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FULL DEVELOPMENT OF TARSKI'S GEOMETRY OF SOLIDS

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Abstract. In this paper we give probably an exhaustive analysis of the geometry of solids which was sketched by Tarski in his short paper [20, 21]. We show that in order to prove theorems stated in [20, 21] one must enrich Tarski's theory with a new postulate asserting that the universe of discourse of the geometry of solids coincides with arbitrary mereological sums of balls, i.e., with solids. We show that once having adopted such a solution Tarski's Postulate 4 can be omitted, together with its versions $4'$ and $4''$. We also prove that the equivalence of postulates 4, $4'$ and $4''$ is not provable in any theory whose domain contains objects other than solids. Moreover, we show that the concentricity relation as defined by Tarski must be transitive in the largest class of structures satisfying Tarski's axioms.

We build a model (in three-dimensional Euclidean space) of the theory of so called $\mathrm{T}^\star$-structures and present the proof of the fact that this is the only (up to isomorphism) model of this theory.

Moreover, we propose different categorical axiomatizations of the geometry of solids. In the final part of the paper we answer the question concerning the logical status (within the theory of $\mathrm{T}^\star$-structures) of the definition of the concentricity relation given by Tarski.

§**1. Introduction.** In his paper "Foundations of the geometry of solids" [20, 21] Alfred Tarski sketched a solution of the following problem:

> "SOME years ago Leśniewski suggested the problem of establishing the foundations of the *geometry of solids* understanding by this term a system of geometry destitute of such geometrical figures as points, lines, and surfaces, and admitting as figures only solids—the intuitive correlates of open (or closed) regular sets of the three-dimensional Euclidean geometry. The specific character of such geometry of solids—in contrast to all point geometries—is shown in particular in the law according to which each figure contains another figure as a proper part."

In his exposition Tarski essentially involved the theory founded by Leśniewski and called by him *mereology* [13].[1] "In this theory the relation of part of

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[1]All details concerning notation within mereology are explained in Appendix A.

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whole (which is treated as a relation between individuals) can be taken as the only primitive notion" [21, p. 25–26]. In terms of this relation we define when an individual is a *mereological sum* (i.e., *collective set*) of some individuals.

Tarski omits the question of philosophical importance of the above problem (in philosophy of nature and philosophy of geometry) and refers a reader to the works of Whitehead and Nicod. We also skip the philosophical aspect of Tarski's theory and focus on formal problems connected with it.

1.1. Primitive notions and the universe of discourse of Tarski's theory. The universe of discourse coincides with the space and its parts («pieces»). Members of the universe may by called *spatial regions* (see e.g., [1, 2, 3, 4, 6, 18]). Thus the notion of *region* is also a primitive one in Tarski's theory (and we venture to say that this notion is implicitly assumed by Tarski).

By means of the notions of *region*, of *relation of part of whole* between regions, and other suitable notions we can define the notion of *point* (see e.g., [3, 4, 8, 18]). However, to construct Euclidean geometry in the set of all points we also need some relation that allow us to compare distances between points. This can be, for example, either the ternary relation of *equidistance of two points from a third* or the quaternary relation of *equidistance* (*congruence of segments*).[2] Thus, after Tarski, we introduce the notion of *mereological ball*[3] thanks to which it is possible to define the ternary relation of *equidistance*.

By means of the three primitive notions Tarski introduced the relational notion of *concentric balls* and the notion of *point*.

> "Definition 6. *A* point *is the class of all spheres* [*balls*] *which are concentric with a given sphere* [*ball*].
> Thus in the geometry of solids, spheres [balls] are treated as individuals, i.e., objects of the first order, while points are classes of spheres [balls] and hence objects of the second order. In ordinary geometry the opposite is true."[4] [21, p. 27]

As we mentioned above, via the notion of *mereological ball* Tarski defines not only the notion of *point*, but also the notion of *equidistance of two points from a third* (see Definition 7, p. 494), which let us speak about point-based geometry. This fact is expressed by Tarski in the following postulate[5]

> "Postulate 1. *The notions of point and of equidistance of two points from a third satisfy all the postulates of ordinary Euclidean geometry of three dimensions.*" [21, p. 28].

[2]For details see e.g., [5, 9, 23].

[3]In [21] the term 'sphere' is used, however in contemporary geometry this term is rather reserved for surfaces of balls.

[4]Notice that in [21] Tarski uses the term 'class' as a synonym for the term 'set', but not the term 'equivalence class'.

[5]Further in the paper we use the words 'postulate' and 'axiom' interchangeably with exactly the same meaning.

In Tarski's theory the notion of *mereological solid* is not a primitive one. We find the following definition

> "DEFINITION 8. *A* solid *is an arbitrary sum of spheres* [*balls*]." [21, p. 27]

Tarski does not assume that all members of the universe of discourse are solids. Despite this he defines the notion of *interior* of a given object only for solids. Moreover, postulates 2 and 4 put some constraints on solids only

> "POSTULATE 2. *If A is a solid, the class α of all interior points of A is a non-empty regular open set.*
> POSTULATE 4. *If A and B are solids, and all interior points of A are at the same time interior to B, then A is a part of B.*"[6] [21, p. 28]

Such a situation cannot be an «oversight» in the sense that Tarski writes about solids, but has all regions on his mind (i.e., members of the universe of discourse). Another two postulates (equivalent to postulate 4 according to Tarski) seem to support this view. The first one takes the following shape

> "POSTULATE 4′. *If A is a solid and B is part of A, then B is also a solid.*" [21, p. 28]

One cannot assume that B is a solid, since in such a case this postulate would be trivial. The same holds when we assume that all regions (i.e., members of the universe of discourse) are solids.

The second postulate has the following form

> "POSTULATE 4″. *If A is a sphere* [*ball*] *and B is part of A, there exists a sphere* [*ball*] *C which is a part of B.*" [21, p. 28]

Again, the assumption that B is a solid does not make much sense, since this makes this postulate superfluous. Indeed, an arbitrary solid is a mereological sum of some balls, so there exists a ball which is part of this solid.

The above examples seem to prove that the notion of *region* is present in Tarski's theory, although it is absent from Tarski's paper.

In §13 we will show that mereology itself together with Definition 8 suffice to prove that postulates 4′ and 4″ are equivalent and that Postulate 4 is their consequence. On the other side, we will prove as well that 4′ and 4″ are stronger than 4 (see Theorem 1.3).

One of the main results obtained in our paper is the fact that the set of postulates 1–3, 4′ and Definition 8 are too weak to prove that all regions are solids.[7] The more so, postulates 1–4 and Definition 8 are too weak as well. One possible way out of this situations is to accept the following version of Postulate 2 (see Theorem 14.3, p. 518)

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[6] In [21] the term 'part' allows for so called improper parts, i.e., whole objects.

[7] Postulate 3 says: "*If the class α of points is a non-empty regular open set, there exists a solid A such that α is the class of all interior points of A*" [21, p. 28]. In this form it is of course stronger than the postulate not asserting that A is a solid.

POSTULATE $2^\star$. *For any region the set of all its interior points is a non-empty regular open set.*

Then from Postulates 1, $2^\star$, 3, 4 and Definition 8 we obtain that the universe of discourse of the geometry of solids coincides with the set of all solids. Tarski seems not to have noticed some important facts. He says directly

> "It should be noticed that the class of all solids (and not that of all spheres [balls]) constitutes what is called the 'universe of discourse' for the geometry of solids. For this reason it may be convenient to adopt the notion of solid as an additional primitive notion. In this case Def. 8 should, of course, be omitted; in its place the fact that the class of solids coincides with that of arbitrary sums of spheres should be stated in a new postulate." [21, p. 28]

Thus it seems that for Tarski the theory with Postulates 1–4 and Definition 8 is equivalent to the theory in which: (a) all regions are *solids*, (b) we take Postulates 1–4 together with

($\star$) *Any region is a mereological sum of some set of balls.*

We will prove that this is not the case. Tarski seems not to have noticed that

- firstly, in the theory with Postulates 1–3 and ($\star$), and Definition 8 we obtain Postulate 4 as its thesis;
- secondly, in the theory with Postulates 1–4 and Definition 8 we do not obtain Postulates $4'$, $4''$ and the sentence ($\star$) as its theses.

Thus the first theory is stronger than the second one.

It is provable that in Tarski's paper [21] one can actually distinguish three theories that include Definition 8 (from the weakest to the strongest one)

(i) with Postulates 1–4 – *the theory of* T*-structures*;
(ii) with Postulates 1–3, $4'$ (resp. $4''$) – *the theory of* T'*-structures*;
(iii) with Postulates 1–3 and ($\star$) – *the theory of* $\mathrm{T}^\star$*-structures*;

i.e., all $\mathrm{T}^\star$-structures are T'-structures and all T'-structures are T-structures (but not vice versa; see below theorems 1.1, 1.2 and 1.3). The theory of $\mathrm{T}^\star$-structures is the one that can be called *the geometry of solids*. However, we will prove that the restriction of any T-structure to the set of all solids is a $\mathrm{T}^\star$-structure (see below Theorem 1.4).

We also analyze different equivalent axiomatizations of the theory of $\mathrm{T}^\star$-structures (see Theorem 14.3). For example, we prove that such a theory can be based on postulates 1, $2^\star$, 3, 4 and Definition 8. Moreover, it is easily provable that sentence ($\star$) in an axiomatization of $\mathrm{T}^\star$-structures can be replaced with the following postulate (see Lemma 3.1)

($\star\star$) *Any region either is a ball or some ball is a* (*proper*) *part of this region.*

We will also show that axioms of T'-structures plus one of the following

- *the mereological sum of all regions is a solid*,
- *the mereological sum of all regions is the sum of all balls*.

form an axiomatization of $\mathrm{T}^\star$-structures (see Lemma 13.4).

1.2. Algebraical formulation of the the geometry of solids. Using the language of contemporary algebra and logic one can say that in his paper [20, 21] Tarski analyzes relational structures of the form $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ such that

(P0) (i) $\langle \mathbb{R}, \sqsubseteq \rangle$ is a mereological structure, i.e., for some non-degenerate complete Boolean algebra $\langle A, +, \cdot, -, \mathbf{0}, \mathbf{1} \rangle$ we have $\mathbb{R} = A \setminus \{\mathbf{0}\}$ and for all $x, y \in \mathbb{R}$: $x \sqsubseteq y$ iff $y = x + y$;

(ii) $\mathbb{B}$ is a distinguished subset of $\mathbb{R}$.

We will call elements of $\mathbb{R}$ *regions* and elements of $\mathbb{B}$ *mereological balls* (or simply *balls* in case it follows from the context that we refer to elements of $\mathbb{B}$). The notions of *region*, *ball* and *being an ingrediens*[8] are primitive notions of the analyzed theory. Let $\mathbb{S}$ be the set of all (mereological) solids, i.e.,

$$\mathbb{S} := \{z \in \mathbb{R} \mid \exists_{X \in 2^{\mathbb{B}}}\, z \text{ sum } X\}. \tag{df $\mathbb{S}$}$$

By the definition of the relation sum for any $z \in \mathbb{R}$ and $X \in 2^{\mathbb{R}}$ we obtain: z sum X iff $X \neq \emptyset$ and $z = \sup_{\sqsubseteq} X$.

By Postulate 1, $\mathbb{B} \neq \emptyset$. So there exists the mereological sum of all balls. We obtain $(\iota\, x)\ x \text{ sum } \mathbb{B} \in \mathbb{S}$, but $(\iota\, x)\ x \text{ sum } \mathbb{B} \notin \mathbb{B}$ (see Fact 8.2). Hence $\mathbb{B} \subsetneq \mathbb{S}$, since for any $x \in \mathbb{R}$: $x \text{ sum } \{x\}$. By (5.2) we obtain: $(\iota\, x)\ x \text{ sum } \mathbb{B} = (\iota\, x)\ x \text{ sum } \mathbb{S}$, i.e., $\sup_{\sqsubseteq} \mathbb{B} = \sup_{\sqsubseteq} \mathbb{S}$.

The class of all T*-structures is not empty. Let $\mathbf{RO}^+_{\mathbb{R}^3}$ and $\mathbf{RC}^+_{\mathbb{R}^3}$ be, respectively, families of all non-empty regular open and closed sets in three-dimensional Cartesian space $\langle \mathbb{R}^3, \varrho \rangle$ (see appendixes C and D), where for any $\bar{x}, \bar{y} \in \mathbb{R}^3$

$$\varrho(\bar{x}, \bar{y}) := \sqrt{\textstyle\sum_{i=1}^{3} (x_i - y_i)^2}.$$

Moreover, let $\mathbf{BO}_{\mathbb{R}^3}$ and $\mathbf{BC}_{\mathbb{R}^3}$ be, respectively, families of all open and closed balls in this space. In §15 we prove the following theorem.

THEOREM 1.1. *$\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq \rangle$ and $\langle \mathbf{RC}^+_{\mathbb{R}^3}, \mathbf{BC}_{\mathbb{R}^3}, \subseteq \rangle$ are T*-structures.*

As we have already said, we will prove further in the paper that the theory of T*-structures is stronger than that of T$'$-structures. This fact will be a consequence of the following theorem, which we will prove in §16 and whose premises are satisfiable by Theorem 1.1 and the construction from Appendix B.

THEOREM 1.2. *Suppose that*

- *$\langle M, \sqsubseteq \rangle$ is a mereological structure,*
- $\mathbb{R} \subsetneq M$,
- $\sqsubseteq_{\mathbb{R}} := \sqsubseteq \cap (\mathbb{R} \times \mathbb{R})$,
- $\forall_{x \in \mathbb{R}}\, \forall_{y \in M \setminus \mathbb{R}}\, y \not\sqsubseteq_{\mathbb{R}} x$,

[8]One object is an ingrediens of another iff it is either its (proper) part or is identical with it (see [13]). The relation of *being an ingrediens* is often called a *part relation*, where the term 'part' allows for so called improper parts, i.e., whole objects.

- $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq_{\mathbb{R}} \rangle$ *is a* $\mathrm{T}^{\star}$*-structure with the relations* $\mathsf{sum}_{\mathbb{R}}$, $\bigcirc_{\mathbb{R}}$ *and* $\wr_{\mathbb{R}}$ *which are included in* $\mathbb{R} \times 2^{\mathbb{R}}$ *and* $\mathbb{R} \times \mathbb{R}$ *respectively, and are determined by* $\sqsubseteq_{\mathbb{R}}$, *i.e.,*

$$z \ \mathsf{sum}_{\mathbb{R}} \ X \overset{\mathrm{df}}{\iff} \forall_{x \in X}\, x \sqsubseteq_{\mathbb{R}} z \wedge \forall_{y \in \mathbb{R}} (y \sqsubseteq_{\mathbb{R}} z \Rightarrow \exists_{x \in X}\, x \bigcirc_{\mathbb{R}} y),$$

$$x \bigcirc_{\mathbb{R}} y \overset{\mathrm{df}}{\iff} \exists_{z \in \mathbb{R}} (z \sqsubseteq_{\mathbb{R}} x \wedge z \sqsubseteq_{\mathbb{R}} y),$$

$$x \wr_{\mathbb{R}} y \overset{\mathrm{df}}{\iff} \neg \exists_{z \in \mathbb{R}} (z \sqsubseteq_{\mathbb{R}} x \wedge z \sqsubseteq_{\mathbb{R}} y).$$

Then

(i) $\langle \mathbb{R}, \sqsubseteq_{\mathbb{R}} \rangle$ *has the following properties*
- $\bigcirc_{\mathbb{R}} = \bigcirc \cap (\mathbb{R} \times \mathbb{R})$, $\wr_{\mathbb{R}} = \wr \cap (\mathbb{R} \times \mathbb{R})$ *and* $\wr_{\mathbb{R}} = (\mathbb{R} \times \mathbb{R}) \setminus \bigcirc_{\mathbb{R}}$;
- $\mathsf{sum}_{\mathbb{R}} = \mathsf{sum} \cap (\mathbb{R} \times 2^{\mathbb{R}}) = \mathsf{sum} \cap (M \times 2^{\mathbb{R}})$.

So also for any $x, y \in \mathbb{R}$
- $x \sqcup_{\mathbb{R}} y = x \sqcup y \in \mathbb{R}$;
- *if* $x \bigcirc y$, *then* $x \sqcap_{\mathbb{R}} y = x \sqcap y \in \mathbb{R}$;
- *if* $x \neq \mathsf{s} := (\iota\, z)\ z \ \mathsf{sum}_{\mathbb{R}}\ \mathbb{B}$, *then* $\mathsf{s} -_{\mathbb{R}} x = \mathsf{s} - x \in \mathbb{S}$.

(ii) $\langle M, \mathbb{B}, \sqsubseteq \rangle$ *is a* T'*-structure in which the set of solids is equal to* $\mathbb{R}$, *so it is not a* $\mathrm{T}^{\star}$*-structure.*

Of course in $\langle M, \mathbb{B}, \sqsubseteq \rangle$ the mereological sum of all members of M is not a solid, i.e., it is not a member of $\mathbb{R}$. In other words, the sum of all members of M is not the sum of all balls. Formally: $\sup_{\sqsubseteq} M \notin \mathbb{R}$ and $\sup_{\sqsubseteq} M \neq \sup_{\sqsubseteq} \mathbb{B}$.

As we mentioned before, we will demonstrate that the theory of T'-structures is stronger than that of T-structures. This fact will follow from Theorem 1.1 and the following theorem, both of which will be proven in §17 (see also Theorems A.1 and A.2, and Lemmas A.3 and A.4).

THEOREM 1.3. *Let* $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ *be a* $\mathrm{T}^{\star}$*-structure and let* $\langle \mathbb{R}^{\mathsf{o}}, \leq, \mathsf{o}, \mathbb{1} \rangle$ *be a non-degenerate complete Boolean lattice obtained from* $\langle \mathbb{R}, \sqsubseteq \rangle$ *by adding zero* o ($x \leq y$ *iff* $x = \mathsf{o}$ *or* $x \sqsubseteq y$). *For* $\mathbb{R}_{\times} := \mathbb{R}^{\mathsf{o}} \times \mathbb{R}^{\mathsf{o}}$ *and* $\leq_{\times} := \{\langle \langle x, y \rangle, \langle u, z \rangle \rangle \mid x \leq u \wedge y \leq z\}$, *the product* $\langle \mathbb{R}_{\times}, \leq_{\times}, \langle \mathsf{o}, \mathsf{o} \rangle, \langle \mathbb{1}, \mathbb{1} \rangle \rangle$ *is also a non-degenerate complete Boolean lattice. We put* $\mathbb{R}^{+}_{\times} := \mathbb{R}_{\times} \setminus \{\langle \mathsf{o}, \mathsf{o} \rangle\}$, $\sqsubseteq_{\times} := \leq_{\times}|_{\mathbb{R}^{+}_{\times}}$ *and* $\mathrm{d}\mathbb{B} := \{\langle b, b \rangle \mid b \in \mathbb{B}\}$.

Then $\langle \mathbb{R}^{+}_{\times}, \mathrm{d}\mathbb{B}, \sqsubseteq_{\times} \rangle$ *is a* T*-structure in which the set of solids is the set* $\mathrm{d}\mathbb{S} := \{\langle s, s \rangle \mid s \in \mathbb{S}\}$, *so it is not a* T'*-structure.*

However, the next theorem shows that the theory of T-structures is sufficiently strong to prove that the restriction of any T-structure to the set of all solids[9] is a $\mathrm{T}^{\star}$-structure (see §14, p. 518).

THEOREM 1.4. *Let* $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ *be a* T*-structure and let* $\sqsubseteq_{\mathbb{S}} := \sqsubseteq \cap (\mathbb{S} \times \mathbb{S})$. *Then* $\langle \mathbb{S}, \mathbb{B}, \sqsubseteq_{\mathbb{S}} \rangle$ *is a* $\mathrm{T}^{\star}$*-structure.*[10]

[9]If $\langle D, R_1, \dots, R_n \rangle$ is a relational structure and $A \subseteq D$, then by *the restriction of* $\langle D, R_1, \dots, R_n \rangle$ *to* A we mean the structure $\langle A, R_1|_A, \dots, R_n|_A \rangle$.

[10]Of course $\langle \mathbb{S}, \sqsubseteq_{\mathbb{S}} \rangle$ is a mereological structure for the relation $\mathsf{sum}_{\mathbb{S}}$ which is included in $\mathbb{S} \times 2^{\mathbb{S}}$ and determined by $\sqsubseteq_{\mathbb{S}}$; see (df $\mathsf{sum}_{\mathbb{S}}$) on p. 504.

1.3. The question of categoricity of the geometry of solids. In [21] Tarski presents without a proof the following theorems: "*The postulate system of the geometry of solids is categorical*", since "*The postulate system of geometry of solids, with the postulates of mereology included, has a model in ordinary three-dimensional geometry. To obtain such a model we interpreted spheres* [*balls*] *as interiors of Euclidian spheres* [*balls*], *and the relation of a part to whole as the inclusion relation restricted to non-empty regular open set*" [21, theorems A and B].

In §12, p. 511, we will prove the following theorem that concerns T-structures.

Theorem 1.5. *For any* T*-structure* $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ *the structure* $\langle \mathbb{S}, \mathbb{B}, \sqsubseteq_{\mathbb{S}} \rangle$ *is isomorphic to* $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq \rangle$.

Thanks to this we will be able to prove the following theorem which together with Theorem 1.1 are strongly connected with the above statements from [21].

Theorem 1.6. *All* $\mathrm{T}^\star$*-structures are isomorphic, since any* $\mathrm{T}^\star$*-structure is isomorphic to* $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq \rangle$.

By Theorems 1.1 and 1.6, the theory of $\mathrm{T}^\star$-structures is categorical. Moreover

Corollary 1.7. $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ *is a* $\mathrm{T}^\star$*-structure iff it is isomorphic to* $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq \rangle$.

Remark 1.1. Since the structures $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq \rangle$ and $\langle \mathbf{RC}^+_{\mathbb{R}^3}, \mathbf{BC}_{\mathbb{R}^3}, \subseteq \rangle$ are isomorphic, from a mereological point of view «real» solids are neither open nor closed (in the sense of topology). The notions of *interior of solid* and of *border* (*fringing*) *of solid* are abstract—they do not refer to solids (first-order objects) but to abstract sets consisted of abstract points. ⊣

Since all $\mathrm{T}^\star$-structures are T'-structures and all T'-structures are T-structures, by Theorem 1.2 and Corollary 1.7 we obtain that the theories of T'-structures and of T-structures are not categorical.

1.4. The problem of transitivity of the relation of concentricity of balls. A point is the set of all balls which are concentric with a given ball. Reflexivity and symmetry of the relation of concentricity of balls are implicit in its definition. A natural question arises at this point: *Is this relation transitive*?

In [21] Tarski does not assume explicitly that the concentricity relation is transitive, i.e., that the points are *equivalence classes* of this relation. Tarski uses the term 'class' as a synonym for the term 'set', but not the term 'equivalence class'. However, we show that the relation of concentricity of balls is transitive in all T-structures (see §11). Nonetheless it must be said that this is by far nontrivial result which we prove after plenty of preparation.

Let us notice that transitivity of the concentricity relation of mereological balls in all T-structures can be proven differently as well. By Theorem 1.6, in all T*-structures this relation is isomorphic to the relation of concentricity of ordinary balls in three-dimensional Euclidean geometry. Thus in all T*-structures the relation of concentricity is transitive and the points are equivalence classes of this relation. Hence, by Theorem 1.4, transitivity holds in all T-structures.

We must say here that there has been some substantial work done in region-based geometries based on Tarski's ideas and the results can be found in [1, 2]. However, our aim is different. There the authors construct their own axiom systems different from those proposed by Tarski. For example, in [1, 2] the authors introduce an axiom saying that the relation of concentricity of balls is transitive. Our goal is to examine Tarski's theory as it is presented in his paper.

1.5. An alternative formulation of the geometry of solids. On the basis of considerations from §1.1 and according to Tarski's intentions, T*-structures can be viewed as structures of the form $\langle \mathbb{S}, \mathbb{B}, \sqsubseteq \rangle$ where

(i) $\langle \mathbb{S}, \sqsubseteq \rangle$ is a mereological structure,
(ii) $\mathbb{B}$ is a distinguished subset of $\mathbb{S}$.

Members of $\mathbb{S}$ are called *solids* and members of $\mathbb{B}$ *balls* (in the mereological sense). An arbitrary T*-structure is to satisfy Postulates 1–3 and members of its universe are to have a property expressed in Definition 8, i.e., the analyzed structure is to satisfy the following condition

($\star_{\mathbb{S}}$) *Any solid is a mereological sum of some balls.*

In §14 we will demonstrate that the two approaches to the theory of T*-structures are definitionally equivalent (see Theorem 14.4).

1.6. On authors' approach to the problem. In this paper we do not confine ourselves to examining the theory of T*-structures (which are suitable for the geometry of solids). On the contrary, we want to comprehensively investigate the theory of T-structures and its relationship to that of T*-structures.

Some of the results obtained for T-structures are still relevant for proving important facts about T*-structures. And although these facts could be obtained in an easier way directly for T*-structures than for T-structures, it is very interesting that they also hold for the latter. Moreover, thanks to these facts we are able to prove some simple consequences concerning T$'$-structures and/or T*-structures.

1.7. On distinct paradigms of geometry. For obvious reasons in this paper we accept the paradigm of contemporary point-based geometry. Thus we construct proper models of theories of, respectively, T-structures, T$'$-structures and T*-structures on the basis of Cartesian model of point-based, three-dimensional Euclidean geometry (see Theorems 1.1, 1.2 and 1.3).

However, one of the aims of this work is to show that geometry could be ruled by a distinct paradigm founded on our «mereo-geometrical» intuitions. In such a case the notions of point-based ordinary geometry could be explained by means of those within pointless geometry of solids. But then the physical space would have to be a model of this theory since it would be difficult to construct a model that is not founded on some point-based construction.

1.8. The logic of the paper. In this work we examine some class of structures by means of the standard methods used in mathematics. These methods are formal but are not formalized. Our approach is similar to the one of a topologist examining some class of topological spaces. She can do this in two ways

1. either treating topological spaces as pairs $\langle X, \mathscr{O}\rangle$, where $\mathscr{O} \in 2^{2^X}$;
2. or treating them as pairs $\langle X, \mathrm{Int}\rangle$ (or $\langle X, \mathrm{Cl}\rangle$), where $\mathrm{Int}, \mathrm{Cl}\colon 2^X \to 2^X$.

In the first case the family $\mathscr{O}$ has the type $((*))$ in the sense of the simple theory of types (individuals, i.e., points, have the type $*$). Therefore in a formalized theory of topological spaces we can use the unary predicate 'is an open set', which applies to variables ranging over objects of the the type $(*)$. In the second case operators Int and Cl have the type $((*),(*))$ in the simple theory of types (they are relations, so they belong to $2^{2^X \times 2^X}$). So in a formalized theory we use the unary function constant 'the interior of the set' (resp. 'the closure of the set') that applies to variables of the second order and of the type $(*)$. It is worth noticing that such a constant produces terms of the same type. Thus we can see that, in both cases, in order to come up with a proper formalized theory we need a monadic second order logic that has variables of the type $*$, and variables and terms of the type $(*)$.

In case we wanted to formalize Tarski's system of geometry of solids, we would need a monadic third-order logic with variables and terms of the types $*$, $(*)$ and $((*))$. Let us remind that we examine the triple $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq\rangle$ where $\mathbb{R}$ is an arbitrary non-empty set, $\mathbb{B}$ is a subset of $\mathbb{R}$ and $\sqsubseteq$ is a binary relation in $\mathbb{R}$. The primitive notions $\mathbb{R}$ and $\mathbb{B}$ have the type $(*)$, while $\sqsubseteq$ has the type $(*,*)$ (in the sense of the simple theory of types). Therefore, in the formalized theory we need to have two unary relation constants 'is a region' ('$\mathbb{R}$') and 'is a ball' ('$\mathbb{B}$'), and one binary relation constant 'is an ingrediens of' ('$\sqsubseteq$') which apply to variables of the type $*$. The axioms of mereology itself require variables of the type $(*)$ (we define the binary predicate 'is a sum of the set' of the type $(*, (*))$; see the relation sum). Once having introduced variables of the second order of the type $(*)$, constants '$\mathbb{R}$' and '$\mathbb{B}$' become terms of the same type. Since points are sets of (concentric) balls, then we express them by means of the variables of the type $(*)$ and the definable relation constant 'is a point' ('Π'). Finally, while using the theory of Pieri's structures we have to speak about different sets of points. Therefore we need variables of the third order and of the type $((*))$. In such a case the constant 'Π' becomes a term of the third order.

Clearly, within such a theory we would use various individual, relation and function constants of types that were created from the types $*$, $(*)$ and $((*))$.

For example, we would have to use the ternary relational constant 'points ... and ... are equidistant from a point ... ' ('Δ^{Π}'), the unary function constant 'the interior of a region' ('$\mathbb{int}$') and the unary relation constant 'is a regular open set' ('$\mathbf{RO}_{\Pi}$'). These would be, respectively, of the types $((*), (*), (*))$, $(*, ((*)))$ and $(((*)))$.

In the whole paper we use the description operator 'ι' forming the expression '$(\iota\, x)\, \varphi(x)$' which is the individual constant 'the only object x such that $\varphi(x)$'. To use it, first we have to prove that there exists exactly one object x such that $\varphi(x)$, formally: $\exists!_x\, \varphi(x)$, i.e., the formula $\varphi(x)$ must fulfill the following two conditions:

$$\exists_x\, \varphi(x),$$
$$\forall_{x,y}\big(\varphi(x) \wedge \varphi(y) \Rightarrow x = y\big).$$

For example, the regions $\mathbb{1}$ and $\mathbb{s}$ are the only sums of all regions and of all balls, respectively. Formally $\mathbb{1} := (\iota\, x)\ x \text{ sum } \mathbb{R}$ and $\mathbb{s} := (\iota\, x)\ x \text{ sum } \mathbb{B}$, by $(\exists! \text{ sum})$ on p. 530, since $\mathbb{R} \neq \emptyset \neq \mathbb{B}$, by (8.1) on p. 501.

