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A SURVEY OF PROPOSITIONAL REALIZABILITY LOGIC

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Abstract. The study of propositional realizability logic was initiated in the 50th of the last century. Some interesting results were obtained in the 60-70th, but many important problems in this area are still open. Now interest to these problems from new generation of researchers is observed. This survey contains an exposition of the results on propositional realizability logic and corresponding techniques. Thus reading this paper can be the start point in exploring and development of constructive logic.

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§1. Introduction. Recursive realizability as a semantics of arithmetical sentences was introduced by S. C. Kleene [18]. It can be considered as a more precise definition of the informal intuitionistic semantics using the exact notion of a partial recursive function instead of the vague intuitionistic concept of an effective operation. Namely Kleene's semantics was the main base of the constructive approach to mathematics worked out by A. A. Markov and his school.

In connection with the development of constructive mathematics it is of interest to study logical principles acceptable from the constructive point of view. In mathematical logic logical principles are expressed by logical formulas. The simplest logical formulas are propositional ones. There are various notions of a realizable propositional formula. They will be considered in this paper.

Every realizable propositional formula can be considered as a constructive logical principle. Thus studying propositional realizability logic was at first important for foundations of mathematics. Later the applications of constructive mathematics in computer science also increased the interest in the subject.

The most intensive development of constructive logic was in the 50th–70th of the last century. Studies in constructive logic in the USSR were initiated by A. N. Kolmogorov, P. S. Novikov, and A. A. Markov. A lot of results were obtained by their students and published in Russian (although many of them are translated into English). Unfortunately, no appropriate description of the class of realizable propositional formulas was found until now; nevertheless some partial results in exploring propositional realizability logic were obtained. Today constructive logic is approached with the help of modern tools. The aim of this paper is an exposition of results in propositional realizability logic, especially of those obtained by Soviet mathematicians, in order to facilitate beginning the study of this logic and to prevent rediscovery of known results.

§2. **Intuitionistic propositional logic.** The realizability semantics originates from the informal intuitionistic semantics of mathematical sentences.

2.1. Informal intuitionistic semantics. From the point of view of intuitionists, a sentence is true if it is proved. Thus the truth of a sentence is connected with its proof. In order to avoid any confusion with formal proofs, we shall use the term "a verification" instead of "a proof". Such understanding of the meaning of sentences leads to an original interpretation of logical connectives and quantifiers stated by L. E. J. Brouwer, A. N. Kolmogorov, and A. Heyting. Namely for every true sentence A we can consider its verification as a text justifying A. Now, if A and B are sentences, then

- a verification of a conjunction $A \;\&\; B$ is a text containing a verification of A and a verification of B;
- a verification of a disjunction $A \vee B$ is a text containing a verification of A or a verification of B and indicating which of them is verified;
- a verification of an implication $A \to B$ is a text describing a general effective operation for obtaining a verification of B from every verification of A;
- a verification of a negative sentence $\neg A$ is a verification of the sentence $A \to \bot$, where $\bot$ is a certainly absurd sentence having no verification.

If $A(x)$ is a predicate with a parameter x over a domain M given in an appropriate way, then

- a verification of an universal sentence $\forall x\, A(x)$ is a text describing a general effective operation which allows to obtain a verification of $A(m)$ for every given $m \in M$;
- a verification of an existential sentence $\exists x\, A(x)$ is a text indicating a concrete $m \in M$ and containing a verification of the sentence $A(m)$.

Of course, this semantics is very informal and is not precise from the mathematical point of view. Nevertheless it is enough for formulating intuitionistically valid logical principles. We shall consider the principles of intuitionistic propositional logic, i.e., those expressible by propositional formulas.

2.2. Intuitionistic propositional calculus. Propositional formulas are constructed in the usual way from a countable set of propositional variables and connectives $\&$, $\vee, \to, \neg$. We use the following convention to eliminate parentheses in the formulas: $\neg$ binds more strongly than $\&$ and $\vee$, which bind more strongly than $\to$; in addition, $\&$ binds more strongly than $\vee$. We shall use $p, q, r, \ldots$ (possibly with subscripts) as metasymbols for propositional variables. $A(p_1, \ldots, p_n)$ will denote a propositional formula containing no variables other than $p_1, \ldots, p_n$. The formula $(A \to B) \;\&\; (B \to A)$ will be denoted by $A \equiv B$.

A propositional formula $A(p_1, \ldots, p_n)$ is called intuitionistically valid if every sentence of the form $A(A_1, \ldots, A_n)$, $A_1, \ldots, A_n$ being arbitrary sentences, is intuitionistically true. Thus intuitionistically valid formulas can be considered as principles of intuitionistic propositional logic. The first attempt of axiomatizing intuitionistic propositional logic was made by A. N. Kolmogorov [20]. He considered from the intuitionistic point of view an axiom system for classical logic in the language with the connectives $\to$ and $\neg$ proposed by D. Hilbert and transformed it into an intuitionistic calculus called now *the Kolmogorov calculus*.

Other more wide axiom systems of intuitionistic propositional logic were later proposed by V. I. Glivenko [5], A. Heyting [7], and G. Gentzen [4]. They are all equivalent in the sense that the same formulas are deducible from each of them and are equivalent to the following axiom system of *intuitionistic propositional calculus* IPC:

1. $A \to (B \to A)$;
2. $(A \to B) \to ((A \to (B \to C)) \to (A \to C))$;
3. $(A \ \& \ B) \to A$;
4. $(A \ \& \ B) \to B$;
5. $(A \to B) \to ((A \to C)) \to (A \to B \ \& \ C))$;
6. $A \to (A \vee B)$;
7. $B \to (A \vee B)$;
8. $(A \to B) \to ((B \to C)) \to ((A \vee B) \to C))$;
9. $(A \to B) \to ((A \to \neg B) \to \neg A)$;
10. $\neg A \to (A \to B)$.

The only inference rule is *modus ponens*.

The calculus with the axioms 1–8 is called *positive propositional calculus* PPC. This calculus is rather carefully studied in [8, chapter III]. The calculus with the axioms 1–9 was considered by I. Johansson [12] and is called *minimal propositional calculus* MPC. It is proved by the author [35] that the Kolmogorov calculus is exactly the $(\to, \neg)$-fragment of MPC. *Classical propositional calculus* CPC is obtained by adding the axiom scheme $\neg\neg A \to A$ to IPC. It is known (see, e.g., [39, chapter IX, §5]) that every formula of the form $\neg A$ deducible in CPC is also deducible in IPC.

If $A_1, \dots, A_n$ are propositional formulas, let $\mathsf{IPC} + A_1 + \cdots + A_n$ be the calculus obtained by adding the formulas $A_1, \dots, A_n$ as axiom schemes to IPC. Formulas A and B are *deductively equivalent* if $\mathsf{IPC} + A \vdash B$ and $\mathsf{IPC} + B \vdash A$.

The problem of completeness of the calculus IPC can not be stated in a precise mathematical form because intuitionistic semantics is very informal. It can be made more precise if we define in a mathematical mode two key notions used in the above description of the informal semantics, namely the notions of a verification and a general effective operation. One way is to consider verifications as formal proofs in a formal system. This was done by S. N. Artemov [1] who introduced *the logic of proofs*. Operations over proofs are also defined in this approach. We consider other interpretations of intuitionistic propositional logic.

2.3. Heyting algebras. A *logical matrix* $\mathfrak{M}$ is a non-empty set M with a distinguished element $1_{\mathfrak{M}}$ equipped with an unary operation $\neg_{\mathfrak{M}}$ and binary operations $\&_{\mathfrak{M}}$, $\vee_{\mathfrak{M}}$, and $\rightarrow_{\mathfrak{M}}$ such that for any elements $x, y \in M$ the following conditions hold:

- if $1_{\mathfrak{M}} \rightarrow_{\mathfrak{M}} x = 1_{\mathfrak{M}}$, then $x = 1_{\mathfrak{M}}$;
- if $x \rightarrow_{\mathfrak{M}} y = y \rightarrow_{\mathfrak{M}} x = 1_{\mathfrak{M}}$, then $x = y$.

A *valuation* on $\mathfrak{M}$ is a function f assigning to each propositional variable p some element $f(p) \in M$. Any valuation is extended to all propositional formulas in a natural way. We say that a formula A is *valid* in $\mathfrak{M}$ and write $\mathfrak{M} \models A$ if $f(A) = 1_{\mathfrak{M}}$ for each valuation f on $\mathfrak{M}$. If $f(A) \neq 1_{\mathfrak{M}}$, we say that f *refutes* A on $\mathfrak{M}$. A formula A is *refutable* on a logical matrix $\mathfrak{M}$ iff there is a valuation refuting A on $\mathfrak{M}$.

A logical matrix $\mathfrak{M}$ is called a *model* of a propositional calculus if all the formulas deducible in the calculus are valid in $\mathfrak{M}$. Models of IPC are called *Heyting algebras* or *pseudo-Boolean algebras*. A detailed exposition of pseudo-Boolean algebras can be found in the book [39]. Every Heyting algebra $\mathfrak{M}$ is partially ordered by the relation $x \leq y \rightleftharpoons x \rightarrow_{\mathfrak{M}} y = 1_{\mathfrak{M}}$. Obviously, $1_{\mathfrak{M}}$ is the greatest element. Each Heyting algebra has the least element $0_{\mathfrak{M}}$. Relative to the order $\leq$ the operations $\&_{\mathfrak{M}}$, $\vee_{\mathfrak{M}}$, $\rightarrow_{\mathfrak{M}}$, and $\neg_{\mathfrak{M}}$ can be characterized in the following way:

- $x \ \&_{\mathfrak{M}} \ y$ is the greatest lower bound of the set $\{x, y\}$;
- $x \vee_{\mathfrak{M}} y$ is the least upper bound of the set $\{x, y\}$;
- $x \rightarrow_{\mathfrak{M}} y$ is the greatest lower bound of the set $\{z \mid x \ \&_{\mathfrak{M}} z \ \leq y\}$;
- $\neg_{\mathfrak{M}} = x \rightarrow_{\mathfrak{M}} 0_{\mathfrak{M}}$.

There are two important ways of constructing Heyting algebras. The *Cartesian product* of Heyting algebras $\mathfrak{M}_1, \dots, \mathfrak{M}_n$ is a Heyting algebra defined as the set $M_1 \times \cdots \times M_n$ equipped with the component-wise operations. The n-th *Cartesian power* $\mathfrak{M}^n$ $(n \geq 1)$ of an algebra $\mathfrak{M}$ is a product of n copies of $\mathfrak{M}$, $\mathfrak{M}^1$ being $\mathfrak{M}$.

Another operation Γ for any Heyting algebra $\mathfrak{M}$ gives an algebra $\mathfrak{N} = \Gamma(\mathfrak{M})$ obtained by adding to $\mathfrak{M}$ a new element $\omega_{\mathfrak{N}}$ with the property: $x \leq \omega_{\mathfrak{N}} \leq 1_{\mathfrak{M}}$ for every $x \neq 1_{\mathfrak{M}}$ in M. Thus, if $\mathfrak{M}$ is not a trivial one-element Heyting algebra, then $1_{\mathfrak{N}} = 1_{\mathfrak{M}}$, $0_{\mathfrak{N}} = 0_{\mathfrak{M}}$. The operations in $\mathfrak{N}$ are defined as follows. If one of the operands is $1_{\mathfrak{N}}$, the definition is obvious. In the converse case, consider the following function $\alpha\colon \mathfrak{M} \rightarrow \mathfrak{N}$:

$$\alpha(x) = \begin{cases} x & \text{if } x \neq 1_{\mathfrak{M}}, \\ \omega_{\mathfrak{N}} & \text{if } x = 1_{\mathfrak{M}}. \end{cases}$$

If $x = \alpha(u)$, $y = \alpha(v)$, then $x \rightarrow_{\mathfrak{N}} y = u \rightarrow_{\mathfrak{M}} v$, $x \ \&_{\mathfrak{N}} \ y = \alpha(u \ \&_{\mathfrak{M}} \ v)$, $x \vee_{\mathfrak{N}} y = \alpha(u \vee_{\mathfrak{M}} v)$, $\neg_{\mathfrak{N}} x = \neg_{\mathfrak{M}} u$.

A Heyting algebra $\mathfrak{N}$ is called *Gödelean* if there is an element $\omega_{\mathfrak{N}}$ in $\mathfrak{N}$ which is the greatest among the elements different from $1_{\mathfrak{N}}$. In this case, the element $\omega_{\mathfrak{N}}$ is called a *Gödelean element* of $\mathfrak{N}$. Obviously, every algebra $\mathfrak{N}$ of the form $\Gamma(\mathfrak{M})$ is Gödelean, $\omega_{\mathfrak{N}}$ being a Gödelean element.

Jaśkowski's sequence of Heyting algebras $\mathfrak{J}_n$ is defined as follows: $\mathfrak{J}_0$ is the two-element algebra with the elements $0, 1$ and the order $0 \leq 1$; $\mathfrak{J}_{n+1} = \Gamma(\mathfrak{J}_n^{n+1})$. Let us assume that a new element added by constructing $\mathfrak{J}_{n+1}$ from $\mathfrak{J}_n^{n+1}$ by means of the operation Γ is $n + 2$. For example, $\mathfrak{J}_1$ consists of the elements $0, 1, 2$ with the order $0 \leq 2 \leq 1$. Thus $0_{\mathfrak{J}_1} = 0$, $\omega_{\mathfrak{J}_1} = 2$, $1_{\mathfrak{J}_1} = 1$. The algebra $\mathfrak{J}_2$ consists of 3 and the elements of the form (a, b), where $a, b \in \{0, 1, 2\}$, with $(a, b) \leq 3 \leq (1, 1)$ for any $(a, b) \neq (1, 1)$. In this case, $0_{\mathfrak{J}_2} = (0, 0)$, $\omega_{\mathfrak{J}_2} = 3$, $1_{\mathfrak{J}_2} = (1, 1)$. The algebra $\mathfrak{J}_3$ consists of the number 4 and the elements of the form (a, b, c), where $a, b, c \in \mathfrak{J}_2$. Here $0_{\mathfrak{J}_3} = ((0, 0), (0, 0), (0, 0))$, $\omega_{\mathfrak{J}_3} = 4$, $1_{\mathfrak{J}_3} = ((1, 1), (1, 1), (1, 1))$. Obviously, Jaśkowski's algebras are Gödelean. The following theorem is important in the study of IPC.

THEOREM 2.1. *If a propositional formula A is not deducible in* IPC, *then there exists n such that A is refutable on the algebra $\mathfrak{J}_n$.*

PROOF. See [11], [41]. ⊣

Let $\mathfrak{N}$ be a finite Gödelean algebra, ω be its Gödelean element. To each element $a \in \mathfrak{N}$ assign in a one-to-one way a propositional variable p_a. Let K be the conjunction of all the formulas $p_{a \circ_{\mathfrak{N}} b} \equiv p_a \circ p_b$ ($\circ \in \{\,\&\,, \vee, \rightarrow\}$) and $p_{\neg_{\mathfrak{N}}} \equiv \neg p_a$ for $a, b \in \mathfrak{N}$. The formula $K \rightarrow p_\omega$ is called a *characteristic formula* for $\mathfrak{N}$ and denoted by $X_{\mathfrak{N}}$. It is obvious that the evaluation $f(p_a) = a$ refutes $X_{\mathfrak{N}}$ on $\mathfrak{N}$. The notion of a characteristic formula was introduced by V. A. Jankov. He proved the following theorem.

THEOREM 2.2. *Let a propositional formula A be refutable on a finite Gödelean algebra $\mathfrak{N}$. Then the formula $X_{\mathfrak{N}}$ is deducible in the calculus* IPC$+A$.

PROOF. See [10]. ⊣

A property of propositional formulas is called intuitionistic if the fact that A has this property implies that every formula deducible in the calculus IPC $+ A$ has this property too.

THEOREM 2.3. *If there exists a propositional formula with an intuitionistic property P refutable on a finite Gödelean algebra $\mathfrak{N}$, then the formula $X_{\mathfrak{N}}$ has the property P.*

PROOF. See [10]. ⊣

This theorem is important in studying various semantics for intuitionistic propositional logic. Let an interpretation (a semantics) of propositional formulas be given. Suppose that IPC is sound with this semantics and we are interested in completeness of IPC. If there exists a formula A valid in the semantics and nondeducible in IPC, then A is refuted on a finite Gödelean algebra (for example, on a Jaśkowski's algebra) $\mathfrak{M}$. In this case, the formula

$X_{\mathfrak{M}}$ is also valid and nondeducible. Thus we can look for an example of this kind among the characteristic formulas.

