mahlo's concept by replacing regular cardinals by inaccessible cardinals. On the other hand, whereas Mahlo had entertained arbitrary closed unbounded subsets, Levy's principle is restricted to definable closed unbounded classes. Be that as it may, it would be through Levy's work that Mahlo's cardinals would come into use in modern set theory cast as the *strongly Mahlo* cardinals, those regular cardinals κ such that the set of smaller *inaccessible* cardinals is stationary in κ.

Mahlo's investigations featured getting a hierarchy of cardinals, the *α-Mahlo cardinals*: 1-Mahlo cardinals are the Mahlo cardinals; $(\alpha + 1)$-cardinals are those cardinals κ such that the set of smaller α-Mahlo cardinals is stationary in κ; and δ-Mahlo cardinals for limit δ are those cardinals which are α-Mahlo for every $\alpha < \delta$. Levy [44] showed how to get these higher Mahlo cardinals through corresponding strengthenings of reflection, beginning with 2-Mahlos arising from reflecting down to V_α where α is itself Mahlo.

To this new investigation based on a new theme, Bernays in a 1961 paper [12], a contribution to a volume of papers commemorating the 70th birthday of Fraenkel, made quite significant advances. He first considered the curtailed reflection schema

$$\varphi \to \exists y(\mathrm{Trans}(y) \ \& \ \varphi^y)$$

for formulas φ without y or any class variables, where $\mathrm{Trans}(y)$ asserts that y is transitive. Levy in his dissertation routinely required such reflecting y's to model ZF, and this necessitated working with a formalized satisfaction predicate. Starting with the observation that set parameters $a_1, \ldots, a_n$ can appear in φ and y can be required to contain them by introducing clauses $\exists x(a_i \in x)$ into φ, Bernays just with his schema established Pair, Union, Infinity, and Replacement (schema)—in effect achieving a remarkably economical presentation of ZF.

Bernays then made his main contribution by introducing classes, and quite precipitously, allowed quantifications $\forall A$ and $\exists A$ of classes. For formulas φ of this expanded language, we can cast his class reflection schema as follows:

$$\varphi(A_1, \ldots, A_r) \longrightarrow \exists y(\mathrm{Trans}(y) \ \ \& \ \ \varphi^y(A_1 \cap y, \ldots, A_r \cap y)).$$

Here the relativization has to be newly and carefully stipulated: First, all defined terms are assumed eliminated in favor of their definitions, this including any class terms of the Bernays 1958 text using the conversion schema. Then set quantification is relativized as before and now class quantification $\forall A$ is replaced by $\forall x \subseteq y$, this being formalizable. Bernays showed that with this crucial move into quantification over classes, all the various Levy reflection principles for getting the higher, α-Mahlo cardinals are at once subsumed. In modern terms, the Mahlo reflection principles amount to asserting the existence of strong closure ordinals for Skolem functions, and with the conceptually stronger $\forall A$ one can get at this by taking an intersection of closed classes.

Given Bernays' modest approach to classes in his 1958 text, how are we to take this incursion by him into what we would now regard as full second-order logic? Gödel in his late sixties, and despite his own earlier pronouncements about how the Continuum Hypothesis is false and how new large cardinal axioms might establish this, worked on "orders of growth" axioms that might actually establish the Continuum Hypothesis.[42] In this Gödel exhibited a remarkable fluidity, to see simply where the mathematics leads. So too Bernays, despite his lack of commitment to classes as "real mathematical objects". In the end Bernays' mathematical instincts manifested themselves and he, stimulated by Levy's work, established a significant result properly of second-order set theory.

Bernays' work itself would soon be subsumed into set theory with V_κ playing the role of the universe V and quantification over classes, etc. carried forth with higher-order quantifiers. In a 1961 abstract William Hanf and Dana Scott [33] formulated the *indescribable cardinals* by thus ascribing reflection properties to domains V_κ. In this context, Bernays' principle amounts to the ascription of second-order indescribability, Π^1_n-indescribability for all n, to the class On of all ordinals. Here, Π^1_n refers to the quantifier complexity for the second-order quantifiers with just n alternations of quantifiers starting with $\forall$. Hanf and Scott characterized the weakly compact cardinals, large cardinals arising from the investigation of infinitary languages, exactly as the Π^1_1-indescribable cardinals, and in fact Bernays' argument for subsuming the Mahlo hierarchy with his reflection