§2. Points and the ternary equidistance relation. By means of three primitive notions we will introduce definitions of some derived notions, among which the most important will be the notions of *point* and of *equidistance of two points from a third.*

REMARK 2.1. From the axiom (P1), i.e., Postulate 1 in [21] (see pp. 482 and 500), it follows that $|\mathbb{B}| \geqslant \mathfrak{c}$, where $|\mathbb{B}|$ is the cardinal number of $\mathbb{B}$ (see (8.1) on p. 501). Let us notice that all definitions presented in this section are correct even if $\mathbb{B} = \emptyset$. In such a case defined notions simply have empty extensions. ⊣

Further we assume the numeration of definitions deriving from [21].

DEFINITION 1. A ball a is *externally tangent* to a ball b iff (i) a is external to b and (ii) for any balls x and y such that a is ingrediens of both x and y, while x and y are external to b, it is the case that x is an ingrediens of y or y is an ingrediens of x (see Figure 1). Formally

$$a \mathrel{\mathbb{ET}} b \overset{\text{df}}{\iff} a, b \in \mathbb{B} \wedge a \wr b \wedge \forall_{x,y \in \mathbb{B}}(a \sqsubseteq x \wr b \wedge a \sqsubseteq y \wr b \Rightarrow x \sqsubseteq y \vee y \sqsubseteq x).$$

REMARK 2.2. In case balls are balls of ordinary geometry then the relation of external tangency is of course symmetrical. It is worth noticing that it does not follow from the above definition and the axioms of mereology that $\mathbb{ET}$ has the property of being symmetrical. Consider the mereological structure generated by four pairwise disjoint non-empty sets A, B, C and D, i.e., $\mathbb{R} := \big\{S \cup X \cup Y \cup Z \mid S, X, Y, Z \in \{A, B, C, D\}\big\}$ and $\sqsubseteq\ :=\ \subseteq$. Of course in this structure: $X \wr Y$ iff $X \cap Y = \emptyset$. Moreover, we put $\mathbb{B} := \{A, B, A \cup C, A \cup D\}$. Clearly, $B \mathrel{\mathbb{ET}} A$, since $B \wr A$ and there are no elements

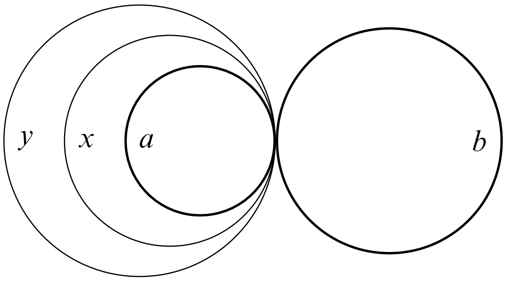


FIGURE 1. a is externally tangent to b.

of $\mathbb{B}$ different from B and such that B is their subset. But it is not the case that $A \,\mathbb{E}\mathbb{T}\, B$, since $A \sqsubseteq A \cup C$, $A \sqsubseteq A \cup D$, $B \wr A \cup C$, $B \wr A \cup D$, $A \cup C \not\sqsubseteq A \cup D$ and $A \cup D \not\sqsubseteq A \cup C$.

Clearly, to be symmetrical $\mathbb{E}\mathbb{T}$ would have to satisfy the following condition

$$a \,\mathbb{E}\mathbb{T} b \iff a, b \in \mathbb{B} \wedge a \wr b \wedge \forall_{x,y \in \mathbb{B}} \big[\big((a \sqsubseteq x \wr b \wedge a \sqsubseteq y \wr b) \vee (b \sqsubseteq x \wr a \wedge b \sqsubseteq y \wr a) \big) \Longrightarrow x \sqsubseteq y \vee y \sqsubseteq x \big].$$

The above construction proves that we should be very careful about assuming that elements of $\mathbb{B}$ share all the properties of ordinary balls known from three-dimensional geometry. They should of course, since Tarski's system is aimed at capturing Euclidean geometry. But, as we show it further in our work, even after introducing other definitions and axioms of Tarski, his system is still too weak to assure that elements of $\mathbb{B}$ are true balls. To do this one has to add an additional axiom. Thus all our drawings serve only as heuristic devices that are to help the reader familiarize herself with complicated definitions and clarify intuitions hidden in them.

As for the logical status of Definitions 1–5 see also Remark 2.3 and §18. ⊣

DEFINITION 2. A ball a is *internally tangent* to a ball b iff (i) a is a part of b and (ii) for any balls x and y, of which a is an ingrediens and which are ingredienses of b, either x is an ingrediens of y or conversely (see Figure 2). Formally

$$a \,\mathbb{I}\mathbb{T}\, b \overset{\mathrm{df}}{\iff} a, b \in \mathbb{B} \wedge a \sqsubset b \wedge \forall_{x,y \in \mathbb{B}} (a \sqsubseteq x \sqsubseteq b \wedge a \sqsubseteq y \sqsubseteq b \Rightarrow x \sqsubseteq y \vee y \sqsubseteq x).$$

DEFINITION 3. Balls a and b are *externally diametrically tangent* to a ball c iff (i) both a and b are externally tangent to c and (ii) for any balls x and y external to c and such that a is an ingrediens of x and b is an ingrediens of y, x is external to y (see Figure 3). Formally

$$ab \,\mathbb{E}\mathbb{D}\, c \overset{\mathrm{df}}{\iff} a, b, c \in \mathbb{B} \wedge a \,\mathbb{E}\mathbb{T}\, c \wedge b \,\mathbb{E}\mathbb{T}\, c \wedge \forall_{x,y \in \mathbb{B}} (a \sqsubseteq x \wr c \wedge b \sqsubseteq y \wr c \Rightarrow x \wr y).$$

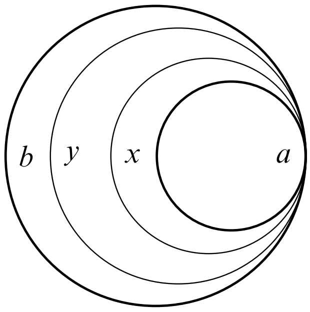


FIGURE 2. a is internally tangent to b.

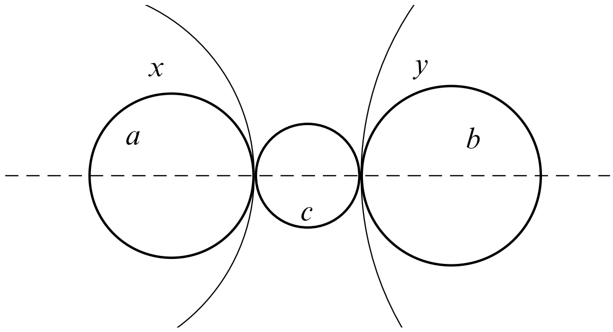


FIGURE 3. a and b are externally diametrically tangent to c.

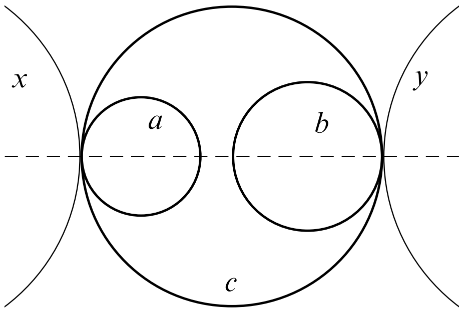


FIGURE 4. a and b are internally diametrically tangent to c.

DEFINITION 4. Balls a and b are *internally diametrically tangent* to a ball c iff (i) both a and b are internally tangent to c and (ii) for any balls x and y external to c and such that a is externally tangent to x and b is externally tangent to y, x is external to y (see Figure 4). Formally

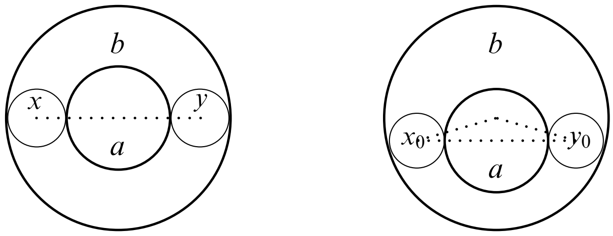
$$\begin{aligned} ab \mathbin{\mathbb{ID}} c \overset{\text{df}}{\iff} {}& a, b, c \in \mathbb{B} \wedge a \mathbin{\mathbb{IT}} c \wedge b \mathbin{\mathbb{IT}} c \\ & \wedge \forall_{x,y \in \mathbb{B}} (x \wr c \wedge y \wr c \wedge a \mathbin{\mathbb{ET}} x \wedge b \mathbin{\mathbb{ET}} y \Rightarrow x \wr y). \end{aligned}$$

DEFINITION 5. A ball a is *concentric* with a ball b iff there holds one of the following (mutually exclusive) conditions

(i) a is identical with b;
(ii) a is a part of b and for any balls x and y, which are both externally tangent to a and internally tangent to b, x and y are internally diametrically tangent to b (see Figure 5);
(iii) b is a part of a and for any balls x and y, which are externally diametrically tangent to b and internally tangent to a, x and y are internally diametrically tangent to a.

Formally

$$\begin{aligned} a \circledcirc b \overset{\mathrm{df}}{\iff} {} & a, b \in \mathbb{B} \wedge \big[a = b \\ & \veebar \big(a \sqsubset b \wedge \forall_{x,y \in \mathbb{B}}(xy \,\mathbb{ED}\, a \wedge x \,\mathbb{IT}\, b \wedge y \,\mathbb{IT}\, b \Rightarrow xy \,\mathbb{ID}\, b)\big) \\ & \quad \veebar \big(b \sqsubset a \wedge \forall_{x,y \in \mathbb{B}}(xy \,\mathbb{ED}\, b \wedge x \,\mathbb{IT}\, a \wedge y \,\mathbb{IT}\, a \Rightarrow xy \,\mathbb{ID}\, a)\big)\big]. \end{aligned}$$



A B

FIGURE 5. (A) a is concentric with b; (B) a is not concentric with b.

By theses of mereology, there are no $x, a \in \mathbb{R}$ such that $x \sqsubset a$ and $x \wr a$. So there are no $x, y, a \in \mathbb{R}$ such that $xy \,\mathbb{ED}\, a$, $x \,\mathbb{IT}\, a$ and $y \,\mathbb{IT}\, a$. Thus for any $a, b \in \mathbb{R}$

$$\begin{aligned} a \circledcirc b \iff {} & a, b \in \mathbb{B} \wedge \big(a \sqsubseteq b \wedge \forall_{x,y \in \mathbb{B}}(xy \,\mathbb{ED}\, a \wedge x \,\mathbb{IT}\, b \wedge y \,\mathbb{IT}\, b \Rightarrow xy \,\mathbb{ID}\, b)\big) \\ & \vee \big(b \sqsubseteq a \wedge \forall_{x,y \in \mathbb{B}}(xy \,\mathbb{ED}\, b \wedge x \,\mathbb{IT}\, a \wedge y \,\mathbb{IT}\, a \Rightarrow xy \,\mathbb{ID}\, a)\big). \end{aligned}$$

Clearly, the relation $\circledcirc$ is reflexive and symmetrical, i.e.,

$$\forall_{b \in \mathbb{B}}\, b \circledcirc b, \tag{$\mathrm{r}_{\circledcirc}$}$$

$$\forall_{a,b \in \mathbb{B}}(a \circledcirc b \iff b \circledcirc a). \tag{$\mathrm{s}_{\circledcirc}$}$$

For the transitivity of $\circledcirc$ see Remark 2.3 and §§1.4 and 11.

DEFINITION 6. By a *point* we mean the set of all those balls that are concentric with a given ball.

Let Π be the set of all points, i.e.,

$$\beta \in \Pi \overset{\mathrm{df}}{\iff} \exists_{b \in \mathbb{B}}\, \beta = \{x \in \mathbb{B} \mid x \circledcirc b\}. \tag{df Π}$$

For any ball b, let π_b be the point determined by b, i.e.,

$$\pi_b := \{x \in \mathbb{B} \mid x \circledcirc b\}.$$

Notice that by $(\mathrm{r}_{\circledcirc})$

$$\forall_{b\in\mathbb{B}}\, b \in \pi_b \in \Pi, \tag{2.1}$$

$$\forall_{\beta\in\Pi}\, \exists_{b\in\mathbb{B}}\, \beta = \pi_b, \tag{2.2}$$

$$\Pi \neq \emptyset \iff \mathbb{B} \neq \emptyset. \tag{2.3}$$

Remark 2.3. Further in the paper—while describing the theory of T-structures—until §11 we will use the facts that the relation $\circledcirc$ is reflexive and symmetrical. We do not assume however anything about its transitivity, since the fact that $\circledcirc$ is transitive will be proven by us not earlier than in §11 and will be based only upon results obtained in §9 and §10.

Moreover, we will make use of the definition of the relation $\circledcirc$ in two places only. Firstly, in the proof of Fact 8.2(i) we use Definition 5, but without any reference to the relations $\mathbb{ET}$, $\mathbb{IT}$ and $\mathbb{ED}$; that is, without using Definitions 1–3. Secondly, in the proof of Theorem 13.2 we use the fact that if $a \circledcirc b$, then either $a \sqsubseteq b$ or $b \sqsubseteq a$.

The reader may thus have the following «philosophical questions» : *What is the role of the accepted definition of the relation* $\circledcirc$, *that is, what is the role of Definitions* 1–5 *in the theory of* T-*structures*? In other words: *What would happen, if we defined* $\circledcirc$ *in a way that is incompatible with our mereo-geometrical intuitions*?

The answer to the above question is: the result would be the empty theory of T-structures, i.e., the theory having no model and thus inconsistent. The theory of structures $\langle\mathbb{R}, \mathbb{B}, \sqsubseteq\rangle$ that were to satisfy only (P0) and (P1) would be inconsistent as well. It is in §15.1 where we use Definitions 1–5 while demonstrating that $\langle\mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq\rangle$ is a T*-structure. Other T*-structures are isomorphic to it. Moreover, using these structures as an initial point we construct models of the theory of T-structures which are not T*-structures.

Of course this problem demands a more precise explanation and we present one in §18. There we prove the theorem which will show that the only way to define the concentricity relation in T-structures is to accept a definition (or definitions) that would be equivalent to Definition 5 (or Definitions 1–5). $\dashv$

Since the function $\mathbb{B} \ni b \mapsto \pi_b \in \Pi$ is *onto* the set Π, we obtain that

$$|\Pi| \leqslant |\mathbb{B}| \leqslant |\mathbb{S}| \leqslant |\mathbb{R}|. \tag{2.4}$$

Definition 7. Points α and β are *equidistant* from a point γ iff $\alpha = \beta = \gamma$ or there exists a ball in γ such that no ball from α or β either is an ingrediens of or is exterior to this ball (see Figure 6).

Formally, let $\Delta^\Pi \subseteq \Pi \times \Pi \times \Pi$ and let for $\alpha, \beta, \gamma \in \Pi$

$$\alpha\beta\, \Delta^\Pi\, \gamma \overset{\mathrm{df}}{\iff} \alpha = \beta = \gamma \vee \exists_{c\in\gamma}\, \neg\exists_{a\in\alpha\cup\beta}\big(a \sqsubseteq c \vee a \wr c\big). \tag{df Δ^Π}$$

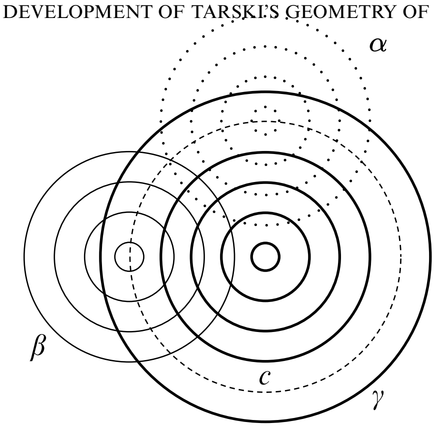


FIGURE 6. α and β are equidistant from γ.

REMARK 2.4. (i) In [21], in the definition of the relation Δ^{Π} Tarski did not take into account the expression '$\alpha = \beta = \gamma$'. His definition is equivalent to the sentence below

$$\alpha\beta\, \Delta^{\Pi}\, \gamma \overset{\text{df}}{\iff} \exists_{c\in\gamma} \neg\exists_{a\in\alpha\cup\beta}\big(a \sqsubseteq c \vee a \wr c\big).$$

Unfortunately, such a definition leads to a contradiction in Tarski's theory. The first one of the specific axioms of the geometry of solids asserts that the set Π together with the equidistance relation Δ^{Π} satisfy all the postulates of Pieri's geometry (cf. Postulate 1 on p. 482, and (P1) on p. 500). According to the third axiom of Pieri the relation of equidistance is reflexive on its two first arguments (cf. [15]), i.e., $\forall_{\alpha,\beta\in\Pi}\, \beta\beta\, \Delta^{\Pi}\, \alpha$. Hence we obtain the following particular instance of the above axiom: $\forall_{\beta\in\Pi}\, \beta\beta\, \Delta^{\Pi}\, \beta$. According to $(\mathrm{df}\,\Delta^{\Pi})$, $\beta\beta\, \Delta^{\Pi}\, \beta$ means that for some $b_0 \in \beta$, no ball from β either is an ingrediens of or is exterior to b_0. However, $b_0 \sqsubseteq b_0$, by $(\mathrm{r}_{\sqsubseteq})$, so we get a contradiction. Thus we should reformulate Tarski's original definition to get rid of the above inconsistency. It seems that including an alternative part '$\alpha = \beta = \gamma$' is the simplest solution.

(ii) From the corrected definition $(\mathrm{df}\,\Delta^{\Pi})$ we obtain the fourth and sixth axioms of Pieri, i.e., for any $\alpha, \beta, \gamma \in \Pi$

$$\alpha\beta\, \Delta^{\Pi}\, \gamma \Longrightarrow \beta\alpha\, \Delta^{\Pi}\, \gamma\,, \tag{IV}$$

$$\alpha\beta\, \Delta^{\Pi}\, \alpha \Longrightarrow \alpha = \beta\,. \tag{VI}$$

(iii) Yet we do not obtain the other axioms; for example, we cannot prove that $\Pi \neq \emptyset$, so by (2.3), we also cannot demonstrate that $\mathbb{B} \neq \emptyset$. ⊣

§3. Balls as ingredienses of regions. For any region x we put

$$\mathbb{B}_x := \{b \in \mathbb{B} \mid b \sqsubseteq x\}, \tag{$\mathrm{df}\,\mathbb{B}_x$}$$

i.e., $\mathbb{B}_x$ is the set of all balls that are ingrediens es of x. We have that

$$\forall_{x\in\mathbb{R}}\big(x \mathrel{\mathsf{sum}} \mathbb{B}_x \iff \exists_{Y\in 2^{\mathbb{B}}}\, x \mathrel{\mathsf{sum}} Y\big). \tag{3.1}$$

Indeed, let $x \mathrel{\mathsf{sum}} Y_0$, for some $Y_0 \subseteq \mathbb{B}$. Then, by (df sum), $\emptyset \neq Y_0 \subseteq \mathbb{B}_x$ and, by ($\exists$ sum), for some $y_0 \in \mathbb{R}$ we have $y_0 \mathrel{\mathsf{sum}} \mathbb{B}_x$. Moreover, $Y_0 \subseteq \mathbb{B}_x \subseteq \{y \in \mathbb{R} \mid y \sqsubseteq x\}$. Thus, by (A.19) and (A.20), we obtain $x \sqsubseteq y_0 \sqsubseteq x$. Hence $x = y_0$, by ($\mathrm{as}_{\sqsubseteq}$).

Lemma 3.1. *The following sentences are equivalent*

($\star$) $\forall_{x\in\mathbb{R}}\,\exists_{Y\in 2^{\mathbb{B}}}\, x \mathrel{\mathsf{sum}} Y$.
($\star\star$) $\forall_{x\in\mathbb{R}}\,\exists_{b\in\mathbb{B}}\, b \sqsubseteq x$, *i.e.*, $\forall_{x\in\mathbb{R}}\,\mathbb{B}_x \neq \emptyset$.
(§) $\forall_{x\in\mathbb{R}}\, x \mathrel{\mathsf{sum}} \mathbb{B}_x$.

Proof. "($\star$) $\Leftrightarrow$ (§)" By (3.1). "($\star$) $\Rightarrow$ ($\star\star$)" By (df sum).

"($\star\star$) $\Rightarrow$ (§)" Assume that for some $x_0 \in \mathbb{R}$ we have $\neg x_0 \mathrel{\mathsf{sum}} \mathbb{B}_{x_0}$. Since it is the case that $\forall_{b\in\mathbb{B}_{x_0}}\, b \sqsubseteq x_0$, then (by (df sum)) for some $y_0 \in \mathbb{R}$: $y_0 \sqsubseteq x_0$ and $y_0 \wr b$, for all $b \in \mathbb{B}_{x_0}$. By ($\star\star$), for some $b_0 \in \mathbb{B}$: $b_0 \sqsubseteq y_0$. Hence $b_0 \sqsubseteq x_0$, by transitivity of $\sqsubseteq$. Thus $b_0 \in \mathbb{B}_{x_0}$; so $y_0 \wr b_0$, which contradicts $b_0 \sqsubseteq y_0$. $\dashv$

§4. Mereological solids. Let us remind that in Tarski's theory the notion of *mereological solid* is defined as follows.

Definition 8. A *solid* is an arbitrary sum of balls.

Let $\mathbb{S}$ be the set of all (mereological) solids as defined by (df $\mathbb{S}$) on page 485.

Notice that $\mathbb{B} \subseteq \mathbb{S}$, because for any $b \in \mathbb{B}$, $b \mathrel{\mathsf{sum}} \{b\}$, by (A.19), and $b \mathrel{\mathsf{sum}} \mathbb{B}_b$, by (3.1). Hence and by (df sum) we obtain

$$\mathbb{B} \neq \emptyset \iff \mathbb{S} \neq \emptyset. \tag{4.1}$$

Moreover, respectively by (df $\mathbb{S}$), (3.1) and (df sum), we have

$$\forall_{s\in\mathbb{S}}\,\exists_{b\in\mathbb{B}}\, b \sqsubseteq s, \qquad \text{i.e., } \forall_{s\in\mathbb{S}}\,\mathbb{B}_s \neq \emptyset, \tag{4.2}$$

$$\forall_{s\in\mathbb{S}}\, s \mathrel{\mathsf{sum}} \mathbb{B}_s. \tag{4.3}$$

Lemma 4.1. *For any* $X \subseteq \mathbb{S}$ *and* $z \in \mathbb{R}$

$$z \mathrel{\mathsf{sum}} X \iff z \mathrel{\mathsf{sum}} \textstyle\bigcup_{s\in X} \mathbb{B}_s.$$

Proof. Let $z \mathrel{\mathsf{sum}} X$, i.e., by (df sum), $\forall_{s\in X}\, s \sqsubseteq z$ and $\forall_{y\in\mathbb{R}}\big(y \sqsubseteq z \Rightarrow \exists_{s\in X}\,\exists_{u\in\mathbb{R}}(u \sqsubseteq y \wedge u \sqsubseteq s)\big)$. Let $Y := \bigcup_{s\in X} \mathbb{B}_s$. We will show that $z \mathrel{\mathsf{sum}} Y$, i.e., $\forall_{b\in Y}\, s \sqsubseteq z$ and $\forall_{y\in\mathbb{R}}\big(y \sqsubseteq z \Rightarrow \exists_{b\in Y}\,\exists_{u\in\mathbb{R}}(u \sqsubseteq y \wedge u \sqsubseteq b)\big)$. Firstly, let $b \in Y$, i.e., for some $s_0 \in X$, $b \in \mathbb{B}_{s_0}$. Then $b \sqsubseteq s_0 \sqsubseteq z$, so by transitivity of $\sqsubseteq$ we have that $b \sqsubseteq z$. Secondly, let $y \sqsubseteq z$. Then for some $s_0 \in X$ and $u_0 \in \mathbb{R}$ we have $u_0 \sqsubseteq y$ and $u_0 \sqsubseteq s_0$. By (4.3), $s_0 \mathrel{\mathsf{sum}} \mathbb{B}_{s_0}$. Therefore, $\forall_{y\in\mathbb{R}}\big(y \sqsubseteq s_0 \Rightarrow \exists_{b\in\mathbb{B}_{s_0}}\exists_{u\in\mathbb{R}}(u \sqsubseteq y \wedge u \sqsubseteq b)\big)$. Hence, because $u_0 \sqsubseteq s_0$, for some $b_1 \in \mathbb{B}_{s_0} \subseteq Y$ and $u_1 \in \mathbb{R}$ we obtain that $u_1 \sqsubseteq u_0 \sqsubseteq y$ and $u_1 \sqsubseteq b_1$. So by transitivity of $\sqsubseteq$ we have $u_1 \sqsubseteq y$. Thus $z \mathrel{\mathsf{sum}} Y$.

Now we suppose that z sum $\bigcup_{s\in X} \mathbb{B}_s$. Then $X \neq \emptyset$ and, by ($\exists!$ sum), there is exactly one x_0 such that x_0 sum X. But also x_0 sum $\bigcup_{s\in X} \mathbb{B}_s$. Hence $x_0 = z$, by (fun sum). ⊣

LEMMA 4.2. (i) *For any* $\emptyset \neq X \subseteq \mathbb{S}$, $(\imath\, z)\ z$ sum X *is a solid.*
(ii) $\mathsf{sum} \cap (\mathbb{R} \times 2^{\mathbb{S}}) = \mathsf{sum} \cap (\mathbb{S} \times 2^{\mathbb{S}})$.

PROOF. (i) Let $\emptyset \neq X \subseteq \mathbb{S}$. Then, by ($\exists!$ sum), there is exactly one $z \in \mathbb{R}$ such that z sum X. Thus, by Lemma 4.1, z sum $\bigcup_{s\in X} \mathbb{B}_s$. So $x \in \mathbb{S}$. (ii) By (i). ⊣

§5. Space and super-space. According to the properties of mereological structures, there exists the unity $\mathbb{1}$ of a structure $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq\rangle$ satisfying (P0).[11] The unity is the mereological sum of all regions from $\mathbb{R}$, i.e.,

$$\mathbb{1} := (\imath\, x)\ x \ \mathsf{sum}\ \mathbb{R} = \sup_{\sqsubseteq} \mathbb{R},$$
$$\forall_{x\in\mathbb{R}}\, x \sqsubseteq \mathbb{1}.$$

In §16 (see also Theorem 1.2 in §1.2) we will construct some T′-structure in which $\mathbb{1} \notin \mathbb{S} \neq \emptyset$. Thus it is reasonable to introduce in the theory of T-structures the following definition

$$\mathbb{s} := (\imath\, x)\ x \ \mathsf{sum}\ \mathbb{B} = \sup_{\sqsubseteq} \mathbb{B}. \tag{df $\mathbb{s}$}$$

Of course $\mathbb{s} \in \mathbb{S}$, as it is the mereological sum of all balls.

Let $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq\rangle$ be an arbitrary structure satisfying (P0), let $\mathbb{B} \neq \emptyset$ and $\mathbb{S} := \{z \in \mathbb{R} \mid \exists_{X\in 2^{\mathbb{B}}}\, z \ \mathsf{sum}\ X\}$. Then, obviously, $\mathbb{s} \sqsubseteq \mathbb{1}$ and when $\mathbb{1} \notin \mathbb{S}$, then $\mathbb{s} \sqsubset \mathbb{1}$, because $\mathbb{s} \in \mathbb{S}$. Moreover

$$\forall_{s\in\mathbb{S}}\, s \sqsubseteq \mathbb{s}, \tag{5.1}$$
$$\mathbb{s}\ \mathsf{sum}\ \mathbb{S}. \tag{5.2}$$

Indeed, if $s \in \mathbb{S}$, then s sum $\mathbb{B}_s$, by (4.3). But $\mathbb{s}$ sum $\mathbb{B}$ and $\mathbb{B}_s \subseteq \mathbb{B}$. Hence $s \sqsubseteq \mathbb{s}$, by (A.20). Moreover, if $x \sqsubseteq \mathbb{s}$, then for some $b \in \mathbb{B} \subseteq \mathbb{S}$: $b \bigcirc x$, since $\mathbb{s}$ sum $\mathbb{B}$. Thus $\mathbb{s}$ sum $\mathbb{S}$, by (df sum).

We will call $\mathbb{1}$ and $\mathbb{s}$, respectively, *super-space* and *space*.

Notice that

$$\mathbb{s} = \mathbb{1} \iff \mathbb{1} \in \mathbb{S}. \tag{5.3}$$

Indeed, if $\mathbb{1} \in \mathbb{S}$, then by (df $\mathbb{S}$), for some $X_0 \subseteq \mathbb{B}$ we have $\mathbb{1}$ sum X_0. Hence $\mathbb{1} \sqsubseteq \mathbb{s}$, by (df $\mathbb{s}$) and (A.20). Therefore, $\mathbb{s} = \mathbb{1}$, by $\mathbb{s} \sqsubseteq \mathbb{1}$ and (as$_{\sqsubseteq}$).

§6. Interior points of regions. Tarski defined the notion of *an interior point* only for solids (cf. [21, p. 27]). Our definition is more general. We do not assume (in this place) that the universe $\mathbb{R}$ contains only solids.

[11] It follows from (P0) that $\mathbb{1} = \mathbf{1}$, where $\mathbf{1}$ is the unity of a Boolean algebra $\langle A, +, \cdot, -, \mathbf{0}, \mathbf{1}\rangle$.

DEFINITION 9. A point β is an *interior point* of a region x iff there exists a ball $b \in \beta$ such that $b \sqsubseteq x$ (see Figure 7).

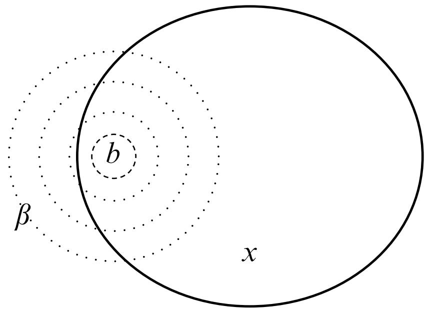


FIGURE 7. β is an interior point of x.