§3. Medvedev Logic. Heyting algebras give a rather formal characterization of intuitionistic propositional logic. An interesting although informal interpretation of that logic was proposed by Kolmogorov [21]. He considers propositional formulas as schemes of problems. Namely let propositional variables mean arbitrary problems. Then $A \ \& \ B$ is the problem "To solve the problems A and B", $A \vee B$ is the problem "To solve the problem A or to solve the problem B", $A \to B$ is the problem "To reduce the problem B to the problem A", i.e., "To solve the problem B assuming that a solution of the problem A is given", the problem $\neg A$ is the problem $A \to \bot$, where $\bot$ is a problem having no solution. A propositional formula $A(p_1, \dots, p_n)$ is considered as a principle of *the logic of problems* if there exists an uniform solution for all the problems of the form $A(A_1, \dots, A_n)$, where $A_1, \dots, A_n$ are arbitrary problems. Kolmogorov shows that the calculus IPC is sound with this interpretation: all the formulas deducible in IPC are principles of the logic of problems.

In order to state the problem of completeness of IPC relative to the interpretation by means of problems, we have to make the notion of a problem more precise from the mathematical point of view. This was done by Yu. T. Medvedev [23] who considered so-called finite problems. A *finite problem* is a pair $\mathfrak{A} = \langle F, X \rangle$, where F is a non-empty set, X is its subset (possibly empty). Intuitively, F is the set of *possible solutions* of the problem, X is the set of its *actual solutions*. Let $\bot$ be a fixed problem without any actual solutions, for example, $\bot = \langle \{0\}, \emptyset \rangle$. For a finite problem $\mathfrak{A} = \langle F, X \rangle$ we denote F by $\varphi(\mathfrak{A})$ and X by $\chi(\mathfrak{A})$. Logical operations on finite problems are defined in the following way:

$$\varphi(\mathfrak{A} \ \& \ \mathfrak{B}) = \varphi(\mathfrak{A}) \times \varphi(\mathfrak{B}), \ \chi(\mathfrak{A} \ \& \ \mathfrak{B}) = \chi(\mathfrak{A}) \times \chi(\mathfrak{B}),$$

where $X \times Y$ means Cartesian product;

$$\varphi(\mathfrak{A} \vee \mathfrak{B}) = \varphi(\mathfrak{A}) \oplus \varphi(\mathfrak{B}), \ \chi(\mathfrak{A} \vee \mathfrak{B}) = \chi(\mathfrak{A}) \oplus \chi(\mathfrak{B}),$$

where $X \oplus Y = (X \times \{0\}) \cup (Y \times \{1\})$;

$$\varphi(\mathfrak{A} \to \mathfrak{B}) = \varphi(\mathfrak{B})^{\varphi(\mathfrak{A})}, \ \chi(\mathfrak{A} \to \mathfrak{B}) = \{f \in \varphi(\mathfrak{B})^{\varphi(\mathfrak{A})} \mid f(\chi(\mathfrak{A})) \subseteq \chi(\mathfrak{B})\},$$

where Y^X means the set of maps $f \colon X \to Y$;

$$\neg\mathfrak{A} = \mathfrak{A} \to \bot.$$

Let $A(p_1, \dots, p_n)$ be a propositional formula, $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ be finite problems. Then $A(\mathfrak{A}_1, \dots, \mathfrak{A}_n)$ is a finite problem obtained by substituting $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ for $p_1, \dots, p_n$ in $A(p_1, \dots, p_n)$.

Let $\mathcal{F}$ be a system of non-empty finite sets $F_1, \dots, F_n$. We say that a propositional formula $A(p_1, \dots, p_n)$ is *valid* over a system $\mathcal{F}$ if there exists an element e such that $e \in \chi(A(\mathfrak{A}_1, \dots, \mathfrak{A}_n))$ for all the finite problems $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ with $\varphi(\mathfrak{A}_i) = F_i$ $(i = 1, \dots, n)$. In other words, a formula $A(p_1, \dots, p_n)$ is valid over $\mathcal{F}$ if there is an uniform actual solution for all the problems of the form $A(\mathfrak{A}, \dots, \mathfrak{A}_n)$, where $\varphi(\mathfrak{A}_i) = F_i$. A propositional formula $A(p_1, \dots, p_n)$ is called *finitely valid* if it is valid over every system $F_1, \dots, F_n$. The set of finitely valid formulas is called *Medvedev Logic*; we shall denote it by ML.

Medvedev [25] proved that this logic has a finite model property: there exists a sequence of finite Heyting algebras Φ_n $(n = 1, 2, \dots)$ such that a propositional formula A is in Medvedev Logic iff it is valid in Φ_n for every n. The sequence Φ_n is defined as follows. Let $I_n = \{1, 2, \dots, n\}$ $(n \geq 1)$, σ^n be the family of its nonempty subsets. For $\sigma \subseteq \sigma^n$, its closure is defined as the family $\sigma^* = \{E \subseteq I_n \mid (\exists E_0 \in \sigma)\, E_0 \subseteq E\}$. A family σ is called closed if $\sigma^* = \sigma$. Let Φ_n be the set of all closed $\sigma \subseteq \sigma^n$. It is easy to prove that Φ_n ordered by the relation $\supseteq$ is a Heyting algebra. Obviously, σ_n is the least element and $\emptyset$ is the greatest element in Φ_n. Note that every algebra Φ_n is Gödelean: $\{I_n\}$ is its Gödelean element. It was proved by Medvedev [25] that a propositional formula A is not finitely valid if and only if there exists n such that A is refuted on Φ_n.

Every formula deducible in IPC is finitely valid. IPC is not complete relative to Medvedev's interpretation. For example, the formulas

$$(\neg p \to (q \vee r)) \to ((\neg p \to q) \vee (\neg p \to r)) \tag{1}$$

and

$$((\neg\neg p \to p) \to (\neg p \vee \neg\neg p)) \to (\neg p \vee \neg\neg p) \tag{2}$$

are finitely valid but are not deducible in IPC.

No axiomatization of ML is known, but it is proved that it can not be axiomatized by any system of axiom schemes with bounded number of variables (see [22]). In particular, this logic is not finitely axiomatizable.

§**4. Kleene's recursive realizability.** Another way of a specification of the informal intuitionistic semantics is to define a mathematically precise notion of a general effective operation. There is such a notion in mathematics, namely the notion of an algorithm. A variant of intuitionistic semantics based on interpreting effective operations as algorithms was proposed by S. C. Kleene [18] who introduced in 1945 the notion of *recursive realizability* for the sentences of first-order arithmetic. The main idea of Kleene was to consider natural numbers as the codes of verifications and partial recursive functions as effective operations. Partial recursive functions are coded by natural numbers by means of the Gödel enumeration. A code of a verification of a sentence is called *a realization* of the sentence.

We shall consider the first-order language of formal arithmetic containing the constant 0, the symbols for all the primitive recursive functions, the predicate symbol $=$ (identity), the propositional connectives $\&$, $\vee$, $\to$, and $\neg$, the universal quantifier symbol $\forall$ and the existential quantifier symbol $\exists$. The closed arithmetical formulas will be called arithmetical sentences.

Elements of the theory of partial recursive functions can be found in [19]. The total partial recursive functions are called general recursive. A conjecture that the partial recursive functions are just the arithmetical computable functions (Church Thesis) is universally recognized in mathematics. An indexing of the partial recursive functions is described in [19]. The unary function whose index is x will be denoted by $\{x\}$. Note that every natural number is an index of an unary partial recursive function. Let $\varphi(x_1, \ldots, x_n)$ be an expression defining an n-ary partial recursive function φ in an evident way. Then a Gödel number of this function can be found. Let $\Lambda x_1, \ldots, x_n.\varphi(x_1, \ldots, x_n)$ denote a Gödel number of φ. If $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is an expression defining an $(m + n)$-ary partial recursive function φ, then

$$\alpha(y_1, \ldots, y_m) = \Lambda x_1, \ldots, x_n.\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

is an m-ary recursive function. For any $k_1, \ldots, k_m$ the value $\alpha(k_1, \ldots, k_m)$ can be found from the Gödel number of φ by using s-m-n-theorem (see [40, §1.8]). For any natural number b let $c_b = \Lambda x.b$.

Let $\pi_0, \pi_1, \pi_2, \ldots$ be the sequence of prime numbers, i.e., $\pi_0 = 2$, $\pi_1 = 3$, $\pi_2 = 5$ and so on. We shall use $(a)_i$ to denote the exponent with which the prime number π_i appears in the decomposition of a number a into prime factors.

The relation "*a natural number e realizes an arithmetical sentence* Φ" is denoted $e\,\mathbf{r}\,\Phi$ and is defined by induction on the number of logical connectives and quantifiers in Φ.

- If Φ is an atomic sentence $t_1 = t_2$, then $e\,\mathbf{r}\,\Phi \rightleftharpoons [e = 0$ and Φ is true$]$.

Let Φ and Ψ be arithmetical sentences. Then

- $e\,\mathbf{r}\,(\Phi \;\&\; \Psi) \rightleftharpoons [e$ is of the form $2^a \cdot 3^b$ and $a\,\mathbf{r}\,\Phi$, $b\,\mathbf{r}\,\Psi]$;
- $e\,\mathbf{r}\,(\Phi \vee \Psi) \rightleftharpoons [e$ is of the form $2^0 \cdot 3^a$ and $a\,\mathbf{r}\,\Phi$ or e is of the form $2^1 \cdot 3^b$ and $b\,\mathbf{r}\,\Psi]$;
- $e\,\mathbf{r}\,(\Phi \to \Psi) \rightleftharpoons [e$ is an index of a partial recursive function ϕ such that for any a, if $a\,\mathbf{r}\,\Phi$, then $\phi(a)\,\mathbf{r}\,\Psi]$;
- $e\,\mathbf{r}\,\neg\Phi \rightleftharpoons [e\,\mathbf{r}\,(\Phi \to 0 = 1)]$.

Let $\Phi(x)$ be an arithmetical formula with the only parameter x. Then

- $e\,\mathbf{r}\,\forall x\,\Phi(x) \rightleftharpoons [e$ is an index of a general recursive function f such that $f(n)\,\mathbf{r}\,\Phi(n)$ for every $n]$;
- $e\,\mathbf{r}\,\exists x\,\Phi(x) \rightleftharpoons [e$ is of the form $2^n \cdot 3^a$ and $a\,\mathbf{r}\,\Phi(n)]$.

An arithmetical sentence Φ is called *realizable* if there is e such that $e \mathbf{r} \Phi$. An arithmetical formula $\Phi(x_1, \dots, x_m)$ with the only parameters $x_1, \dots, x_m$ is called realizable if the sentence $\forall x_1 \dots \forall x_m \Phi(x_1, \dots, x_m)$ is realizable. Obviously, a formula $\Phi(x_1, \dots, x_m)$ is realizable iff there exists an m-ary recursive function f such that $f(k_1, \dots, k_m) \mathbf{r} \Phi(k_1, \dots, k_m)$ for every $k_1, \dots, k_m$. In this case, we say that the function f realizes the formula $\Phi(x_1, \dots, x_m)$.

It follows from the definition of recursive realizability that a sentence of the form $\neg\Phi$ is realizable iff Φ is not realizable. Obviously, if Φ is not realizable, then every number is a realization of $\neg\Phi$, in particular, $0 \mathbf{r} \neg\Phi$. It is also evident that if $b \mathbf{r} \Psi$, then $c_b \mathbf{r} (\Phi \to \Psi)$ for every sentence Φ.

A Σ_1-formula is an arithmetical formula of the form $\exists x_1 \dots \exists x_n \Phi$, Φ being a quantifier-free formula. It can be proved that a closed Σ_1-formula is realizable iff it is true from the classical point of view in the standard interpretation of the language of arithmetic. Note that the predicate $\{x\}(y) = z$ can be expressed by a Σ_1-formula $\mathrm{R}(x, y, z)$. For any $n = 0, 1, 2, \dots$ let $\mathrm{R}_n(x)$ be the formula $\mathrm{R}(x, x, n)$. Thus for any i the formula $\mathrm{R}_n(i)$ is realizable iff $\{i\}(i) = n$.

The following theorem is proved by D. Nelson.

THEOREM 4.1. *Every formula deducible in intuitionistic arithmetic* HA *is realizable.*

PROOF. See [26]. ⊣

The Nelson theorem means that intuitionistic arithmetic is sound with recursive realizability.

From the point of view of constructive semantics, a sentence is true iff it is realizable.

Note that the notion of realizability can be formulated as a translation from the language of arithmetic into itself. Thus propositions on realizability of sentences are arithmetical formulas. This formulas can be considered from the point of view of some arithmetical theory (which may be constructive or not). In this case, realizability is the subject of proof theory.

Although the subject of this paper is constructive semantics, we shall consider it from the classical point of view, i.e., the metatheory used here is classical. Constructive meaning of propositions is explained by explicit using of appropriate concepts in the recursion theory.

§5. Realizable propositional formulas. Let A be a propositional formula with variables $p_1, \dots, p_n$; denote it by $A(p_1, \dots, p_n)$. An arithmetical instance of A is a formula $A(\Phi_1, \dots \Phi_n)$ obtained by substituting arithmetical formulas $\Phi_1, \dots, \Phi_n$ for the variables $p_1, \dots, p_n$ into A. A technical feature of Nelson's proof of Theorem 4.1 deserves to be mentioned: for any propositional formula A deducible in IPC there exists a number a such that a realizes every closed arithmetical instance of A. It follows that every arithmetical instance of A is realized by a constant function of appropriate arity with the only value a. Moreover, the following theorem holds.

THEOREM 5.1. *Let an inference in* IPC *of a formula* $A(p_1, \dots, p_n)$ *from formulas* $A_1(p_1, \dots, p_n), \dots, A_m(p_1, \dots, p_n)$ *be given. Then one can effectively find an* m*-ary partial recursive function* ϱ *such that for any arithmetical formulas* $\Phi_1(\bar{x}), \dots, \Phi_n(\bar{x})$, *where* $\bar{x}$ *is* $x_1, \dots, x_k$, *and any list* $\bar{a}$ *of natural numbers* $a_1, \dots, a_k$, *if* $\alpha_i \,\mathbf{r}\, A_i(\Phi_1(\bar{a}), \dots, \Phi_n(\bar{a}))$ $(i = 1, \dots, m)$, *then* $\varrho(\alpha_1, \dots, \alpha_m) \,\mathbf{r}\, A(\Phi_1(\bar{a}), \dots, \Phi_n(\bar{a}))$.

It would be rather natural to define a realizable propositional formula as a formula $A(p_1, \dots, p_n)$ such that every one of its arithmetical instances is realizable. But this definition admits several variants. First we can distinguish arbitrary arithmetical instances and only closed ones. Second we can distinguish constructive and nonconstructive treatment of the definition depending on the existence of an algorithm giving a realization for every arithmetical instance. Thus we have a priori four variants of realizability for propositional formulas. Two nonconstructive variants are following.

(1) A propositional formula is *weakly realizable* if every one of its closed arithmetical instances is realizable. It is easy to prove that a propositional formula is weakly realizable iff it is a classical tautology. In particular, the law of excluded middle $p \vee \neg p$ is weakly realizable. Thus this notion of realizability for propositional formulas is not interesting and will not be considered later.

(2) A propositional formula is *irrefutable* if every one of its arithmetical instances is realizable. That is, let $A(p_1, \dots, p_n)$ be a propositional formula, $\Phi_1(\bar{x}), \dots, \Phi_n(\bar{x})$ be a list of arithmetical formulas with the parameters in the list $\bar{x}$ such that the formula $A(\Phi_1(\bar{x}), \dots, \Phi_n(\bar{x}))$ is not realizable. In this case, we say that the list $\Phi_1(\bar{x}), \dots, \Phi_n(\bar{x})$ *refutes* the formula $A(p_1, \dots, p_n)$ or is *a refutation* of this formula. Thus a propositional formula is irrefutable iff it has no refutation. Obviously, if a formula is refutable, then there exists a refutation of it with a single parameter, since we have pairing. Clearly, the formula $p \vee \neg p$ is refutable. Indeed, let $\mathrm{R}_0(x)$ be a Σ_1-formula expressing the undecidable predicate $\{x\}(x) = 0$. Then realizability of the formula $\forall x\, (\mathrm{R}_0(x) \vee \neg \mathrm{R}_0(x))$ would mean decidability of the predicate $\{x\}(x) = 0$.