[42]See Kanamori [42] §8.

schema requires only universal class quantification. This raises a historically interesting point:

In his Berkeley Ph.D. work done by 1960, Hanf had in effect shown through their incipient infinitary language formulation that weakly compact cardinals κ are α-Mahlo for every $\alpha < \kappa$.[43] This in a strong sense had answered a pivotal, classical question of Tarski in the theory of large cardinals: Can the least inaccessible cardinal be measurable? The answer is most decidedly no, as measurable cardinals are weakly compact and α-Mahlo cardinals are highly inaccessible. This result was greeted by Abraham Robinson [60] p. 78 as "a spectacular success" for metamathematical methods. Hanf's work radically altered size intuitions about problems that were coming to be understood in terms of large cardinals.

Bernays' work, given the Π^1_1-indescribability of weakly compact cardinals, established Hanf's result. Moreover, Bernays' work was probably done earlier, for he announced [11] his results at a September 1959 conference. And, it would be through reflection phenomena rather than involved infinitary language approaches that such transcendence results for large cardinals would henceforth be approached. Had Bernays looked in different directions for sufficient hypotheses to derive his class reflection schema or had he applied already known results about measurability, he would have been first to solve Tarski's problem. Gödel in a letter of 11 August 1961 latterly mentioned new work on Tarski's problem,[44] but this did not seem to elicit a response.

Bernays at the ETH had over a dozen students, including J. Richard Büchi and Erwin Engeler, and as colleagues active in set theory, Ernst Specker and his student Hans Läuchli. In 1946 Bernays together with the philosophers Ferdinand Gonseth and Gaston Bachelard founded the now-prominent journal ***Dialectica***. Bernays produced lucid and incisive, wide-ranging philosophical essays and reviews over several decades.[45] In 1976 there appeared a celebratory volume on Bernays' set-theoretic work edited by Müller. In his late eighties, Bernays passed away in Zürich on 18 September 1977, four months before Gödel.

§**5. Envoi.** From the point of view of modern set theory Bernays can be considered an important transitional figure in two respects. First, Bernays recast von Neumann's work to provide a viable first-order axiomatization of sets and classes that, like Zermelo's late axiomatization, incorporated Replacement and Foundation. Gödel then made crucial use of Bernays' axiomatization to formally cast logical operations and definability. However, classes would lose this role with ZFC becoming the standard axiomatization and the satisfaction predicate become domesticated in set theory. Classes

[43] Hanf's work appeared in print only years later in [32].
[44] See Gödel [29] p. 193ff.
[45] See Sieg and Tait [65].

would remain very much a part of the practice of set theory, but they would come to be regarded as informal constructs, usually extensions of formulas. This transition is already in evidence in Bernays' final development [10] of set theory.

Second, Bernays evidently took classes as partaking in a *conceptual* analysis of sets, this being worked out through axiomatization. However varying his ontological commitment to classes, they participate in the process of providing mathematical sense to sets. This way of thinking would recede in the 1960s after Paul Cohen's discovery of forcing, as set theory became more like group theory, with a profusion of models for various combinatorial propositions and the development of more and more methods for investigating a range of hypotheses and their relative consistencies.[46]

In a 1965 international colloquium on the philosophy of mathematics, Bernays [13] maintained that the recent results of Cohen (p. 109) "do not directly concern set theory itself, but rather the axiomatization of set theory", and took forcing extension models to have the "character of non-standard models". After alluding to the fact stressed by Kreisel that the Continuum Hypothesis *is* decided in second-order logic, Bernays averred (p. 111ff): "Our inability to deal successfully with the continuum problem is certainly connected with the circumstance that our explicit knowledge of the continuum is very restricted."

The advent of forcing would be looked upon more favorably as an opening up of set theory and the beginning of its transformation into a modern, sophisticated field of mathematics, one that through elaboration of method would provide new insights into issues even like the continuum problem. Forcing itself went a considerable distance in downgrading any formal theory of classes because of the added encumbrance of having to specify the classes of generic extensions.

But there would be an eternal return. As methods evolved ever more sophisticated, new levels of understanding have been considered to have been achieved about "the concept of set" in the unbridled formulations about classes, particularly in the work of Hugh Woodin on the Continuum Hypothesis. Of this at least, Bernays would have approved.

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