Let $\mathtt{int}\colon \mathbb{R} \to 2^{\Pi}$ be the operation which ascribes to every object from $\mathbb{R}$ the set of its interior points, i.e., for any $x \in \mathbb{R}$

$$\mathtt{int}(x) := \{\beta \in \Pi \mid \exists_{b \in \beta}\, b \sqsubseteq x\}. \tag{df int}$$

REMARK 6.1. Clearly, one should not confound the operation $\mathtt{int}$ with the ordinary Int operation of point-based topology. The first one ascribes to first-order objects (members of $\mathbb{R}$) sets of points, which are sets of sets of elements of $\mathbb{R}$ (members of $2^{2^{\mathbb{R}}}$). In an ordinary topology an operation Int attributes to sets of points (subsets of the universe) sets of points, i.e., it is of the form $\mathrm{Int}\colon 2^X \to 2^X$. ⊣

Since $\sqsubseteq$ is transitive and reflexive, then for any $x, y \in \mathbb{R}$

$$x \sqsubseteq y \Rightarrow \mathtt{int}(x) \subseteq \mathtt{int}(y), \tag{6.1}$$

$$\mathtt{int}(x) \cup \mathtt{int}(y) \subseteq \mathtt{int}(x \sqcup y), \tag{6.2}$$

$$x \bigcirc y \Longrightarrow \mathtt{int}(x \sqcap y) \subseteq \mathtt{int}(x) \cap \mathtt{int}(y), \tag{6.3}$$

where $\sqcup$ and $\sqcap$ are binary operators in $\mathbb{R}$ introduced by (df$\sqcup$) and (df$\sqcap$). Indeed, $x \sqsubseteq x \sqcup y$, $y \sqsubseteq x \sqcup y$, $x \sqcap y \sqsubseteq x$ and $x \sqcap y \sqsubseteq y$.

Since $\sqsubseteq$ is reflexive, for any $x \in \mathbb{R}$

$$\mathtt{int}(x) \neq \emptyset \iff \mathbb{B}_x \neq \emptyset. \tag{6.4}$$

Indeed, let $\mathtt{int}(x) \neq \emptyset$. Then for some $\beta_0 \in \Pi$, $\beta_0 \in \mathtt{int}(x)$. Hence for some $b_0 \in \beta_0$, $b_0 \sqsubseteq x$. Thus $\mathbb{B}_x \neq \emptyset$. Conversely, let $\mathbb{B}_x \neq \emptyset$. Then for some $b_0 \in \mathbb{B}$, $b_0 \sqsubseteq x$. Hence, by (r$_\sqsubseteq$), $\pi_{b_0} \in \mathtt{int}(x)$. So $\mathtt{int}(x) \neq \emptyset$.

Moreover, by (4.2), (6.4) and (r$_\sqsubseteq$)

$$\forall_{s \in \mathbb{S}}\ \mathtt{int}(s) \neq \emptyset, \tag{6.5}$$

$$\forall_{\beta \in \Pi} \forall_{b \in \beta}\, \beta \in \mathtt{int}(b), \tag{6.6}$$

$$\forall_{b \in \mathbb{B}}\, \pi_b \in \mathtt{int}(b). \tag{6.7}$$

LEMMA 6.1. *The following sentences are equivalent.*

($\star\star$) $\forall_{x\in\mathbb{R}}\,\exists_{b\in\mathbb{B}}\, b \sqsubseteq x$, *i.e.*, $\forall_{x\in\mathbb{R}}\,\mathbb{B}_x \neq \emptyset$.

(\$) $\forall_{x\in\mathbb{R}}\,\mathtt{int}(x) \neq \emptyset$.

(¶) $\forall_{x,y\in\mathbb{R}}(x \bigcirc y \Rightarrow \mathtt{int}(x) \cap \mathtt{int}(y) \neq \emptyset)$.

PROOF. ($\star\star$) $\Leftrightarrow$ (\$) By (6.4).

(\$) $\Leftrightarrow$ (¶) Let us assume (\$) and $x \bigcirc y$. Then for some $z_0 \in \mathbb{R}$: $z_0 \sqsubseteq x$ and $z_0 \sqsubseteq y$. By (6.1) and (\$) we obtain that $\emptyset \neq \mathtt{int}(z_0) \subseteq \mathtt{int}(x) \cap \mathtt{int}(y)$. Conversely, assume (¶). For any $x \in \mathbb{R}$, by ($\mathrm{r}_\bigcirc$), we have that $x \bigcirc x$, so $\mathtt{int}(x) \neq \emptyset$. ⊣

LEMMA 6.2. *If* $\forall_{x,y\in\mathbb{R}}(\mathtt{int}(x) \subseteq \mathtt{int}(y) \Rightarrow x \sqsubseteq y)$, *then either* (\$) *or* $|\mathbb{R}| = 1$.

PROOF. If $\forall_{x,y\in\mathbb{R}}(\mathtt{int}(x) \subseteq \mathtt{int}(y) \Rightarrow x \sqsubseteq y)$ and for some $z_0 \in \mathbb{R}$, $\mathtt{int}(z_0) = \emptyset$; then $\forall_{y\in\mathbb{R}}\, z_0 \sqsubseteq y$. So $|\mathbb{R}| = 1$, by ($\nexists 0$). ⊣

From suitable definitions it follows that

$$\mathtt{int}(\mathbb{s}) = \Pi = \mathtt{int}(\mathbb{1}), \tag{6.8}$$

i.e., the set of all interior points of $\mathbb{1}$ and the set of all interior points of $\mathbb{s}$ coincide with the set of all mereological points.

§7. Fringe points of regions.

A definition of a fringe point of a given object is absent from Tarski's paper. But it is a «natural» consequence of definitions and notions from Tarski's work. It was introduced by Jaśkowski (cf. [11]).

DEFINITION 10. A point β is a *fringe point* of a region x from $\mathbb{R}$ iff no ball in β is either ingrediens of or is exterior to x (see Figure 8).

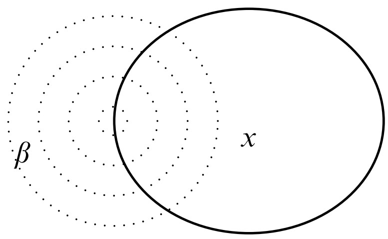


FIGURE 8. β is a fringe point of x.

Formally, we define the operation $\mathfrak{fr}\colon \mathbb{R} \to 2^{\Pi}$ such that for any $x \in \mathbb{R}$

$$\begin{aligned} \mathfrak{fr}(x) &:= \{\beta \in \Pi \mid \neg\exists_{b\in\beta}(b \sqsubseteq x \vee b \wr x)\} \\ &= \{\beta \in \Pi \mid \forall_{b\in\beta}(b \not\sqsubseteq x \wedge b \bigcirc x)\}. \end{aligned} \tag{df $\mathfrak{fr}$}$$

The following equivalence emphasizes the intuitive sense of (df Δ^{Π})

$$\alpha\beta\,\Delta^{\Pi}\,\gamma \iff \alpha = \beta = \gamma \vee \exists_{c\in\gamma}\big(\alpha \in \mathfrak{fr}(c) \wedge \beta \in \mathfrak{fr}(c)\big). \tag{df$'$ Δ^{Π}}$$

Directly from (df $\mathtt{int}$) and (df $\mathtt{fr}$) it follows that for any $x \in \mathbb{R}$

$$\mathtt{fr}(x) \cap \mathtt{int}(x) = \emptyset. \tag{7.1}$$

Hence, by (6.8), we have that

$$\mathtt{fr}(\mathbb{1}) = \emptyset = \mathtt{fr}(\mathtt{s}). \tag{7.2}$$

Moreover, for any $x, y \in \mathbb{R}$

$$\big(\mathtt{fr}(x) \cup \mathtt{fr}(y)\big) \setminus \mathtt{int}(x \sqcup y) \subseteq \mathtt{fr}(x \sqcup y). \tag{7.3}$$

Indeed, let $\beta \in \mathtt{fr}(x) \cup \mathtt{fr}(y)$ and $\beta \notin \mathtt{int}(x \sqcup y)$. Then $\forall_{b\in\beta}\, b \not\sqsubseteq x \sqcup y$ and either $\forall_{b\in\beta}\, b \bigcirc x$ or $\forall_{b\in\beta}\, b \bigcirc y$. So also $\forall_{b\in\beta}\, b \bigcirc x \sqcup y$. Thus $\beta \in \mathtt{fr}(x \sqcup y)$.

FACT 7.1. *For any* $x \in \mathbb{R}$

(i) *if* $x \neq \mathbb{1}$, *then* $\mathtt{fr}(x) = \mathtt{fr}(-x)$;
(ii) *if* $x \sqsubset \mathtt{s}$, *then* $\mathtt{fr}(x) = \mathtt{fr}(\mathtt{s} - x)$.

PROOF. (i) Let x be an arbitrary region such that $x \neq \mathbb{1}$. For any ball b: $b \not\sqsubseteq x$ iff $b \bigcirc -x$, and $b \bigcirc x$ iff $b \not\sqsubseteq -x$. Hence $\beta \in \mathtt{fr}(x)$ iff $\beta \in \mathtt{fr}(-x)$.

(ii) Let x be an arbitrary region such that $x \sqsubset \mathtt{s}$. Then there exist regions $-x$ and $\mathtt{s} - x$. Since $b \sqsubseteq \mathtt{s}$, for any ball b we obtain: $b \not\sqsubseteq x$ iff $b \bigcirc -x$ iff $b \bigcirc \mathtt{s} \sqcap -x = \mathtt{s} - x$, and $b \bigcirc x$ iff $b \not\sqsubseteq \mathtt{s} - x$. Hence $\beta \in \mathtt{fr}(x)$ iff $\beta \in \mathtt{fr}(\mathtt{s} - x)$. ⊣

§8. The specific postulates for T-structures. In this section we present the specific postulates of Tarski's geometry of solids together with some of their consequences that are relevant from a point of view of further development of this paper. In §13 we present an alternative version of Tarski's theory (suggested by him in [21]), and prove that it is stronger than the one presented below (see also §1.1).

8.1. The first postulate. The first one of the specific axioms of the geometry of solids states that points defined as sets of concentric balls are points of an ordinary point-based geometry, and that the relation Δ^{Π} is an ordinary equidistance relation. Formally we can put this axiom in the following way

$$\langle \Pi, \Delta^{\Pi} \rangle \text{ is a Pieri's structure.} \tag{P1}$$

Directly from the above axiom and Fact D.1 we obtain the following corollary.[12]

LEMMA 8.1. *The structure* $\langle \Pi, \Delta^{\Pi} \rangle$ *is isomorphic to the structure* $\langle \mathbb{R}^3, \Delta^{\mathbb{R}^3} \rangle$, *where for any* $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^3$ (*for* ϱ *see p.* 485)

$$\bar{x}\,\bar{y}\, \Delta^{\mathbb{R}^3}\, \bar{z} \overset{\text{df}}{\iff} \varrho(\bar{x}, \bar{z}) = \varrho(\bar{y}, \bar{z}).$$

[12] All the details concerning notation within Pieri's structures are explained in Appendix D.

By (P1) we have that the set of all mereological points has the power of continuum, i.e., $|\Pi| = \mathfrak{c}$. Hence, by (2.4), it is the case that

$$\mathfrak{c} = |\Pi| \leqslant |\mathbb{B}| \leqslant |\mathbb{S}| \leqslant |\mathbb{R}|. \tag{8.1}$$

Moreover, by $(\exists \wr)$, we have that

$$\exists_{x,y\in\mathbb{R}}\, x \wr y,$$
$$\exists_{a,b\in\mathbb{B}} \neg\, a \circledcirc b.$$

FACT 8.2. *Let* $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ *be a structure satisfying* (P0) *and* (P1). *Then*

(i) *the mereological sum of all balls is not a ball, i.e.,* $\mathbb{s} \notin \mathbb{B}$;

(ii) *for any* $b \in \mathbb{B}$ *it is the case that* $b \sqsubset \mathbb{s}$;

(iii) *the unity of the structure is not a ball, i.e.,* $\mathbb{1} \notin \mathbb{B}$.

PROOF. (i) Suppose that $\mathbb{s} \in \mathbb{B}$. We will consider two logical possibilities.

Firstly, suppose that $\neg a \circledcirc \mathbb{s}$ for some $a \in \mathbb{B}$, i.e., $a \sqsubset \mathbb{s}$ and for some $b, c \in \mathbb{B}$ we have $bc \,\mathbb{ED}\, a$, and $b \,\mathbb{IT}\, \mathbb{s}$, and $c \,\mathbb{IT}\, \mathbb{s}$, but $\neg bc \,\mathbb{ID}\, \mathbb{s}$. Since $\forall_{x\in\mathbb{B}}\, x \sqsubseteq \mathbb{s}$, then $\neg\, x \wr \mathbb{s}$ and it is the case that $\forall_{x,y\in\mathbb{B}}(x \wr \mathbb{s} \wedge y \wr \mathbb{s} \wedge b \,\mathbb{ET}\, x \wedge c \,\mathbb{ET}\, y \Rightarrow x \wr y)$. Thus, by the definition of $\mathbb{ID}$, either $\neg\, b \,\mathbb{IT}\, \mathbb{s}$ or $\neg\, c \,\mathbb{IT}\, \mathbb{s}$, a contradiction.

Secondly, suppose that for any $b \in \mathbb{B}$ we have $b \circledcirc \mathbb{s}$, i.e., $\pi_{\mathbb{s}} = \mathbb{B}$. Let $\alpha, \beta \in \Pi$ be arbitrary distinct points such that $\alpha \neq \pi_{\mathbb{s}} \neq \beta$. Since $\langle \Pi, \Delta^{\Pi} \rangle$ is a Pieri's structure, so for some $\gamma_0 \in \Pi$ we have $\alpha\beta\, \Delta^{\Pi}\, \gamma_0$. Hence for some $c_0 \in \gamma_0$, $\neg\exists_{a\in\alpha\cup\beta}\big(a \sqsubseteq c_0 \vee a \wr c_0\big)$. But $c_0 \in \pi_{\mathbb{s}}$, so $\alpha\beta\, \Delta^{\Pi}\, \pi_{\mathbb{s}}$. It follows that $\exists_{\gamma\in\Pi}\forall_{\alpha,\beta\in\Pi}(\alpha \neq \beta \Rightarrow \alpha\beta\, \Delta^{\Pi}\, \gamma)$, but this sentence is false in all Pieri's structures.

Thus $\mathbb{s} \notin \mathbb{B}$.

(ii) For any $b \in \mathbb{B}$, $b \sqsubseteq \mathbb{s}$. Moreover, by (i), we have $b \neq \mathbb{s}$; so $b \sqsubset \mathbb{s}$.

(iii) If $\mathbb{1} \in \mathbb{B}$, then $\mathbb{1} \sqsubseteq \mathbb{s}$, since $\mathbb{s}$ sum $\mathbb{B}$; so $\mathbb{1} = \mathbb{s} \notin \mathbb{B}$, by (i). Thus $\mathbb{1} \notin \mathbb{B}$. ⊣

8.2. Topology in the set of points. Since, by (P1), $\langle \Pi, \Delta^{\Pi} \rangle$ is a Pieri's structure, we can define the family of all open balls $\mathbf{BO}_{\Pi}$ in $\langle \Pi, \Delta^{\Pi} \rangle$ and by means of it we can introduce in Π the family $\mathscr{O}_{\Pi}$ of open sets and three standard topological operations Int_{Π}, Cl_{Π} and $\mathrm{Fr}_{\Pi} \colon 2^{\Pi} \to 2^{\Pi}$, where $\mathrm{Fr}_{\Pi} A := \mathrm{Cl}_{\Pi} A \setminus \mathrm{Int}_{\Pi} A$, for any $A \subseteq \Pi$. Thus $\langle \Pi, \mathscr{O}_{\Pi} \rangle$ is a topological space. Moreover, it has all topological properties of three-dimensional Euclidean space. In $\langle \Pi, \mathscr{O}_{\Pi} \rangle$ we introduce the family $\mathbf{RO}_{\Pi}$ of all regular open sets (open domains), i.e., the family of sets that are equal to the interior of their closure. Let $\mathbf{RO}^{+}_{\Pi} := \mathbf{RO}_{\Pi} \setminus \{\emptyset\}$.

LEMMA 8.3. *If* $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ *satisfies* (P0) *and* (P1), *then* $\langle \mathbf{RO}^{+}_{\Pi}, \mathbf{BO}_{\Pi}, \subseteq \rangle$ *is isomorphic to* $\langle \mathbf{RO}^{+}_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq \rangle$.

PROOF. Let us take an arbitrary structure $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ which satisfies (P0) and (P1) and let $\langle \Pi, \Delta^{\Pi} \rangle$ be a Pieri's structure for it. By Lemma 8.1 there is an isomorphism I from $\langle \Pi, \Delta^{\Pi} \rangle$ onto $\langle \mathbb{R}^3, \Delta^{\mathbb{R}^3} \rangle$. Moreover, I is a homeomorphism from $\langle \Pi, \mathscr{O}_\Pi \rangle$ onto $\langle \mathbb{R}^3, \varrho \rangle$ with a standard topology, by Fact D.5. Notice that for any $U \in \mathbf{RO}^+_\Pi$, $I[U] \in \mathbf{RO}^+_{\mathbb{R}^3}$. We define a bijection $\Im\colon \mathbf{RO}^+_\Pi \to \mathbf{RO}^+_{\mathbb{R}^3}$ such that for $U \in \mathbf{RO}_\Pi$

$$\Im(U) := I[U].$$

$\Im$ is an isomorphism from $\langle \mathbf{RO}^+_\Pi, \mathbf{BO}_\Pi, \subseteq \rangle$ onto $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq \rangle$. Firstly, by (D.1), for any $U \in \mathbf{RO}_\Pi$: $U \in \mathbf{BO}_\Pi$ iff $\Im(U) := I[U] \in \mathbf{BO}_{\mathbb{R}^3}$. Secondly, for any $U, V \in \mathbf{RO}_\Pi$: $U \subseteq V$ iff $I[U] \subseteq I[V]$ iff $\Im[U] \subseteq \Im[V]$, because I is injective. $\dashv$

8.3. The second postulate. Thanks to the above constructions we can introduce the second axiom of Tarski which states that the set of interior points of any solid is a non-empty open domain in Π

$$\forall_{s \in \mathbb{S}}\ \mathtt{int}(s) \in \mathbf{RO}^+_\Pi. \tag{P2}$$

Remark 8.1. Lemmas 3.1 and 6.1 show that strengthening axiom (P2) would give us the equality $\mathbb{R} = \mathbb{S}$. As it can be seen in the appropriate lemma, by strengthening (P2) we mean changing the range of quantifiers to the whole universe $\mathbb{R}$. $\dashv$

Lemma 8.4. *If* $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ *satisfies* (P0)–(P2), *then for any* $x \in \mathbb{R}$ *the set* $\mathtt{int}(x)$ *is open in* $\langle \Pi, \mathscr{O}_\Pi \rangle$.

Proof. If $\mathtt{int}(x) = \emptyset$, then clearly $\mathtt{int}(x)$ is an open set. Let us suppose that $\beta \in \mathtt{int}(x) \neq \emptyset$. Then for some $b_0 \in \beta$ we have $b_0 \sqsubseteq x$ and $\beta \in \mathtt{int}(b_0)$. But, by (P2), $\mathtt{int}(b_0) \in \mathbf{RO}^+_\Pi$. We will show that $\mathtt{int}(b_0) \subseteq \mathtt{int}(x)$. Let α be an arbitrary point from $\mathtt{int}(b_0)$. Again by (df $\mathtt{int}$) we have that there exists a ball $a_0 \in \alpha$ such that $a_0 \sqsubseteq b_0$. Consequently $a_0 \sqsubseteq x$, so $\alpha \in \mathtt{int}(x)$ and we conclude that $\mathtt{int}(b_0) \subseteq \mathtt{int}(x)$. It follows that every point in $\mathtt{int}(x)$ has an open neighborhood which is a subset of $\mathtt{int}(x)$. Thus $\mathtt{int}(x) \in \mathscr{O}_\Pi$. $\dashv$

8.4. The third postulate. The third axiom says that for any non-empty regular open subset of Π there exists a solid such that this subset is the set of all interior points of this solid, i.e.,

$$\forall_{U \in \mathbf{RO}^+_\Pi} \exists_{s \in \mathbb{S}}\ \mathtt{int}(s) = U. \tag{P3}$$

8.5. The fourth postulate. According to the fourth axiom, for any solids s_1 and s_2, if the set of all interior points of s_1 is a subset of the set of all interior points of s_2, then s_1 is an ingrediens of s_2, i.e.,

$$\mathtt{int}(s_1) \subseteq \mathtt{int}(s_2) \Longrightarrow s_1 \sqsubseteq s_2. \tag{P4}$$

Remark 8.2. Lemmas 3.1 and 6.2 together with (8.1) show that strengthening (P4) would give us the equality $\mathbb{R} = \mathbb{S}$. As it can be seen in the appropriate lemma by strengthening (P4) we mean, again (see Remark 8.1), changing the range of quantifiers to $\mathbb{R}$. $\dashv$

Now we prove the stronger version of (P4).

THEOREM 8.5. *If* $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ *satisfies* (P0) *and* (P4), *and* $\mathbb{S} := \{z \in \mathbb{R} \mid \exists_{X \in 2^{\mathbb{B}}} z \textsf{ sum } Y\}$, *then*

$$\forall_{s \in \mathbb{S}} \forall_{x \in \mathbb{R}} \big(\mathtt{int}(s) \subseteq \mathtt{int}(x) \Longrightarrow s \sqsubseteq x\big).$$

PROOF. Let $s \in \mathbb{S}$ and $x \in \mathbb{R}$. Assume that $\mathtt{int}(s) \subseteq \mathtt{int}(x)$. By (6.5), $\mathtt{int}(s) \neq \emptyset$. Therefore $\mathtt{int}(x) \neq \emptyset$, which means that $\mathbb{B}_x \neq \emptyset$. Let $s_0 \textsf{ sum } \mathbb{B}_x$. By (A.19), (A.20) and the fact that $\mathbb{B}_x \subseteq \{y \in \mathbb{R} \mid y \sqsubseteq x\}$, we obtain $s_0 \sqsubseteq x$. We will show now that $\mathtt{int}(s) \subseteq \mathtt{int}(s_0)$. If $\beta \in \mathtt{int}(s)$, then $\beta \in \mathtt{int}(x)$, i.e., for some $b_0 \in \beta$, $b_0 \sqsubseteq x$. Thus $b_0 \in \mathbb{B}_x$, so $b_0 \sqsubseteq s_0$. Hence $\beta \in \mathtt{int}(s_0)$. Since $\mathtt{int}(s) \subseteq \mathtt{int}(s_0)$, by (P4) we have $s \sqsubseteq s_0 \sqsubseteq x$. We obtain $s \sqsubseteq x$ by transitivity of $\sqsubseteq$. ⊣

By Theorem 8.5, (6.1), (as$_{\sqsubseteq}$) and (df $\sqsubset$) the following facts hold.

$$\forall_{s \in \mathbb{S}} \forall_{x \in \mathbb{R}} \big(\mathtt{int}(s) \subseteq \mathtt{int}(x) \iff s \sqsubseteq x\big), \tag{8.2}$$

$$\forall_{s,s' \in \mathbb{S}} \big(\mathtt{int}(s) = \mathtt{int}(s') \iff s = s'\big), \tag{8.3}$$

$$\forall_{s \in \mathbb{S}} \forall_{x \in \mathbb{R}} \big(\mathtt{int}(s) \subsetneq \mathtt{int}(x) \Longrightarrow s \sqsubset x\big), \tag{8.4}$$

$$\forall_{x \in \mathbb{R}} \forall_{s \in \mathbb{S}} \big(x \sqsubset s \Longrightarrow \mathtt{int}(x) \subsetneq \mathtt{int}(s)\big), \tag{8.5}$$

$$\forall_{s,s' \in \mathbb{S}} \big(\mathtt{int}(s) \subsetneq \mathtt{int}(s') \iff s \sqsubset s'\big). \tag{8.6}$$

Let us remind here that in [21] Tarski wrote

> "The specific character of such a geometry of solids—in contrast to all point geometries—is shown in particular in the law according to which each figure contains another figure as a proper part."

We can thus say that the following fact (8.7) goes along Tarski's intentions. Let us notice however, that the scope of the quantifiers in this fact is the set $\mathbb{S}$ and, as we will prove it further, one cannot prove the property mentioned in the above quotation about *all* objects from $\mathbb{R}$ on the basis of Tarski's axioms.[13]

$$\forall_{s \in \mathbb{S}} \exists_{s' \in \mathbb{S}} \, s' \sqsubset s. \tag{8.7}$$

Indeed, let s be an arbitrary solid. By (P2) it is the case that $\mathtt{int}(s) \in \mathbf{RO}^+_\Pi$. Using (P1) and Fact D.6 we have that in $\mathbf{RO}^+_\Pi$ there exists U such that $U \subsetneq \mathtt{int}(s)$. From (P3) we infer that there exists $s_0 \in \mathbb{S}$ such that $\mathtt{int}(s_0) = U$, that is $\mathtt{int}(s_0) \subsetneq \mathtt{int}(s)$. Hence, by (8.6), we achieve that $s_0 \sqsubset s$.

8.6. T-structures. *A Tarski's structure* (in short: T-*structure*) is any structure of the form $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ satisfying postulates (P0)–(P4) together with (df $\mathbb{S}$).

Let $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ be a T-structure. We put

$$\sqsubseteq_{\mathbb{S}} := \sqsubseteq \cap (\mathbb{S} \times \mathbb{S}). \tag{df $\sqsubseteq_{\mathbb{S}}$}$$

[13] In Theorem 12.9 we will prove that $\mathbf{s}$ has no atomic parts, i.e., $\forall_{x \in \mathbb{R}}(x \sqsubset \mathbf{s} \Rightarrow \exists_{y \in \mathbb{R}} \, y \sqsubset x)$. However, in §16 we will point to a T-structure in which $\mathbf{s} \sqsubset \mathbb{1}$ and the object $\mathbb{1} - \mathbf{s}$ has atomic parts.

and we define the following relations which are included in $\mathbb{S} \times \mathbb{S}$ and $\mathbb{S} \times 2^{\mathbb{S}}$, respectively, and are determined by $\sqsubseteq_{\mathbb{S}}$, i.e.,

$$x \bigcirc_{\mathbb{S}} y \overset{\text{df}}{\iff} \exists_{z \in \mathbb{S}}(z \sqsubseteq_{\mathbb{S}} x \wedge z \sqsubseteq_{\mathbb{S}} y), \tag{df $\bigcirc_{\mathbb{S}}$}$$

$$x \wr_{\mathbb{S}} y \overset{\text{df}}{\iff} \neg\exists_{z \in \mathbb{S}}(z \sqsubseteq_{\mathbb{S}} x \wedge z \sqsubseteq_{\mathbb{S}} y), \tag{df $\wr_{\mathbb{S}}$}$$

$$z \ \mathsf{sum}_{\mathbb{S}}\ X \overset{\text{df}}{\iff} \forall_{s \in X}\, s \sqsubseteq_{\mathbb{S}} z \wedge \forall_{y \in \mathbb{S}}(y \sqsubseteq_{\mathbb{S}} z \Rightarrow \exists_{s \in X}\, s \bigcirc_{\mathbb{S}} y). \tag{df $\mathsf{sum}_{\mathbb{S}}$}$$

FACT 8.6. *For any* T-*structure* $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$

(i) *the mapping* $\mathtt{int}|_{\mathbb{S}}$ *is an isomorphism from* $\langle \mathbb{S}, \sqsubseteq_{\mathbb{S}} \rangle$ *onto* $\langle \mathbf{RO}^+_{\Pi}, \subseteq \rangle$,

(ii) $\langle \mathbb{S}, \sqsubseteq_{\mathbb{S}} \rangle$ *is a mereological structure for* $\mathsf{sum}_{\mathbb{S}}$ *and* $\bigcirc_{\mathbb{S}}$, *and with the unity* $\mathbb{s}$.

PROOF. (i) By (P2), (P3) and (8.3), $\mathtt{int}|_{\mathbb{S}}$ is a bijection from $\mathbb{S}$ onto $\mathbf{RO}^+_{\Pi}$. Now we use (P4) and (6.1).

(ii) $\langle \mathbf{RO}^+_{\Pi}, \subseteq \rangle$ is a mereological structure with the unity Π (see Corollary C.2). Thus, by (i), $\langle \mathbb{S}, \sqsubseteq_{\mathbb{S}} \rangle$ is a mereological structure for $\mathsf{sum}_{\mathbb{S}}$ and $\bigcirc_{\mathbb{S}}$. The solid $\mathbb{s}$ is the unity of this structure, since $\mathtt{int}(\mathbb{s}) = \Pi$. ⊣

REMARK 8.3. Further we will prove stronger facts concerning T-structures. Firstly, we are going to show that the mapping $\mathtt{int}|_{\mathbb{S}}$ is an isomorphism from $\langle \mathbb{S}, \mathbb{B}, \sqsubseteq_{\mathbb{S}} \rangle$ onto $\langle \mathbf{RO}^+_{\Pi}, \mathbf{BO}_{\Pi}, \subseteq \rangle$ (see Fact 12.1). Secondly, we will prove that $\langle \mathbb{S}, \sqsubseteq_{\mathbb{S}} \rangle$ is a mereological substructure of $\langle \mathbb{R}, \sqsubseteq \rangle$ in the following sense (see Fact 12.8)

- $\bigcirc_{\mathbb{S}} = \bigcirc \cap (\mathbb{S} \times \mathbb{S})$ and $\wr_{\mathbb{S}} = \wr \cap (\mathbb{S} \times \mathbb{S})$,
- $\mathsf{sum}_{\mathbb{S}} = \mathsf{sum} \cap (\mathbb{S} \times 2^{\mathbb{S}}) = \mathsf{sum} \cap (\mathbb{R} \times 2^{\mathbb{S}})$.