Now consider constructive variants of the notion of realizable propositional formula.

(3) A propositional formula A is called *effectively realizable* if there exists an algorithm that for every closed arithmetical instance of A produces a realization for it. Obviously, every effectively realizable formula is irrefutable. Thus the formula $p \vee \neg p$ is not effectively realizable. Any example of an irrefutable propositional formula which is not effectively realizable is not found until now. On the other hand, it is not known whether these two notions coincide.

(4) The fourth notion of realizability for propositional formulas means that for any (not necessary closed) arithmetical instance of a propositional

formula we can effectively find a Gödel number of a general recursive function realizing this instance. It is rather obvious that this notion coincides with the notion of effectively realizable propositional formula.

The next variant of the notion of realizable propositional formula is implied by the mentioned above fact that every formula A deducible in IPC is uniformly realizable in the sense that there exists a common realization for all closed arithmetical instances of A. This observation leads to the following notion.

(5) A propositional formula A is called *uniformly realizable* if there exists a natural number realizing every closed arithmetical instance of A. Evidently, every uniformly realizable formula is effectively realizable, hence irrefutable. No example of effectively realizable propositional formula which is not uniformly realizable is known. However it is open whether these two notions coincide.

Let us summarize our attempts to give a definition of realizable propositional formula. We have three notions: a propositional formula A is

- irrefutable if every arithmetical instance of A is realizable;
- effectively realizable if there exists an algorithm producing a realization for every closed arithmetical instance of A;
- uniformly realizable if there exists a natural number realizing every closed arithmetical instance of A.

As it was mentioned above, every uniformly realizable propositional formula is effectively realizable and every effectively realizable formula is irrefutable. It is not known whether these inclusions are strict. The problem of describing the class of realizable (in any sense) propositional formulas is still open.

§6. Incompleteness results. In this section, we shall consider relations between propositional realizability logic on the one hand and the calculus IPC and Medvedev Logic ML on the other hand.

6.1. The Rose formula. Let D be the propositional formula $\neg p \vee \neg q$, $\mathbf{J}$ (and $\mathbf{J}(p, q)$) be the formula

$$((\neg\neg D \to D) \to (\neg D \vee \neg\neg D)) \to (\neg D \vee \neg\neg D).$$

This formula was proposed by G. F. Rose [41].

THEOREM 6.1. *The formula* $\mathbf{J}$ *is uniformly realizable but is not deducible in* IPC.

PROOF. The formula $\mathbf{J}$ is not deducible in IPC because it is refuted on Jaśkowski's algebra $\mathfrak{J}_3$ by the evaluation

$$f(p) = ((0,1),(1,1),(1,1)),$$
$$f(q) = ((1,0),(1,1),(1,1)).$$

It is easy to prove that $f(\mathbf{J}) = 4$.

Let us prove that **J** is uniformly realizable. Let arithmetical sentences A and B be given. If c_i is a Gödel number of an unary constant function with the only value i, then the arithmetical formula $\mathbf{J}(\mathrm{A}, \mathrm{B})$ is realizable by the number $\Lambda x.\varphi(x)$, where $\varphi(x)$ is defined as follows:

$$\varphi(x) = \begin{cases} 1 & \text{if } (\{x\}(c_1))_0 = (\{x\}(c_2))_0 = 0, \\ 2 & \text{if } (\{x\}(c_1))_1 \text{ or } (\{x\}(c_2))_0 = 1, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Let C be the formula $\neg\mathrm{A} \vee \neg\mathrm{B}$. We have to prove that for any x realizing the premise of the formula $\mathbf{J}(\mathrm{A}, \mathrm{B})$ the number $\varphi(x)$ realizes its conclusion.

Let $x \,\mathbf{r}\, ((\neg\neg\mathrm{C} \to \mathrm{C}) \to (\neg\mathrm{C} \vee \neg\neg\mathrm{C}))$. We have to prove that

$$\varphi(x) \,\mathbf{r}\, (\neg\mathrm{C} \vee \neg\neg\mathrm{C}).$$

Classically, we consider two cases: C is realizable or not.

(1) C is realizable. In this case, $\neg\mathrm{A}$ is realizable and $0 \,\mathbf{r}\, \neg\mathrm{A}$ or $\neg\mathrm{B}$ is realizable and $0 \,\mathbf{r}\, \neg\mathrm{B}$. Then obviously, $1 \,\mathbf{r}\, \mathrm{C}$ or $2 \,\mathbf{r}\, \mathrm{C}$. Thus

$$c_1 \,\mathbf{r}\, (\neg\neg\mathrm{C} \to \mathrm{C}) \text{ or } c_2 \,\mathbf{r}\, (\neg\neg\mathrm{C} \to \mathrm{C}).$$

Let $c_1 \,\mathbf{r}\, (\neg\neg\mathrm{C} \to \mathrm{C})$. Then $\{x\}(c_1) \,\mathbf{r}\, (\neg\mathrm{C} \vee \neg\neg\mathrm{C})$. As $\neg\mathrm{C}$ is not realizable, $(\{x\}(c_1))_0 = 1$. In this case, $\varphi(x) = 2$ and indeed,

$$2 \,\mathbf{r}\, (\neg\mathrm{C} \vee \neg\neg\mathrm{C}).$$

Thus $\varphi(x)$ realizes the conclusion of the formula $\mathbf{J}(\mathrm{A}, \mathrm{B})$.

Similarly, if $c_2 \,\mathbf{r}\, (\neg\neg\mathrm{C} \to \mathrm{C})$, then

$$\{x\}(c_2) \,\mathbf{r}\, (\neg\mathrm{C} \vee \neg\neg\mathrm{C}),$$

$(\{x\}(c_2))_0 = 1$, and $\varphi(x) = 2$, i.e., $\varphi(x)$ realizes the conclusion of the formula $\mathbf{J}(\mathrm{A}, \mathrm{B})$.

(2) C is not realizable. Then clearly, c_1 and c_2 both are realizations of the formula $\neg\neg\mathrm{C} \to \mathrm{C}$ and $\{x\}(c_1) \,\mathbf{r}\, (\neg\mathrm{C} \vee \neg\neg\mathrm{C})$, $\{x\}(c_2) \,\mathbf{r}\, (\neg\mathrm{C} \vee \neg\neg\mathrm{C})$, i.e., $(\{x\}(c_1))_0 = (\{x\}(c_2))_0 = 0$. In this case, $\varphi(x) = 1$ and in fact,

$$1 \,\mathbf{r}\, (\neg\mathrm{C} \vee \neg\neg\mathrm{C}).$$

Thus $\varphi(x)$ realizes the conclusion of the formula $\mathbf{J}(\mathrm{A}, \mathrm{B})$. ⊣

6.2. Realizability and Medvedev Logic. We consider relations between propositional logic of realizability and Medvedev Logic.

Theorem 6.2. *There are refutable propositional formulas in Medvedev Logic.*

Proof. The formulas (1) and (2) considered in §3 are finitely valid and refutable. The formula (1) is refuted by substituting the formulas $\neg\exists y\, \mathrm{R}(x, x, y)$, $\mathrm{R}_0(x)$, and $\neg\mathrm{R}_0(x)$ for p, q, r respectively. A refutation of the formula (2) is proposed in [9]. ⊣

Theorem 6.3. *There are uniformly realizable propositional formulas which are not finitely valid.*

PROOF. Consider the following formula:

$$\begin{aligned}(\neg\neg p_1 \equiv \neg p_2 \ \&\ \neg p_3) \ \&\ (\neg\neg p_2 \equiv \neg p_3 \ \&\ \neg p_1) \ \&\ (\neg\neg p_3 \equiv \neg p_1 \ \&\ \neg p_2)&\\ \&\ ((\neg p_1 \to (\neg p_1 \vee \neg p_3)) \to (\neg p_2 \vee \neg p_3))&\\ \&\ ((\neg p_2 \to (\neg p_3 \vee \neg p_1)) \to (\neg p_3 \vee \neg p_1))&\\ \&\ ((\neg p_3 \to (\neg p_1 \vee \neg p_2)) \to (\neg p_1 \vee \neg p_2))&\\ \to (\neg p_1 \vee \neg p_2 \vee \neg p_3).&\end{aligned}$$

Denote it by $\mathbf{L}$ (and $\mathbf{L}(p_1, p_2, p_3)$). We prove that $\mathbf{L}$ is uniformly realizable but is not finitely valid.

The formula $\mathbf{L}$ was found by the author (see [28], [29]). In fact, it is deductively equivalent to the characteristic formula for the algebra Φ_3 in the sequence of Heyting algebras, mentioned in §3, related to Medvedev Logic. Thus it is easy to prove that $\mathbf{L}$ is not finitely valid. Namely it is not valid over the system $\mathcal{F} = F, F, F$, where $F = \{0\}$. Let $\mathfrak{A}_0$ be the finite problem $\langle F, \emptyset\rangle$, $\mathfrak{A}_1$ be $\langle F, F\rangle$. It can be proved that there is no common solution for the problems $\mathbf{L}(\mathfrak{A}_1, \mathfrak{A}_0, \mathfrak{A}_0)$, $\mathbf{L}(\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_0)$, $\mathbf{L}(\mathfrak{A}_0, \mathfrak{A}_0, \mathfrak{A}_1)$.

LEMMA 6.4. *The formula $\mathbf{L}$ is uniformly realizable.*

PROOF. See Appendix 6.4. ⊣

This lemma completes the proof of the theorem. ⊣

6.3. Other underivable realizable formulas. Other realizable but undeducible in IPC propositional formulas were found. The relations between them are completely studied by D. P. Skvortsov [42]. He proved that all the known realizable propositional formulas are deducible in the calculus $\mathsf{IPC} + \mathbf{L} + \mathbf{C}^* + \mathbf{r} + \mathbf{P}$, where $\mathbf{L}$ is the formula considered above, $\mathbf{P}$ is the formula

$$\begin{aligned}(\neg(p_1 \ \&\ p_2) \ \&\ \neg\neg(q_1 \vee q_2)&\\ \&\ (((p_1 \to q_1) \vee (p_1 \to q_2)) \to ((p_2 \to q_1) \vee (p_2 \to q_2))))&\\ \to ((p_2 \to q_1) \vee (p_2 \to q_2))&\end{aligned}$$

proposed by the author [29], $\mathbf{r}$ is the formula

$$\begin{aligned}\neg(p \ \&\ q) \ \&\ \neg(\neg p \ \&\ \neg q)&\\ \&\ ((\neg\neg p \to p) \to p \vee \neg p) \ \&\ ((\neg\neg q \to q) \to q \vee \neg q)&\\ \to (\neg p \vee \neg q)&\end{aligned}$$

proposed by V. A. Jankov [9], $\mathbf{C}^*$ is the formula

$$\begin{aligned}(\neg(p_1 \ \&\ p_2) \ \&\ (\neg p_1 \to (q_1 \vee r)) \ \&\ (\neg p_2 \to (q_2 \vee r)))&\\ \to ((\neg p_1 \to q_1) \vee (\neg p_1 \to r) \vee (\neg p_2 \to q_2) \vee (\neg p_2 \to r))&\end{aligned}$$

proposed by D. P. Skvortsov [42] as a modified variant of a sequence of formulas introduced by G. S. Tseitin [43].

§7. **Disjunction property.** The calculus IPC has an important property called a *disjunction property* discovered by K. Gödel [6].

THEOREM 7.1. *If a disjunction $A \vee B$ is deducible in* IPC, *then at least one of the formulas A, B is deducible.*

PROOF. See [39]. ⊣

A. A. Markov stated a conjecture that the realizability logic has the disjunction property too. This hypothesis is confirmed for all the variants of realizability for propositional formulas. The disjunction property is evident in the case of the uniform realizability. Indeed, if a number a realizes every arithmetical instance of a propositional formula $A \vee B$, then obviously, $(a)_1$ realizes every arithmetical instance of A (if $(a)_0 = 0$) or of B (if $(a)_0 = 1$). Thus one of the formulas A, B is uniformly realizable. Other variants of realizability were considered by F. L. Varpakhovskii and M. M. Kipnis.

The following theorem was proved by F. L. Varpakhovskii [45].

THEOREM 7.2. *A disjunction of refutable formulas is refutable.*

PROOF. See Appendix B. ⊣

It follows from Theorem 7.2 that if a formula $A_1 \vee A_2$ is irrefutable, then at least one of the formulas A_1, A_2 must be irrefutable. Unfortunately, the proof does not give any method for determining which of these formulas is really irrefutable. It is open whether the proof of the disjunction property for irrefutable formulas can be made effective.

The same situation obtains in the case of the effectively realizable propositional formulas.

The following theorem was proved by M. M. Kipnis [15].

THEOREM 7.3. *If a propositional formula $A_1 \vee A_2$ is effectively realizable, then at least one of the formulas A_1 and A_2 is effectively realizable.*

PROOF. See Appendix C. ⊣

The proof of this theorem proposed by M. M. Kipnis is nonconstructive in the sense that it does not give any effective method for indicating an effectively realizable disjunct in the effectively realizable disjunction. It is not known whether such a method exists.

§8. **Results on partial completeness.** Although the calculus IPC is not complete relative to recursive realizability, it was proved that every irrefutable propositional formula without one of the connectives $\rightarrow$, $\neg$, $\vee$ is deducible in IPC.

8.1. Completeness of the implication-free fragment. The following theorem was proved by G. Rose [41].

THEOREM 8.1. *Every irrefutable propositional formula without $\to$ is deducible in* IPC.

PROOF. We prove that if a formula is not deducible in IPC, then it is refutable. First we prove that if a $\to$-free formula A is not deducible IPC, then A is equivalent in IPC to a disjunction $A_1 \vee \cdots \vee A_n$ $(n \geq 1)$ such that each of the formulas $A_1, \dots, A_n$ is not a classical tautology. The proof is by induction on the number of logical connectives in A. The statement is evident if A is a propositional variable. Now let the statement be true for every $\to$-free formula containing $\leq i$ connectives and A be a $\to$-free formula containing $i + 1$ connectives. Then A is of the form $\neg B$, $B \vee C$ or $B \ \& \ C$, where B and C are such that the inductive hypothesis holds for them.

Let A be a formula $\neg B$. If A is not deducible in IPC, then A is not a classical tautology because by the fact mentioned in the subsection 2.2, every classical tautology of the form $\neg B$ is deducible in IPC. Thus the statement is true for A.

Let A be of the form $B \vee C$. If A is not deducible in IPC, then the formulas B, C are not deducible and, by the inductive hypothesis, are equivalent in IPC to disjunctions B' and C' of the non-tautologies. Then obviously, A is equivalent to the disjunction $B' \vee C'$, and the statement is true for A.

Let A be of the form $B \ \& \ C$. If A is not deducible in IPC, then at least one of the formulas B, C is not deducible. For example, let B be not deducible. By the inductive hypothesis, B is equivalent in IPC to a disjunction of non-tautologies $B_1, \dots, B_n$. Then, by distributivity, A is equivalent in IPC to the disjunction of the non-tautologies $B_j \ \& \ C$ $(j = 1, \dots, n)$.

Now the theorem follows from Theorem 7.2 and a trivial fact that every non-tautology is refutable. ⊣

Theorem 8.1 was strengthened be F. A. Kabakov:

THEOREM 8.2. *There are no propositional formulas A and B without $\to$ such that the formula $A \to B$ is irrefutable but is not deducible in* IPC.

PROOF. See [13]. ⊣

8.2. Completeness of the negation-free fragment. Completeness of the positive propositional calculus PPC relative to recursive realizability was first announced by Yu. T. Medvedev [24] as a corollary of a similar result for Medvedev Logic and a theorem proved in [24] that any realizable propositional formula is finitely valid. Unfortunately, the proof of that theorem has a defect and the theorem itself is incorrect as it was shown in the subsection 6.2. Nevertheless the completeness result holds.

THEOREM 8.3. *Every irrefutable propositional formula without $\neg$ is deducible in* IPC.

PROOF. A *critical implication* is a positive propositional formula of the form

$$\bigwedge_{i=1}^{k}((P_i \to Q_i) \to Q_i) \to R, \tag{3}$$

where P_i are iterated conjunctions of variables, Q_i and R are iterated disjunctions of variables, and for each i, P_i and Q_i have no variables in common. The following statement announced without any proof by Medvedev [23] and proved later by E. Z. Skvortsova (see [2]) plays a crucial role in the proof: *if a positive formula A is not deducible in* IPC, *then a critical implication is deducible in the calculus* $\mathsf{IPC} + A$. Thus it is enough to prove the theorem only for the critical implications.