So also for any $s, s_1, s_2 \in \mathbb{S}$ (see the proof of Theorem 1.2, p. 523)

- $s_1 \sqcup_{\mathbb{S}} s_2 = s_1 \sqcup s_2 \in \mathbb{S}$;
- if $s_1 \bigcirc s_2$, then $s_1 \sqcap_{\mathbb{S}} s_2 = s_1 \sqcap s_2 \in \mathbb{S}$;
- if $s \sqsubset \mathbb{s}$, then $\mathbb{s} -_{\mathbb{S}} s = \mathbb{s} - s \in \mathbb{S}$.

We will also prove some facts that concern all regions, not only solids. ⊣

COROLLARY 8.7. *For any solids* $s_1, s_2 \in \mathbb{S}$

$$\mathtt{int}(s_1) \cap \mathtt{int}(s_2) \neq \emptyset \iff s_1 \bigcirc_{\mathbb{S}} s_2.$$

PROOF. Let $s_1, s_2 \in \mathbb{S}$. If $\mathtt{int}(s_1) \cap \mathtt{int}(s_2) \neq \emptyset$, then $\mathtt{int}(s_1) \cap \mathtt{int}(s_2) \in \mathbf{RO}^+_{\Pi}$, by (P2). Hence, by (P3), for some $s_0 \in \mathbb{S}$ it is the case that $\mathtt{int}(s_0) = \mathtt{int}(s_1) \cap \mathtt{int}(s_2)$. So $\mathtt{int}(s_0) \subseteq \mathtt{int}(s_1)$ and $\mathtt{int}(s_0) \subseteq \mathtt{int}(s_2)$. Hence $s_0 \sqsubseteq_{\mathbb{S}} s_1$ and $s_0 \sqsubseteq_{\mathbb{S}} s_2$, by Fact 8.6(i). Let $s_1 \bigcirc_{\mathbb{S}} s_2$, i.e., for some $s_0 \in \mathbb{S}$: $s_0 \sqsubseteq s_1$ and $s_0 \sqsubseteq s_2$. Then $\emptyset \neq \mathtt{int}(s_0) \subseteq \mathtt{int}(s_1) \cap \mathtt{int}(s_2)$, by (6.1) and (P2). ⊣

§9. Topological properties of T-structures. It is obvious that all the properties of $\mathtt{int}$ and $\mathtt{fr}$ presented in §§6–8 hold in all T-structures. Now we are going to prove some new properties that are entailed by postulates (P0)–(P4).

All facts from this section hold for any T-structure $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$, where $\mathtt{s}$ sum $\mathbb{B}$ and $\mathbb{S} := \{z \in \mathbb{R} \mid \exists_{X \in 2^{\mathbb{B}}} z \ \mathsf{sum}\ X\}$. Below is what we can call *the main theorem* for T-structures.

THEOREM 9.1. *For any $x, y \in \mathbb{R}$ and $\beta \in \Pi$*

$$\beta \in \mathtt{int}(x) \cap \mathtt{int}(y) \iff \exists_{b \in \beta}(b \sqsubseteq x \wedge b \sqsubseteq y).$$

PROOF. Suppose that $\beta \in \mathtt{int}(x) \cap \mathtt{int}(y)$. By Lemma 8.4, $\mathtt{int}(x), \mathtt{int}(y) \in \mathscr{O}_\Pi$, so $\mathtt{int}(x) \cap \mathtt{int}(y) \in \mathscr{O}_\Pi$. Hence for some $B_0 \in \mathbf{BO}_\Pi$ we have $\beta \in B_0 \subseteq \mathtt{int}(x) \cap \mathtt{int}(y)$, because $\mathbf{BO}_\Pi$ is a base for $\langle \Pi, \mathscr{O}_\Pi \rangle$. But $\mathbf{BO}_\Pi \subseteq \mathbf{RO}_\Pi$, so by (P3), for some $s_0 \in \mathbb{S}$: $B_0 = \mathtt{int}(s_0)$. So $\beta \in \mathtt{int}(s_0) \subseteq \mathtt{int}(x) \cap \mathtt{int}(y)$. Thus $s_0 \sqsubseteq x$ and $s_0 \sqsubseteq y$, by Theorem 8.5. Moreover, for some $b_0 \in \beta$ we have $b_0 \sqsubseteq s_0$. Hence $b_0 \sqsubseteq x$ and $b_0 \sqsubseteq y$, by ($\mathrm{t}_\sqsubseteq$). ⊣

COROLLARY 9.2. *For any $x, y \in \mathbb{R}$ such that $x \bigcirc y$*

$$\mathtt{int}(x \sqcap y) = \mathtt{int}(x) \cap \mathtt{int}(y).$$

PROOF. Let $x \sqcap y$. Then $\mathtt{int}(x \sqcap y) \subseteq \mathtt{int}(x) \cap \mathtt{int}(y)$, by (6.3). If $\beta \in \mathtt{int}(x) \cap \mathtt{int}(y)$, then, by Theorem 9.1, for some $b_0 \in \beta$: $b_0 \sqsubseteq x$ and $b_0 \sqsubseteq y$; so also $b_0 \sqsubseteq x \sqcap y$. Hence $\beta \in \mathtt{int}(x \sqcap y)$. ⊣

COROLLARY 9.3. $\forall_{\gamma \in \Pi} \forall_{a,b \in \gamma} \exists_{c \in \gamma}(c \sqsubseteq a \wedge c \sqsubseteq b)$.

PROOF. Suppose that $a, b \in \gamma$. Then $\gamma \in \mathtt{int}(a) \cap \mathtt{int}(b)$. So, by Theorem 9.1, there is $c \in \gamma$ such that $c \sqsubseteq a$ and $c \sqsubseteq b$. ⊣

REMARK 9.1. In §11 we will show that in all T-structures: $\forall_{\gamma \in \Pi} \forall_{a,b \in \gamma}\, a \circledcirc b$. On the base of symmetry of $\circledcirc$, this fact is equivalent to transitivity of the concentricity relation.

COROLLARY 9.4. *For any $x, y \in \mathbb{R}$*

$$\mathtt{fr}(x \sqcup y) = \big(\mathtt{fr}(x) \cup \mathtt{fr}(y)\big) \setminus \mathtt{int}(x \sqcup y).$$

PROOF. Let $\beta \in \mathtt{fr}(x \sqcup y)$. Then $\forall_{b \in \beta}\big(b \not\sqsubseteq x \sqcup y \wedge b \bigcirc x \sqcup y\big)$. Therefore $\beta \notin \mathtt{int}(x \sqcup y)$. Suppose towards a contradiction, that $\beta \notin \mathtt{fr}(x)$ and $\beta \notin \mathtt{fr}(y)$. Then for some $a_0 \in \beta$ either $a_0 \sqsubseteq x$ or $a_0 \wr x$, and for some $b_0 \in \beta$ either $b_0 \sqsubseteq y$ or $b_0 \wr y$. Thus $a_0 \wr x$ and $b_0 \wr y$. By Corollary 9.3, for some $c_0 \in \beta$ we have $c_0 \sqsubseteq a_0$ and $c_0 \sqsubseteq b_0$. Hence $c_0 \wr x \sqcup y$ and we get a contradiction. So $\beta \in \mathtt{fr}(x) \cup \mathtt{fr}(y)$. The converse inclusion we obtain by (7.3). ⊣

COROLLARY 9.5. *For any $x, y \in \mathbb{R}$ such that $x \bigcirc y$*

$$\mathtt{fr}(x \sqcap y) \subseteq \mathtt{fr}(x) \cup \mathtt{fr}(y),$$

$$\mathtt{fr}(x \sqcap y) \subseteq (\mathtt{fr}(x) \cap \mathtt{int}(y)) \cup (\mathtt{fr}(y) \cap \mathtt{int}(x)) \cup (\mathtt{fr}(x) \cap \mathtt{fr}(y)).$$

PROOF. Let $x \bigcirc y$ and $\beta \in \mathtt{fr}(x \sqcap y)$.

Firstly, suppose that $\beta \notin \mathtt{fr}(x)$ and $\beta \notin \mathtt{fr}(y)$. Then $\forall_{b \in \beta}\big(\neg(b \sqsubseteq x \wedge b \sqsubseteq y) \wedge b \bigcirc x \wedge b \bigcirc y\big)$, for some $a_0 \in \beta$ either $a_0 \sqsubseteq x$ or $a_0 \wr x$, and for some $b_0 \in \beta$ either $b_0 \sqsubseteq x$ or $b_0 \wr x$. Therefore $a_0 \sqsubseteq x$ and $b_0 \sqsubseteq y$, so

$\beta \in \mathtt{int}(x)$ and $\beta \in \mathtt{int}(y)$. By Theorem 9.1 we have $\exists_{b\in\beta}(b \sqsubseteq x \wedge b \sqsubseteq y)$, a contradiction.

Secondly, suppose that either $\beta \notin \mathtt{fr}(x)$ or $\beta \notin \mathtt{fr}(y)$. In the first case we obtain that $\beta \in \mathtt{fr}(y)$. Moreover, $\forall_{b\in\beta}\, b \bigcirc x$ and for some $a_0 \in \beta$ either $a_0 \sqsubseteq x$ or $a_0 \wr x$. Therefore $a_0 \sqsubseteq x$, so $\beta \in \mathtt{int}(x)$. Thus $\beta \in \mathtt{fr}(y) \cap \mathtt{int}(x)$. Similarly, in the second case we obtain that $\beta \in \mathtt{fr}(x) \cap \mathtt{int}(y)$. ⊣

Corollary 9.6. $\forall_{x\in\mathbb{R}}\, \forall_{\beta\in\mathtt{int}(x)}\, \forall_{b\in\beta}\, \exists_{a\in\beta}(a \sqsubseteq x \wedge a \sqsubseteq b)$.

Proof. If $\beta \in \mathtt{int}(x)$ and $b \in \beta$, we use Theorem 9.1, since $\beta \in \mathtt{int}(b)$, by (6.6). ⊣

Lemma 9.7. *For any $x, y \in \mathbb{R}$ such that $x \not\sqsubseteq y$, there exist regions $-y$ and $x - y$, and*

(i) $\mathtt{int}(x - y) \cap \mathtt{int}(y) = \emptyset$,
(ii) $\mathtt{int}(x) \subseteq \mathtt{int}(x - y) \cup \mathtt{fr}(x - y) \cup \mathtt{int}(y)$,
(iii) $\mathtt{int}(x) = \mathtt{int}(x - y) \cup \big(\mathtt{fr}(x - y) \cap \mathtt{int}(x)\big) \cup \big(\mathtt{int}(y) \cap \mathtt{int}(x)\big)$,
(iv) $\mathtt{fr}(x - y) \cap \mathtt{int}(x) \subseteq \mathtt{fr}(y)$ *and* $\mathtt{fr}(y) \cap \mathtt{int}(x) \subseteq \mathtt{fr}(x - y)$.

Proof. (i) From (A.4) we have $y \wr x - y$, i.e., there is no $z \in \mathbb{R}$ such that $z \sqsubseteq y$ and $z \sqsubseteq x - y$. Thus $\mathtt{int}(y) \cap \mathtt{int}(x - y) = \emptyset$, by Theorem 9.1.

(ii) Suppose that $\beta \in \mathtt{int}(x)$, $\beta \notin \mathtt{int}(x - y)$ and $\beta \notin \mathtt{int}(y)$. Then $\forall_{b\in\beta}\, b \not\sqsubseteq x - y$ and $\forall_{b\in\beta}\, b \not\sqsubseteq y$. Suppose that $b \in \beta$. Then, by Corollary 9.6, for some $b_0 \in \beta$ we have $b_0 \sqsubseteq x$ and $b_0 \sqsubseteq b$. Since $b_0 \not\sqsubseteq y$, then $b_0 \bigcirc x - y$, so also $b \bigcirc x - y$. Thus $\forall_{b\in\beta}\, b \bigcirc x - y$, i.e., $\beta \in \mathtt{fr}(x - y)$.

(iii) Since $x - y \sqsubseteq x$, we have $\mathtt{int}(x - y) \subseteq \mathtt{int}(x)$.

(iv) Assume that $\beta \in \mathtt{int}(x)$ and $\beta \in \mathtt{fr}(x - y)$. Then for some $b_0 \in \beta$ we have $b_0 \sqsubseteq x$ and $\forall_{b\in\beta}\, b \not\sqsubseteq y$. Moreover suppose, towards a contradiction, that for some $b_1 \in \beta$ we have $b_1 \wr y$. Then $b_1 \not\sqsubseteq -y$. By Corollary 9.3, for some $c_0 \in \beta$ we have $c_0 \sqsubseteq b_0 \sqsubseteq x$ and $c_0 \sqsubseteq b_1 \sqsubseteq -y$. Thus $c_0 \sqsubseteq x - y$ and $\beta \in \mathtt{int}(x - y)$. So we get a contradiction by (7.1). Therefore, for any $b \in \beta$, $b \bigcirc y$. Thus $\beta \in \mathtt{fr}(x)$.

Assume that $\beta \in \mathtt{int}(x)$ and $\beta \in \mathtt{fr}(y)$. Then for some $b_0 \in \beta$ we have $b_0 \sqsubseteq x$ and $\forall_{b\in\beta}(b \not\sqsubseteq y \wedge b \bigcirc y)$. Let $b \in \beta$. Thus $b \not\sqsubseteq x - y$, since $b \bigcirc y$. By Corollary 9.3, for some $c_0 \in \beta$ we have $c_0 \sqsubseteq b_0 \sqsubseteq x$ and $c_0 \sqsubseteq b$. Since $c_0 \sqsubseteq -y$ we have $c_0 \bigcirc x - y$. Hence also $b \bigcirc x - y$. Thus $\beta \in \mathtt{fr}(x - y)$. ⊣

Corollary 9.8. (i) *For any $x \in \mathbb{R}$ such that $x \neq \mathbb{1}$, $\mathtt{int}(x) \cap \mathtt{int}(-x) = \emptyset$.*
(ii) *For any $x \in \mathbb{R}$ such that $x \sqsubset \mathtt{s}$, $\mathtt{int}(x) \cap \mathtt{int}(\mathtt{s} - x) = \emptyset$.*

Proof. (i) If $x \neq \mathbb{1}$, then $\mathbb{1} \not\sqsubseteq x$ and $-x = \mathbb{1} \sqcap -x = \mathbb{1} - x$. Thus $\mathtt{int}(x) \cap \mathtt{int}(-x) = \emptyset$, by Lemma 9.7(i).

(ii) If $x \sqsubset \mathtt{s}$, then $\mathtt{s} \not\sqsubseteq x$. Thus $\mathtt{int}(x) \cap \mathtt{int}(\mathtt{s} - x) = \emptyset$, by Lemma 9.7(i). ⊣

Corollary 9.9. (i) *For any $x \in \mathbb{R}$ such that $x \neq \mathbb{1}$, $\mathtt{int}(x) \cup \mathtt{fr}(x) \cup \mathtt{int}(-x) = \Pi$.*
(ii) *For any $x \in \mathbb{R}$ such that $x \sqsubset \mathtt{s}$, $\mathtt{int}(x) \cup \mathtt{fr}(x) \cup \mathtt{int}(\mathtt{s} - x) = \Pi$.*

PROOF. (i) If $x \neq \mathbb{1}$, then $\mathbb{1} \not\sqsubseteq x$ and $-x = \mathbb{1} - x$. Thus; by (6.8), Lemma 9.7(ii) and Fact 7.1(i); we obtain $\Pi \subseteq \mathtt{int}(-x) \cup \mathtt{fr}(-x) \cup \mathtt{int}(x) = \mathtt{int}(x) \cup \mathtt{fr}(x) \cup \mathtt{int}(-x) \subseteq \Pi$.

(ii) If $x \sqsubset \mathbb{s}$, then $\mathbb{s} \not\sqsubseteq x$. Thus; by (6.8), Lemma 9.7(ii) and Fact 7.1(ii); we have $\Pi \subseteq \mathtt{int}(\mathbb{s}-x) \cup \mathtt{fr}(\mathbb{s}-x) \cup \mathtt{int}(x) = \mathtt{int}(x) \cup \mathtt{fr}(x) \cup \mathtt{int}(\mathbb{s}-x) \subseteq \Pi$. ⊣

COROLLARY 9.10. *In* $\langle \Pi, \mathscr{O}_\Pi \rangle$ *for any* $x \in \mathbb{R}$

(i) $\mathrm{Cl}_\Pi\, \mathtt{int}(x) \subseteq \mathtt{int}(x) \cup \mathtt{fr}(x)$, *so* $\mathrm{Fr}_\Pi\, \mathtt{int}(x) \subseteq \mathtt{fr}(x)$;

(ii) $\mathtt{fr}(x)$ *is closed.*

PROOF. (i) Let $\beta \in \mathrm{Cl}_\Pi\, \mathtt{int}(x)$. Then, since $\mathbf{RO}_\Pi$ is a base for $\langle \Pi, \mathscr{O}_\Pi \rangle$, for any $U \in \mathbf{RO}_\Pi$: if $\beta \in U$ then $U \cap \mathtt{int}(x) \neq \emptyset$. Hence for any $b \in \beta$, by (P2) and (6.6), we have $\mathtt{int}(b) \cap \mathtt{int}(x) \neq \emptyset$; so $b \bigcirc x$, by Theorem 9.1. Suppose that $\beta \notin \mathtt{int}(x)$. Then $b \not\sqsubseteq x$, for any $b \in \beta$. Thus $\beta \in \mathtt{fr}(x)$.

Moreover, $\mathrm{Fr}_\Pi\, \mathtt{int}(x) := \mathrm{Cl}_\Pi\, \mathtt{int}(x) \backslash \mathrm{Int}_\Pi\, \mathtt{int}(x) = \mathrm{Cl}_\Pi\, \mathtt{int}(x) \backslash \mathtt{int}(x) \subseteq (\mathtt{int}(x) \cup \mathtt{fr}(x)) \setminus \mathtt{int}(x) = \mathtt{fr}(x)$, by Lemma 8.4 and (7.1).

(ii) By Corollary 9.9(i), (7.1), and Fact 7.1(i), $\mathtt{fr}(x) = \Pi \setminus (\mathtt{int}(x) \cup \mathtt{int}(-x))$. By Lemma 8.4, $\mathtt{int}(x), \mathtt{int}(-x) \in \mathscr{O}_\Pi$, so also $\mathtt{int}(x) \cup \mathtt{int}(-x) \in \mathscr{O}_\Pi$. ⊣

COROLLARY 9.11. (i) *For any* $x \in \mathbb{R}$ *such that* $x \sqsubset \mathbb{s}$, $\mathtt{fr}(x) \neq \emptyset$.

(ii) *For any* $s \in \mathbb{S}$, $s \sqsubset \mathbb{s}$ *iff* $\mathtt{fr}(s) \neq \emptyset$.

(iii) *For any* $b \in \mathbb{B}$, $\mathtt{fr}(b) \neq \emptyset$.

PROOF. (i) If $x \in \mathbb{R}$ and $x \sqsubset \mathbb{s}$, then $\mathtt{int}(x), \mathtt{int}(\mathbb{s} - x) \in \mathscr{O}_\Pi \setminus \{\Pi\}$, by Lemma 8.4, (6.8) and (8.5). Suppose, towards a contradiction, that $\mathtt{fr}(x) = \emptyset$. Then $\Pi = \mathtt{int}(x) \cup \mathtt{int}(\mathbb{s} - x)$, by Corollary 9.9(ii). Hence $\mathtt{int}(x) \neq \emptyset \neq \mathtt{int}(\mathbb{s} - x)$. But, by Corollary 9.8(ii), $\mathtt{int}(x) \cap \mathtt{int}(\mathbb{s} - x) = \emptyset$. Thus we obtain that $\langle \Pi, \mathscr{O}_\Pi \rangle$ is not connected,[14] which is a contradiction.

(ii) By (i). Moreover, if $\mathtt{fr}(s) \neq \emptyset$ then $s \neq \mathbb{s}$, because $\mathtt{fr}(\mathbb{s}) = \emptyset$.

(iii) By Fact 8.2(ii), $b \sqsubset \mathbb{s}$. So $\mathtt{fr}(b) \neq \emptyset$, by (i). ⊣

COROLLARY 9.12. *For any* x *such that* $x \sqsubset \mathbb{s}$; *in* $\langle \Pi, \mathscr{O}_\Pi \rangle$, $\mathrm{Cl}_\Pi\, \mathtt{int}(x) \cap \mathtt{int}(\mathbb{s} - x) = \emptyset = \mathtt{int}(x) \cap \mathrm{Cl}_\Pi\, \mathtt{int}(\mathbb{s} - x)$.

PROOF. $\mathrm{Cl}_\Pi\, \mathtt{int}(x) \cap \mathtt{int}(\mathbb{s}-x) \subseteq (\mathtt{int}(x) \cup \mathtt{fr}(x)) \cap \mathtt{int}(\mathbb{s}-x) = (\mathtt{int}(x) \cap \mathtt{int}(\mathbb{s} - x)) \cup (\mathtt{fr}(x) \cap \mathtt{int}(\mathbb{s} - x)) = \emptyset \cup (\mathtt{fr}(\mathbb{s} - x) \cap \mathtt{int}(\mathbb{s} - x)) = \emptyset$, by Corollary 9.10(i), (7.1) and Fact 7.1(ii). In an analogous way we prove that $\mathtt{int}(s) \cap \mathrm{Cl}_\Pi\, \mathtt{int}(\mathbb{s} - s) = \emptyset$. ⊣

COROLLARY 9.13. *Let* $x \neq \mathbb{1}$ (*resp.* $x \sqsubset \mathbb{s}$) *and* $\mathscr{O}^x_\Pi := \{(\Pi \setminus \mathtt{fr}(x)) \cap U \mid U \in \mathscr{O}_\Pi\}$. *Then* $\langle \Pi \setminus \mathtt{fr}(x), \mathscr{O}^x_\Pi \rangle$ *is a topological subspace of* $\langle \Pi, \mathscr{O}_\Pi \rangle$. *Moreover,* $\mathtt{int}(x)$ *and* $\mathtt{int}(-x)$ (*resp.* $\mathtt{int}(\mathbb{s} - x)$) *are clopen, i.e., closed and open, in this subspace.*

PROOF. Let $x \neq \mathbb{1}$. By (7.1), Fact 7.1(i) and Lemma 8.4 we have that $\mathtt{int}(x), \mathtt{int}(-x) \in \mathscr{O}_\Pi$ and $\mathtt{int}(x), \mathtt{int}(-x) \in \mathscr{O}^x_\Pi$. Therefore it suffices

[14]See Fact D.3. Let us remind here that a topological space $\langle X, \mathscr{O} \rangle$ is connected iff there are no non-empty open subsets U and V of X such that $U \cap V = \emptyset$ and $X = U \cup V$; in the other words, iff the only clopen (closed and open) subsets of $\langle X, \mathscr{O} \rangle$ are $\emptyset$ and X.

to show that these sets are closed in $\langle \Pi \setminus \mathtt{fr}(x), \mathscr{O}^x_\Pi \rangle$. In the following transformations we use the fact that $\mathtt{int}(x)$, $\mathtt{int}(-x)$ and $\mathtt{fr}(x)$ are pairwise disjoint (see Corollary 9.8(ii), (7.1) and Fact 7.1(i)) and a standard definition of the closure operator Cl in $\langle \Pi \setminus \mathtt{fr}(x), \mathscr{O}^x_\Pi \rangle$. By Corollary 9.9(i), (7.1), Fact 7.1(i), Corollary 9.8(i) and Lemma 8.4 we obtain: $\mathrm{Cl}\,\mathtt{int}(x) = (\Pi \setminus \mathtt{fr}(x)) \setminus \mathrm{Int}[(\Pi \setminus \mathtt{fr}(x)) \setminus \mathtt{int}(x)] = (\Pi \setminus \mathtt{fr}(x)) \setminus \mathrm{Int}\,\mathtt{int}(-x) = (\Pi \setminus \mathtt{fr}(x)) \setminus \mathtt{int}(-x) = \mathtt{int}(x)$. In an analogous way we prove that $\mathrm{Cl}\,\mathtt{int}(-x) = \mathtt{int}(-x)$.

If $x \sqsubset \mathtt{s}$, then also in an analogous way we demonstrate that in $\langle \Pi \setminus \mathtt{fr}(x), \mathscr{O}^x_\Pi \rangle$ it is the case that $\mathrm{Cl}\,\mathtt{int}(x) = \mathtt{int}(x)$ and $\mathrm{Cl}\,\mathtt{int}(\mathtt{s} - x) = \mathtt{int}(\mathtt{s} - x)$. ⊣

§10. Properties of mereological balls in T-structures. Let $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ be a T-structure. For arbitrary different points α and β from Π we put

$$\mathbb{B}^\beta_\alpha := \{b \in \mathbb{B} \mid b \in \alpha \wedge \beta \in \mathtt{fr}(b)\},$$

i.e., $\mathbb{B}^\beta_\alpha$ is the set of all mereological balls being an element of α and having β as its fringe point. Moreover, we take

$$\mathrm{S}^\beta_\alpha := \{\gamma \in \Pi \mid \gamma\beta \, \Delta^\Pi \, \alpha\}$$

i.e., S^β_α is the sphere in $\langle \Pi, \Delta^\Pi \rangle$ whose the center point is α and such that β is its element. Since the third of axiom of Pieri asserts that $\beta\beta \, \Delta^\Pi \, \alpha$, so $\beta \in \mathrm{S}^\beta_\alpha$.

LEMMA 10.1. *For any $\alpha, \beta \in \Pi$ such that $\alpha \neq \beta$*

(i) $\mathbb{B}^\beta_\alpha \neq \emptyset$,

(ii) *for any $b \in \mathbb{B}^\beta_\alpha$, $\emptyset \neq \mathtt{fr}(b) \subseteq \mathrm{S}^\beta_\alpha$.*

PROOF. (i) The third axiom of Pieri asserts that $\beta\beta \, \Delta^\Pi \, \alpha$, thus by (df′ Δ^Π) we have that there exists a ball b such that $b \in \alpha$ and $\beta \in \mathtt{fr}(b)$.

(ii) Let $b \in \mathbb{B}^\beta_\alpha$. Then, by Corollary 9.11(iii), $\mathtt{fr}(b) \neq \emptyset$. Moreover, suppose that $\gamma \in \mathtt{fr}(b)$. Then, by the facts that $b \in \alpha$, $\beta \in \mathtt{fr}(b)$ and (df′ Δ^Π) we have that $\gamma\beta \, \Delta^\Pi \, \alpha$; therefore $\gamma \in \mathrm{S}^\beta_\alpha$. Thus $\emptyset \neq \mathtt{fr}(b) \subseteq \mathrm{S}^\beta_\alpha$. ⊣

LEMMA 10.2. *For any $b \in \mathbb{B}$ there exists a point β such that $\beta \neq \pi_b$ and $b \in \mathbb{B}^\beta_{\pi_b}$.*

PROOF. By Corollary 9.11(iii), for some β_0 we have $\beta_0 \in \mathtt{fr}(b)$. Since $b \in \pi_b$, we obtain $b \in \mathbb{B}^{\beta_0}_{\pi_b}$. Moreover, $\pi_b \neq \beta_0$, because $\pi_b \in \mathtt{int}(b)$ and $\mathtt{int}(b) \cap \mathtt{fr}(b) = \emptyset$. ⊣

LEMMA 10.3. *For any $b \in \mathbb{B}$ we have $\mathtt{int}(\mathtt{s} - b) \neq \emptyset$.*

PROOF. By Lemmas 10.1(ii) and 10.2 there exists a point β such that $\emptyset \neq \mathtt{fr}(b) \subseteq \mathrm{S}^\beta_{\pi_b}$. Suppose that $\mathtt{int}(\mathtt{s} - b) = \emptyset$. Then $\mathtt{int}(b) = \Pi \setminus \mathtt{fr}(b)$, by Corollary 9.9(ii) and Fact 7.1(ii). Now notice that $\Pi \setminus \mathtt{fr}(b) \notin \mathbf{RO}_\Pi$, since $\mathrm{Cl}_\Pi(\Pi \setminus \mathtt{fr}(b)) = \Pi$, and $\mathrm{Int}_\Pi \mathrm{Cl}_\Pi(\Pi \setminus \mathtt{fr}(b)) = \Pi$. So we get a contradiction. ⊣

LEMMA 10.4. *For any $b \in \mathbb{B}$ the set $\Pi \setminus \mathtt{fr}(b)$ is not connected in $\langle \Pi, \mathscr{O}_\Pi \rangle$.*[15]

PROOF. By (7.1), Fact 7.1(ii) and Corollary 9.9(ii), we have $\Pi \setminus \mathtt{fr}(b) = \mathtt{int}(b) \cup \mathtt{int}(\mathtt{s} - b)$. But $\mathtt{int}(b) \neq \emptyset \neq \mathtt{int}(\mathtt{s} - b)$, by (P2) and Lemma 10.3. Moreover, $\mathrm{Cl}_\Pi \, \mathtt{int}(b) \cap \mathtt{int}(\mathtt{s} - b) = \emptyset = \mathtt{int}(b) \cap \mathrm{Cl}_\Pi \, \mathtt{int}(\mathtt{s} - b)$, by Corollary 9.12. ⊣

LEMMA 10.5. *For any $\alpha, \beta \in \Pi$ such that $\alpha \neq \beta$ and for any $b \in \mathbb{B}^\beta_\alpha$ we have $\mathtt{fr}(b) = \mathrm{S}^\beta_\alpha$.*

PROOF. By Lemma 10.1(ii), $\emptyset \neq \mathtt{fr}(b) \subseteq \mathrm{S}^\beta_\alpha$. By Lemma 10.4 the set $\Pi \setminus \mathtt{fr}(b)$ is not connected in $\langle \Pi, \mathscr{O}_\Pi \rangle$. So $\mathtt{fr}(b) = \mathrm{S}^\beta_\alpha$. ⊣

In the proof of the next lemma we use known properties of three-dimensional Euclidean space expressed in the following lemma.