We prove that any critical implication is refutable. Let a critical implication (3) be given and $p_0, \dots, p_n$ are the variables occurring in it. Replace any variable p_j by the arithmetical formula $\neg \mathrm{R}_j(x)$, where $\mathrm{R}_j(x)$ is a Σ_1-formula expressing the predicate $\{x\}(x) = j$. Then (3) is transformed into an arithmetical formula of the form

$$\bigwedge_{i=1}^{k}((\mathrm{P}_i(x) \to \mathrm{Q}_i(x)) \to \mathrm{Q}_i(x)) \to \mathrm{R}(x).$$

We prove that the universal closure of this formula is not realizable. Obviously, it is enough to prove that

(1) the formulas $\forall x\, ((\mathrm{P}_i(x) \to \mathrm{Q}_i(x)) \to \mathrm{Q}_i(x))$ $(i = 1, \dots, k)$ are realizable and

(2) the formula $\forall x\, \mathrm{R}(x)$ is not realizable.

To prove (1) we have to describe a computable function f such that for any a the number $f(a)$ is a Gödel number of a partial recursive function which takes every realization of $\mathrm{P}_i(a) \to \mathrm{Q}_i(a)$ to a realization of $\mathrm{Q}_i(a)$. As the reasoning does not depend on i, we omit the subscripts. Let $\mathrm{P}(a)$ be the formula $\neg \mathrm{R}_{s_1}(a) \;\;\&\;\; \cdots \;\;\&\;\; \neg \mathrm{R}_{s_l}(a)$ and $\mathrm{Q}(a)$ be the formula $\neg \mathrm{R}_{r_1}(a) \vee \cdots \vee \neg \mathrm{R}_{r_m}(a)$, the numbers $s_1, \dots, s_l, r_1, \dots, r_m$ being distinct. We assume conjuncts in $\mathrm{P}(a)$ and disjuncts in $\mathrm{Q}(a)$ associated to the left. As every realizable negative sentence is realized by 0, it is easy to find a number b_l depending only on l and realizing the formula $\mathrm{P}(a)$ if it is realizable. Namely $b_1 = 0$, $b_{i+1} = 2^{b_i} \cdot 3^0 = 2^{b_i}$. Similarly, for any $j = 1, \dots, m$ there is a number c_{jm} realizing the formula $\mathrm{Q}(a)$ if its jth disjunct is realizable. Namely, $c_{11} = 0$, $c_{12} = 2^0 \cdot 3^0 = 1$, $c_{22} = 2^1 \cdot 3^0 = 2$ and so on (a recurrent formula for c_{mn} can be easily found). The function φ with the Gödel number $f(a)$ is computed in the following way. Let a number d be given. Run parallel computing $\{a\}(a)$ and $\{d\}(b_l)$. If the first computation terminates before the second one does, we find j such that $r_j \neq \{a\}(a)$ and set $\varphi(d) = c_{jm}$. If the second computation terminates before the first one does, we set $\varphi(d)$ to be $\{d\}(b_l)$ if $\{d\}(b_l) = c_{jm}$ for some $j = 1, \dots, m$ and c_{1m} else. Let us show that if

$$d \,\mathbf{r}\, (\mathrm{P}(a) \to \mathrm{Q}(a)), \tag{4}$$

then

$$\varphi(d) \,\mathbf{r}\, \mathrm{Q}(a). \tag{5}$$

Let (4) hold. Consider two cases: $\{a\}(a)$ is defined or not.

In the first case, if the computation $\{a\}(a)$ terminates as the first, there exists j such that $r_j \neq \{a\}(a)$. This is obvious if $m \geq 2$. If $m = 1$ and $r_1 = \{a\}(a)$, then the sentence $\mathrm{Q}(a)$ is not realizable but $\mathrm{P}(a)$ is realizable. But this is impossible because of (4). It is obvious that (5) holds. If the computation $\{d\}(b_l)$ terminates as the first and b_l realizes the formula $\mathrm{P}(a)$, (5) is evident. Assume that $\mathrm{P}(a)$ is not realizable. This means that $\{a\}(a)$ is one of the numbers $r_1, \dots, r_l$ but then every disjunct in $\mathrm{Q}(a)$ is realizable, thus every number c_{jm} for $j = 1, \dots, m$ realizes $\mathrm{Q}(a)$. Thus (5) holds in this case too.

If $\{a\}(a)$ is not defined, then the first computation does not terminate but the second one does because $\mathrm{P}(a)$ is evidently realizable. As $b_l \,\mathbf{r}\, \mathrm{P}(a)$, it is obvious that (5) holds.

To prove (2) assume the converse, i.e., the formula $\forall x\, \mathrm{R}(x)$ is realizable. Let $\mathrm{R}(x)$ be the formula

$$\neg \mathrm{R}_{t_0}(x) \vee \cdots \vee \neg \mathrm{R}_{t_k}(x).$$

Obviously, for any a a realizable disjunct in $\mathrm{R}(a)$ can be effectively found. Therefore there is a general recursive function f such that for any a the value $f(a)$ is one of the numbers $t_0, \dots, t_k$ such that the formula $\neg \mathrm{R}_{t_j}(a)$ is realizable. Let b be a Gödel number of f and $f(b) = \{b\}(b) = t_j$. By the definition of f, the formula $\mathrm{R}_{t_j}(a)$ is realizable. But it is impossible because this formula just means that $\{b\}(b) \neq t_j$. This shows that in fact the formula $\forall x\, \mathrm{R}(x)$ is not realizable and (2) is proved. ⊣

8.3. Completeness of the disjunction-free fragment. Completeness of the disjunction-free fragment of IPC relative to recursive realizability was announced by Yu. T. Medvedev [24] as a corollary of a similar result for Medvedev Logic and the mentioned above incorrect theorem [24] that any realizable propositional formula is finitely valid. A correct proof of the completeness result was provided by the author [29], [28].

THEOREM 8.4. *Every irrefutable propositional formula without* $\vee$ *is deducible in* IPC.

PROOF. We introduce some notions analogous to those considered by F. A. Kabakov [14]. Let $\mathfrak{M}$ be a finite Heyting algebra. We say that a system of arithmetical formulas $\mathbf{F} = \{\Phi_j \mid j \in \mathfrak{M}\}$ with the only free variable x *weakly simulates* $\mathfrak{M}$ *by realizability* if for any $a, b, c \in \mathfrak{M}$ the following conditions hold:

- if $a = \neg_{\mathfrak{M}} b$, then the formula $\Phi_a(x) \equiv \neg \Phi_b(x)$ is realizable;
- if $a = b \circ_{\mathfrak{M}} c$, where $\circ$ is $\to$ or $\&$, then $\Phi_a(x) \equiv \Phi_b(x) \circ \Phi_c(x)$ is realizable;
- $\Phi_a(x)$ is realizable if and only if $a = 1_{\mathfrak{M}}$.

$\mathfrak{M}$ is *weakly simulated by realizability* if some system of arithmetical formulas weakly simulates $\mathfrak{M}$ by realizability.

Obviously, if a $\vee$-free propositional formula A is refuted on a finite Heyting algebra weakly simulated by realizability, then A is refutable. If a propositional formula is not deducible in IPC, then it is refutable on a Jaśkowski's algebra $\mathfrak{J}_n$ for some $n = 0, 1, 2, \ldots$.

LEMMA 8.5. *For any n the algebra $\mathfrak{J}_n$ is weakly simulated by realizability.*

PROOF. We prove this lemma by induction on n. If $n = 0$, the algebra $\mathfrak{J}_n$ consists of the elements $0, 1$ with the order $0 < 1$. We put

$$\Phi_0(x) \rightleftharpoons \neg\, x = x, \quad \Phi_1(x) \rightleftharpoons x = x.$$

It is easy to prove that the system $\mathbf{F}_0 = \{\Phi_0(x), \Phi_1(x)\}$ weakly simulates $\mathfrak{J}_0$ by realizability.

Let us consider the case $n = 1$. The algebra $\mathfrak{J}_1$ consists of the elements $0, 1, 2$ with the order $0 < 2 < 1$. We put

$$\Phi_0(x) \rightleftharpoons \neg\, x = x, \quad \Phi_1(x) \rightleftharpoons x = x, \quad \Phi_2(x) \rightleftharpoons \mathrm{R}_0(x) \vee \neg \mathrm{R}_0(x)$$

(as in the previous subsection, $\mathrm{R}_j(x)$ is a formula expressing the predicate $\{x\}(x) = j$). It can be proved that the system $\mathbf{F}_1 = \{\Phi_0(x), \Phi_1(x), \Phi_2(x)\}$ weakly simulates $\mathfrak{J}_1$ by realizability.

Suppose that for a given $k \geq 1$ a system $\mathbf{F}_k = \{\Phi_j(x) \mid j \in \mathfrak{J}_k\}$ weakly simulates $\mathfrak{J}_k$. A system $\mathbf{F}_{k+1}$ weakly simulating $\mathfrak{J}_{k+1}$ is defined as follows. The elements of the algebra $\mathfrak{J}_{k+1}$ are the number $k+2$ and the finite sequences $(a_0, \ldots, a_k)$, where $a_i \in \mathfrak{J}_k$ $(i = 0, \ldots, k)$. Let $\Psi_i(x)$ be the formula $\mathrm{R}_i(x)$ for $i = 0, \ldots, k-1$ and $\Psi_k(x)$ be $\neg \bigvee_{i=0}^{k-1} \Psi_i(x)$.

We put for $a \in \mathfrak{J}_{k+1}$

$$\Phi_a(x) \rightleftharpoons \begin{cases} \bigwedge_{i=0}^{k} (\Psi_i((x)_0) \to \Phi_{a_i}((x)_1)) & \text{if } a = (a_0, \ldots, a_k), \\ \bigvee_{i=0}^{k} (\Psi_i((x)_0) \to \Phi_{k+1}((x)_1)) & \text{if } a = k+2. \end{cases}$$

Let us show that the system $\mathbf{F}_{k+1} = \{\Phi_b(x) \mid b \in \mathfrak{J}_{k+1}\}$ weakly simulates $\mathfrak{J}_{k+1}$. First of all note that $1_{\mathfrak{J}_{k+1}} = (1_{\mathfrak{J}_k}, \ldots, 1_{\mathfrak{J}_k})$, the formula $\Phi_{1_{\mathfrak{J}_k}}(x)$ being realizable. As $\Phi_{1_{\mathfrak{J}_{k+1}}}(x)$ is $\bigwedge_{i=0}^{k} (\Psi_i((x)_0) \to \Phi_{1_{\mathfrak{J}_k}}((x)_1))$, it is obvious that $\Phi_{1_{\mathfrak{J}_{k+1}}}(x)$ is realizable too. On the other hand, if $a \in \mathfrak{J}_{k+1}$ is such that $a = (a_0, \ldots, a_k) \neq 1_{\mathfrak{J}_{k+1}}$, then $a_j \neq 1_{\mathfrak{J}_k}$ for some j $(0 \leq j \leq k)$. By the inductive hypothesis, the formula $\Phi_{a_j}(x)$ is not realizable. This implies that the formula $\Phi_a(x)$ is not realizable too. Indeed, if $\Phi_a(x)$ is realizable, then its conjunct $\Psi_j((x)_0) \to \Phi_{a_j}((x)_1)$ is realizable. This yields that there is a partial recursive function f such that

$$f(2^l \cdot 3^m)\, \mathbf{r}\, (\Psi_j(l) \to \Phi_{a_j}(m)) \tag{6}$$

for any l, m. Obviously, we can choose l and e such that $e\,\mathbf{r}\,\Psi_j(l)$. Then it follows from (6) that $\{f(2^l \cdot 3^m)\}(e)\,\mathbf{r}\,\Phi_{a_j}(m)$ for every m. This means that the function $g(x) = \{f(2^l \cdot 3^x)\}(e)$ realizes the formula $\Phi_{a_j}(x)$ contrary to the assumption. The formula $\Phi_{k+2}(x)$ is not realizable too. In fact, if it is realizable, then there exist partial recursive functions f and g such that for any l, m

(1) $f(2^l \cdot 3^m)$ is $j \in \{0, \ldots, k\}$ such that the formula $\Psi_j(l) \to \Phi_{k+1}(m)$ is realizable and

(2) $g(2^l \cdot 3^m)\,\mathbf{r}\,(\Psi_j(l) \to \Phi_{k+1}(m))$.

Let m be fixed, $l = \Lambda x.f(2^x \cdot 3^m)$, $f(2^l \cdot 3^m) = j$. Then

$$g(2^l \cdot 3^m)\,\mathbf{r}\,(\Psi_j(l) \to \Phi_{k+1}(m)).$$

Note that the formula $\Psi_j(l)$ means $\{l\}(l) = j$ if $0 \leq j \leq k-1$ and $\{l\}(l) \notin \{0, \ldots, k-1\}$ if $j = k$. As $\{l\}(l) = f(2^l \cdot 3^m) = j$, we see that the formula $\Psi_j(l)$ is realizable. Obviously, the numbers l, j and a realization of the formula $\Psi_j(l)$ can be found effectively. As a realization of the formula $\Psi_j(l) \to \Phi_{k+1}(m)$ is given too, we can find a realization of the formula $\Phi_{k+1}(m)$. This means that the formula $\Phi_{k+1}(x)$ is realizable contrary to the inductive hypothesis. Therefore the formula $\Phi_{k+2}(x)$ is not realizable.

Now we have to consider the conditions of weak simulation for each of the connectives $\neg, \to, \&$. It is rather evident that the conditions hold if one of the operands is $1_{\mathfrak{J}_{k+1}}$. Consider the alternative cases. Let $a, b \in \mathfrak{J}_{k+1}$ be such that $a = \alpha(u)$, $b = \alpha(v)$ for some $u, v \in \mathfrak{J}_k^{k+1}$. Let $u = (a_0, \ldots, a_k)$, $v = (b_0, \ldots, b_k)$, a_i, b_i $(i = 0, \ldots, k)$ being in $\mathfrak{J}_k$. Let us denote $\mathfrak{J}_k$ by $\mathfrak{M}$ and $\mathfrak{J}_{k+1}$ by $\mathfrak{N}$.

Consider the connective $\to$. We have

$$a \to_{\mathfrak{N}} b = (a_0 \to_{\mathfrak{M}} b_0, \ldots, a_k \to_{\mathfrak{M}} b_k).$$

Note that $\Phi_a(x)$ is the formula

$$\bigwedge_{i=0}^{k} (\Psi_i((x)_0) \to \Phi_{a_i}((x)_1)), \tag{7}$$

$\Phi_b(x)$ is the formula

$$\bigwedge_{i=0}^{k} (\Psi_i((x)_0) \to \Phi_{b_i}((x)_1)), \tag{8}$$

and $\Phi_{a \to_{\mathfrak{N}} b}(x)$ is the formula

$$\bigwedge_{i=0}^{k} (\Psi_i((x)_0) \to \Phi_{a_i \to_{\mathfrak{M}} b_i}((x)_1)).$$

We have to prove that the formulas $\Phi_{a \to_{\mathfrak{N}} b}(x)$ and $\Phi_a(x) \to \Phi_b(x)$ are equivalent in the sense that the formula $\Phi_{a \to_{\mathfrak{N}} b} \equiv (\Phi_a(x) \to \Phi_b(x))$ is realizable. By the inductive hypothesis, for any $i = 0, \dots, k$ the formulas $\Phi_{a_i \to_{\mathfrak{N}} b_i}(x)$ and $\Phi_{a_i}(x) \to \Phi_{b_i}(x)$ are equivalent. In particular, this means that for each m one can find effectively a realization of the formula $\Phi_{a_i \to_{\mathfrak{N}} b_i}(m)$ from any realization of the formula $\Phi_{a_i}(m) \to \Phi_{b_i}(m)$ and vice versa. Thus we have to prove that the formulas

$$\bigwedge_{i=0}^{k}(\Psi_i((x)_0) \to (\Phi_{a_i}((x)_1) \to \Phi_{b_i}((x)_1))$$

and

$$\bigwedge_{i=0}^{k}(\Psi_i((x)_0) \to \Phi_{a_i}((x)_1)) \to \bigwedge_{i=0}^{k}(\Psi_i((x)_0) \to \Phi_{b_i}((x)_1))$$

are equivalent. Obviously, it is enough to prove that for each l, m one can effectively find a realization of the formula

$$\bigwedge_{i=0}^{k}(\Psi_i(l) \to (\Phi_{a_i}(m) \to \Phi_{b_i}(m))) \tag{9}$$

from any realization of the formula

$$\bigwedge_{i=0}^{k}(\Psi_i(l) \to \Phi_{a_i}(m)) \to \bigwedge_{i=0}^{k}(\Psi_i(l) \to \Phi_{b_i}(m)) \tag{10}$$

and vice versa. Let a realization of the formula (10) be given. For any $i = 0, \dots, k$ a realization of the formula $\Psi_i(l) \to (\Phi_{a_i}(m) \to \Phi_{b_i}(m))$ must be the number $\Lambda y.\Lambda z.f(y, z)$, where f is a partial recursive function such that if $c\,\mathbf{r}\,\Psi_i(l)$, $d\,\mathbf{r}\,\Phi_{a_i}(m)$, then $f(c, d)\,\mathbf{r}\,\Phi_{b_i}(m))$. A function f can be computed by the following algorithm. Let numbers c, d be given. Note that the formula $\Psi_i(l) \to \Phi_{a_i}(m)$ is realized by the number $\Lambda y.d$ and other conjuncts in the premise of the formula (10) are realized by 0 as implications with nonrealizable premises. Thus we can find a realization of the premise of the formula (10). As a realization of (10) is given, we can find a realization of its conclusion and further a realization of the formula $\Psi_i(l) \to \Phi_{b_i}(m)$. As a realization of its premise $\Psi_i(l)$ is given, we can find a realization $f(c, d)$ of the conclusion $\Phi_{b_i}(m)$. The fact that a realization of the formula (10) can be found from any realization of the formula (9) is trivial because (9) is deducible from (10) by means of IPC.