LEMMA 10.6. *Let S be a sphere in $\langle \Pi, \Delta^\Pi \rangle$ and let $\mathscr{O}^S_\Pi := \{(\Pi \setminus S) \cap U \mid U \in \mathscr{O}_\Pi\}$. Then $\langle \Pi \setminus S, \mathscr{O}^S_\Pi \rangle$ is a topological subspace of $\langle \Pi, \mathscr{O}_\Pi \rangle$.*

Let $B \in \mathbf{BO}_\Pi \subsetneq \mathbf{RO}^+_\Pi$ be the only open ball such that $S = \mathrm{Fr}_\Pi \, B$, $S \cap B = \emptyset$ and $\mathrm{Cl}_\Pi \, B = B \cup S$. Then B and $\Pi \setminus \mathrm{Cl}_\Pi \, B$ are the only non-empty and clopen proper subsets in $\langle \Pi \setminus S, \mathscr{O}^S_\Pi \rangle$. Moreover, $B \cup (\Pi \setminus \mathrm{Cl}_\Pi \, B) = \Pi \setminus S$.

For arbitrary different points α and β from Π we put

$$\mathrm{B}^\beta_\alpha := (\iota\, B)\ (B \in \mathbf{BO}_\Pi \wedge \mathrm{Fr}_\Pi \, B = \mathrm{S}^\beta_\alpha),$$

i.e., B^β_α is an open ball in $\langle \Pi, \Delta^\Pi \rangle$ such that S^β_α is its surface, in other words, α is the center of B^β_α and β lies on the surface of B^β_α. Of course $\mathrm{B}^\beta_\alpha \in \mathbf{RO}^+_\Pi$.

LEMMA 10.7. *For any different points α and β*

(i) *for any mereological ball $b \in \mathbb{B}^\beta_\alpha$ we have $\mathtt{int}(b) = \mathrm{B}^\beta_\alpha$; moreover,*

$$\mathtt{int}(\mathtt{s} - b) = \Pi \setminus (\mathrm{B}^\beta_\alpha \cup \mathrm{S}^\beta_\alpha) = \Pi \setminus \mathrm{Cl}_\Pi \, \mathrm{B}^\beta_\alpha = \mathrm{Int}_\Pi(\Pi \setminus \mathrm{B}^\beta_\alpha)\,;$$

(ii) *there is exactly one ball in $\mathbb{B}^\beta_\alpha$.*

PROOF. (i) Suppose that $b \in \mathbb{B}^\beta_\alpha$. By Lemma 10.5 we have $\mathtt{fr}(b) = \mathrm{S}^\beta_\alpha$. By Corollary 9.9(ii) and Fact 7.1(ii) we obtain

$$\Pi \setminus \mathrm{S}^\beta_\alpha = \mathtt{int}(b) \cup \mathtt{int}(\mathtt{s} - b).$$

By Corollary 9.8(ii), $\mathtt{int}(b) \cap \mathtt{int}(\mathtt{s} - b) = \emptyset$. By (P2), $\mathtt{int}(b) \in \mathbf{RO}^+_\Pi$. By Lemma 10.3, $\mathtt{int}(\mathtt{s} - b) \neq \emptyset$. Hence $\mathtt{int}(b) \subsetneq \Pi \setminus \mathrm{S}^\beta_\alpha$ and $\mathtt{int}(\mathtt{s} - b) \subsetneq \Pi \setminus \mathrm{S}^\beta_\alpha$.

Next, by Corollary 9.13, $\mathtt{int}(b)$ and $\mathtt{int}(\mathtt{s} - b)$ are clopen in $\langle \Pi \setminus \mathrm{S}^\beta_\alpha, \mathscr{O}^b_\Pi \rangle$. From Lemma 10.6 we achieve that B^β_α and $\Pi \setminus (\mathrm{B}^\beta_\alpha \cup \mathrm{S}^\beta_\alpha)$ are the only non-empty and clopen subsets in $\langle \Pi \setminus \mathrm{S}^\beta_\alpha, \mathscr{O}^b_\Pi \rangle$. Thus either $\mathtt{int}(b) = \mathrm{B}^\beta_\alpha$ or $\mathtt{int}(b) = \Pi \setminus (\mathrm{B}^\beta_\alpha \cup \mathrm{S}^\beta_\alpha)$.

However, since it is the case that $\alpha \in \mathrm{B}^\beta_\alpha$ and $\alpha \in \mathtt{int}(b)$, the second equality cannot hold, what reduces to the fact that $\mathtt{int}(b) = \mathrm{B}^\beta_\alpha$. Hence

[15] Any subset Y of X is connected in $\langle X, \mathscr{O} \rangle$ iff there are no non-empty subsets X_1 and X_2 of X such that $\mathrm{Cl}\, X_1 \cap X_2 = \emptyset = X_1 \cap \mathrm{Cl}\, X_2$ and $Y = X_1 \cup X_2$.

we immediately have that $\mathtt{int}(\mathsf{s}-b)=\Pi\setminus(\mathrm{S}^{\beta}_{\alpha}\cup\mathrm{B}^{\beta}_{\alpha})=\mathrm{Int}_{\Pi}(\Pi\setminus\mathrm{B}^{\beta}_{\alpha})=\Pi\setminus\mathrm{Cl}_{\Pi}\,\mathrm{B}^{\beta}_{\alpha}$.

(ii) By Lemma 10.1(i), there is $b_0\in\mathbb{B}^{\beta}_{\alpha}$. Hence $\mathtt{int}(b_0)=\mathrm{B}^{\beta}_{\alpha}$, by (i). Moreover, by (i), $\mathtt{int}(b)=\mathrm{B}^{\beta}_{\alpha}$, for any $b\in\mathbb{B}^{\beta}_{\alpha}$. Thus uniqueness is a consequence of (8.3). ⊣

Corollary 10.8. *If* $\mathbb{B}^{\beta_1}_{\alpha_1}\ni b\in\mathbb{B}^{\beta_2}_{\alpha_2}$, *then* $\mathbb{B}^{\beta_1}_{\alpha_1}=\{b\}=\mathbb{B}^{\beta_2}_{\alpha_2}$ *and* $\alpha_1=\alpha_2$.

Proof. By Lemma 10.7, if $\mathbb{B}^{\beta_1}_{\alpha_1}\ni b\in\mathbb{B}^{\beta_2}_{\alpha_2}$, then $\mathrm{B}^{\beta_1}_{\alpha_1}=\mathtt{int}(b)=\mathrm{B}^{\beta_2}_{\alpha_2}$. So $\alpha_1=\alpha_2$. Moreover, $\mathbb{B}^{\beta_1}_{\alpha_1}=\{b\}=\mathbb{B}^{\beta_2}_{\alpha_2}$. ⊣

By Lemmas 10.2, 10.5 and 10.7(i) we obtain the following theorem.

Theorem 10.9. *For any* $b\in\mathbb{B}$ *there exists a point* β *such that* $\beta\neq\pi_b$, $b\in\mathbb{B}^{\beta}_{\pi_b}$, $\mathtt{fr}(b)=\mathrm{S}^{\beta}_{\pi_b}$ *and* $\mathtt{int}(b)=\mathrm{B}^{\beta}_{\pi_b}$.

Directly from Lemma 10.7 we obtain the following fact.

Corollary 10.10. *For any different points* α *and* β *there is exactly one ball* $b\in\mathbb{B}^{\beta}_{\alpha}$ *such that* $\mathtt{int}(b)=\mathrm{B}^{\beta}_{\alpha}$.

Theorem 10.11. *For any* $B\in\mathbf{BO}_{\Pi}$ *there exists exactly one mereological ball* $b\in\mathbb{B}$ *such that* $\mathtt{int}(b)=B$.

Proof. For any $B\in\mathbf{BO}_{\Pi}$ there are different points α and β such that $B=\mathrm{B}^{\beta}_{\alpha}$. By Corollary 10.10 there is exactly one ball $b\in\mathbb{B}^{\beta}_{\alpha}$ such that $\mathtt{int}(b)=\mathrm{B}^{\beta}_{\alpha}=B$. ⊣

§11. Transitivity of the concentricity relation. Now we can show that relation $\circledcirc$ is transitive in all T-structures, which means that points from Π are equivalence classes of concentric balls

$$\forall_{a,b,c\in\mathbb{B}}\big(a\circledcirc b\wedge b\circledcirc c\Longrightarrow a\circledcirc c\big).\qquad(\mathrm{t}_{\circledcirc})$$

Indeed, suppose that $a\circledcirc b$ and $b\circledcirc c$. Then $a,c\in\pi_b$, by $(\mathrm{s}_{\circledcirc})$. Moreover, by Corollary 9.11(iii), for some $\alpha_0,\gamma_0\in\Pi$ we have: $\pi_b\neq\alpha_0\in\mathtt{fr}(a)$ and $\pi_b\neq\gamma_0\in\mathtt{fr}(c)$. Thus $a\in\mathbb{B}^{\alpha_0}_{\pi_b}$ and $c\in\mathbb{B}^{\gamma_0}_{\pi_b}$. Now, by Corollary 10.2, for some $\alpha_1,\gamma_1\in\Pi$ such that $\alpha_1\neq\pi_a$, $\gamma_1\neq\pi_c$ we have $a\in\mathbb{B}^{\alpha_1}_{\pi_a}$ and $c\in\mathbb{B}^{\gamma_1}_{\pi_c}$. So it must be that $\pi_a=\pi_b=\pi_c$, by Corollary 10.8. In consequence $a\circledcirc c$, by $(\mathrm{r}_{\circledcirc})$.

Corollary 11.1. $\forall_{\gamma\in\Pi}\forall_{a,b\in\gamma}\,a\circledcirc b$ (*see Remark* 9.1).

From $(\mathrm{r}_{\circledcirc})$, $(\mathrm{s}_{\circledcirc})$ and $(\mathrm{t}_{\circledcirc})$ we obtain that

$$\forall_{a,b\in\mathbb{B}}\big(a\circledcirc b\iff\pi_a=\pi_b\big),\qquad(11.1)$$

$$\Pi=\mathbb{B}/_{\circledcirc},\qquad(11.2)$$

i.e., points from Π are equivalence classes of concentric balls. Moreover

$$\forall_{\beta\in\Pi}\forall_{b\in\mathbb{B}}\big(b\in\beta\iff\pi_b=\beta\big).\qquad(11.3)$$

Indeed, let $b \in \beta$, i.e., $\beta = \pi_a$ and $b \circledcirc a$, for some $a \in \mathbb{B}$. Then $\pi_b = \pi_a = \beta$, by (11.1). Conversely, if $\pi_b = \beta$, then $b \in \beta$, because $b \in \pi_b$.

COROLLARY 11.2. *For any $\beta \in \Pi$ the image $\mathtt{int}[\beta]$ is the family of all open balls in $\langle \Pi, \Delta^{\Pi} \rangle$ such that β is their center.*

PROOF. Let $b \in \beta$. Then $\beta = \pi_b$, by (11.3). By Theorem 10.9, there exists a point α such that $\alpha \neq \beta$ and $\mathtt{int}(b) = \mathrm{B}^{\alpha}_{\beta}$. Conversely, if B is a ball such that β is its center, then there exists a point α such that $\alpha \neq \beta$ and $B = \mathrm{B}^{\alpha}_{\beta}$. By Corollary 10.10, there is exactly one ball $b \in \mathbb{B}^{\alpha}_{\beta}$ such that $\mathtt{int}(b) = \mathrm{B}^{\alpha}_{\beta}$. ⊣

§12. Main facts for T-structures. All facts from this section hold for any T-structure $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ in which $\mathtt{s}$ sum $\mathbb{B}$ and $\mathbb{S} := \{z \in \mathbb{R} \mid \exists_{X \in 2^{\mathbb{B}}}\, z \text{ sum } X\}$.

In Fact 8.6(i) we proved that the mapping $\mathtt{int}|_{\mathbb{S}}$ is an isomorphism from $\langle \mathbb{S}, \sqsubseteq_{\mathbb{S}} \rangle$ onto $\langle \mathbf{RO}^{+}_{\Pi}, \subseteq \rangle$. Now we are going to strengthen this result.

FACT 12.1. *The mapping $\mathtt{int}|_{\mathbb{S}}$ is an isomorphism from $\langle \mathbb{S}, \mathbb{B}, \sqsubseteq_{\mathbb{S}} \rangle$ onto $\langle \mathbf{RO}^{+}_{\Pi}, \mathbf{BO}_{\Pi}, \subseteq \rangle$; so the mapping $\mathtt{int}|_{\mathbb{B}}$ is a bijection from $\mathbb{B}$ onto $\mathbf{BO}_{\Pi}$.*

PROOF. By Fact 8.6(i), $\mathtt{int}|_{\mathbb{S}}$ is an isomorphism from $\langle \mathbb{S}, \sqsubseteq_{\mathbb{S}} \rangle$ onto $\langle \mathbf{RO}^{+}_{\Pi}, \subseteq \rangle$. So it suffices to show that $\mathtt{int}[\mathbb{B}] = \mathbf{BO}_{\Pi}$. Firstly, if $b \in \mathbb{B}$, then $\mathtt{int}(b) \in \mathbf{BO}_{\Pi}$, because by Theorem 10.9 for some $\beta_0 \in \Pi$ we have $\beta_0 \neq \pi_b$ and $\mathtt{int}(b) = \mathrm{B}^{\beta_0}_{\pi_x}$. Thus $\mathtt{int}[\mathbb{B}] \subseteq \mathbf{BO}_{\Pi}$. Moreover, by Theorem 10.11, $\mathbf{BO}_{\Pi} \subseteq \mathtt{int}[\mathbb{B}]$. ⊣

From the above fact and Lemma 8.3 we obtain Theorem 1.5, which was formulated in §1.3: *The structure $\langle \mathbb{S}, \mathbb{B}, \sqsubseteq_{\mathbb{S}} \rangle$ is isomorphic to $\langle \mathbf{RO}^{+}_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq \rangle$.*

Now we will prove that for all solids the scope of the existential quantifier in (df $\bigcirc$) may by limited to $\mathbb{B}$, i.e., two overlapping solids have some ball as common part. This enables us to prove that any product of overlapping solids is a solid.[16] But first we will show that for all balls the scope of the existential quantifier in (df $\bigcirc$) may be also limited to $\mathbb{B}$.

THEOREM 12.2. *For any $a, b \in \mathbb{B}$*

$$a \bigcirc b \iff \exists_{c \in \mathbb{B}}(c \sqsubseteq a \wedge c \sqsubseteq b).$$

PROOF. Let $a, b \in \mathbb{B}$. Of course if $\exists_{c \in \mathbb{B}}(c \sqsubseteq a \wedge c \sqsubseteq b)$, then $a \bigcirc b$, by (df $\bigcirc$).

Conversely, suppose that $a \bigcirc b$. If $a \sqsubseteq b$ or $b \sqsubseteq a$, then $\exists_{c \in \mathbb{B}}(c \sqsubseteq a \wedge c \sqsubseteq b)$. Thus let $a \not\sqsubseteq b$ and $b \not\sqsubseteq a$. In such a case there exist regions $a \sqcap b$, $a - b$ and $b - a$. Since $a - b \sqsubset a$ and $b - a \sqsubset b$, then, by (8.5), $\mathtt{int}(a - b) \subsetneq \mathtt{int}(a)$ and $\mathtt{int}(b - a) \subsetneq \mathtt{int}(b)$. Moreover, by Theorem 10.9, $\mathtt{int}(a)$ and $\mathtt{int}(b)$ are balls from $\mathbf{BO}_{\Pi}$, and $\mathtt{fr}(a)$ and $\mathtt{fr}(b)$ are non-degenerate spheres from $\mathbf{S}^{\circ}_{\Pi}$. Let $B_a := \mathtt{int}(a)$, $B_b := \mathtt{int}(b)$, $S_a := \mathtt{fr}(a)$ and $S_b := \mathtt{fr}(b)$.

[16]Notice that these facts are equivalent neither to (P4′) which says that any part of any solid is a solid, nor to (P4″) which says the any part of any ball has a part which is a ball as well.

Assume towards a contradiction, that $\neg\exists_{c\in\mathbb{B}}(c \sqsubseteq a \wedge c \sqsubseteq b)$, i.e., $\neg\exists_{c\in\mathbb{B}}\, c \sqsubseteq a \sqcap b$, by (A.15). Hence, by (6.4) and Corollary 9.2, $B_a \cap B_b = \mathtt{int}(a) \cap \mathtt{int}(b) = \mathtt{int}(a \sqcap b) = \emptyset$. By Lemma 9.7(iii) and (iv) we obtain

$$B_a = \mathtt{int}(a - b) \cup \big(\mathtt{fr}(a - b) \cap B_a\big) = \mathtt{int}(a - b) \cup (S_b \cap B_a),$$
$$B_b = \mathtt{int}(b - a) \cup \big(\mathtt{fr}(b - a) \cap B_b\big) = \mathtt{int}(b - a) \cup (S_a \cap B_b).$$

Since $B_a \cap B_b = \emptyset$, then also $B_a \cap S_b = \emptyset = S_a \cap B_b$. Thus $\mathtt{int}(a) = \mathtt{int}(a-b)$ and $\mathtt{int}(b) = \mathtt{int}(b - a)$, and, by Theorem 8.5, $a \sqsubseteq a - b$ and $b \sqsubseteq b - a$. So we get a contradiction. ⊣

THEOREM 12.3. *For any $s_1, s_2 \in \mathbb{S}$*

$$s_1 \bigcirc s_2 \iff \exists_{c\in\mathbb{B}}(c \sqsubseteq s_1 \wedge c \sqsubseteq s_2)$$
$$\iff \exists_{s\in\mathbb{S}}(s \sqsubseteq s_1 \wedge s \sqsubseteq s_2).$$

PROOF. Suppose that $s_1 \bigcirc s_2$. Then for some $x_0 \in \mathbb{R}$ we have $x_0 \sqsubseteq s_1$ and $x_0 \sqsubseteq s_2$. For some $X_1, X_2 \subseteq \mathbb{B}$ it is the case that s_1 sum X_1 and s_2 sum X_2. Since $s_1 \sqcap s_2 \sqsubseteq s_1$, so for some $b_1 \in X_1$, $b_1 \bigcirc (s_1 \sqcap s_2)$. It follows that $b_1 \bigcirc s_2$. Hence, by (A.22), for some $b_2 \in X_2$ we have $b_1 \bigcirc b_2$. Thus, by Theorem 12.2, for some $c_0 \in \mathbb{B}$ we obtain $c_0 \sqsubseteq b_1 \sqcap b_2$. Since $b_1 \sqsubseteq s_1$ and $b_2 \sqsubseteq s_2$, so $c_0 \sqsubseteq s_1 \sqcap s_2$. ⊣

THEOREM 12.4. *For any $s_1, s_2 \in \mathbb{S}$ such that $s_1 \bigcirc s_2$, the product $s_1 \sqcap s_2$ is a solid.*

PROOF. Suppose that $s_1 \bigcirc s_2$. By Theorem 12.3, $\mathtt{int}(s_1 \sqcap s_2) \neq \emptyset$. By Corollary 9.2, $\mathtt{int}(s_1 \sqcap s_2) = \mathtt{int}(s_1) \cap \mathtt{int}(s_2)$. Since $\mathtt{int}(s_1) \cap \mathtt{int}(s_2) \in \mathbf{RO}_\Pi$, then $\mathtt{int}(s_1 \sqcap s_2) \in \mathbf{RO}^+_\Pi$. So for some $s_0 \in \mathbb{S}$ we have $\mathtt{int}(s_0) = \mathtt{int}(s_1 \sqcap s_2)$. By Theorem 8.5, $s_0 \sqsubseteq s_1 \sqcap s_2$.

Now notice that s_0 sum $\{x \in \mathbb{S} \mid x \sqsubseteq s_1 \sqcap s_2\}$. Indeed, $\{s_0\} \subseteq \{x \in \mathbb{S} \mid x \sqsubseteq s_1 \sqcap s_2\} \subseteq \{x \in \mathbb{R} \mid x \sqsubseteq s_1 \sqcap s_2\}$, for some $s_2 \in \mathbb{S}$ we have s_2 sum $\{x \in \mathbb{S} \mid x \sqsubseteq s_1 \sqcap s_2\}$, and $\mathtt{int}(s_0) \subseteq \mathtt{int}(s_2) \subseteq \mathtt{int}(s_1 \sqcap s_2)$. So $\mathtt{int}(s_0) = \mathtt{int}(s_2) = \mathtt{int}(s_1 \sqcap s_2)$ and $s_0 = s_2$, by (8.6).

Suppose towards a contradiction that $s_0 \sqsubset s_1 \sqcap s_2$. Then there is $x \in \mathbb{R}$ such that $x \sqsubseteq s_1$, $x \sqsubseteq s_2$ and $x \wr s_0$. Hence for some $b_1 \in \mathbb{B}$ we have $b_1 \sqsubseteq s_1$ and $b_1 \bigcirc x$, because s_1 sum $\mathbb{B}_{s_1}$. Since $b_1 \bigcirc s_2$, so for some $b_2 \in \mathbb{B}$ we have $b_2 \sqsubseteq s_2$ and $b_1 \bigcirc b_2$, because s_2 sum $\mathbb{B}_{s_2}$ by (A.22). So, by Theorem 12.2, for some $b_0 \in \mathbb{S}$ we have $b_0 \sqsubseteq b_1 \sqcap b_2 \sqsubseteq s_1 \sqcap s_2$. On the other hand, there is no ball b such that $b \sqsubseteq s_1 \sqcap s_2$ and $b \bigcirc x$, since s_0 sum $\{x \in \mathbb{S} \mid x \sqsubseteq s_1 \sqcap s_2\}$. So $b_0 \wr x \sqsubseteq s_1 \sqcap s_2$, a contradiction. Thus $s_0 \not\sqsubset s_1 \sqcap s_2$ and $s_1 \sqcap s_2 = s_0 \in \mathbb{S}$. ⊣

COROLLARY 12.5. *In $\langle \Pi, \mathscr{O}_\Pi \rangle$ for any $s \in \mathbb{S}$*

$$\mathrm{Cl}_\Pi\, \mathtt{int}(s) = \mathtt{int}(s) \cup \mathtt{fr}(s),$$
$$\mathrm{Fr}_\Pi\, \mathtt{int}(s) = \mathtt{fr}(s).$$

Thus the set $\mathtt{int}(s) \cup \mathtt{fr}(s)$ is closed in $\langle \Pi, \mathscr{O}_\Pi \rangle$.

PROOF. $\mathrm{Cl}_\Pi\, \mathtt{int}(s) \subseteq \mathtt{int}(s) \cup \mathtt{fr}(s)$, by Corollary 9.10(i). Since $\mathtt{int}(s) \subseteq \mathrm{Cl}_\Pi\, \mathtt{int}(s)$, it is enough to show that $\mathtt{fr}(s) \subseteq \mathrm{Cl}_\Pi\, \mathtt{int}(s)$. Let $\beta \in \mathtt{fr}(s)$ and let U be an arbitrary set from $\mathbf{RO}_\Pi$ such that $\beta \in U$. We will demonstrate that $U \cap \mathtt{int}(s) \neq \emptyset$. Indeed, by (df $\mathtt{fr}$), for any $b \in \beta$ we have $b \bigcirc s$. Moreover, by (P3), for some $s_U \in \mathbb{S}$ we have $U = \mathtt{int}(s_U) \ni \beta$. Hence for some $b_0 \in \beta$, $b_0 \sqsubseteq s_U$. Therefore $\mathtt{int}(b_0) \subseteq U$ and $b_0 \bigcirc s$, so $\emptyset \neq \mathtt{int}(b_0) \cap \mathtt{int}(s) \subseteq U \cap \mathtt{int}(s)$, by Theorem 12.3 and Corollary 9.2. Thus $\beta \in \mathrm{Cl}_\Pi\, \mathtt{int}(s)$.

Moreover, $\mathrm{Fr}_\Pi\, \mathtt{int}(s) = \mathrm{Cl}_\Pi \setminus \mathrm{Int}_\Pi\, \mathtt{int}(s) = (\mathtt{int} \cup \mathtt{fr}(s)) \setminus \mathtt{int}(s) = \mathtt{fr}(s)$, because of the fact that $\mathtt{int}(s) \cap \mathtt{fr}(s) = \emptyset$. ⊣

Now we turn to a description of the partial operator $\perp$ from the set $\{s \in \mathbb{S} \mid s \sqsubset \mathbb{s}\}$ into itself. Let $s \in \mathbb{S}$ and $s \sqsubset \mathbb{s}$. By (P2) and (8.3), $\mathtt{int}(s) \in \mathbf{RO}_\Pi \setminus \{\emptyset, \Pi\}$. Notice that in the Boolean algebra of regular open subsets of Π, by (df $-^\circ$) we have

$$-^\circ\, \mathtt{int}(s) := \mathrm{Int}_\Pi(\Pi \setminus \mathtt{int}(s)) \in \mathbf{RO}_\Pi \setminus \{\emptyset, \Pi\}\,.$$

Indeed, since in this algebra $\mathbf{1} := \Pi \neq \mathtt{int}(s)$ and $\mathbf{0} := \emptyset \neq \mathtt{int}(s)$, so $\emptyset \neq -^\circ\, \mathtt{int}(s) \neq \Pi$. Notice that

$$\mathrm{Int}_\Pi(\Pi \setminus \mathtt{int}(s)) = \Pi \setminus \mathrm{Cl}_\Pi\, \mathtt{int}(s).$$

Therefore, $\mathrm{Cl}_\Pi\, \mathtt{int}(s) \neq \Pi$. By (P3) and (8.3) there is exactly one solid $s^\perp$ such that

$$\begin{aligned} s^\perp &:= (\iota\, x)(x \in \mathbb{S} \wedge \mathtt{int}(x) = \mathrm{Int}_\Pi(\Pi \setminus \mathtt{int}(s))),\\ \mathtt{int}(s^\perp) &= \Pi \setminus \mathrm{Cl}_\Pi\, \mathtt{int}(s),\\ \mathrm{Cl}_\Pi\, \mathtt{int}(s) &= \Pi \setminus \mathtt{int}(s^\perp). \end{aligned}$$

Hence, by (7.1), Fact 7.1(ii) and Corollaries 9.9(ii), 9.8(ii) and 12.5, and Theorem 8.5, we obtain that

$$\mathtt{int}(s^\perp) = \mathtt{int}(\mathbb{s} - s), \tag{12.1}$$

$$s^\perp \sqsubseteq \mathbb{s} - s. \tag{12.2}$$

Notice that

$$\begin{aligned} \mathbb{B}_{s^\perp} &= \{b \in \mathbb{B} \mid b \sqsubseteq \mathbb{s} - s\},\\ \{x \in \mathbb{S} \mid x \sqsubseteq s^\perp\} &= \{x \in \mathbb{S} \mid x \sqsubseteq \mathbb{s} - s\}. \end{aligned}$$

Indeed, by (12.2), $\mathbb{B}_{s^\perp} \subseteq \{b \in \mathbb{B} \mid x \sqsubseteq \mathbb{s} - s\}$ and $\{x \in \mathbb{S} \mid x \sqsubseteq s^\perp\} \subseteq \{x \in \mathbb{S} \mid x \sqsubseteq \mathbb{s} - s\}$. Moreover, for any $x \in \mathbb{S}$ such that $x \sqsubseteq \mathbb{s} - s$ we have $\mathtt{int}(x) \subseteq \mathtt{int}(\mathbb{s} - s) = \mathtt{int}(s^\perp)$. Hence $x \sqsubseteq s^\perp$, by (P4). Thus, by (4.3), we obtain

$$s^\perp \ \mathsf{sum}\ \{b \in \mathbb{B} \mid b \sqsubseteq \mathbb{s} - s\}. \tag{12.3}$$

By (A.21), $s^{\perp}$ sum $\{x \in \mathbb{S} \mid x \sqsubseteq s^{\perp}\}$; so

$$s^{\perp} \ \mathsf{sum}\ \{x \in \mathbb{S} \mid x \sqsubseteq \mathbb{s} - s\}. \tag{12.4}$$

Now we can prove that

$$s^{\perp} = \mathbb{s} - s. \tag{12.5}$$

Suppose that $s^{\perp} \sqsubset \mathbb{s} - s$. In such a case there exists a region $(\mathbb{s} - s) - s^{\perp}$. By (12.3) no ball is an ingrediens of this region. Moreover, $s \sqcup s^{\perp} \sqsubset \mathbb{s}$ and $(\mathbb{s} - s) - s^{\perp} = \mathbb{s} - (s \sqcup s^{\perp})$. By Lemma 4.2(i), $s \sqcup s^{\perp} \in \mathbb{S}$. Thus there exists $(s \sqcup s^{\perp})^{\perp}$ and $(s \sqcup s^{\perp})^{\perp} \sqsubseteq \mathbb{s} - (s \sqcup s^{\perp})$. Since $(s \sqcup s^{\perp})^{\perp}$ is a solid, so for some $b_0 \in \mathbb{B}$ we have $b_0 \sqsubseteq (s \sqcup s^{\perp})^{\perp}$. Therefore also $b_0 \sqsubseteq \mathbb{s} - (s \sqcup s^{\perp})$, a contradiction. Thus $s^{\perp} \not\sqsubset \mathbb{s} - s$ and $s^{\perp} = \mathbb{s} - s$ by (12.2).

COROLLARY 12.6. *For any $s \in \mathbb{S}$ such that $s \sqsubset \mathbb{s}$*

(i) $\mathbb{s} - s \in \mathbb{S}$,
(ii) $\mathtt{int}(-s) = \mathtt{int}(\mathbb{s} - s) = \mathrm{Int}_{\Pi}(\Pi \setminus \mathtt{int}(s)) \in \mathbf{RO}^{+}_{\Pi}$,
(iii) $\Pi \setminus \mathtt{fr}(s)$ *is not connected in* $\langle \Pi, \mathscr{O}_{\Pi} \rangle$.

PROOF. (i) By (12.5).