The case of the connective $\neg$ is reduced to the previous case because $\neg_{\mathfrak{N}} a = a \to_{\mathfrak{N}} 0_{\mathfrak{N}}$, the formula $\neg\Phi_a(x)$ is equivalent to $\Phi_a(x) \to \neg\, x = x$, and it is easy to prove that $\Phi_{0_{\mathfrak{N}}}$ is equivalent to $\neg\, x = x$.

Let us consider the connective &. Remember that $a \;\&_{\mathfrak{N}}\; b = \alpha(u \;\&_{\mathfrak{M}}\; v)$. If $u \;\&_{\mathfrak{M}}\; v = 1_{\mathfrak{M}}$, then $u = v = 1_{\mathfrak{M}}$, $a = b = \omega_{\mathfrak{N}} = k + 2$, and the condition of weak simulation holds trivially because it means that the formula $\Phi_{k+2}(x) \;\&\; \Phi_{k+2}(x) \equiv \Phi_{k+2}(x)$ is realizable. If $u \;\&_{\mathfrak{M}}\; v \neq 1_{\mathfrak{M}}$, then

$a \mathbin{\&_{\mathfrak{N}}} b = (a_0 \mathbin{\&_{\mathfrak{M}}} b_0, \dots, a_k \mathbin{\&_{\mathfrak{M}}} b_k)$. Note that $\Phi_a(x)$ is the formula (7), $\Phi_b(x)$ is the formula (8), and $\Phi_{a \mathbin{\&_{\mathfrak{N}}} b}(x)$ is the formula

$$\bigwedge_{i=0}^{k} (\Psi_i((x)_0) \to \Phi_{a_i \mathbin{\&_{\mathfrak{M}}} b_i}((x)_1)).$$

We have to prove that the formulas $\Phi_{a \mathbin{\&_{\mathfrak{N}}} b}(x)$ and $\Phi_a(x) \;\&\; \Phi_b(x)$ are equivalent. By the inductive hypothesis, for any $i = 0, \dots, k$ the formulas $\Phi_{a_i \mathbin{\&_{\mathfrak{M}}} b_i}(x)$ and $\Phi_{a_i}(x) \;\&\; \Phi_{b_i}(x)$ are equivalent. Thus we have to prove that the formulas

$$\bigwedge_{i=0}^{k} (\Psi_i((x)_0) \to (\Phi_{a_i}((x)_1) \;\&\; \Phi_{b_i}((x)_1))$$

and

$$\bigwedge_{i=0}^{k} (\Psi_i((x)_0) \to \Phi_{a_i}((x)_1)) \;\&\; \bigwedge_{i=0}^{k} (\Psi_i((x)_0) \to \Phi_{b_i}((x)_1))$$

are equivalent, but this is evident because each of them is deducible from another by means of IPC. ⊣

The theorem is an evident consequence of Lemma 8.5. ⊣

§9. **Partial results.** Consider some attempts to describe the class of realizable propositional formulas.

9.1. Varpakhovskii's calculus. F. L. Varpakhovskii [46] considers a decidable class of realizable propositional formulas including all such formulas known at that time. All the formulas in Varpakhovskii's class are in Medvedev Logic. Thus an example of a realizable formula which is not in Medvedev Logic considered in the subsection 6.2 shows that Varpakhovskii's class is not complete. Later Varpakhovskii [47] proposed an axiomatic class of realizable formulas in an extended propositional language. Namely two additional connectives called strong implication and conditional disjunction are added to the propositional and arithmetical languages. The notion of recursive realizability is generalized to the new connectives in the following way.

Let Φ and Ψ be closed arithmetical formulas. Strong implication from Φ to Ψ is denoted by $\Phi \Rightarrow \Psi$. Let e be a natural number. Then $e \,\mathbf{r}\, (\Phi \Rightarrow \Psi)$ iff for any a such that $a \,\mathbf{r}\, \Phi$ the value $\{e\}(a)$ is defined and for any a, if $\{e\}(a)$ is defined, then $\{e\}(a) \,\mathbf{r}\, \Psi$.

The definition of conditional disjunction is rather complicated. Let θ be a list of arithmetical formulas $\Phi_1, \dots, \Phi_m$, θ_i $(i = 1, \dots, n)$ be some sublist $\Phi_{i,1}, \dots, \Phi_{i,m_i}$, and $\Psi_1, \dots, \Psi_n$ be arithmetical formulas. Then the expression $(\theta(\theta_1\Psi_1 \nabla \dots \nabla \theta_n \Psi_n))$ (abridged notation is $\left(\theta \left(\nabla_{i=1}^{n} \theta_i \Psi_i\right)\right)$) is called conditional disjunction of the formulas $\Psi_1, \dots, \Psi_n$ with the conditions

$\theta, \theta_1, \dots, \theta_n$. Let the formulas $\Phi_1, \dots, \Phi_m, \Psi_1, \dots, \Psi_n$ be closed, e be a natural number. Then $e\,\mathbf{r}\,\left(\theta\left(\nabla_{i=1}^{n}\theta_i\Psi_i\right)\right)$ iff

(1) e is of the form $\prod_{i=0}^{n} \pi_i^{e_i}$,

(2) for any sequence of natural numbers $\bar{a} = a_1, \dots, a_m$ there exists $i \in \{1, \dots, n\}$ such that if $a_j\,\mathbf{r}\,\Phi_j$ for any $j = 1, \dots, m$, then $\{e_0\}(\bar{a}) = i$ and the value $\{e_i\}(a_{i,1}, \dots, a_{i,m_i})$ is defined,

(3) for any $i \in \{1, \dots, n\}$ and any $\bar{a} = a_1, \dots, a_m$, if $a_{i,j}\,\mathbf{r}\,\Phi_{i,j}$ for any $j = 1, \dots, m_i$, and the value $\{e_i\}(a_{i,1}, \dots, a_{i,m_i})$ is defined, then $\{e_i\}(a_{i,1}, \dots, a_{i,m_i})\,\mathbf{r}\,\Psi_i$.

The notion of an uniformly realizable propositional formula in the extended language is defined in an obvious way. Varpakhovskii proposed a propositional calculus in the extended language obtained from IPC by adding the following axiom schemes, where Ss, As, and Bs are arbitrary propositional formulas, θ is an arbitrary list of propositional formulas $A_1, \dots, A_m$, and $\theta_1, \theta_2, \dots$ are its sublists:

1. $\neg\neg S \to ((S \Rightarrow A) \to A)$;
2. $(S_1 \Rightarrow A_1)\ \&\ (S_2 \Rightarrow A_2) \to ((S_1 \vee S_2) \Rightarrow (A_1 \vee A_2))$;
3. $(S \Rightarrow A_1)\ \&\ (A_1 \to A_2) \to (S \Rightarrow A_2)$;
4. $(S \Rightarrow A_1)\ \&\ (S \Rightarrow A_2) \to (S \Rightarrow (A_1\ \&\ A_2))$;
5. $(A \to B) \equiv ((A)((A)B))$;
6. $(\theta(\dots\nabla\theta_i B_i \nabla\theta_{i+1}B_{i+1}\nabla\dots)) \equiv (\theta(\dots\nabla\theta_{i+1}B_{i+1}\nabla\theta_i B_i\nabla\dots))$;
7. $(\theta(\theta_1(B' \vee B'')\nabla\dots)) \to (\theta(\theta_1 B'\nabla\theta_1 B''\nabla\dots))$;
8. $(\theta(\theta_1(A \to B)\nabla\dots)) \to (\theta^*(\theta_1^* B\nabla\dots))$, where θ^* and θ_1^* are obtained by adding A to θ and θ_1 as the last element;
9. $(A \to A_p)\ \&\ \left(\theta\left(\nabla_{i=1}^{n}\theta_i B_i\right)\right) \to \left(\theta'\left(\nabla_{i=1}^{n}\theta_i' B_i\right)\right)$, where $1 \le p \le m$, θ' and θ_i' are obtained from θ and θ_i by replacing A_p by A;
10. $(B_1 \to B)\ \&\ (\theta(\theta_1 B_1\nabla\dots)) \to (\theta(\theta_1 B\nabla\dots))$;
11. $\theta\left(\left(\nabla_{i=1}^{r}\theta_i B_i\right)\nabla\left(\nabla_{i=r+1}^{n}\theta_i(\neg S)\right)\right) \to (S \Rightarrow \bigvee_{i=1}^{r}(\neg\neg T^i \to B_i))$, where $1 \le r \le n$, all the elements of θ are of the form $\neg S \to \bigvee_{t=1}^{l}\neg\neg T_t$ or $\neg T_1 \Rightarrow S$, T^i is a conjunction of formulas T_t chosen in one subformula from every element of the list θ_i.

Varpakhovskii proves that any propositional formula deducible in this calculus is uniformly realizable. Moreover, all the known realizable propositional formulas are deducible. As it follows from the Skvortsov result mentioned in the subsection 6.3, it is enough to check that four concrete formulas listed there are deducible. Varpakhovskii observes that his calculus gives an uniform formalization of the principles used in the proving realizability of propositional formulas. The problem of completeness of Varpakhovskii's calculus is still open.

9.2. Relation to Russian constructivism. The definition of recursive realizability can be formulated as a translation from the language of arithmetic into itself. Namely for any arithmetical formula Φ, a formula $x \,\mathsf{R}\, \Phi$ with an additional parameter x is inductively defined in such a way that $x \,\mathsf{R}\, \Phi$ means $x \,\mathbf{r}\, \Phi$ (see [18], [26]). One can construct the formula $x \,\mathsf{R}\, \Phi$ such that it does not contain $\vee$ and contains $\exists$ only in the subformulas of the form $\exists x_1 \ldots \exists x_n \Phi$, where Φ is a quantifier-free formula. Formulas of this kind are called *almost negative*.

Let T be an axiomatic arithmetical theory. The propositional logic of T is the set $\mathcal{PL}(\mathsf{T})$ of propositional formulas such that all their arithmetical instances are deducible in T. The author [29] considered a formal system S obtained from HA by adding the axioms $\Phi \equiv \exists x \,\mathsf{R}\, \Phi$ for any arithmetical formula Φ and the axiom scheme

$$\forall x\, (\Phi(x) \vee \neg\Phi(x)) \;\;\&\;\; \neg\neg\exists x\, \Phi(x) \rightarrow \exists x\, \Phi(x) \qquad \text{(MP)}$$

called the Markov Principle. It was proved that all the known realizable propositional formulas are in the logic $\mathcal{PL}(\mathsf{S})$.

Extended Church's Thesis is the following scheme of axioms, where $\Psi(x)$ is an almost negative formula:

$$\forall x\, (\Psi(x) \rightarrow \exists y\, \Phi(x, y)) \rightarrow \exists e\, \forall x\, (\Psi(x) \rightarrow \exists y\, (\mathrm{R}(e, x, y) \;\;\&\;\; \Phi(x, y))). \qquad \text{(ECT)}$$

The system S is equivalent to the system of "Russian" or "traditional" constructivism (see [3]) HA+MP+ECT. A. Visser [48] calls this system Markov's Arithmetic and denotes it by MA. Thus the result just mentioned means that any known realizable propositional formula is in the logic $\mathcal{PL}(\mathsf{MA})$.

9.3. Restricted interpretations. The Kleene-Mostowski hierarchy of arithmetical predicates classifies arithmetical formulas according to the quantifier prefix in their prenex normal form. Namely the classes Σ_0 and Π_0 consist of quantifier-free formulas. If $\Phi(v)$ is a prenex formula in the class Σ_n then the formula $\forall v\, \Phi(v)$ is in the class Π_{n+1}. If $\Phi(v)$ is in the class Π_n, then the formula $\exists v\, \Phi(v)$ is in the class Σ_{n+1}. Any arithmetical formula is equivalent to a formula in the class Σ_n or Π_n for an appropriate n. A set $A \subseteq \boldsymbol{N}$ is in the class Π_n if the predicate $x \in A$ is representable by an arithmetical formula in the class Π_n.

M. M. Kipnis [16] proposed a constructive classification of arithmetical predicates similar to the Kleene-Mostowski hierarchy. The main idea of the Kipnis approach was that any arithmetical formula Φ is constructively (i.e., from the point of view of recursive realizability) equivalent to the formula $\exists x\, x \,\mathsf{R}\, \Phi$, where $x \,\mathsf{R}\, \Phi$ is almost negative. On the other hand, classical transformation of the almost negative formula to the prenex normal form is constructively valid in the presence of the Markov Principle (which is admissible from the constructive point of view) if we replace $\exists v$ by $\neg\neg\exists v$ (or $\neg\forall v\, \neg$). Let us call such "prenex" formulas weakly prenex. Thus any formula Φ can be transformed to a formula of the form $\exists x\,\Psi$, where Ψ is weakly prenex.

A formal definition of the Kipnis classification is the following. At first the classes Σ_n^0 and Π_n^0 are defined.

- If the formula $\Phi \vee \neg\Phi$ is realizable, then $\neg\Phi \in \Sigma_0^0$, $\neg\neg\Phi \in \Pi_0^0$.
- If $\Phi \in \Pi_k^0$, then $\neg\exists x\,\Phi \in \Pi_{k+1}^0$.
- If $\Phi \in \Sigma_k^0$, then $\neg\forall x\,\Phi \in \Sigma_{k+1}^0$.

The class Σ_k (Π_k) consists of the formulas of the form $\exists x\,\Phi$, where $\Phi \in \Sigma_k^0$ (respectively $\Phi \in \Pi_k^0$). For any arithmetical formula Φ there exist k and a formula Ψ in the class Σ_k or Π_k such that the formula $\Phi \equiv \Psi$ is realizable.

Let C_k be Σ_k or Π_k. For any $k \geq 1$ and $n \geq 0$ there exists a formula $\mathrm{D}(x, x_1, \dots, x_n)$ in the class C_k universal for the formulas with free variables $x_1, \dots, x_n$ in the same class. Namely for any formula $\Phi(x_1, \dots, x_n)$ in C_k there exists a such that the formulas $\mathrm{D}(a, x_1, \dots, x_n)$ and $\Phi(x_1, \dots, x_n)$ are completely equivalent in the sense that for any $e, a_1, \dots, a_n$, $e\,\mathbf{r}\,\mathrm{D}(a, a_1, \dots, a_n)$ if and only if $e\,\mathbf{r}\,\Phi(a_1, \dots, a_n)$. In particular, for any $k \geq 1$ there exists a formula $\mathrm{D}(x)$ universal for closed formulas in the class C_k. Note that classically any closed formula Φ is equivalent to a formula in the class Σ_0 or Π_0, but from the constructive point of view the situation is different. In order to find an equivalent formula in one of this classes, we have to decide if Φ is realizable or not. Nevertheless, by syntactic analysis of Φ, we can find k and an appropriate equivalent formula in a class C_k.

Let K be a class of arithmetical formulas. One can consider variants of the notions of realizability for propositional formulas obtained by interpreting propositional variables only by the formulas in the class K. A propositional formula $A(p_1, \dots, p_n)$ is called K-irrefutable iff for any arithmetical formulas $\Phi_1, \dots, \Phi_n$ in the class K, the formula $A(\Phi_1, \dots, \Phi_n)$ is realizable. $A(p_1, \dots, p_n)$ is called K-uniformly realizable iff there exists a such that $a\,\mathbf{r}\,A(\Phi_1, \dots, \Phi_n)$ for any closed formulas $\Phi_1, \dots, \Phi_n$ in the class K. Relations between the notions of K-irrefutability and K-uniform realizability for the classes Σ_k and Π_k as K were studied by M. M. Kipnis [17]. The following theorem is important in studying realizability logic.

THEOREM 9.1. *Let* K *be a class* Σ_k *or* Π_k. *If a propositional formula is not* K*-uniformly realizable, then it is refutable.*

PROOF. See Appendix D. ⊣

We see that if a propositional formula A is not K-uniformly realizable for a class K consisting of arithmetical formulas of bounded logical complexity, then A is refutable. Let L be the class of arithmetical formulas of the form $\exists x\,(x + 1 = n)$ $(n = 0, 1, 2, \dots)$. The main property of these formulas is that each of them has at most one realization. Usually a nonrealizable propositional formula is not L-uniformly realizable. There was a conjecture that this holds for any nonrealizable formula. This conjecture is disproved by the following example. Let **M** be the formula

$$(((p \vee \neg p) \to (\neg q \vee \neg\neg q)) \to (\neg q \vee \neg\neg q)) \\ \to (((\neg q \vee \neg\neg q) \to (p \vee \neg p)) \to (p \vee \neg p)).$$

THEOREM 9.2. *The formula* **M** *is refutable and* L*-uniformly realizable.*

PROOF. See [37]. ⊣

9.4. Formulas of one variable. Any known realizable propositional formula undeducible in IPC contains at least two variables. But what about the formulas of one variable? It was proved by I. Nishimura [27] that any propositional formula of one variable is equivalent in IPC to a formula in the sequence $A_n(p)$ $(n = \infty, 0, 1, 2, \dots)$, where $A_\infty(p)$ is $p \to p$, $A_0(p)$ is $p \ \& \ \neg p$, $A_1(p)$ is p, $A_2(p)$ is $\neg p$, and for $n \geq 0$, $A_{2n+3}(p)$ is $A_{2n+1}(p) \vee A_{2n+2}(p)$, $A_{2n+4}(p)$ is $A_{2n+2}(p) \to A_{2n+1}(p)$.

Obviously, for any natural n the formula $A_n(p)$ is not deducible in IPC. There is a hypothesis that these formulas are not realizable. In view of the disjunction property of the class of realizable propositional formulas, it is enough to prove this hypothesis only for even values of n. Non-realizability of the formulas $A_n(p)$ for even $n \leq 8$ is rather evident. The formula $A_{10}(p)$ is equivalent in IPC to the "skeleton" of the Rose formula $((\neg\neg p \to p) \to (\neg p \vee \neg\neg p)) \to (\neg p \vee \neg\neg p)$. It was proved by V. A. Jankov [9] that this formula is refutable. Thus the hypothesis is valid at least for $n \leq 11$.

§10. On predicate realizability logic. Predicate formulas are first-order formulas constructed in the usual way from predicate variables A_i^j $(i, j = 0, 1, 2, \dots)$ and individual variables $x_0, x_1, x_2, \dots$. We say that A_n^m is an m-ary predicate variable. The 0-ary predicate variables are propositional variables.

Let $\mathcal{A}$ be a predicate formula, $P_1, \dots, P_n$ be all the predicate variables in $\mathcal{A}$, P_i being an m_i-ary predicate variable. Denote $\mathcal{A}$ by $\mathcal{A}(P_1, \dots, P_n)$. A list of arithmetical formulas $\Phi_1, \dots, \Phi_n$ is admissible (quasiadmissible) for substitution in $\mathcal{A}(P_1, \dots, P_n)$ if for any i $(i = 1, \dots, n)$ Φ_i does not contain any free variables other than $x_1, \dots, x_{m_i}$ (resp. $x_0, x_1, \dots, x_{m_i}$). If a list $\Phi_1, \dots, \Phi_n$ is admissible or quasiadmissible for substitution in $\mathcal{A}(P_1, \dots, P_n)$, let $\mathcal{A}(\Phi_1, \dots, \Phi_n)$ be an arithmetical formula obtained by replacing atomic subformulas $P_i(y_1, \dots, y_{m_i})$ by $\Phi_i[y_1/x_1, \dots, y_{m_i}/x_{m_i}]$. In this case, the formula $\mathcal{A}(\Phi_1, \dots, \Phi_n)$ is called an arithmetical instance of the predicate formula $\mathcal{A}(P_1, \dots, P_n)$. A closed predicate formula $\mathcal{A}$ is called

- *weakly realizable* if every closed arithmetical instance of $\mathcal{A}$ is realizable;
- *irrefutable* if every arithmetical instance of $\mathcal{A}$ is realizable;

- *effectively realizable* if there exists an algorithm producing a realization for every closed arithmetical instance of $\mathcal{A}$;
- *uniformly realizable* if there exists a natural number realizing every closed arithmetical instance of $\mathcal{A}$.

Obviously, if a predicate formula is uniformly realizable, then it is effectively realizable, every effectively realizable predicate formula is irrefutable, and every irrefutable formula is weakly realizable. These notions are essentially different. It is evident that the law of excluded middle is weakly realizable but is not irrefutable. The following theorems were proved by the author [31], [33].

THEOREM 10.1. *There exists an irrefutable predicate formula which is not effectively realizable.*

THEOREM 10.2. *There exists an effectively realizable predicate formula which is not uniformly realizable.*

The following theorem is proved by the author [28], [32].

THEOREM 10.3. *The set of* (*the Gödel numbers of*) *irrefutable* (*effectively realizable, uniformly realizable*) *predicate formulas is not arithmetical.*

This means that the set of the Gödel numbers of the formulas in each of these classes of predicate formulas is not defined by an arithmetical formula. It follows that these classes are not recursively enumerable and are not recursively axiomatizable.

A more detailed description is obtained for irrefutable predicate formulas. Let V denote the set of the Gödel numbers of the realizable closed arithmetical formulas, $'$ is the jump operation in the recursion theory (see [40, §13.1]).

The following theorem is proved by the author [38].

THEOREM 10.4. *The set of the Gödel numbers of irrefutable predicate formulas is recursively isomorphic to the complement of the set* V'.

Note that in the case of predicate logic the notion of weakly realizable formula is not as trivial as in the case of propositional logic. Namely the set of weakly realizable predicate formulas is recursively isomorphic to the set of irrefutable predicate formulas. Thus Theorem 10.4 is valid for both these notions.

The predicate logic of an axiomatic arithmetical theory T is the set $\mathcal{QL}(\mathsf{T})$ of predicate formulas such that all their arithmetical instances are deducible in T. The following theorem is proved by the author [36].

THEOREM 10.5. *The set of the Gödel numbers of the predicate formulas in the logic* $\mathcal{QL}(\mathsf{MA})$ *is* Π_2*-complete.*

§11. Absolute realizability. Realizable (in any sense) predicate formulas can be considered as principles of constructive logic in the field of arithmetical sentences. In general, it is natural to represent a predicate formula as a principle of constructive logic if it becomes a constructively true sentence for any interpretation of the predicate variables as concrete predicates. Let us call such formulas *constructively valid*. Unfortunately, significant difficulties arise when we attempt to give a mathematically precise definition of a constructively valid predicate formula. In constructive mathematics a predicate is commonly understood to be a parametric proposition of an appropriate language. The concept of a predicate can be made precise (and unavoidably narrowed) by choosing a particular language. Fixing a first-order language L, we can define an n-place predicate as a formula in the language L with free variables $x_1, \ldots, x_n$. If a kind of constructive semantics is defined for L, a closed predicate formula is called constructively valid with respect to the language L if any its L-instance is true in the considered semantics. Of special interest for constructive logic are the predicate formulas constructively valid with respect to any conceivable first-order language with a constructive semantics. Therefore the notion of realizable predicate formula is a partial variant of a general notion of a constructively valid predicate formula.

The author has shown [30] that constructive predicate logic can not be identified with the theory of recursive realizability of predicate formulas. Consider a language L obtained by adding an unary predicate symbol T to the first-order language of formal arithmetic. The predicate $T(x)$ is interpreted as the truth predicate for the usual language of arithmetic. We generalize the notion of realizability to this extended language by letting $e\,\mathbf{r}\,T(n)$ iff $e\,\mathbf{r}\,\Phi_n$, where Φ_n is an arithmetical sentence with the Gödel number n. It is proved in [30] that there are uniformly realizable predicate formulas which are refutable in the language L. The process of extending languages by adding the truth predicate can be continued by a transfinite induction over the constructive ordinals, and more restrictive predicate logics will appear at every stage.

The fact that the concept of a realizable predicate formula depends on the language in which we formulate the predicates admissible for interpreting predicate variables sets a problem of developing a concept of realizability for predicate formulas such that it would include interpretations relative to individual languages as special cases. Such a concept was proposed by the author [34] and is called *absolute realizability*. The main idea was that in constructive mathematics the meaning of a sentence is connected with a construction verifying its truth. In classical logic a sentence is identified with its truth value. Similarly, we can identify the constructive meaning of a sentence with the set of (the Gödel numbers of) the objects verifying this sentence. Thus we come to the idea of interpreting propositional variables as arbitrary sets of naturals, the logical operations being defined according to recursive realizability:

- $A \;\&\; B \fallingdotseq \{2^a \cdot 3^b \mid a \in A,\ b \in B\}$;
- $A \vee B \fallingdotseq \{2^0 \cdot 3^a \mid a \in A\} \cup \{2^1 \cdot 3^b \mid b \in B\}$;

- $A \to B \leftrightharpoons \{x \mid \forall a\,(a \in A \Rightarrow \{x\}(a) \in B)\}$;
- $\neg A \leftrightharpoons A \to \emptyset$.

The notion of a predicate is also generalized. Namely a k-place generalized predicate is a function (in the set-theoretical sense) of type $\boldsymbol{N}^k \to 2^{\boldsymbol{N}}$, i.e., a function defined on $\boldsymbol{N}^k$ whose values are sentences (sets of naturals). Application of quantifiers to the generalized predicates are defined in a natural way. Then analogs of irrefutable and uniformly realizable predicate formulas are defined and it is proved that they coincide. This yields the concept of an absolutely realizable predicate formula. Two important properties of the predicate logic of absolute realizability are proved in [34].

THEOREM 11.1. *The set of the Gödel numbers of the absolutely realizable predicate formulas is complete in the class Π^1_1 of the analytical hierarchy.*

THEOREM 11.2. *If a predicate formula is not absolutely realizable, then it is refuted by predicates in the arithmetical language with an additional Π^1_1-complete predicate $W(x)$.*

A generalized predicate $P(x_1, \dots, x_n)$ is said to be hyperarithmetical if the relation $y \in P(x_1, \dots, x_n)$ is hyperarithmetical, i.e., expressible both by a Σ^1_1-formula and by a Π^1_1-formula in the language of second-order arithmetic.

THEOREM 11.3. *There is a predicate formula which is not absolutely realizable but can not be refuted by hyperarithmetical generalized predicates.*

§12. Summary. In this section, we collect results in the study of propositional realizability logic considered above.

- There are three natural notions of realizability for the propositional formulas: irrefutability, effective realizability, and uniform realizability (see §5). It is not known whether theese notions are different. As it follows from the definitions, uniform realizability implies effective realizability and effective realizability implies irrefutability.
- The intuitionistic propositional calculus IPC is sound with realizability: every propositional formula deducible in IPC is uniformly realizable, i.e., realizable in the strongest sense. Moreover, if a propositional formula A is realizable in some sense (i.e., irrefutable, effectively realizable or uniformly realizable), then every propositional formula deducible in the calculus $\mathsf{IPC} + A$ is realizable in the same sense.
- The calculus IPC is not complete relative to realizability: there exist uniformly realizable propositional formulas which are not deducible in IPC (see §6). The following four formulas play an important role at the present stage of research.

1. The formula **L**:

$$
\begin{aligned}
((\neg\neg p_1 \equiv \neg p_2 \;\&\; \neg p_3)
&\;\&\; (\neg\neg p_2 \equiv \neg p_3 \;\&\; \neg p_1)\\
&\;\&\; (\neg\neg p_3 \equiv \neg p_1 \;\&\; \neg p_2)\\
&\;\&\; ((\neg p_1 \to (\neg p_1 \vee \neg p_3)) \to (\neg p_2 \vee \neg p_3))\\
&\;\&\; ((\neg p_2 \to (\neg p_3 \vee \neg p_1)) \to (\neg p_3 \vee \neg p_1))\\
&\;\&\; ((\neg p_3 \to (\neg p_1 \vee \neg p_2)) \to (\neg p_1 \vee \neg p_2)))\\
&\to (\neg p_1 \vee \neg p_2 \vee \neg p_3).
\end{aligned}
$$

2. The formula $\mathbf{C}^*$:

$$
\begin{aligned}
(\neg(p_1 \;\&\; p_2) \;\&\; (\neg p_1 \to (q_1 \vee r)) \;\&\; (\neg p_2 \to (q_2 \vee r)))&\\
\to ((\neg p_1 \to q_1) \vee (\neg p_1 \to r) \vee (\neg p_2 \to q_2) \vee (\neg p_2 \to r)).&
\end{aligned}
$$

3. The formula **r**:

$$
\begin{aligned}
\neg(p \;\&\; q) \;\&\; \neg(\neg p \;\&\; \neg q)&\\
\;\&\; ((\neg\neg p \to p) \to p \vee \neg p) \;\&\; ((\neg\neg q \to q) \to q \vee \neg q)&\\
\to (\neg p \vee \neg q).&
\end{aligned}
$$

4. The formula **P**:

$$
\begin{aligned}
(\neg(p_1 \;\&\; p_2) \;\&\; \neg\neg(q_1 \vee q_2) \;\&\; ((p_1 \to q_1) \vee (p_1 \to q_2))&\\
\to ((p_2 \to q_1) \vee (p_2 \to q_2)))&\\
\to ((p_2 \to q_1) \vee (p_2 \to q_2)).&
\end{aligned}
$$

All the known realizable propositional formulas are deducible in the calculus $\mathsf{IPC} + \mathbf{L} + \mathbf{C}^* + \mathbf{r} + \mathbf{P}$. The formula **L** was proposed by the author as an example of a formula which is uniformly realizable but is not in Medvedev Logic (see §6.2). The formula $\mathbf{C}^*$ was proposed be D. P. Skvortsov [42] as a modified variant of a sequence of formulas introduced by G. S. Tseitin [43]. The formula **P** was proposed by the author [29], the formula **r** was proposed by V. A. Jankov [9].

- The propositional logic of realizability and Medvedev Logic of finite problems are not included one to another (see §6.2). The formulas

$$(\neg p \to (q \vee r)) \to ((\neg p \to q) \vee (\neg p \to r))$$

and

$$((\neg\neg p \to p) \to (\neg p \vee \neg\neg p)) \to (\neg p \vee \neg\neg p)$$

are in Medvedev Logic but are refutable, i.e., are not realizable in the weakest sense. The formula **L** just mentioned is uniformly realizable, i.e., realizable in the strongest sense, but is not finitely valid.

- The classes of irrefutable, effectively realizable, and uniformly realizable propositional formulas have the disjunction property: if a disjunction $A \vee B$ is in one of these classes, then at least one of the formulas A and B is in the same class (see §7). The known proofs of the disjunction property for the irrefutable and effective realizable formulas are nonconstructive: they do not give any method for determining which of the formulas A and B is in the corresponding class.
- The following variant of partial completeness of IPC relative to realizability holds (see §8): if a propositional formula A does not contain at least one of the connectives $\rightarrow$, $\neg$, $\vee$ and is irrefutable, i.e., realizable in the weakest sense, then A is deducible in IPC.