(ii) By (P2) and (12.1) $\mathtt{int}(\mathbb{s} - s) \in \mathbf{RO}^{+}_{\Pi}$. Of course $\mathbb{s} \bigcirc -s$ and $\mathbb{s} - s = \mathbb{s} \sqcap -s$. By (6.8) and Corollary 9.2, $\mathtt{int}(\mathbb{s} \sqcap -s) = \mathtt{int}(\mathbb{s}) \cap \mathtt{int}(-s) = \Pi \cap \mathtt{int}(-s) = \mathtt{int}(-s)$. Moreover, $\mathtt{int}(-s) = \mathtt{int}(\mathbb{s} - s) = \Pi \setminus (\mathtt{fr}(s) \cup \mathtt{int}(s)) = \Pi \setminus \mathrm{Cl}_{\Pi}\, \mathtt{int}(s) = \mathrm{Int}_{\Pi}(\Pi \setminus \mathtt{int}(s))$.

(iii) By Corollary 9.12, $\mathrm{Cl}_{\Pi}\, \mathtt{int}(s) \cap \mathtt{int}(\mathbb{s}-s) = \emptyset = \mathtt{int}(s) \cap \mathrm{Cl}_{\Pi}\, \mathtt{int}(\mathbb{s}-s)$. But, by (7.1), Fact 7.1(ii) and Corollary 9.9(ii), we have $\Pi \setminus \mathtt{fr}(s) = \mathtt{int}(s) \cup \mathtt{int}(\mathbb{s} - s)$. ⊣

COROLLARY 12.7. *For any $s_1, s_2 \in \mathbb{S}$ such that $s_1 \not\sqsubseteq s_2$, $s_1 - s_2 \in \mathbb{S}$ and $\mathtt{int}(s_1 - s_2) = \mathtt{int}(s_1) \cap \mathtt{int}(-s_2) \in \mathbf{RO}^{+}_{\Pi}$.*

PROOF. If $s_1 \not\sqsubseteq s_2$, then: $s_1 - s_2 := s_1 \sqcap -s_2 = s_1 \sqcap \mathbb{s} \sqcap -s_2 = s_1 \sqcap (\mathbb{s} - s_2)$. So, by Corollary 12.6(i) and Theorem 12.4, we obtain that $s_1 - s_2 \in \mathbb{S}$. Moreover, by Corollaries 9.2 and 12.6, $\mathtt{int}(s_1 - s_2) = \mathtt{int}(s_1 \sqcap -s_2) = \mathtt{int}(s_1) \cap \mathtt{int}(-s_2)$. ⊣

FACT 12.8. (i) $\bigcirc_{\mathbb{S}} = \bigcirc \cap (\mathbb{S} \times \mathbb{S})$ *and* $\wr_{\mathbb{S}} = \wr \cap (\mathbb{S} \times \mathbb{S})$.
(ii) $\mathsf{sum}_{\mathbb{S}} = \mathsf{sum} \cap (\mathbb{S} \times 2^{\mathbb{S}}) = \mathsf{sum} \cap (\mathbb{R} \times 2^{\mathbb{S}})$.

PROOF. (i) By Theorem 12.3.

(ii) By Lemma 4.2(ii) we have $\mathsf{sum} \cap (\mathbb{R} \times 2^{\mathbb{S}}) = \mathsf{sum} \cap (\mathbb{S} \times 2^{\mathbb{S}})$. We will show that $\mathsf{sum} \cap (\mathbb{S} \times 2^{\mathbb{S}}) \subseteq \mathsf{sum}_{\mathbb{S}}$. Let $\langle z, X \rangle \in \mathsf{sum} \cap (\mathbb{S} \times 2^{\mathbb{S}})$. Then $z \in \mathbb{S}$, $X \subseteq \mathbb{S}$, $\forall_{s \in X}\, s \sqsubseteq_{\mathbb{S}} z$ and for any $y \in \mathbb{R}$ such that $y \sqsubseteq z$ we have $\exists_{s \in X}\, s \bigcirc y$. Suppose that $y \sqsubseteq_{\mathbb{S}} z$. Then $y, z \in \mathbb{S}$ and $\exists_{s \in X}\, s \bigcirc_{\mathbb{S}} y$, by (i). Hence z $\mathsf{sum}_{\mathbb{S}}$ X.

Now we will show that $\mathsf{sum}_{\mathbb{S}} \subseteq \mathsf{sum} \cap (\mathbb{S} \times 2^{\mathbb{S}})$. Let z $\mathsf{sum}_{\mathbb{S}}$ X. Then $z \in \mathbb{S}$ and $\emptyset \neq X \subseteq \mathbb{S}$. Hence $\forall_{s \in X}\, s \sqsubseteq z$. Moreover, z sum $\mathbb{B}_z$ and for some $s_0 \in \mathbb{S}$ we have s_0 sum $\bigcup_{s \in X} \mathbb{B}_s$. Since $\forall_{s \in X}\, s \sqsubseteq z$, so $\bigcup_{s \in X} \mathbb{B}_s \subseteq \mathbb{B}_z$, by ($\mathrm{t}_{\sqsubseteq}$). Hence $s_0 \sqsubseteq z$, by (A.20). Suppose, towards a contradiction, that

$s_0 \sqsubset z$. Then, by Corollary 12.7, there is a solid $z - s_0$. But $z - s_0 \sqsubseteq_{\mathbb{S}} z$, so $s_1 \bigcirc_{\mathbb{S}} z - s_0$, for some $s_1 \in X$. Since $s_1 \sqsubseteq s_0$, so $s_0 \bigcirc_{\mathbb{S}} z - s_0$, a contradiction. Thus $s_0 = z$, i.e., z sum $\bigcup_{s \in X} \mathbb{B}_s$. Hence z sum X, by Lemma 4.1. $\dashv$

REMARK 12.1. By Corollary 12.7 and Fact 12.8 we obtain another proof of the fact that $\langle \mathbb{S}, \sqsubseteq_{\mathbb{S}} \rangle$ is a mereological structure.

Indeed, $\sqsubseteq_{\mathbb{S}}$ is reflexive, antisymmetric and transitive, respectively, by $(\mathrm{r}_{\sqsubseteq})$, $(\mathrm{as}_{\sqsubseteq})$ and $(\mathrm{t}_{\sqsubseteq})$. We will show that the partial order $\langle \mathbb{S}, \sqsubseteq_{\mathbb{S}} \rangle$ satisfies $(\mathrm{sep}_{\sqsubseteq})$. Let $x, y \in \mathbb{S}$ and $x \not\sqsubseteq_{\mathbb{S}} y$. Then $x - y \in \mathbb{S}$, by Corollary 12.7. We have $x - y = x \sqcap -y \sqsubseteq_{\mathbb{S}} x$ and there is no $u \in \mathbb{S}$ such that $u \sqsubseteq_{\mathbb{S}} x - y$ and $u \sqsubseteq_{\mathbb{S}} y$.

From $(\mathrm{t}_{\sqsubseteq})$, $(\mathrm{as}_{\sqsubseteq})$, $(\mathrm{sep}_{\sqsubseteq})$ and (df$\,\mathsf{sum}$) we obtain (fun$\,\mathsf{sum}$) for $\langle \mathbb{S}, \sqsubseteq_{\mathbb{S}} \rangle$, i.e., for any $X \in 2^{\mathbb{S}}$ and $z, y \in \mathbb{S}$: if z $\mathsf{sum}_{\mathbb{S}}$ X and y $\mathsf{sum}_{\mathbb{S}}$ X, then $z = y$.

By $(\exists!\,\mathsf{sum})$ for $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$, for any $X \subseteq \mathbb{S} \setminus \{\emptyset\}$ there is exactly one $z_0 \in \mathbb{R}$ such that z_0 sum X. But also z_0 $\mathsf{sum}_{\mathbb{S}}$ X, by Fact 12.8(ii). Thus $\langle \mathbb{S}, \sqsubseteq_{\mathbb{S}} \rangle$ satisfies $(\exists\,\mathsf{sum})$. $\dashv$

In §8.5 we demonstrated that no solid is an atom. This fact can be strengthen to the following theorem.

THEOREM 12.9. *No ingrediens of* $\mathtt{s}$ *is an atom, i.e.,*

$$\forall_{x \in \mathbb{R}} (x \sqsubseteq \mathtt{s} \Longrightarrow \exists_{y \in \mathbb{R}}\, y \sqsubset x).$$

PROOF. 1. Let a be an atom such that $a \sqsubseteq \mathtt{s}$, i.e., $a \sqsubseteq \mathtt{s}$ and $\neg \exists_{x \in \mathbb{R}}\, x \sqsubset a$. (Of course, by (8.7) $a \notin \mathbb{S}$, so $a \sqsubset \mathtt{s}$.) Then, by Corollary 9.11(i), for some $\beta_0 \in \Pi$ we have $\beta_0 \in \mathtt{fr}(a)$. Hence $\forall_{b \in \beta_0}\, b \bigcirc a$. But since a is an atom, so $\forall_{b \in \beta_0}\, a \sqsubseteq b$. (Of course $\forall_{b \in \beta_0}\, a \sqsubset b$, because $a \notin \mathbb{B}$). Thus $\{x \in \mathbb{R} \mid \forall_{b \in \beta_0}\, x \sqsubseteq b\} \neq \emptyset$.

2. For some $z_0 \in \mathbb{R}$ we have that z_0 sum $\{x \in \mathbb{R} \mid \forall_{b \in \beta_0}\, x \sqsubseteq b\}$, by 1 and $(\exists!\,\mathsf{sum})$. Moreover, by (A.1) we obtain that $z_0 = \sup_{\sqsubseteq}\{x \in \mathbb{R} \mid \forall_{b \in \beta_0}\, x \sqsubseteq b\}$, so $z_0 = \inf_{\sqsubseteq} \beta_0$. Therefore, $a \sqsubseteq z_0$ and $\forall_{b \in \beta_0}\, z_0 \sqsubseteq b$.

3. $\mathtt{int}(z_0) = \emptyset$. Indeed, suppose towards a contradiction that for some $b_0 \in \mathbb{B}$ we have $b_0 \sqsubseteq z_0$. Then $\forall_{b \in \beta_0}\, b_0 \sqsubseteq b$, by 2 and $(\mathrm{t}_{\sqsubseteq})$. Thus $\forall_{b \in \beta_0}\, \mathtt{int}(b_0) \subseteq \mathtt{int}(b)$. By Theorem 10.9, $\mathtt{int}(b_0)$ is a ball in $\langle \Pi, \Delta^{\Pi} \rangle$. By Corollary 11.2, the image $\mathtt{int}[\beta_0]$ is the family of all open balls in $\langle \Pi, \Delta^{\Pi} \rangle$ such that β_0 is their center. Hence we get a contradiction. Thus $\mathtt{int}(z_0) = \emptyset$, by (6.4).

4. $\forall_{b \in \beta_0}\, z_0 \sqsubset b$, since $z_0 \notin \mathbb{S}$, by 3 and (P2).

5. $z_0 \sqsubset \mathtt{s}$. Indeed, by Fact 8.2(ii), $\forall_{b \in \mathbb{B}}\, b \sqsubset \mathtt{s}$; so $z_0 \sqsubset \mathtt{s}$, by 4 and $(\mathrm{t}_{\sqsubset})$.

6. $\mathtt{fr}(z_0) = \{\beta_0\}$. Firstly, $\beta_0 \in \mathtt{fr}(z_0)$. Indeed, by 4 $\forall_{b \in \beta_0}\, b \not\sqsubseteq z_0$. Moreover, $\forall_{b \in \beta_0}\, b \bigcirc z_0$, because by 1 and 2 $a \sqsubseteq z_0$ and $a \sqsubseteq b$, for any $b \in \beta_0$. Secondly, suppose that $\alpha \in \mathtt{fr}(z_0)$ and $\alpha \neq \beta_0$. Then in $\langle \Pi, \Delta^{\Pi} \rangle$ for some point $\gamma_0 \in \Pi$ such that $\alpha \neq \gamma_0 \neq \beta_0$ we have $\mathrm{B}^{\gamma_0}_{\alpha} \cap \mathrm{B}^{\gamma_0}_{\beta_0} = \emptyset$. By Corollary 10.10 there are balls $b_1 \in \mathbb{B}^{\gamma_0}_{\alpha}$ and $b_2 \in \mathbb{B}^{\gamma_0}_{\beta_0}$ for which $\mathtt{int}(b_1) = \mathrm{B}^{\gamma_0}_{\alpha}$ and $\mathtt{int}(b_2) = \mathrm{B}^{\gamma_0}_{\beta_0}$. Since $b_1 \in \alpha$ and $b_2 \in \beta_0$, so by the assumption and by 4 we obtain that $z_0 \bigcirc b_1$ and $z_0 \sqsubset b_2$. Hence $b_1 \bigcirc b_2$ and, by Theorem 12.4, the product

$b_1 \sqcap b_2$ is a solid. So $\mathrm{B}^{\gamma_0}_{\alpha} \cap \mathrm{B}^{\gamma_0}_{\beta_0} = \mathtt{int}(b_\alpha) \cap \mathtt{int}(b_{\beta_0}) = \mathtt{int}(b_\alpha \sqcap b_{\beta_0}) \in \mathbf{RO}^+_\Pi$, by Corollary 9.2 and (P2). Thus we have a contradiction.

7. $\mathbb{s} - z_0 \notin \mathbb{S}$. By 3, 5, 6, (7.1), Fact 7.1 and Corollary 9.9(ii) we obtain that $\mathtt{int}(\mathbb{s} - z_0) = \Pi \setminus \{\beta_0\} \notin \mathbf{RO}^+_\Pi$. So $\mathbb{s} - z_0 \notin \mathbb{S}$, by (P2).

8. $\mathbb{s} - z_0 \ \mathsf{sum}\ \{\mathbb{s} - b \mid b \in \beta_0\}$. By Fact 8.2(ii) and by 4, $\forall_{b \in \beta_0}\, b \sqsubset \mathbb{s}$ and $z_0 \sqsubset \mathbb{s}$. Moreover, for any $b \in \beta_0$ we obtain that $\mathbb{s} - b \sqsubseteq \mathbb{s} - z_0$, since $z_0 \sqsubseteq b$, by 2. Suppose that $y \sqsubseteq \mathbb{s} - z_0$ and $\forall_{b \in \beta_0}\, y \wr \mathbb{s} - b$. Then $y \sqsubset \mathbb{s}$ and $y \wr z_0$, so $\forall_{b \in \beta_0}\, y \sqsubseteq b$. Hence $y \sqsubseteq z_0$, because $z_0 = \inf_{\sqsubseteq} \beta_0$, by 2. So we get a contradiction.

9. $\mathbb{s} - z_0 \in \mathbb{S}$. For any $b \in \beta_0$ we have that $\mathbb{s} - b \in \mathbb{S}$, by Corollary 12.6(i). So $\mathbb{s} - z_0 \in \mathbb{S}$, by 8 and Lemma 4.2(i).

10. Now notice that 7 and 9 are contradictory. So no ingrediens of $\mathbb{s}$ is an atom. ⊣

§13. Alternative axioms. T′-structures. In [21] Tarski mentioned as well two following axioms

$$\forall_{x \in \mathbb{R}} \forall_{s \in \mathbb{S}} (x \sqsubseteq s \Rightarrow x \in \mathbb{S}), \tag{P4$'$}$$

$$\forall_{x \in \mathbb{R}} \forall_{b \in \mathbb{B}} (x \sqsubseteq b \Rightarrow \exists_{a \in \mathbb{B}}\, a \sqsubseteq x), \tag{P4$''$}$$

suggesting that they are equivalent to (P4) on the basis of the remaining axioms. We will show now that (P4′) is equivalent to (P4″) and that (P4) follows from both of them.

Theorem 13.1. *Let* $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ *satisfy* (P0) *and let* $\mathbb{S} := \{z \in \mathbb{R} \mid \exists_{X \in 2^{\mathbb{B}}}\, z \ \mathsf{sum}\ X\}$. *Then in* $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ *the sentences* (P4′) *and* (P4″) *have the same value.*

Proof. Suppose that $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ satisfies (P4′). Then, by (P4′), for any $x \in \mathbb{R}$ and $b \in \mathbb{B}$ such that $x \sqsubseteq b$, $x \in \mathbb{S}$. Thus, by (4.2), $\exists_{a \in \mathbb{B}}\, a \sqsubseteq x$.

Now suppose that $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ satisfies (P4″), $x \in \mathbb{R}$, $s \in \mathbb{S}$ and $x \sqsubseteq s$. Since $s \ \mathsf{sum}\ \mathbb{B}_s$, by (df sum) for some $b_0 \in \mathbb{B}_s$ and $x_0 \in \mathbb{R}$ we have $x_0 \sqsubseteq x$ and $x_0 \sqsubseteq b_0$. Hence, by (P4″), for some $a_0 \in \mathbb{B}$, $a_0 \sqsubseteq x_0$. Therefore, $a_0 \sqsubseteq x$, by transitivity of $\sqsubseteq$. Thus $\mathbb{B}_x \neq \emptyset$. Hence, by ($\exists$ sum), for some $s_0 \in \mathbb{S}$ we have $s_0 \ \mathsf{sum}\ \mathbb{B}_x$. However, since $\mathbb{B}_x \subseteq \{y \in \mathbb{R} \mid y \sqsubseteq x\}$, then $s_0 \sqsubseteq x$, by (A.19) and (A.20).

Assume, towards a contradiction, that $x \not\sqsubseteq s_0$. Thus there is a region $x - s_0$. Since $x \sqsubseteq s$, also $x - s_0 \sqsubseteq s$. Hence, by (df sum), for some $c_0 \in \mathbb{B}_s$ and $y_0 \in \mathbb{R}$ we have $y_0 \sqsubseteq x - s_0$ and $y_0 \sqsubseteq c_0$. Consequently, by (P4″), for some $d_0 \in \mathbb{B}$, $d_0 \sqsubseteq y_0$. Since $y_0 \sqsubseteq x - s_0 \sqsubseteq x$, then $d_0 \sqsubseteq x$, and we conclude that $d_0 \in \mathbb{B}_x$. From the fact that $s_0 \ \mathsf{sum}\ \mathbb{B}_x$ we have that $d_0 \sqsubseteq s_0$. So we get a contradiction, because also $d_0 \sqsubseteq x - s_0$. As a result $x \sqsubseteq s_0$. Thus, by ($\mathrm{as}_{\sqsubseteq}$), $s_0 = x$ and $x \in \mathbb{S}$. ⊣

Theorem 13.2. *Let* $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ *satisfy* (P0) *and* (P4′), *and let* $\mathbb{S} := \{z \in \mathbb{R} \mid \exists_{X \in 2^{\mathbb{B}}}\, z \ \mathsf{sum}\ X\}$. *Then* $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ *also satisfies* (P4).

PROOF. For arbitrary solids s and s' such that $s \not\sqsubseteq s'$ there is the solid $s - s'$, by (P4′) (since $s - s' \sqsubseteq s$). Hence for some $b_0 \in \mathbb{B}$ we have $b_0 \sqsubseteq s - s'$. Furthermore, since $b_0 \sqsubseteq s - s' \sqsubseteq s$ and $s - s' \wr s'$, then $b_0 \sqsubseteq s$ and $b_0 \wr s'$, by (A.12). We have $\pi_{b_0} \in \mathtt{int}(s)$, by (df $\mathtt{int}$), and the facts that $b_0 \sqsubseteq s$ and $b_0 \in \pi_{b_0}$.

Let b be an arbitrary ball from π_{b_0}. Then $b \circledcirc b_0$, so either $b \sqsubseteq b_0$ or $b_0 \sqsubseteq b$. Hence, in both cases, $b \not\sqsubseteq s'$. Thus $\pi_{b_0} \notin \mathtt{int}(s')$, so $\mathtt{int}(s) \not\subseteq \mathtt{int}(s')$. ⊣

A T′-*structure* is any structure of the form $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ satisfying postulates (P0)–(P3) and (P4′), where $\mathbb{S} := \{z \in \mathbb{R} \mid \exists_{X \in 2^{\mathbb{B}}}\, z \mathrel{\mathsf{sum}} X\}$. In §17 we will construct an example of a T-structure which is not a T′-structure (see also Theorem 1.3 in §1.2). From Theorem 13.2 we obtain

COROLLARY 13.3. *All* T′-*structures are* T-*structures.*

LEMMA 13.4. *In all* T′-*structures*

$$\mathbb{1} \in \mathbb{S} \iff \mathbb{s} = \mathbb{1} \iff \mathbb{R} = \mathbb{S}.$$

PROOF. If $\mathbb{1} \in \mathbb{S}$, then $\forall_{x \in \mathbb{R}}(x \sqsubseteq \mathbb{1} \Rightarrow x \in \mathbb{S})$, by (P4′). Since $\forall_{x \in \mathbb{R}}\, x \sqsubseteq \mathbb{1}$, we have $\mathbb{R} = \mathbb{S}$. See also (5.3) and remember that $\mathbb{s} \in \mathbb{S}$. ⊣

Moreover, we obtain

LEMMA 13.5. *Let* $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ *satisfy* (P0) *and let* $\mathbb{S} := \{z \in \mathbb{R} \mid \exists_{X \in 2^{\mathbb{B}}}\, z \mathrel{\mathsf{sum}} X\}$. *Then in* $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ *the sentence* (P4′) *has the same value as the following sentence*[17]

$$\forall_{x \in \mathbb{R}} \forall_{s \in \mathbb{S}} (x \bigcirc s \Rightarrow x \sqcap s \in \mathbb{S}).$$

PROOF. Let $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ satisfy (P4′), $x \in \mathbb{R}$, $s \in \mathbb{S}$ and $x \bigcirc s$. Then there is $x \sqcap s$ such that $x \sqcap s \mathrel{\mathsf{sum}} \{y \in \mathbb{R} \mid y \sqsubseteq x \wedge y \sqsubseteq s\}$. We have $\emptyset \neq \{y \in \mathbb{R} \mid y \sqsubseteq x \wedge y \sqsubseteq s\} \subseteq \mathbb{S}$, by (P4′). So $x \sqcap s \in \mathbb{S}$, by Lemma 4.2(i).

Let $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ satisfy $\forall_{x \in \mathbb{R}} \forall_{s \in \mathbb{S}} (x \bigcirc s \Rightarrow x \sqcap s \in \mathbb{S})$. Then for any $x \in \mathbb{R}$, $s \in \mathbb{S}$ such that $x \sqsubseteq s$ we have $x \bigcirc s$ and $x = x \sqcap s$, by ($\mathrm{r}_{\sqsubseteq}$), (df $\bigcirc$) and (A.16). Thus $x = x \sqcap s \in \mathbb{S}$. ⊣

We can compare Fact 12.8 to the following fact.

FACT 13.6. *Let* $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ *satisfy* (P0) *and* (P4′), *and let* $\mathbb{B} \neq \emptyset$ *and* $\mathbb{S} := \{z \in \mathbb{R} \mid \exists_{X \in 2^{\mathbb{B}}}\, z \mathrel{\mathsf{sum}} X\}$. *Then*

(i) $\bigcirc_{\mathbb{S}} = \bigcirc \cap (\mathbb{S} \times \mathbb{S})$ *and* $\wr_{\mathbb{S}} = \wr \cap (\mathbb{S} \times \mathbb{S})$,

(ii) $\mathsf{sum}_{\mathbb{S}} = \mathsf{sum} \cap (\mathbb{S} \times 2^{\mathbb{S}}) = \mathsf{sum} \cap (\mathbb{R} \times 2^{\mathbb{S}})$.

PROOF. (i) By (P4′).

(ii) By Lemma 4.2(ii), $\mathsf{sum} \cap (\mathbb{R} \times 2^{\mathbb{S}}) = \mathsf{sum} \cap (\mathbb{S} \times 2^{\mathbb{S}})$. We will show that $\mathsf{sum}_{\mathbb{S}} \subseteq \mathsf{sum} \cap (\mathbb{S} \times 2^{\mathbb{S}})$. Let $z \mathrel{\mathsf{sum}_{\mathbb{S}}} X$. Then $z \in \mathbb{S}$ and $X \subseteq \mathbb{S}$. Suppose that $y \in \mathbb{R}$ and $y \sqsubseteq z$. Then, by (P4′), $y \in \mathbb{S}$, and so $y \sqsubseteq_{\mathbb{S}} z$. Thus, by the proper assumption, $\exists_{s \in X}\, s \bigcirc_{\mathbb{S}} y$. Also by the proper assumption, $\forall_{s \in X}\, s \sqsubseteq_{\mathbb{S}} z$. Thus $\langle z, X \rangle \in \mathsf{sum} \cap (\mathbb{S} \times 2^{\mathbb{S}})$.

[17] Notice that in the theory of T-structures we have only the following thesis (see Theorem 12.4): $\forall_{s_1, s_2 \in \mathbb{S}} (s_1 \bigcirc s_2 \Rightarrow s_1 \sqcap s_2 \in \mathbb{S})$.

Now we will show that $\mathsf{sum}\cap(\mathbb{S}\times 2^{\mathbb{S}}) \subseteq \mathsf{sum}_{\mathbb{S}}$. Let $\langle z, X\rangle \in \mathsf{sum}\cap(\mathbb{S}\times 2^{\mathbb{S}})$. Then $z \in \mathbb{S}$, $X \subseteq \mathbb{S}$, $\forall_{s\in X}\, s \sqsubseteq_{\mathbb{S}} z$ and for any $y \in \mathbb{R}$ such that $y \sqsubseteq z$ we have $\exists_{s\in X}\, s \bigcirc y$. Suppose the $y \sqsubseteq_{\mathbb{S}} z$. Then $y, z \in \mathbb{S}$ and $\exists_{s\in X}\, s \bigcirc_{\mathbb{S}} y$, by (i). Hence $z\ \mathsf{sum}_{\mathbb{S}}\ X$. ⊣

§**14. T*-structures.** Since the following sentence

$$\forall_{x\in\mathbb{R}}\, \exists_{Y\in 2^{\mathbb{B}}}\, x\ \mathsf{sum}\ Y \qquad \text{i.e., } \forall_{x\in\mathbb{R}}\, x \in \mathbb{S}, \text{ i.e., } \mathbb{R} = \mathbb{S}, \tag{$\star$}$$

entails (P4′), where $\mathbb{S} := \{z \in \mathbb{R} \mid \exists_{X\in 2^{\mathbb{B}}}\, z\ \mathsf{sum}\ X\}$, so along Tarski's intentions (see Introduction, p. 484) proper structures for the geometry of solids are those of the form $\langle\mathbb{R}, \mathbb{B}, \sqsubseteq\rangle$ and satisfying (P0)–(P3) plus ($\star$).

By Corollary 13.3 we obtain

COROLLARY 14.1. (i) *All* T**-structures are* T′*-structures.*
(ii) *All* T**-structures are* T*-structures.*

Now we turn to the proof of Theorem 1.4, which was stated in §1.2: *For any* T*-structure* $\langle\mathbb{R}, \mathbb{B}, \sqsubseteq\rangle$, $\langle\mathbb{S}, \mathbb{B}, \sqsubseteq_{\mathbb{S}}\rangle$ *is a* T**-structure.*[18]

PROOF OF THEOREM 1.4. Let $\langle\mathbb{R}, \mathbb{B}, \sqsubseteq\rangle$ be a T-structure. Then, by Fact 8.6(ii), $\langle\mathbb{S}, \sqsubseteq_{\mathbb{S}}\rangle$ is a mereological structure. Therefore $\langle\mathbb{S}, \mathbb{B}, \sqsubseteq_{\mathbb{S}}\rangle$ satisfies (P0) and ($\star$). By Fact 12.8 the relation $\circledcirc_{\mathbb{S}}$ for $\langle\mathbb{S}, \mathbb{B}, \sqsubseteq_{\mathbb{S}}\rangle$ coincides with the relation $\circledcirc$ for $\langle\mathbb{R}, \mathbb{B}, \sqsubseteq\rangle$. Thus both structures have the same set of points Π and the relation Δ^{Π}. In consequence $\langle\mathbb{S}, \mathbb{B}, \sqsubseteq_{\mathbb{S}}\rangle$ satisfies (P1)–(P4), since they are satisfied by $\langle\mathbb{R}, \mathbb{B}, \sqsubseteq\rangle$. ⊣

Let us notice now that directly from Theorem 1.5 (which was proved in §12) via Corollary 14.1(ii) and ($\star$) we obtain Theorem 1.6: *Any* T**-structure is isomorphic to* $\langle\mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq\rangle$, *so all* T**-structures are isomorphic.*

Clearly, the above theorem is also a consequence of Lemma 8.3 and the following fact which we obtain via Fact 12.1, Corollary 14.1(ii) and ($\star$).

FACT 14.2. *The mapping* $\mathtt{int}$ *is an isomorphism from a* T**-structure* $\langle\mathbb{R}, \mathbb{B}, \sqsubseteq_{\mathbb{S}}\rangle$ *onto* $\langle\mathbf{RO}^+_{\Pi}, \mathbf{BO}_{\Pi}, \subseteq\rangle$.

Of course the theorems presented in this section do not say anything about existence of any T-structure. This will be assured in the next section where we will prove that $\langle\mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq\rangle$ is a T*-structure. Only in such a case we will be entitled to say that the theory of T*-structures is categorical.

The following theorem presents equivalent sets of postulates for T*-structures.

THEOREM 14.3. *For any structure* $\langle\mathbb{R}, \mathbb{B}, \sqsubseteq\rangle$ *the following conditions are equivalent*

(a) *it is a* T**-structure,*
(b) *it is a* T*-structure satisfying* ($\star$),
(c) *it is a* T*-structure satisfying* ($\star\star$), *i.e.,* $\forall_{x\in\mathbb{R}}\, \exists_{b\in\mathbb{B}}\, b \sqsubseteq x$,

[18] This theorem can also be obtained directly from Theorem 1.5, which was proved in §12, and from Theorem 1.1, which we will prove further in §15.1.