§13. Open problems. In this section, we state open problems in the study of propositional realizability logic.

PROBLEM 1. *Relations between the notions of irrefutable, effectively realizable, and uniformly realizable propositional formula* (see §5).

- Are there irrefutable propositional formulas which are not effectively realizable?
- Are there effectively realizable propositional formulas which are not uniformly realizable?

Positive answers to the similar questions for predicate formulas are given by Theorems 10.1 and 10.2.

PROBLEM 2. *Description of the class of realizable propositional formulas.*

- Are the classes of irrefutable, effectively realizable, uniformly realizable propositional formulas decidable?
- Are these classes recursively enumerable?
- Is the Varpakhovskii calculus (see the subsection 9.1) complete relative to realizability?
- Does the propositional realizability logic coincide with the propositional logic of Markov's Arithmetic MA (see the subsection 9.2)?

Negative answers to the first two questions for predicate formulas are given by Theorem 10.3. Negative answer to the last question for predicate formulas is given by Theorems 10.3 and 10.5.

PROBLEM 3. *Effectiveness of the disjunction property* (see §7).

- Is there an algorithm for indicating an effectively realizable disjunct in an effectively realizable disjunction of propositional formulas?
- Is there an algorithm for indicating an irrefutable disjunct in an irrefutable disjunction of propositional formulas?

PROBLEM 4. *Formulas of one variable* (see the subsection 9.4).

- Is there a propositional formula of one variable which is irrefutable (effectively realizable, uniformly realizable) but nondeducible in IPC?

- Are the formulas $A_n(p)$ $(n \geq 12, n \neq \infty)$ in the Nishimura sequence refutable?

PROBLEM 5. *Absolutely realizable propositional formulas.*

- Is the class of absolutely realizable propositional formulas decidable?
- Is this class recursively enumerable?
- Is every realizable (in any sense) propositional formula absolutely realizable?

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Appendix A. Proof of Lemma 6.4. Let $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3$ be arithmetical sentences. We prove that there exists a number e independent of $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3$ such that e realizes the arithmetical formula $\mathbf{L}(\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3)$, i.e., the formula

$$\begin{aligned}(\neg\neg\mathrm{A}_1 \equiv \neg\mathrm{A}_2 \ \& \ \neg\mathrm{A}_3)& \\ \& \ (\neg\neg\mathrm{A}_2 \equiv \neg\mathrm{A}_3 \ \& \ \neg\mathrm{A}_1)& \\ \& \ (\neg\neg\mathrm{A}_3 \equiv \neg\mathrm{A}_1 \ \& \ \neg\mathrm{A}_2)& \\ \& \ ((\neg\mathrm{A}_1 \to \neg\mathrm{A}_1 \vee \neg\mathrm{A}_3) \to \neg\mathrm{A}_2 \vee \neg\mathrm{A}_3)& \\ \& \ ((\neg\mathrm{A}_2 \to \neg\mathrm{A}_3 \vee \neg\mathrm{A}_1) \to \neg\mathrm{A}_3 \vee \neg\mathrm{A}_1)& \\ \& \ ((\neg\mathrm{A}_3 \to \neg\mathrm{A}_1 \vee \neg\mathrm{A}_2) \to \neg\mathrm{A}_1 \vee \neg\mathrm{A}_2)& \\ \to \neg\mathrm{A}_1 \vee \neg\mathrm{A}_2 \vee \neg\mathrm{A}_3).& \qquad (11)\end{aligned}$$

Begin with some auxiliary statements.

LEMMA A.1. *Let f be a general recursive function with values* 0 *and* 1. *Then there exists a number x such that*

$$(\{x\}(x))_0 = 0 \textit{ and } f(x) = 1$$

or

$$(\{x\}(x))_0 = 1 \textit{ and } f(x) = 0.$$

PROOF. Assume that the opposite case holds. Then for any x

$$(\{x\}(x))_0 = 0 \Rightarrow f(x) = 0, \qquad (12)$$

$$(\{x\}(x))_0 = 1 \Rightarrow f(x) = 1. \qquad (13)$$

Define a function g by letting

$$g(x) = \begin{cases} 2 & \text{if } f(x) = 0, \\ 1 & \text{if } f(x) = 1. \end{cases}$$

Let $g = \{m\}$. Then

$$(\{m\}(m))_0 = 0 \Rightarrow (g(m))_0 = 0 \Rightarrow g(m) = 1 \Rightarrow f(m) = 1.$$

This contradicts (12) for $x = m$. Similarly,

$$(\{m\}(m))_0 = 1 \Rightarrow (g(m))_0 = 1 \Rightarrow g(m) = 2 \Rightarrow f(m) = 0.$$

This contradicts (13) for $x = m$. We see that $(\{m\}(m))_0$ is 0 or 1 but both cases are impossible. Thus the conditions (12) and (13) are not consistent and lemma is proved. Note that we can find the corresponding x effectively if an algorithm for computing f is given. ⊣

LEMMA A.2. *Let the formula*

$$(\neg\neg A_1 \equiv \neg A_2 \ \& \ \neg A_3) \\ \& \ (\neg\neg A_2 \equiv \neg A_3 \ \& \ \neg A_1) \\ \& \ (\neg\neg A_3 \equiv \neg A_1 \ \& \ \neg A_2) \quad (14)$$

be realizable and numbers a_1, a_2, a_3 be such that

$$a_1 \, \boldsymbol{r} \, ((\neg A_1 \to \neg A_2 \vee \neg A_3) \to \neg A_2 \vee \neg A_3),$$

$$a_2 \, \boldsymbol{r} \, ((\neg A_2 \to \neg A_3 \vee \neg A_1) \to \neg A_3 \vee \neg A_1),$$

$$a_3 \, \boldsymbol{r} \, ((\neg A_3 \to \neg A_1 \vee \neg A_2) \to \neg A_1 \vee \neg A_2).$$

For any x and $i = 1, 2, 3$ let

$$f_i(x) = (\{a_i\}(x))_0, \ \alpha(x) = \Lambda y.2^{(\{x\}(x))_0} \cdot 3^0.$$

Then there exist x and $i \in \{1, 2, 3\}$ such that

$$(\{x\}(x))_0 = 0 \text{ and } f_i(\alpha(x)) = 1$$

or

$$(\{x\}(x))_0 = 1 \text{ and } f_i(\alpha(x)) = 0.$$

PROOF. It follows from realizability of the formula (14) that exactly two of the formulas $\neg A_1, \neg A_2, \neg A_3$ are realizable. As the situation is absolutely symmetric, let us consider the case when $\neg A_1$ and $\neg A_2$ are realizable and $\neg A_3$ is not. Then the formula $\neg A_3 \to \neg A_1 \vee \neg A_2$ is realized by every number, thus $\{a_3\}$ is a general recursive function. The function α is also general recursive. Therefore the function $f(x) = f_3(\alpha(x))$ is general recursive, and we obtain the proposition under the proof as immediate consequence of Lemma A.1. ⊣

Let a realization of the premise of the formula (11) be given. It follows that the formula (14) is realizable and the numbers a_1, a_2, a_3 as in Lemma A.2 can be found effectively. Then, by Lemma A.2, there exist x and $i \in \{1, 2, 3\}$ such that $(\{x\}(x))_0 = 0$ and $f_i(\alpha(x)) = 1$ or $(\{x\}(x))_0 = 1$ and $f_i(\alpha(x)) = 0$, where f_i and α are the same functions as in Lemma A.2. Note that the number x can be found effectively. Now we can describe an algorithm which allows to find a realization of the conclusion of the formula (11). Obviously, it is enough to indicate a realizable member in the disjunction $\neg A_1 \vee \neg A_2 \vee \neg A_3$. We state that:

- if $i = 1$ and $f_1(\alpha(x)) = 0$, then $\neg A_2$ is realizable;
- if $i = 1$ and $f_1(\alpha(x)) = 1$, then $\neg A_3$ is realizable;
- if $i = 2$ and $f_2(\alpha(x)) = 0$, then $\neg A_3$ is realizable;
- if $i = 2$ and $f_2(\alpha(x)) = 1$, then $\neg A_1$ is realizable;
- if $i = 3$ and $f_3(\alpha(x)) = 0$, then $\neg A_1$ is realizable;
- if $i = 3$ and $f_3(\alpha(x)) = 1$, then $\neg A_2$ is realizable.

Consider the first case, namely $f_1(\alpha(x)) = 0$ for some x such that $(\{x\}(x))_0 = 1$. Note that $\{x\}(x)$ is defined and $\alpha(x)$ is the Gödel number of the constant function whose only value is $2^{(\{x\}(x))_0} \cdot 3^0$. We have to prove that $\neg A_2$ is realizable. Assume that $\neg A_2$ is not realizable. Then $\neg A_1$ and $\neg A_3$ are realizable. In this case, the number $\alpha(x)$ is a realization of the formula $\neg A_1 \to \neg A_2 \vee \neg A_3$. Indeed, every number a realizes $\neg A_1$. Note that $\{\alpha(x)\}(a) = 2^{(\{x\}(x))_0} \cdot 3^0 = 2^1 \cdot 3^0$ and in fact, this number realizes the formula $\neg A_2 \vee \neg A_3$. Therefore $\{a_1\}(\alpha(x))$ realizes the formula $\neg A_2 \vee \neg A_3$ and $(\{a_1\}(\alpha(x))_0 = 1$, that is $f_1(\alpha(x)) = 0$. But this is not the case. The contradiction means that the formula $\neg A_2$ is realizable. Other cases are considered in the same way.

Thus we have an algorithm for finding a realizable member in the disjunction $\neg A_1 \vee \neg A_2 \vee \neg A_3$ if a realization of the premise of the formula (11) is given. This means that the formula (11) is realizable. As the realization does not depend on the sentences A_1, A_2, A_3, the formula **L** is uniformly realizable. Lemma 6.4 is proved.

Appendix B. Proof of Theorem 7.2. Let propositional formulas $A_1(p_1, \ldots, p_n)$ and $A_2(p_1, \ldots, p_n)$ be refutable and let lists of arithmetical formulas $\Phi_{11}(x_1), \ldots, \Phi_{1n}(x_1)$ and $\Phi_{21}(x_2), \ldots, \Phi_{2n}(x_2)$ be their refutations. Recall that $R_0(x)$ is an arithmetical Σ_1-formula expressing the predicate $\{x\}(x) = 0$. Let $M_i(x, x_1, x_2)$ $(i = 1, \ldots, n)$ be the formula

$$\Phi_{1i}(x_1) \;\&\; \neg\neg R_0(x) \vee \Phi_{2i}(x_1) \;\&\; \neg R_0(x).$$

Let B_i be the propositional formula $q_{1i} \;\&\; \neg\neg r \vee q_{2i} \;\&\; \neg r$ $(i = 1, \ldots, n)$. Note that the following deducibilities hold in IPC:

$$\neg\neg r \vdash B_i \equiv q_{1i},$$
$$\neg r \vdash B_i \equiv q_{2i}.$$

In this case, it can be shown in the usual way that

$$\neg\neg r \vdash A_1(B_1, \ldots, B_n) \equiv A_1(q_{11}, \ldots, q_{1n}),$$
$$\neg r \vdash A_2(B_1, \ldots, B_n) \equiv A_2(q_{21}, \ldots, q_{2n}).$$

In particular,

$$\neg\neg r, A_1(B_1, \dots, B_n) \vdash A_1(q_{11}, \dots, q_{1n}), \tag{15}$$

$$\neg r, A_2(B_1, \dots, B_n) \vdash A_2(q_{21}, \dots, q_{2n}). \tag{16}$$

Replace in (15) the variables $q_{11}, \dots, q_{1n}$ by the arithmetical formulas $\Phi_{11}(x_1), \dots, \Phi_{1n}(x_1)$ and r by $\mathrm{R}_0(x)$. Then B_i is transformed into M_i. By Theorem 5.1, there exists a partial recursive function ϱ of two variables such that for any natural a, a_1, a_2 if

$$\alpha_1 \,\mathbf{r}\, \neg\neg\mathrm{R}_0(a), \qquad \alpha_2 \,\mathbf{r}\, A_1(\mathrm{M}_1(a, a_1, a_2), \dots, \mathrm{M}_n(a, a_1, a_2)),$$

then

$$\varrho(\alpha_1, \alpha_2) \,\mathbf{r}\, A_1(\Phi_{11}(a_1), \dots, \Phi_{1n}(a_1)).$$

Let a be such that $\{a\}(a) = 0$. Then the formula $\mathrm{R}_0(a)$ is realizable and $0 \,\mathbf{r}\, \neg\neg\mathrm{R}_0(a)$. In this case, if $\alpha \,\mathbf{r}\, A_1(\mathrm{M}_1(a, a_1, a_2), \dots, \mathrm{M}_n(a, a_1, a_2))$, then $\varrho(0, \alpha) \,\mathbf{r}\, A_1(\Phi_{11}(a_1), \dots, \Phi_{1n}(a_1))$. Let $\varrho_1(\alpha) = \varrho(0, \alpha)$. Thus the partial recursive function ϱ_1 has the following property: if $\{a\}(a) = 0$, then for any a_1, a_2, α such that $\alpha \,\mathbf{r}\, A_1(\mathrm{M}_1(a, a_1, a_2), \dots, \mathrm{M}_n(a, a_1, a_2))$ we get $\varrho_1(\alpha) \,\mathbf{r}\, A_1(\Phi_{11}(a_1), \dots, \Phi_{1n}(a_1))$.

Similarly, replacing in (16) $q_{21}, \dots, q_{2n}$ by $\Phi_{21}(x_2), \dots, \Phi_{2n}(x_2)$ and r by $\mathrm{R}_0(x)$ and arguing as above, we see that there exists a partial recursive function ϱ_2 with the following property: if $\{a\}(a) = 0$ is false, then for any a_1, a_2, α such that $\alpha \,\mathbf{r}\, A_2(\mathrm{M}_1(a, a_1, a_2), \dots, \mathrm{M}_n(a, a_1, a_2))$ we get $\varrho_2(\alpha) \,\mathbf{r}\, A_2(\Phi_{21}(a_1), \dots, \Phi_{2n}(a_1))$.

Now we can prove that the list of arithmetical formulas $\mathrm{M}_1, \dots, \mathrm{M}_n$ refutes the formula $A_1 \vee A_2$. Assume the converse. Then there exists a general recursive function f such that for any a, a_1, a_2 the number $f(a, a_1, a_2)$ realizes the formula

$$A_1(\mathrm{M}_1(a, a_1, a_2), \dots, \mathrm{M}_n(a, a_1, a_2)) \\ \vee A_2(\mathrm{M}_1(a, a_1, a_2), \dots, \mathrm{M}_n(a, a_1, a_2)). \tag{17}$$

It follows from the definition of realizability of a disjunction that

$$f(x, x_1, x_2) = 2^{f_0(x, x_1, x_2)} \cdot 3^{f_1(x, x_1, x_2)},$$

the functions f_0, f_1 being general recursive.

Let $\psi(x_1, x_2) = \Lambda x. f_0(x, x_1, x_2)$. The function ψ is general recursive. Obviously, for any a_1, a_2 the number $f(\psi(a_1, a_2), a_1, a_2)$ realizes the formula (17) for $a = \psi(a_1, a_2)$. It follows from the definition of realizability of a disjunction that if $f_0(\psi(a_1, a_2), a_1, a_2) = 0$, then

$$f_1(\psi(a_1, a_2), a_1, a_2) \,\mathbf{r}\, A_1(\mathrm{M}_1(\psi(a_1, a_2), a_1, a_2), \dots, \mathrm{M}_n(\psi(a_1, a_2), a_1, a_2)).$$

Suppose that there exists a_2 such that $f_0(\psi(a_1, a_2), a_1, a_2) = 0$ for every a_1. Then obviously, for any a_1 the number $f_1(\psi(a_1, a_2), a_1, a_2)$ realizes the formula

$$A_1(\mathrm{M}_1(\psi(a_1, a_2), a_1, a_2), \dots, \mathrm{M}_n(\psi(a_1, a_2), a_1, a_2))$$

and if $a = \psi(a_1, a_2)$, then $\{a\}(a) = f_0(\psi(a_1, a_2), a_1, a_2) = 0$. It follows from the property of ϱ_1 that for any a_1 the number $\varrho_1(f_1(\psi(a_1, a_2), a_1, a_2))$ realizes the formula $A_1(\Phi_{11}(a_1), \dots, \Phi_{1n}(a_1))$. This means that the function $g(x_1) = \varrho_1(f_1(\psi(x_1, a_2), x_1, a_2))$ realizes $A_1(\Phi_{11}(x_1), \dots, \Phi_{1n}(x_1))$, but it is impossible because $\Phi_{11}(x_1), \dots, \Phi_{1n}(x_1)$ is just a refutation of A_1.

We have proved that for any a_2 there exists a_1 such that

$$f_0(\psi(a_1, a_2), a_1, a_2) = 1.$$

In this case, the functions

$$\chi(x_2) = \mu x_1[f_0(\psi(x_1, x_2), x_1, x_2) = 1]$$

and

$$h(x_2) = \varrho_2(f_1(\chi(x_2), \psi(\chi(x_2), x_2), x_2))$$

are general recursive. We prove that h realizes $A_2(\Phi_{21}(x_2), \dots, \Phi_{2n}(x_2))$. Indeed, if a number a_2 is given, we have that the number $a_1 = \chi(a_2)$ is such that $f_0(\psi(a_1, a_2), a_1, a_2) = 1$. Then $f_1(\chi(a_2), \psi(\chi(a_2), a_2), a_2)$ realizes the formula $A_2(\mathrm{M}_1(\psi(a_1, a_2), a_1, a_2), \dots, \mathrm{M}_n(\psi(a_1, a_2), a_1, a_2))$. At the same time, if $a = \psi(a_1, a_2)$, then $\{a\}(a) = f_0(\psi(a_1, a_2), a_1, a_2) = 1$. It follows from the property of the function ϱ_2 that $\varrho_2(\psi(\chi(a_2), a_2), a_2))$ realizes the formula $A_2(\Phi_{21}(a_2), \dots, \Phi_{2n}(a_2))$. It means that h realizes $A_2(\Phi_{21}(x_2), \dots, \Phi_{2n}(x_2))$, but it is impossible as $\Phi_{21}(x_2), \dots, \Phi_{2n}(x_2)$ is a refutation of A_2.

We have proved that the list $\mathrm{M}_1, \dots, \mathrm{M}_n$ refutes the formula $A_1 \vee A_2$. Theorem 7.2 is proved.

Appendix C. Proof of Theorem 7.3. In order to prove the disjunction property for the effectively realizable propositional formulas, let us make the definition of this kind of realizability more precise. Namely let for any $n \geq 1$ a Gödel numbering of the n-tuples of closed arithmetical formulas be fixed. Then Φ_k^j $(j = 1, \dots, n)$ denotes the jth formula of the kth n-tuple. A propositional formula $A(p_1, \dots, p_n)$ is effectively realizable if there exists a one-place general recursive function φ such that for any k the number $\varphi(k)$ realizes the formula $A(\Phi_k^1, \dots, \Phi_k^n)$. In this case, we say that the function φ realizes the formula $A(p_1, \dots, p_n)$.

Let a propositional formula $A_1(p_1, \dots, p_n) \vee A_2(p_1, \dots, p_n)$ be effectively realizable. Let Φ_{im}^j $(1 \leq j \leq n,\ i, m = 0, 1, 2, \dots)$ be the formula $\bigwedge_{k=1}^{m} (\neg\neg \mathrm{R}_k(i) \to \Phi_k^j)$. For any $m, j \geq 1$ let B_m^j be the propositional formula

$\bigwedge_{k=1}^{m} (\neg\neg r_k \to p_{jk})$. Remark that if we replace in B_m^j the variables r_k by the formulas $\mathrm{R}_k(i)$ and the variables p_{jk} by Φ_k^j, then the formula Φ_{im}^j is obtained.

Obviously, for any $k \leq m$ we have in IPC

$$\bigwedge_{1\leq u<v\leq m} \neg(r_u \ \& \ r_v),\ \neg\neg r_k \vdash B_m^j \equiv p_{jk}.$$

In this case, it can be shown in the usual way that for $t = 1, 2$

$$\bigwedge_{1\leq u<v\leq m} \neg(r_u \ \& \ r_v),\ \neg\neg r_k \vdash A_t(B_m^1, \dots, B_m^n) \equiv A_t(p_{1k}, \dots, p_{nk}).$$

In particular,

$$\bigwedge_{1\leq u<v\leq m} \neg(r_u \ \& \ r_v),\ \neg\neg r_k, A_t(B_m^1, \dots, B_m^n) \vdash A_t(p_{1k}, \dots, p_{nk}). \qquad (18)$$

In view of the remark above it follows from Theorem 5.1 that for any k and $t = 1, 2$ there exists a partial recursive function ϱ_{mkt} of three variables such that for any i, if

$$\alpha_0\,\mathbf{r} \bigwedge_{1\leq u<v\leq m} \neg(\mathrm{R}_u(i) \ \& \ \mathrm{R}_v(i)),\ \alpha_1\,\mathbf{r}\,\neg\neg\mathrm{R}_k(i),\ \alpha_2\,\mathbf{r}\,A_t(\Phi_{im}^1, \dots, \Phi_{im}^n),$$

then

$$\varrho_{mkt}(\alpha_0, \alpha_1, \alpha_2)\,\mathbf{r}\,A_t(\Phi_k^1, \dots, \Phi_k^n).$$

If i is such that $\{i\}(i) = k$, then $0\,\mathbf{r}\,\neg\neg\mathrm{R}_k(i)$. On the other hand, the formula $\bigwedge_{1\leq u<v\leq m} \neg(\mathrm{R}_u(i) \ \& \ \mathrm{R}_v(i))$ is realized by a number b_m independent of i. Therefore, if $\alpha\,\mathbf{r}\,A_t(\Phi_{im}^1, \dots, \Phi_{im}^n)$, then $\varrho_{mkt}(b_m, 0, \alpha)\,\mathbf{r}\,A_t(\Phi_k^1, \dots, \Phi_k^n)$. Let $\sigma_{mkt}(\alpha) = \varrho_{mkt}(b_m, 0, \alpha)$. Thus the partial recursive function σ_{mkt} has the following property: if $\{i\}(i) = k$ and $\alpha\,\mathbf{r}\,A_t(\Phi_{im}^1, \dots, \Phi_{im}^n)$, then $\sigma_{mkt}(\alpha)\,\mathbf{r}\,A_t(\Phi_k^1, \dots, \Phi_k^n)$. As for any m and k the inference (18) can be found effectively, we may argue that the algorithm for σ_{mkt} effectively depends on m and k, i.e., there exists a general recursive function τ_t such that σ_{mkt} is $\{\tau_t(m, k)\}$.

Let ρ be a general recursive function of two variables such that for any i, m the number $\rho(i, m)$ is the Gödel number of the n-tuple $\Phi_{im}^1, \dots, \Phi_{im}^n$.

Now let a general recursive function φ realize the formula (18).

LEMMA C.1. *It is impossible that no number s exists with the property*:

(1) $s = 0$ *or* $s = 1$;

(2) *for any k there exist natural numbers i, m, c such that $\{i\}(i) = k$, $m \geq k$, and $\varphi(\rho(i, m)) = 2^s \cdot 3^c$.*

Proof. Recall that sets of natural numbers A, B are called *recursively separable* if there exists a recursive set C such that $A \subseteq C$, $B \subseteq \bar{C}$. Let $M_k = \{i \mid \{i\}(i) = k\}$. It is easy to prove (see the proof of Theorem XII in [40]) that the sets M_k and M_l are not recursively separable if $k \neq l$. Now assume that for $s = 0$ and $s = 1$ the statement (2) does not hold. Then there exist natural numbers k_0 and k_1 such that for any natural numbers i, m, c, if $\{i\}(i) = k_s$ and $m \geq k_s$, then $\varphi(\rho(i, m)) \neq 2^s \cdot 3^c$. Let $m = \max(k_0, k_1)$. Note that the number $\varphi(\rho(i, m))$ realizes the formula

$$A_1(\Phi^1_{im}, \ldots, \Phi^n_{im}) \vee A_2(\Phi^1_{im}, \ldots, \Phi^n_{im}),$$

thus $\varphi(\rho(i, m))$ is of the form $2^0 \cdot 3^c$ or $2^0 \cdot 3^c$ for some c. Therefore for any i we have:

$$\begin{aligned} i \in M_{k_0} &\Rightarrow (\varphi(\rho(i, m)))_0 = 1; \\ i \in M_{k_1} &\Rightarrow (\varphi(\rho(i, m)))_0 = 0. \end{aligned} \tag{19}$$

Let $C = \{i \mid (\varphi(\rho(i, m)))_0 = 1\}$, then $\bar{C} = \{i \mid (\varphi(\rho(i, m)))_0 = 0\}$. Obviously, C is a recursive set and it follows from (19) that $M_{k_0} \subseteq C$, $M_{k_1} \subseteq \bar{C}$. This contradicts inseparability of M_{k_0} and M_{k_1}. ⊣

Thus classically, for $s = 0$ or $s = 1$ the statement (2) is true. Consider the case $s = 0$. Obviously, for any k the corresponding numbers i, m, c can be found effectively. In particular, there are general recursive functions μ, ς such that $m = \mu(k)$, $c = \varsigma(k)$. It follows from the definition of realizability of a disjunction that the number $\varsigma(k)$ realizes the formula $A_1(\Phi^1_{im}, \ldots, \Phi^n_{im})$. Using the property of the function σ_{mk1} we obtain that the number $\sigma_{mk1}(\varsigma(k))$ realizes the formula $A_1(\Phi^1_k, \ldots, \Phi^n_k)$. But $\sigma_{mk1}(\varsigma(k)) = \{\tau_1(\mu(k), k)\}(\varsigma(k))$. Thus we have proved that for any k the number $\{\tau_1(\mu(k), k)\}(\varsigma(k))$ realizes the formula $A_1(\Phi^1_k, \ldots, \Phi^n_k)$. This means that the function

$$\psi(x) = \{\tau_1(\mu(x), x)\}(\varsigma(x))$$

realizes the formula $A_1(p_1, \ldots, p_n)$.

By the same argument one can prove that in the case $s = 1$, the function ψ defined in a similar way realizes the formula $A_2(p_1, \ldots, p_n)$. Theorem 7.3 is proved.

Appendix D. Proof of Theorem 9.1. Let a propositional formula $A(p_0, \ldots, p_n)$ be not K-uniformly realizable. Let $\mathrm{D}(x) \in \mathrm{K}$ be a universal formula for the closed formulas in the class K. By the definition of the classes Σ_k and Π_k, the formula $\mathrm{D}(x)$ is of the form $\exists z\, \mathrm{B}(x, z)$, where $\mathrm{B}(x, z)$ is in the class K^0. Note that the formulas in the class K^0 begin with $\neg$. Let $\mathrm{C}(x, y)$ be the formula

$$\{y\}(y)\, \mathsf{R}\, A(\exists z\, \mathrm{B}((x)_0, z), \ldots, \exists z\, \mathrm{B}((x)_n, z)),$$

$\mathrm{F}(x, y)$ be the formula

$$\neg\mathrm{C}(x, y) \ \ \& \ \ \forall v\, (v < x \to \mathrm{C}(v, y)).$$

Then for any i $(0 \le i \le n)$ let $\Phi_i(x)$ be the formula

$$\exists z\, \neg\exists y\, (\mathrm{F}(y, x) \ \ \& \ \ \neg\mathrm{B}((y)_i, z)).$$

We prove that the list $\Phi_0(x), \dots, \Phi_n(x)$ is a refutation of $A(p_0, \dots, p_n)$. Assume the converse. Then there exists b such that

$$b\,\mathbf{r}\,\forall x\, A(\Phi_0(x), \dots, \Phi_n(x)).$$

It follows from the definition of recursive realizability that

$$\{b\}(b)\,\mathbf{r}\,A(\Phi_0(b), \dots, \Phi_n(b)). \tag{20}$$

We prove that

$$\{b\}(b)\,\mathbf{r}\,A(\Psi_0, \dots, \Psi_n) \tag{21}$$

for any closed formulas $\Psi_0, \dots, \Psi_n$ in the class K, in contrast with the assumption that $A(p_0, \dots, p_n)$ is not K-uniformly realizable. This contradiction will prove the theorem.

Let (20) hold. Suppose that there exist closed formulas $\Psi_0, \dots, \Psi_n$ in the class K such that (21) does not hold. Obviously, for any $i = 0, \dots, n$ there exists a_i such that the formula Ψ_i is completely equivalent to the formula $\exists z\, \mathrm{B}(a_i, z)$. Let $a = 2^{a_0} \cdot \dots \cdot \pi_n^{a_n}$ and g be the minimal possible value of a. Now take the formulas $\Psi_0, \dots, \Psi_n$ corresponding to this value. Thus the formulas Ψ_i are completely equivalent to the formulas $\exists z\, \mathrm{B}((g)_i, z)$ $(i = 0, \dots, n)$. It is evident that the formulas $A(\Psi_0, \dots, \Psi_n)$ and $A(\exists z\, \mathrm{B}((a)_0, z), \dots, \exists z\, \mathrm{B}((a)_n, z))$ are also completely equivalent. Therefore

$$\{b\}(b)\,\mathbf{r}\,A(\exists z\, \mathrm{B}((a)_0, z), \dots, \exists z\, \mathrm{B}((a)_n, z)) \tag{22}$$

does not hold for $a = g$, but holds for any $a < g$. Let us prove that for $i = 0, \dots, n$ the formulas $\exists z\, \mathrm{B}((g)_i, z)$ and $\Phi_i(b)$ are completely equivalent. This will imply that (20) does not hold contrary to the assumption.

Let $e\,\mathbf{r}\,\exists z\, \mathrm{B}((g)_i, z)$, then $(e)_1\,\mathbf{r}\,\mathrm{B}((g)_i, (e)_0)$. We have to prove that $e\,\mathbf{r}\,\Phi_i(b)$, i.e., $e\,\mathbf{r}\,\exists z\, \neg\exists y\, (\mathrm{F}(y, b) \ \ \& \ \ \neg\mathrm{B}((y)_i), z)$. This means that

$$(e)_1\,\mathbf{r}\,\neg\exists y\, (\mathrm{F}(y, b) \ \ \& \ \ \neg\mathrm{B}((y)_i, (e)_0)).$$

As the negation of a sentence is realized by any number if the sentence is not realizable, it is enough to prove that $\exists y\, (\mathrm{F}(y, b) \ \ \& \ \ \neg\mathrm{B}((y)_i, (e)_0))$ is not realizable. Assume the converse, i.e., this formula is realizable. In this case, there exists a such that the formula

$$\mathrm{F}(a, b) \ \ \& \ \ \neg\mathrm{B}((a)_i, (e)_0) \tag{23}$$

is realizable, in particular, the formula $\mathrm{F}(a, b)$ is realizable. The structure of the formula $\mathrm{F}(x, y)$ is such that the the first conjunct in (23) is realizable if and only if $a = g$, but the formula $\mathrm{B}((g)_i, (e)_0)$ is realizable and the second conjunct in (23) is not realizable if $a = g$. Thus any realization of the formula $\exists z\, \mathrm{B}((g)_i, z)$ is a realization of the formula $\Phi_i(b)$.

Now let $e\, \mathbf{r}\, \Phi_i(b)$. This means that $(e)_1\, \mathbf{r}\, \neg\exists y\, (\mathrm{F}(y, b)\ \ \&\ \ \neg\mathrm{B}((y)_i, (e)_0))$, i.e., the formula

$$\exists y\, (\mathrm{F}(y, b)\ \ \&\ \ \neg\mathrm{B}((y)_i, (e)_0)) \tag{24}$$

is not realizable. We have to prove that $e\, \mathbf{r}\, \exists z\, \mathrm{B}((g)_i, z)$. This means $(e)_1\, \mathbf{r}\, \mathrm{B}((g)_i, (e)_0)$. As the formula $\mathrm{B}((g)_i, (e)_0)$ is a negation of a sentence, it is realized by any number if it is realizable. Thus it is enough to prove that the formula $\mathrm{B}((g)_i, (e)_0)$ is realizable. Assume the converse, i.e., this formula is not realizable. Then $\neg\mathrm{B}((g)_i, (e)_0)$ is realizable. The structure of the formula $\mathrm{F}(x, y)$ is such that the the formula $\mathrm{F}(g, b)$ is realizable. Thus the formula $\mathrm{F}(g, b)\ \ \&\ \ \neg\mathrm{B}((g)_i), (e)_0)$ is realizable in spite of the assumption that the formula (24) is not realizable. Theorem 9.1 is proved.

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