(d) *it is a* T*-structure satisfying* (§), *i.e.*, $\forall_{x\in\mathbb{R}}\, x \;\mathsf{sum}\; \mathbb{B}_x$,
(e) *it is a* T*-structure satisfying* (\$), *i.e.*, $\forall_{x\in\mathbb{R}}\, \mathtt{int}(x) \neq \emptyset$,
(f) *it is a* T*-structure satisfying* (¶),
i.e., $\forall_{x,y\in\mathbb{R}}(x \bigcirc y \Rightarrow \mathtt{int}(x) \cap \mathtt{int}(y) \neq \emptyset)$,
(g) *it is a* T*-structure satisfying* $\forall_{x,y\in\mathbb{R}}\big(\mathtt{int}(x) \subseteq \mathtt{int}(y) \Rightarrow x \sqsubseteq y\big)$,
(h) *it is a* T′*-structure satisfying* $\mathbb{1} \in \mathbb{S}$,
(i) *it is a* T′*-structure satisfying* $\mathbb{s} = \mathbb{1}$,
(j) *it is a* T′*-structure satisfying* $\forall_{x\in\mathbb{R}}\,\forall_{s\in\mathbb{S}}\big(\mathtt{int}(x) \subseteq \mathtt{int}(s) \Rightarrow x \sqsubseteq s\big)$,
(k) *it satisfies* (P0), (P1), (P3), (P4) *and*

$$\forall_{x\in\mathbb{R}}\, \mathtt{int}(x) \in \mathbf{RO}^{+}_{\Pi}. \qquad (\mathrm{P2}^{\star})$$

Proof. (a) ⇔ (b) ⇔ ··· ⇔ (g) By Corollary 14.1, Lemmas 3.1, 6.1 and 6.2, and (8.1).

(a) ⇔ (h) ⇔ (i) By Lemma 13.4 and Corollary 14.1.

(b) ⇒ (j) From ($\star$) and (P4) we obtain $\forall_{x\in\mathbb{R}}\,\forall_{s\in\mathbb{S}}\big(\mathtt{int}(x) \subseteq \mathtt{int}(s) \Rightarrow x \sqsubseteq s\big)$.

(j) ⇒ (i) Since $\mathbb{s} \sqsubseteq \mathbb{1}$, $\mathbb{s} \in \mathbb{S}$ and $\mathtt{int}(\mathbb{1}) = \mathtt{int}(\mathbb{s})$, so $\mathbb{1} \sqsubseteq \mathbb{s}$, by (j). Thus $\mathbb{s} = \mathbb{1}$.

(b) ⇒ (k) From ($\star$) and (P2) we obtain (P2*).

(k) ⇒ (e) By logic. ⊣

In §1.5 we tried to persuade the reader that pondering—instead of T*-structures—structures of the form $\langle \mathbb{S}, \mathbb{B}, \sqsubseteq\rangle$ such that

- $\langle \mathbb{S}, \sqsubseteq\rangle$ is a mereological structure,
- $\mathbb{B}$ is a distinguished subset of $\mathbb{S}$,

and satisfying postulates (P1)–(P3) plus

$$\forall_{x\in\mathbb{S}}\,\exists_{Y\in 2^{\mathbb{B}}}\, x \;\mathsf{sum}\; Y \qquad (\star_{\mathbb{S}})$$

goes along Tarski's intentions. Thus, members of the universe $\mathbb{S}$ are to have property expressed in (df $\mathbb{S}$). Therefore elements of $\mathbb{S}$ are called *solids* and members of $\mathbb{B}$ *balls* (in the mereological sense).

The two approaches to the theory of T*-structures are definitionally equivalent in the following sense.

Theorem 14.4. *Let* R *and* S *be the classes of all* T**-structures in the first and the second sense, respectively. Then* $\mathsf{R} = \mathsf{S}$.

§15. Two examples of T*-structures. We will show that the class of T*-structures is not empty by constructing such a structure from regular open sets and open balls in three-dimensional Cartesian space $\langle \mathbb{R}^3, \varrho\rangle$ (see p. 485 and Appendixes C and D). The fact that two complete Boolean algebras of all regular open sets and of all regular closed sets in a given topological space are isomorphic (see Fact C.7) will entail that the universe of discourse of any T*-structure can also consist of regular closed subsets in $\langle \mathbb{R}^3, \varrho\rangle$.

In this section we will prove Theorem 1.1: *$\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq\rangle$ and $\langle \mathbf{RC}^+_{\mathbb{R}^3}, \mathbf{BC}_{\mathbb{R}^3}, \subseteq\rangle$ are* T**-structures.* The former one will be called *open*, the latter *closed*.

15.1. Open T*-structure. Let $\mathbb{R} := \mathbf{RO}^+_{\mathbb{R}^3}$, $\mathbb{B} := \mathbf{BO}_{\mathbb{R}^3}$ and $\sqsubseteq := \{\langle U, V\rangle \in \mathbb{R} \times \mathbb{R} \mid U \subseteq V\}$. By Corollary C.5, $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \subseteq\rangle$ is a mereological structure. Hence $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq\rangle$ satisfies (P0).

Notice that, by the accepted interpretation of $\sqsubseteq$ and by properties of sets from $\mathbf{RO}^+_{\mathbb{R}^3}$, for any $U, V \in \mathbf{RO}^+_{\mathbb{R}^3}$

$$U \bigcirc V \iff \exists_{W \in \mathbf{RO}^+_{\mathbb{R}^3}} W \subseteq U \cap V \iff U \cap V \neq \emptyset, \tag{15.1}$$

$$U \wr V \iff \neg\exists_{W \in \mathbf{RO}^+_{\mathbb{R}^3}} W \subseteq U \cap V \iff U \cap V = \emptyset. \tag{15.2}$$

Indeed, for $U, V \in \mathbf{RO}^+_{\mathbb{R}^3}$ we have $U \cap V \in \mathbf{RO}_{\mathbb{R}^3}$; so $U \cap V \neq \emptyset$ iff $U \cap V \in \mathbf{RO}^+_{\mathbb{R}^3}$.

Lemma 15.1. *In open structure the set of all solids coincides with the set $\mathbf{RO}^+_{\mathbb{R}^3}$, which entails the fact that the domain of open structure contains all solids and only them, i.e., $\mathbb{S} = \mathbb{R} := \mathbf{RO}^+_{\mathbb{R}^3}$. Hence $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq\rangle$ satisfies ($\star$).*

Proof. Let $U \in \mathbf{RO}^+_{\mathbb{R}^3}$. Since the family $\mathbf{BO}_{\mathbb{R}^3}$ is a base for $\langle \mathbb{R}^3, \varrho\rangle$, so for some $\mathcal{B}_U \subseteq \mathbf{BO}_{\mathbb{R}^3}$ we have $U = \bigcup \mathcal{B}_U$ and $U = \operatorname{Int}\operatorname{Cl} U = \operatorname{Int}\operatorname{Cl}\bigcup \mathcal{B}_U =: \sup_{\subseteq} \mathcal{B}_U$. Thus $U \operatorname{\mathsf{sum}}_{\subseteq} \mathcal{B}_U$. So U is a solid in $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq\rangle$.

Of course if $A \operatorname{\mathsf{sum}}_{\subseteq} \mathcal{B}$, where $\mathcal{B} \subseteq \mathbf{RO}^+_{\mathbb{R}^3}$, then $A = \operatorname{Int}\operatorname{Cl}\bigcup\mathcal{B}$. So $A \in \mathbf{RO}^+_{\mathbb{R}^3}$. ⊣

Notice that in open structure we have

$$\mathbb{1} = \mathsf{s} = \mathbb{R}^3 \in \mathbb{S}. \tag{15.3}$$

We introduce the convention that for any ball B from $\mathbf{BO}_{\mathbb{R}^3}$, m_B and r_B are respectively the middle point and the radius of B. We accept the standard definitions of the relations of internal and external tangency in $\langle \mathbb{R}^3, \varrho\rangle$ together with the definitions of external and internal diametrical tangency plus the concentricity relation for $A, B, C \in \mathbf{BO}_{\mathbb{R}^3}$

$$\begin{aligned}
A \mathbin{\mathbb{et}} B &\overset{\text{df}}{\iff} \varrho(\mathrm{m}_A, \mathrm{m}_B) = \mathrm{r}_A + \mathrm{r}_B,\\
A \mathbin{\mathbb{it}} B &\overset{\text{df}}{\iff} \mathrm{r}_B > \mathrm{r}_A \wedge \varrho(\mathrm{m}_A, \mathrm{m}_B) = \mathrm{r}_B - \mathrm{r}_A,\\
AB \mathbin{\mathbb{ed}} C &\overset{\text{df}}{\iff} A \mathbin{\mathbb{et}} B \wedge B \mathbin{\mathbb{et}} C \wedge \varrho(\mathrm{m}_A, \mathrm{m}_B) = \varrho(\mathrm{m}_A, \mathrm{m}_C) + \varrho(\mathrm{m}_C, \mathrm{m}_B),\\
AB \mathbin{\mathbb{id}} C &\overset{\text{df}}{\iff} A \mathbin{\mathbb{it}} C \wedge B \mathbin{\mathbb{it}} C \wedge \varrho(\mathrm{m}_A, \mathrm{m}_B) = \varrho(\mathrm{m}_A, \mathrm{m}_C) + \varrho(\mathrm{m}_C, \mathrm{m}_B),\\
A \circledc B &\overset{\text{df}}{\iff} \mathrm{m}_A = \mathrm{m}_B\,.
\end{aligned}$$

Lemma 15.2. *In open structure we have that $\mathbb{et} = \mathbb{ET}$, $\mathbb{it} = \mathbb{IT}$, $\mathbb{ed} = \mathbb{ED}$, $\mathbb{id} = \mathbb{ID}$ and $\circledc = \circledcirc$.*

Proof. It is provable that in $\langle \mathbb{R}^3, \varrho\rangle$ for any A, B and C from $\mathbf{BO}_{\mathbb{R}^3}$, by (15.2) we obtain.

$A \mathbb{ET} B$ iff $A \cap B = \emptyset$ and $\forall_{X,Y\in\mathbf{BO}_{\mathbb{R}^3}}(A \subseteq X \wedge A \subseteq Y \wedge X \cap B = \emptyset = Y \cap B \Rightarrow X \subseteq Y \vee Y \subseteq X)$ iff $A \mathbb{et} B$.

$A \mathbb{IT} B$ iff $A \subsetneq B$ and $\forall_{X,Y\in\mathbf{BO}_{\mathbb{R}^3}}(A \subseteq X \wedge A \subseteq Y \wedge X \subseteq B \wedge Y \subseteq B \Rightarrow X \subseteq Y \vee Y \subseteq X)$ iff $A \mathbb{it} B$.

$AB \mathbb{ED} C$ iff $A \mathbb{ET} C$ and $B \mathbb{ET} C$ and $\forall_{X,Y\in\mathbf{BO}_{\mathbb{R}^3}}(X \cap C = \emptyset = Y \cap C \wedge A \subseteq X \wedge B \subseteq Y \Rightarrow X \cap Y = \emptyset)$ iff $AB \mathbb{ed} C$.

$AB \mathbb{ID} C$ iff $A \mathbb{ET} C$ and $B \mathbb{ET} C$ and $\forall_{X,Y\in\mathbf{BO}_{\mathbb{R}^3}}(X\cap B = \emptyset = Y\cap B\wedge A \subseteq X \wedge B \subseteq Y \Rightarrow X \cap Y = \emptyset)$ iff $AB \mathbb{id} C$.

$A \circledcirc B$ iff either $A \subseteq B$ and $\forall_{X,Y\in\mathbf{BO}_{\mathbb{R}^3}}(XY \mathbb{ED} A \wedge X \mathbb{IT} B \wedge Y \mathbb{IT} B \Rightarrow XY \mathbb{ID} B)$ or $B \subseteq A$ and $\forall_{X,Y\in\mathbf{BO}_{\mathbb{R}^3}}(XY \mathbb{ED} B \wedge X \mathbb{IT} A \wedge Y \mathbb{IT} A \Rightarrow XY \mathbb{ID} A)$ iff $A \copyright B$. ⊣

In particular the above corollary shows that the relation of concentricity of balls in $\mathbb{R}^3$ coincides in open structure with the relation $\circledcirc$ characterized by means of mereological notions. Clearly $\copyright$ is reflexive, symmetrical and transitive, so this is an equivalence relation in $\mathbf{BO}_{\mathbb{R}^3} \times \mathbf{BO}_{\mathbb{R}^3}$. According to (def Π) mereological points in open structure are sets of concentric balls. Since $\circledcirc = \copyright$ and $\copyright$ is an equivalence relation, we have that every point is the equivalence class of some ball with respect to $\copyright$, i.e.,

$$\Pi = \mathbf{BO}_{\mathbb{R}^3}/_{\copyright}. \tag{15.4}$$

Hence for any $\beta \in \Pi$

$$A, B \in \beta \Longrightarrow \text{the middle point of } A = \text{the middle point of } B. \tag{15.5}$$

Thus, we define the function mid: $\Pi \to \mathbb{R}^3$ in the following way

$$\mathrm{mid}(\beta) := \text{the middle point of some (every) ball from } \beta. \tag{df mid}$$

Of course for any $\beta \in \Pi$

$$\mathrm{mid}(\beta) = \mathrm{m}_B \iff B \in \beta. \tag{15.6}$$

To facilitate notation and to improve its clarity let us accept that the middle point of any ball B from β will be sometimes denoted by 'm_β'. Thus, symbols '$\mathrm{mid}(\beta)$' and 'm_β' denote the same triple of numbers from $\mathbb{R}^3$. The following fact is obvious.

Lemma 15.3. *The mapping* mid *is a bijection from* Π *onto* $\mathbb{R}^3$.

According to (df Δ^Π), by the accepted interpretation of $\sqsubseteq$, (15.1) and by basic properties of $\mathrm{Fr}_{\mathbb{R}^3}$, for any $\alpha, \beta, \gamma \in \Pi$

$$\alpha\beta\, \Delta^\Pi \gamma \iff \alpha = \beta = \gamma \vee \exists_{C\in\gamma} \forall_{B\in\alpha\cup\beta}(B \nsubseteq C \wedge B \cap C \neq \emptyset) \tag{15.7}$$

$$\iff \alpha = \beta = \gamma \vee \exists_{C\in\gamma}\, \mathrm{m}_\alpha, \mathrm{m}_\beta \in \mathrm{Fr}_{\mathbb{R}^3} C\,. \tag{15.8}$$

LEMMA 15.4. *For any* $\alpha, \beta, \gamma \in \Pi$

$$\alpha\beta \, \Delta^{\Pi} \, \gamma \iff \mathrm{m}_\alpha \mathrm{m}_\beta \, \Delta^{\mathbb{R}^3} \, \mathrm{m}_\gamma .$$

PROOF. Assume that $\alpha\beta \, \Delta^{\Pi} \, \gamma$. Thus, by (15.8), either $\alpha = \beta = \gamma$ or for some $B_0 \in \gamma$ we have $\mathrm{m}_\alpha, \mathrm{m}_\beta \in S_0$, where S_0 is the sphere in $\langle \mathbb{R}^3, \varrho \rangle$ which is the surface of B_0. In the first case we have that $\varrho(\mathrm{m}_\alpha, \mathrm{m}_\gamma) = 0 = \varrho(\mathrm{m}_\beta, \mathrm{m}_\gamma)$. In the second case, since $B_0 \in \gamma$, we infer that $\mathrm{m}_\gamma = \mathrm{m}_{B_0}$. Let r_0 be a radius of B_0. Thus, $\varrho(\mathrm{m}_\alpha, \mathrm{m}_\gamma) = r_0 = \varrho(\mathrm{m}_\beta, \mathrm{m}_\gamma)$. Conversely, assume now that $\mathrm{m}_\alpha \mathrm{m}_\beta \, \Delta^{\mathbb{R}^3} \, \mathrm{m}_\gamma$, i.e., $\varrho(\mathrm{m}_\alpha, \mathrm{m}_\gamma) = \varrho(\mathrm{m}_\beta, \mathrm{m}_\gamma)$. If $\varrho(\mathrm{m}_\alpha, \mathrm{m}_\gamma) = 0$, then clearly $\mathrm{m}_\alpha = \mathrm{m}_\beta = \mathrm{m}_\gamma$, so $\alpha = \beta = \gamma$. From this and from (15.8) we obtain that $\alpha\beta \, \Delta^{\Pi} \, \gamma$. If $\varrho(\mathrm{m}_\alpha, \mathrm{m}_\gamma) \neq 0$, then for some $r > 0$ we have $\varrho(\mathrm{m}_\alpha, \mathrm{m}_\gamma) = r = \varrho(\mathrm{m}_\beta, \mathrm{m}_\gamma)$. Hence there exists $\mathrm{S}(\mathrm{m}_\gamma, r)$ such that $\mathrm{m}_\alpha, \mathrm{m}_\beta \in \mathrm{S}(\mathrm{m}_\gamma, r)$. Clearly, in $\mathbb{R}^3$ there is a ball B_0 such that $\mathrm{Fr}_{\mathbb{R}^3} B_0 = \mathrm{S}(\mathrm{m}_\gamma, r)$. The point m_γ is its middle point, so $B_0 \in \gamma$. Concluding, $B_0 \in \gamma$ and $\mathrm{m}_\alpha, \mathrm{m}_\beta \in \mathrm{Fr}_{\mathbb{R}^3} B_0$, so by (15.8) we obtain that $\alpha\beta \, \Delta^{\Pi} \, \gamma$. ⊣

From Lemmas 15.3 and 15.4, and Fact D.2, we obtain

COROLLARY 15.5. (i) mid *is an isomorphism from* $\langle \Pi, \Delta^{\Pi} \rangle$ *onto* $\langle \mathbb{R}^3, \Delta^{\mathbb{R}^3} \rangle$.
(ii) $\langle \Pi, \Delta^{\Pi} \rangle$ *is a Pieri's structure, so* $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq \rangle$ *satisfies* (P1).

Via Lemmas 8.3 and 15.5, we obtain that the bijection $\Im \colon \mathbf{RO}^+_{\Pi} \to \mathbf{RO}^+_{\mathbb{R}^3}$

$$\Im(U) := \mathrm{mid}[U],$$

is an isomorphism from $\langle \mathbf{RO}^+_{\Pi}, \mathbf{BO}_{\Pi}, \subseteq \rangle$ onto $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq \rangle$.

LEMMA 15.6. *For any* $U \in \mathbf{RO}^+_{\mathbb{R}^3}$

$$\mathrm{mid}^{-1}[U] = \{\beta \in \Pi \mid \exists_{B \in \beta} \, B \subseteq U\} =: \mathtt{int}(U).$$

PROOF. Let $U \in \mathbf{RO}^+_{\mathbb{R}^3}$. Assume that $\beta \in \mathrm{mid}^{-1}[U]$, i.e., $\mathrm{mid}(\beta) \in U$. Then for some ball $B \in \mathbf{BO}_{\mathbb{R}^3}$ we have $\mathrm{m}_B = \mathrm{mid}(\beta)$ and $B \subseteq U$. Moreover $B \in \beta$, by (15.6). Conversely, let $B \in \beta$ be such that $B \subseteq U$. Then, by (df mid), we have $\mathrm{mid}(\beta) = \mathrm{m}_B$. Hence $\mathrm{mid}(\beta) \in B \subseteq U$, which means that $\beta \in \mathrm{mid}^{-1}[U]$. ⊣

COROLLARY 15.7. (i) *For any* $U \in \mathbf{RO}^+_{\mathbb{R}^3}$, $\mathtt{int}(U) \in \mathbf{RO}_{\Pi}$.
(ii) *For any* $V \in \mathbf{RO}^+_{\Pi}$ *there is* $U \in \mathbf{RO}^+_{\mathbb{R}^3}$ *such that* $\mathtt{int}(U) = V$.
Hence $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq \rangle$ *satisfies* (P2) *and* (P3).

PROOF. (i) $\mathtt{int}(U) = \mathrm{mid}^{-1}[U] = \Im^{-1}(U) \in \mathbf{RO}_{\Pi}$.
(ii) We put $U := \Im(V)$. Then $\mathtt{int}(U) = \Im^{-1}(\Im(V)) = V$. ⊣

Since earlier we proved that open structure satisfies (P0), (P1) and ($\star$), then it is a T*-structure.

15.2. Closed T*-structure. We assume that $\mathbb{R} := \mathbf{RC}^+_{\mathbb{R}^3}$, $\mathbb{B} := \mathbf{BC}_{\mathbb{R}^3}$ and $\sqsubseteq := \{\langle F, G\rangle \in \mathbb{R} \times \mathbb{R} \mid F \subseteq G\}$. Let $\mathrm{Cl}_{\mathbb{R}^3}, \mathrm{Int}_{\mathbb{R}^3} \colon 2^{\mathbb{R}^3} \to 2^{\mathbb{R}^3}$ be the operations of interior and closure in $\langle \mathbb{R}^3, \varrho\rangle$. By Facts C.3, C.6 and C.7 we obtain

(i) $\mathrm{Cl}_{\mathbb{R}^3}$ is an isomorphism from $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq\rangle$ onto $\langle \mathbf{RC}^+_{\mathbb{R}^3}, \mathbf{RC}_{\mathbb{R}^3}, \subseteq\rangle$,
(ii) $\mathrm{Int}_{\mathbb{R}^3}$ is an isomorphism from $\langle \mathbf{RC}^+_{\mathbb{R}^3}, \mathbf{BC}_{\mathbb{R}^3}, \subseteq\rangle$ onto $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq\rangle$.

Indeed, for any $A \subseteq \mathbb{R}^3$: $A \in \mathbf{BO}_{\mathbb{R}^3}$ iff $\mathrm{Cl}_{\mathbb{R}^3} A \in \mathbf{BC}_{\mathbb{R}^3}$, and for any $A \subseteq \mathbb{R}^3$: $A \in \mathbf{BC}_{\mathbb{R}^3}$ iff $\mathrm{Int}_{\mathbb{R}^3} A \in \mathbf{BO}_{\mathbb{R}^3}$.

In effect we immediately have that $\langle \mathbf{RC}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq\rangle$ is a T*-structure.

§16. An example of T$'$-structure which is not a T*-structure. By Theorem 1.1, $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq\rangle$ is a T*-structure. For $\mathcal{F} := \mathbf{RO}^+_{\mathbb{R}^3}$ we construct the family $\mathcal{F}_{\mathsf{a}}$ such that $\mathcal{F} \subsetneq \mathcal{F}_{\mathsf{a}}$, $\forall_{A\in\mathcal{F}} \forall_{B\in\mathcal{F}_{\mathsf{a}}\setminus\mathcal{F}}\, B \nsubseteq A$ and $\langle \mathcal{F}_{\mathsf{a}}, \subseteq\rangle$ is a mereological structure (see Appendix B). Thus $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq\rangle = \langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq\rangle$ and the structure $\langle M, \sqsubseteq\rangle = \langle \mathcal{F}_{\mathsf{a}}, \subseteq\rangle$ satisfies the premises of Theorem 1.2 (see §1.2, p. 485). Thus, by the conclusion of this theorem, $\langle \mathcal{F}_{\mathsf{a}}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq\rangle$ is a T$'$-structure which is not a T*-structure.

Proof of Theorem 1.2. Of course $\langle M, \mathbb{B}, \sqsubseteq\rangle$ satisfies (P0) and thus $\langle \mathbb{R}, \sqsubseteq_{\mathbb{R}}\rangle$ is a mereological structure with $\mathsf{sum}_{\mathbb{R}}$. Since $\forall_{x\in\mathbb{R}} \forall_{y\in M\setminus\mathbb{R}}\, y \not\sqsubseteq_{\mathbb{R}} x$, we obtain $\bigcirc_{\mathbb{R}} = \bigcirc \cap (\mathbb{R}\times\mathbb{R})$, $\wr_{\mathbb{R}} = \wr \cap (\mathbb{R}\times\mathbb{R})$ and $\wr_{\mathbb{R}} = (\mathbb{R}\times\mathbb{R}) \setminus \bigcirc_{\mathbb{R}}$. We will prove that $\mathsf{sum}_{\mathbb{R}} = \mathsf{sum} \cap (\mathbb{R}\times 2^{\mathbb{R}}) = \mathsf{sum} \cap (M \times 2^{\mathbb{R}})$.

Suppose that $z\ \mathsf{sum}_{\mathbb{R}}\ X$. Then $z \in \mathbb{R}$ and $X \subseteq \mathbb{R}$. Notice that for any $y \in M$ such that $y \sqsubseteq z$, it must be the case that $y \in \mathbb{R}$. So if $y \sqsubseteq z$, then $y \sqsubseteq_{\mathbb{R}} z$ and $\exists_{x\in X} \exists_{u\in\mathbb{R}}(u \sqsubseteq_{\mathbb{R}} x \wedge u \sqsubseteq_{\mathbb{R}} y)$. Moreover, $\forall_{x\in X}\, x \sqsubseteq_{\mathbb{R}} z$. Hence $\langle z, X\rangle \in \mathsf{sum} \cap (\mathbb{R}\times 2^{\mathbb{R}})$. Thus, $\mathsf{sum}_{\mathbb{R}} \subseteq \mathsf{sum} \cap (\mathbb{R}\times 2^{\mathbb{R}}) \subseteq \mathsf{sum} \cap (M\times 2^{\mathbb{R}})$.

Now suppose that $z\ \mathsf{sum}\ X$ and $X \subseteq \mathbb{R}$. Then $X \neq \emptyset$ and, by ($\exists!\,\mathsf{sum}$), there is exactly one $z_0 \in \mathbb{R}$ such that $z_0\ \mathsf{sum}_{\mathbb{R}}\ X$, because $\langle \mathbb{R}, \sqsubseteq_{\mathbb{R}}\rangle$ is a mereological structure. But also $z_0\ \mathsf{sum}\ X$. Hence $z_0 = z$, by (fun sum). Thus, $\mathsf{sum} \cap (M \times 2^{\mathbb{R}}) \subseteq \mathsf{sum}_{\mathbb{R}}$.

By the above result, for any $x, y \in \mathbb{R}$ we obtain: $x \sqcup y := (\iota z)\ z\ \mathsf{sum}\ \{x, y\} = (\iota z)\ z\ \mathsf{sum}_{\mathbb{R}}\ \{x, y\} =: x \sqcup_{\mathbb{R}} y$; if $x \bigcirc y$, then $x \sqcap y := (\iota z)\ z\ \mathsf{sum}\ \{u \in M : u \sqsubseteq x \wedge u \sqsubseteq y\} = (\iota z)\ z\ \mathsf{sum}_{\mathbb{R}}\ \{u \in M : u \sqsubseteq_{\mathbb{R}} x \wedge u \sqsubseteq_{\mathbb{R}} y\} =: x \sqcap_{\mathbb{R}} y$; and if $x \sqsubset \mathsf{s} := (\iota z)\ z\ \mathsf{sum}_{\mathbb{R}}\ \mathbb{R}$, then $\mathsf{s} - x = (\iota z)\ z\ \mathsf{sum}\ \{y \in M \mid y \wr x\} = (\iota z)\ z\ \mathsf{sum}_{\mathbb{R}}\ \{y \in \mathbb{R} \mid y \wr_{\mathbb{R}} x\} = \mathsf{s} -_{\mathbb{R}} x$.

Now we notice that $\mathbb{S}_M := \{z \in M \mid \exists_{X\in 2^{\mathbb{B}}}\, z\ \mathsf{sum}\ X\} = \{z \in \mathbb{R} \mid \exists_{X\in 2^{\mathbb{B}}}\, z\ \mathsf{sum}_{\mathbb{R}}\ X\} =: \mathbb{S} = \mathbb{R}$. Thus $\langle M, \mathbb{B}, \sqsubseteq\rangle$ satisfies (P4$'$), because $\forall_{x\in\mathbb{R}} \forall_{y\in M}(y \sqsubseteq_{\mathbb{R}} x \Rightarrow y \in \mathbb{R})$.

Since $\bigcirc_{\mathbb{R}} = \bigcirc \cap (\mathbb{R}\times\mathbb{R})$ and $\wr_{\mathbb{R}} = \wr \cap (\mathbb{R}\times\mathbb{R})$, then $\langle M, \mathbb{B}, \sqsubseteq\rangle$ and $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq_{\mathbb{R}}\rangle$ have the same: the relation $\circledcirc$, the set of points Π and the relation Δ^{Π}. Hence $\langle M, \mathbb{B}, \sqsubseteq\rangle$ satisfies (P1)–(P3), since $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq_{\mathbb{R}}\rangle$ satisfies these postulates. Thus, $\langle M, \mathbb{B}, \sqsubseteq\rangle$ is a T$'$-structure. Since $\mathbb{S}_M = \mathbb{R} \neq M$, then $\langle M, \mathbb{B}, \sqsubseteq\rangle$ is not a T*-structure. $\dashv$

§17. An example of T-structure which is not a T′-structure. By Theorem 1.1, $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq\rangle$ is a T*-structure. Our example will be obtained from this fact and Theorem 1.3 (see §1.2, p. 486).

Proof of Theorem 1.3. Let $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq\rangle$ be a T*-structure. Let $\langle \mathbb{R}^{\mathsf{o}}, \leq, \mathsf{o}, \mathbb{1}\rangle$ be a non-degenerate complete Boolean lattice which we obtain from $\langle \mathbb{R}, \sqsubseteq\rangle$ by adding zero o ($x \leq y$ iff $x = \mathsf{o}$ or $x \sqsubseteq y$). We put $\mathbb{R}_\times := \mathbb{R}^{\mathsf{o}} \times \mathbb{R}^{\mathsf{o}}$ and $\leq_\times := \{\langle\langle x, y\rangle, \langle u, z\rangle\rangle \mid x \leq u \wedge y \leq z\}$. The product $\langle \mathbb{R}_\times, \leq_\times, \langle \mathsf{o}, \mathsf{o}\rangle, \langle \mathbb{1}, \mathbb{1}\rangle\rangle$ is also a non-degenerate complete Boolean lattice. We define $\mathbb{R}^+_\times := \mathbb{R}_\times \setminus \{\langle \mathsf{o}, \mathsf{o}\rangle\}$, $\sqsubseteq_\times := \leq_\times|_{\mathbb{R}^+_\times}$ and $\mathrm{d}\mathbb{B} := \{\langle b, b\rangle \mid b \in \mathbb{B}\}$.

Generally, for any set X, $\mathrm{d}X := \{\langle x, x\rangle \mid x \in X\}$.

1. By Lemma A.4, $\langle \mathbb{R}^+_\times, \sqsubseteq_\times\rangle$ is a mereological structure. So $\langle \mathbb{R}^+_\times, \mathrm{d}\mathbb{B}, \sqsubseteq_\times\rangle$ satisfies (P0).

2. For any $\langle x_1, x_2\rangle$ and $\langle y_1, y_2\rangle \in \mathbb{R}^+_\times$ we obtain

$$\begin{aligned} \langle x_1, x_2\rangle \bigcirc_\times \langle y_1, y_2\rangle &\iff x_1 \bigcirc y_1 \vee x_2 \bigcirc y_2, \\ \langle x_1, x_2\rangle \wr_\times \langle y_1, y_2\rangle &\iff x_1 \wr y_1 \wedge x_2 \wr y_2. \end{aligned}$$

Indeed, if $\langle x_1, x_2\rangle \bigcirc_\times \langle y_1, y_2\rangle$, then there is $\langle z_1, z_2\rangle \in \mathbb{R}^+_\times$ such that $z_1 \leq x_1$, $z_1 \leq y_1$, $z_2 \leq x_2$ and $z_2 \leq y_2$. Either $z_1 \neq \mathsf{o}$ or $z_2 \neq \mathsf{o}$. If $z_i \neq \mathsf{o}$, then $x_i \neq \mathsf{o} \neq y_i$. So $x_i \bigcirc y_i$ (for $i = 1, 2$). To prove the reverse implication let us assume that $x_1 \bigcirc y_1$, i.e., some $z_0 \in \mathbb{R}$ is such that $z_0 \sqsubseteq x_1$ and $z_0 \sqsubseteq y_1$. By the definition of $\mathbb{R}^+_\times$ it is the case that $\langle z_0, \mathsf{o}\rangle \in \mathbb{R}^+_\times$. Clearly $\langle z_0, \mathsf{o}\rangle \sqsubseteq_\times \langle x_1, x_2\rangle$ and $\langle z_0, \mathsf{o}\rangle \sqsubseteq_\times \langle y_1, y_2\rangle$, so $\langle x_1, x_2\rangle \bigcirc_\times \langle y_1, y_2\rangle$. The proof of the second case is similar.

3. By suitable definitions and by 2, for any $a, b \in \mathbb{B}$ we obtain

$$\begin{aligned} \langle a, a\rangle \sqsubseteq_\times \langle b, b\rangle &\iff a \sqsubseteq b, \\ \langle a, a\rangle \bigcirc_\times \langle b, b\rangle &\iff a \bigcirc b, \\ \langle a, a\rangle \wr_\times \langle b, b\rangle &\iff a \wr b, \\ \langle a, a\rangle \circledcirc_\times \langle b, b\rangle &\iff a \circledcirc b. \end{aligned}$$

4. $\mathrm{d}\mathbb{S}$ is the set of all solids in $\langle \mathbb{R}^+_\times, \mathrm{d}\mathbb{B}, \sqsubseteq_\times\rangle$, where—according to (df $\mathbb{S}$)—solids in $\langle \mathbb{R}^+_\times, \mathrm{d}\mathbb{B}, \sqsubseteq_\times\rangle$ are mereological sums of non-empty subsets of $\mathrm{d}\mathbb{B}$.

Indeed, suppose that $\langle s, s\rangle \in \mathrm{d}\mathbb{S}$, where $s \in \mathbb{S}$, i.e., $s \mathrel{\mathsf{sum}} X$, for some $\emptyset \neq X \subseteq \mathbb{B}$. Notice that $X = \mathrm{pr}_1[\mathrm{d}X] = \mathrm{pr}_2[\mathrm{d}X]$, where $\mathrm{pr}_i \colon \mathbb{R}^+_\times \to \mathbb{R}$ is the projection map defined by $\mathrm{pr}_i(\langle x_1, x_2\rangle) := x_i$, for $i = 1, 2$. Then, by Lemma A.4, $\langle s, s\rangle \mathrel{\mathsf{sum}_\times} \mathrm{d}X$. Of course $\mathrm{d}X \subseteq \mathrm{d}\mathbb{B}$. So $\langle s, s\rangle$ is a solid in $\langle \mathbb{R}^+_\times, \mathrm{d}\mathbb{B}, \sqsubseteq_\times\rangle$.

Now suppose that $\langle x, y\rangle$ is a solid in $\langle \mathbb{R}^+_\times, \mathrm{d}\mathbb{B}, \sqsubseteq_\times\rangle$, i.e., for some $\emptyset \neq \mathsf{X} \subseteq \mathrm{d}\mathbb{B}$, $\langle x, y\rangle \mathrel{\mathsf{sum}_{\sqsubseteq_\times}} \mathsf{X}$. Then $\emptyset \neq \mathrm{pr}_1[\mathsf{X}] = \mathrm{pr}_2[\mathsf{X}] \subseteq \mathbb{B}$. Since $\mathrm{pr}_1[\mathsf{X}] \neq \{\mathsf{o}\} \neq \mathrm{pr}_2[\mathsf{X}]$, by Lemma A.4 we obtain that $x \mathrel{\mathsf{sum}} \mathrm{pr}_1[\mathsf{X}]$ and $y \mathrel{\mathsf{sum}} \mathrm{pr}_2[\mathsf{X}]$. Thus, $x = y \in \mathbb{S}$, by (fun sum) and (df $\mathbb{S}$). So $\langle x, y\rangle \in \mathrm{d}\mathbb{S}$.

5. Clearly $\mathrm{d}\mathbb{S} \subsetneq \mathbb{R}^+_\times$ and for any $s \in \mathbb{R}$, $\langle \mathrm{o}, s\rangle \notin \mathrm{d}\mathbb{S}$. But $\langle \mathrm{o}, s\rangle \sqsubseteq_\times \langle s, s\rangle$. So $\langle \mathbb{R}^+_\times, \mathrm{d}\mathbb{B}, \sqsubseteq_\times\rangle$ does not satisfy (P4′).

6. Since $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq\rangle$ is a T*-structure, the relation $\circledcirc$ is an equivalence relation in $\mathbb{B} \times \mathbb{B}$. Hence, by 3, the relation $\circledcirc_\times$ is an equivalence relation in $\mathrm{d}\mathbb{B} \times \mathrm{d}\mathbb{B}$. Thus, the set $\Pi_\times$ of all points in $\langle \mathbb{R}^+_\times, \mathrm{d}\mathbb{B}, \sqsubseteq_\times\rangle$ has the following form

$$\Pi_\times := \mathrm{d}\mathbb{B}/_{\circledcirc_\times}.$$

So, by suitable definitions and by 3

$$\begin{aligned}\boldsymbol{\beta} \in \Pi_\times &\iff \exists_{\boldsymbol{b}\in \mathrm{d}\mathbb{B}}\, \boldsymbol{\beta} = \{\boldsymbol{x} \in \mathrm{d}\mathbb{B} \mid \boldsymbol{x} \circledcirc_\times \boldsymbol{b}\}\\ &\iff \exists_{b\in\mathbb{B}}\, \boldsymbol{\beta} = \{\langle x, x\rangle \in \mathrm{d}\mathbb{B} \mid x \circledcirc b\}.\end{aligned}$$

Moreover,

$$\begin{aligned}\boldsymbol{\beta} \in \Pi_\times &\iff \boldsymbol{\beta} \subseteq \mathrm{d}\mathbb{B} \wedge \mathrm{pr}_1[\boldsymbol{\beta}] = \mathrm{pr}_2[\boldsymbol{\beta}] \in \Pi\\ &\iff \exists_{\beta\in\Pi}\, \boldsymbol{\beta} = \mathrm{d}\beta.\end{aligned}$$

Indeed, if $\boldsymbol{\beta} \in \Pi_\times$, then $\boldsymbol{\beta} \subseteq \mathrm{d}\mathbb{B}$ and $\mathrm{pr}_1[\boldsymbol{\beta}] = \mathrm{pr}_2[\boldsymbol{\beta}] \in \Pi$. So for some $b_0 \in \mathbb{B}$ we have $\mathrm{pr}_1[\boldsymbol{\beta}] = \mathrm{pr}_2[\boldsymbol{\beta}] = \{b \in \mathbb{B} \mid b \circledcirc b_0\}$. Thus, $\mathrm{pr}_1[\boldsymbol{\beta}] = \mathrm{pr}_2[\boldsymbol{\beta}] = \pi_{b_0}$, and so $\mathrm{d}\pi_{b_0} := \{\langle b, b\rangle \mid b \in \pi_{b_0}\} = \boldsymbol{\beta}$. Conversely, suppose that for $\beta \in \Pi$ it is the case that $\mathrm{d}\beta = \boldsymbol{\beta}$. Then $\boldsymbol{\beta} \subseteq \mathrm{d}\mathbb{B}$ and $\mathrm{pr}_1[\boldsymbol{\beta}] = \mathrm{pr}_2[\boldsymbol{\beta}] = \beta$. So $\boldsymbol{\beta} \in \Pi_\times$.

Thus, we obtain

$$\Pi_\times = \{\mathrm{d}\beta \mid \beta \in \Pi\}. \tag{17.1}$$

7. The function $I : \Pi \to \Pi_\times$ such that

$$I(\beta) := \mathrm{d}\beta$$

is an isomorphism of structures $\langle \Pi, \Delta^\Pi\rangle$ and $\langle \Pi_\times, \Delta^{\Pi_\times}\rangle$. Thus, $\langle \Pi_\times, \Delta^{\Pi_\times}\rangle$ is a Pieri's structure, so $\langle \mathbb{R}^+_\times, \mathrm{d}\mathbb{B}, \sqsubseteq_\times\rangle$ satisfies (P1).

Indeed, directly from (17.1), I is surjective. To prove that I is one-to-one let us assume that $I(\beta) = I(\alpha)$, i.e., $\mathrm{d}\beta = \mathrm{d}\alpha$. Clearly, for any $b \in \mathbb{B}$: $b \in \beta$ iff $\langle b, b\rangle \in \mathrm{d}\beta$ iff $\langle b, b\rangle \in \mathrm{d}\alpha$ iff $b \in \alpha$. So $\alpha = \beta$. Thus, I is a bijection. Moreover, by 3 and 6, for any $\alpha, \beta, \gamma \in \Pi$

$$\begin{aligned}\alpha\beta\,\Delta^\Pi\,\gamma &\iff \alpha = \beta = \gamma \vee \exists_{b\in\gamma}\forall_{a\in\alpha\cup\beta}\big(a \not\sqsubseteq b \wedge a \bigcirc b\big)\\ &\iff \mathrm{d}\alpha = \mathrm{d}\beta = \mathrm{d}\gamma \vee \exists_{\boldsymbol{b}\in \mathrm{d}\gamma}\forall_{\boldsymbol{a}\in \mathrm{d}\,\alpha\cup \mathrm{d}\beta}\big(\boldsymbol{a} \not\sqsubseteq_\times \boldsymbol{b} \wedge \boldsymbol{a} \bigcirc_\times \boldsymbol{b}\big)\\ &\iff I(\alpha) = I(\beta) = I(\gamma) \vee \exists_{\boldsymbol{b}\in I(\gamma)}\forall_{\boldsymbol{a}\in I(\alpha)\cup I(\beta)}\big(\boldsymbol{a} \not\sqsubseteq_\times \boldsymbol{b} \wedge \boldsymbol{a} \bigcirc_\times \boldsymbol{b}\big)\\ &\iff I(\alpha)I(\beta)\,\Delta^{\Pi_\times}\,I(\gamma).\end{aligned}$$

8. For $s \in \mathbb{S}$, $I[\mathbb{int}(s)] = \mathbb{int}_\times(\langle s, s\rangle)$. Indeed, $\boldsymbol{\beta} \in I[\mathbb{int}(s)]$ iff $\exists_{\beta\in\mathbb{int}(s)}\, \boldsymbol{\beta} = I(\beta)$ iff $\exists_{\beta\in\Pi}\big(\boldsymbol{\beta} = \mathrm{d}\beta \wedge \exists_{b\in\mathbb{B}}\, b \sqsubseteq s\big)$ iff $\exists_{\beta\in\Pi}\big(\mathrm{d}\beta = \boldsymbol{\beta} \wedge \exists_{\boldsymbol{b}\in\boldsymbol{\beta}}\exists_{b\in\beta}\, \boldsymbol{b} = \langle b, b\rangle \sqsubseteq_\times \langle s, s\rangle\big)$ iff $\exists_{\boldsymbol{b}\in\boldsymbol{\beta}}\, \boldsymbol{b} \sqsubseteq_\times \langle s, s\rangle$ iff $\boldsymbol{\beta} \in \mathbb{int}_\times(\langle s, s\rangle)$.

9. Since $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ satisfies (P2), so for any $s \in \mathbb{S}$ we have $\mathbb{int}(s) \in \mathbf{RO}^{+}_{\Pi}$. Hence, by (D.6) and by 8, $\mathbb{int}_{\times}(\langle s, s \rangle) = I[\mathbb{int}(s)] \in \mathbf{RO}_{\Pi_{\times}} \setminus \{\emptyset\}$. Thus the structure $\langle \mathbb{R}^{+}_{\times}, \mathrm{d}\mathbb{B}, \sqsubseteq_{\times} \rangle$ satisfies (P2).

10. Assume that $\boldsymbol{U} \in \mathbf{RO}_{\Pi_{\times}} \setminus \{\emptyset\}$. Then, by (D.3) and (D.5), $I^{-1}[\boldsymbol{U}] = I^{-1}[\mathrm{Int}_{\Pi_{\times}} \mathrm{Cl}_{\Pi_{\times}} \boldsymbol{U}] = \mathrm{Int}_{\Pi} \mathrm{Cl}_{\Pi} I^{-1}[\boldsymbol{U}]$. So $I^{-1}[\boldsymbol{U}] \in \mathbf{RO}^{+}_{\Pi}$. Since $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ satisfies (P3), so for some $s_0 \in \mathbb{S}$ we have $I^{-1}[\boldsymbol{U}] = \mathbb{int}(s_0)$. Moreover, $\boldsymbol{U} = I[I^{-1}[\boldsymbol{U}]] = I[\mathbb{int}(s_0)] = \mathbb{int}_{\times}(\langle s_0, s_0 \rangle)$. So $\langle \mathbb{R}^{+}_{\times}, \mathrm{d}\mathbb{B}, \sqsubseteq_{\times} \rangle$ satisfies (P3).

11. Let $\mathbb{int}_{\times}(\langle s_1, s_1 \rangle) \subseteq \mathbb{int}_{\times}(\langle s_2, s_2 \rangle)$. Then, by 8, $I[\mathbb{int}(s_1)] \subseteq I[\mathbb{int}(s_2)]$. Since I is a bijection, we obtain that $\mathbb{int}(s_1) \subseteq \mathbb{int}(s_2)$. Hence $s_1 \sqsubseteq s_2$, because $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ satisfies (P4). So also $\langle s_1, s_1 \rangle \sqsubseteq_{\times} \langle s_2, s_2 \rangle$. It follows that $\langle \mathbb{R}^{+}_{\times}, \mathrm{d}\mathbb{B}, \sqsubseteq_{\times} \rangle$ satisfies (P4) as well.

To sum up, by 1, 7 and 9–11, $\langle \mathbb{R}^{+}_{\times}, \mathrm{d}\mathbb{B}, \sqsubseteq_{\times} \rangle$ is a T-structure in which $\mathrm{d}\mathbb{S}$ is the set of all solids. So, by 5, it is not a T′-structure. $\dashv$

§18. Logical status of Definitions 1–5. Now we go back to the problem briefly sketched in Remark 2.3. To answer the questions raised therein we will give short but exhaustive analysis of structures $\langle \mathbb{R}, \mathbb{B}, \circledcirc, \sqsubseteq \rangle$ in which $\mathbb{R}$ is a non-empty set, $\mathbb{B}$ is its subset while $\circledcirc$ and $\sqsubseteq$ are binary relations in $\mathbb{B}$ and $\mathbb{R}$, respectively. We assume that $\langle \mathbb{R}, \mathbb{B}, \circledcirc, \sqsubseteq \rangle$ satisfies (P0). Thus, as before, we can define the set of all solids by (df $\mathbb{S}$).

Analogously as in the case of T-structures, in $\langle \mathbb{R}, \mathbb{B}, \circledcirc, \sqsubseteq \rangle$ by a *point* we mean the set of all those balls that are concentric with a given ball. Again let Π be the set of all points, which we obtain by (df Π). Moreover, we introduce: the ternary relation Δ^{Π} in Π by (df Δ^{Π}); the operations $\mathbb{int}, \mathbb{fr} \colon \mathbb{R} \to 2^{\Pi}$ by (df $\mathbb{int}$) and (df $\mathbb{fr}$); and the solid $\mathbb{s}$ by (df $\mathbb{s}$). We assume that $\langle \Pi, \Delta^{\Pi} \rangle$ is a Pieri's structure, i.e., $\langle \mathbb{R}, \mathbb{B}, \circledcirc, \sqsubseteq \rangle$ satisfies (P1).

In our considerations carried out in §§2–9, starting from p. 493, the only facts obtained from the definition of the relation $\circledcirc$ (Definition 5) were its reflexivity and symmetry (we did not use its transitivity). Moreover, using (P1) and Definitions 4 and 5 we proved Fact 8.2(i) which says that

$$\mathbb{s} \notin \mathbb{B}. \tag{+}$$

In consequence we also had that $\mathbb{1} \notin \mathbb{B}$ (see the proof of Fact 8.2(iii)).

Further, we construct the topological space $\langle \Pi, \mathscr{O}_{\Pi} \rangle$. In this space we have the family of regular open sets $\mathbf{RO}_{\Pi}$ and it makes sense to speak about satisfaction of postulates (P2), (P3) and (P4).

Now we want to answer the following question: *What is the logical status of Definitions* 1–5 *in the theory of* T-*structures*? In light of what was written just above and of Remark 2.3, we used Definitions 1–5 in our constructions very few times. The most important results obtained by means of properties of the definitions at question were Fact 8.2(i) and Theorem 13.2. Thus we will try to examine structures $\langle \mathbb{R}, \mathbb{B}, \circledcirc, \sqsubseteq \rangle$ satisfying the same axioms as T*-structures except for the facts that we put either two or three axioms on the relation $\circledcirc$ (including no definitions) and we add one additional axiom which was obtained in Fact 8.2 by means (among others) the definitions currently absent.

Definition 11. We say that $\langle \mathbb{R}, \mathbb{B}, \circledcirc, \sqsubseteq \rangle$ is a $\mathrm{T}_{\circledcirc}$-*structure* iff it satisfies (P0)–(P4), $(\mathrm{r}_{\circledcirc})$, $(\mathrm{s}_{\circledcirc})$ and $(+)$. Moreover, a $\mathrm{T}_{\circledcirc}$-structure is called $\mathrm{T}^{\star}_{\circledcirc}$-*structure* iff it satisfies $(\star)$.

For $\mathrm{T}_{\circledcirc}$-structures we have all the facts from §§2–12, starting from p. 493. We will show that $\mathrm{T}^{\star}_{\circledcirc}$-structures are essentially the same as $\mathrm{T}^{\star}$-structures (with obvious changes).

Let $\langle \mathbb{R}, \mathbb{B}, \circledcirc, \sqsubseteq \rangle$ be a $\mathrm{T}_{\circledcirc}$-structure. Since $\langle \Pi, \Delta^{\Pi} \rangle$ is a Pieri's structure, then for balls from $\mathbf{BO}_{\Pi}$ we can define the relation $©^{\Pi}$ in the same way as we did it on p. 520 in case of the relation © for balls from $\mathbf{BO}_{\mathbb{R}^3}$, i.e., for any $A, B \in \mathbf{BO}_{\Pi}$

$$A \,©^{\Pi}\, B \stackrel{\mathrm{df}}{\iff} \mathrm{m}_A = \mathrm{m}_B,$$

where m_A and m_B are middle points of A and B, respectively.

Since $\langle \mathbf{RO}^{+}_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \subseteq \rangle$ is a $\mathrm{T}^{\star}$-structure in which $\mathbb{s} = \mathbb{R}^3 \notin \mathbf{BO}_{\mathbb{R}^3}$, we obtain

Theorem 18.1. *$\langle \mathbf{RO}^{+}_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, ©, \subseteq \rangle$ is a $\mathrm{T}^{\star}_{\circledcirc}$-structure.*

Lemma 18.2. *$\langle \mathbf{RO}^{+}_{\Pi}, \mathbf{BO}_{\Pi}, ©^{\Pi}, \subseteq \rangle$ is isomorphic to $\langle \mathbf{RO}^{+}_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, ©, \subseteq \rangle$.*

Proof. We refer back to the proof of Lemma 8.3 and use the fact that for any α, β from Π, $I[\mathrm{B}^{\beta}_{\alpha}] = \mathrm{B}^{I(\beta)}_{I(\alpha)}$ (which is routinely verified). Thus, for any $A, B \in \mathbf{BO}_{\Pi}$: $A \,©^{\Pi}\, B$ iff $\mathrm{m}_A = \mathrm{m}_B$ iff $\mathrm{m}_{I[A]} = \mathrm{m}_{I[B]}$ iff $I[A] \,©\, I[B]$. ⊣

Fact 18.3. *$\mathtt{int}|_{\mathbb{S}}$ is an isomorphism from $\langle \mathbb{S}, \mathbb{B}, \circledcirc, \sqsubseteq_{\mathbb{S}} \rangle$ onto $\langle \mathbf{RO}^{+}_{\Pi}, \mathbf{BO}_{\Pi}, ©^{\Pi}, \subseteq \rangle$; so $\mathtt{int}|_{\mathbb{B}}$ is a bijection from $\mathbb{B}$ onto $\mathbf{BO}_{\Pi}$.*

Proof. By Fact 12.1 the mapping $\mathtt{int}|_{\mathbb{S}}$ is an isomorphism from $\langle \mathbb{S}, \mathbb{B}, \sqsubseteq_{\mathbb{S}} \rangle$ onto $\langle \mathbf{RO}^{+}_{\Pi}, \mathbf{BO}_{\Pi}, \subseteq \rangle$. Thus we only have to prove that for any $a, b \in \mathbb{B}$

$$a \circledcirc b \iff \mathtt{int}(a) \,©^{\Pi}\, \mathtt{int}(b).$$

Let $a \circledcirc b$. Then, by $(\mathrm{r}_{\circledcirc})$, $a, b \in \pi_b$, so $\pi_b \in \mathtt{int}(a)$ and $\pi_b \in \mathtt{int}(b)$. Moreover, by (7.1) and Corollary 9.11(iii), for some $\alpha_0, \beta_0 \in \Pi$ such that $\alpha_0 \neq \pi_b \neq \beta_0$ we have $\alpha_0 \in \mathtt{fr}(a)$ and $\beta_0 \in \mathtt{fr}(b)$. Hence $a \in \mathbb{B}^{\alpha_0}_{\pi_b}$ and $b \in \mathbb{B}^{\beta_0}_{\pi_b}$. Therefore, by Lemma 10.7(i), $\mathtt{int}(a) = \mathrm{B}^{\alpha_0}_{\pi_b}$ and $\mathtt{int}(b) = \mathrm{B}^{\beta_0}_{\pi_b}$. Now we immediately obtain that $\mathtt{int}(a) \,©\, \mathtt{int}(b)$. Conversely, let $\mathtt{int}(a) \,©\, \mathtt{int}(b)$. Then, by Theorem 10.9, for some $\alpha_0, \beta_0 \in \Pi$ such that $\alpha_0 \neq \pi_a$ and $\beta_0 \neq \pi_b$ we have $a \in \mathbb{B}^{\alpha_0}_{\pi_a}$, $b \in \mathbb{B}^{\beta_0}_{\pi_b}$, $\mathtt{int}(a) = \mathrm{B}^{\alpha}_{\pi_a}$ and $\mathtt{int}(b) = \mathrm{B}^{\beta}_{\pi_b}$. Thus $\mathrm{B}^{\alpha}_{\pi_a} \,©\, \mathrm{B}^{\beta}_{\pi_b}$, which means that $\pi_a = \pi_b$; so $a \circledcirc b$, by $(\mathrm{r}_{\circledcirc})$. ⊣

Corollary 18.4. *Any $\mathrm{T}^{\star}_{\circledcirc}$-structure $\langle \mathbb{R}, \mathbb{B}, \circledcirc, \sqsubseteq \rangle$*

(i) *is isomorphic to $\langle \mathbf{RO}^{+}_{\Pi}, \mathbf{BO}_{\Pi}, ©^{\Pi}, \subseteq \rangle$,*

(ii) *is isomorphic to $\langle \mathbf{RO}^{+}_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, ©, \subseteq \rangle$.*

Notice that the following conditions are equivalent

(a) $\langle \mathbb{R}, \mathbb{B}, \circledcirc, \sqsubseteq \rangle$ is a $\mathrm{T}^{\star}_{\circledcirc}$-structure,

(b) $\langle \mathbb{R}, \mathbb{B}, \circledcirc, \sqsubseteq \rangle$ satisfies (P0)–(P3), $(\mathrm{r}_{\circledcirc})$, $(\mathrm{s}_{\circledcirc})$, $(+)$, $(\star)$ and

$$\forall_{a,b\in\mathbb{B}}\big(a \circledcirc b \Longrightarrow a \sqsubseteq b \vee b \sqsubseteq a\big). \tag{++}$$

Indeed, because $(++)$ is true in $\langle \mathbf{RO}^{+}_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \copyright, \subseteq \rangle$, so it is true in every $\mathrm{T}^{\star}_{\circledcirc}$-structure. Conversely, from $(\star)$ we have $(\mathrm{P4}')$. Moreover, thanks to (P0), $(\mathrm{P4}')$ and $(++)$ we get (P4) (see the proof of Theorem 13.2).

Let us notice that $(++)$ was the fourth (and the last one) fact inferred from Definition 5 which we used in §§2–14. Therefore, for the relation $\circledcirc$ to be definable in $\langle \mathbb{R}, \mathbb{B}, \circledcirc, \sqsubseteq \rangle$, we have to take as its definition a sentence which is true in $\langle \mathbf{RO}^{+}_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \copyright, \subseteq \rangle$. Since there is such a sentence, then we obtain the following theorem.

THEOREM 18.5. *The relation* $\circledcirc$ *is definable in every* $\mathrm{T}_{\circledcirc}$*-structure.*

Clearly, an acceptance of some definition of the concentricity relation that stands in contradiction to geometric intuitions would result in an inconsistent theory. Such a definition would have to be satisfied in every $\mathrm{T}_{\circledcirc}$-structure, in particular in $\langle \mathbf{RO}^{+}_{\mathbb{R}^3}, \mathbf{BO}_{\mathbb{R}^3}, \copyright, \subseteq \rangle$. But in such a case it would contradict some sentence that is true of $\copyright$.

FACT 18.6. *For any* $\mathrm{T}_{\circledcirc}$*-structure the relation* $\circledcirc$ *is the same as the one which we obtain by means of Definitions* 1–5.

But this is only one of possible solutions to the problem. Another one was presented by Jaśkowski in [10, 11].

For any structure $\langle \mathbb{R}, \mathbb{B}, \sqsubseteq \rangle$ satisfying (P0), in [11] Jaśkowski defined two notions: *being externally saturated* and *being internally concentric*.

DEFINITION 12. A ball a is *externally saturated* to balls b and c iff (i) a is external to b and c, and (ii) every ball such that a is its part overlaps either b or c (see Figure 9). Formally

$$a \,\mathbb{E}\$\, bc \;\overset{\mathrm{df}}{\Longleftrightarrow}\; a, b, c \in \mathbb{B} \wedge a \wr b \wedge a \wr c \wedge \forall_{x\in\mathbb{B}}(a \sqsubset x \Rightarrow x \bigcirc b \vee x \bigcirc c).$$

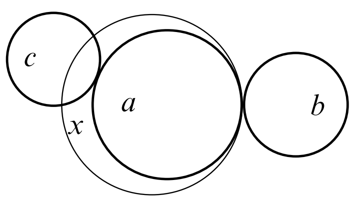


FIGURE 9. a is externally saturated to b and c.

Of course in three-dimensional Euclidean space for open (resp. closed) balls relations $\mathbb{E}\$$ and $\mathbb{ET}$ have suitable interpretations. It is routinely verified that

FACT 18.7. (i) *For any* A, B, C *from* $\mathbf{BO}_{\mathbb{R}^3}$ (*resp.* $\mathbf{BC}_{\mathbb{R}^3}$)

$$A \,\mathbb{ES}\, BC \iff B \nsubseteq C \wedge C \nsubseteq B \wedge A \,\mathbb{et}\, B \wedge A \,\mathbb{et}\, C\,.$$

(ii) *In three-dimensional Euclidean space for any open* (*resp. closed*) *balls* A, B, C

$$A \,\mathbb{ES}\, BC \iff B \nsubseteq C \wedge C \nsubseteq B \wedge A \,\mathbb{ET}\, B \wedge A \,\mathbb{ET}\, C\,.$$ [19]

FACT 18.8. *In any* T*-structure* (*resp.* $\mathrm{T}_{\circledcirc}$*-structure*) *for any* a, b, c *from* $\mathbb{B}$

$$a \,\mathbb{ES}\, bc \iff b \not\sqsubseteq c \wedge c \not\sqsubseteq b \wedge a \,\mathbb{ET}\, b \wedge a \,\mathbb{ET}\, c.$$

PROOF. By Fact 18.7, Lemma 8.3 and Fact 12.1 (resp. Lemma 18.2 and Fact 18.3). ⊣

DEFINITION 13. A ball a is *internally concentric* to a ball b iff

(i) a is an ingrediens of b, and
(ii) for all balls x, y, z: if
- x is external to b,
- y is externally saturated to a and x, and y is an ingrediens of b,
- z is external to y and a is an ingrediens of z,

then x is external to z (see Figure 10).

Formally

$$a \,\mathbb{IC}\, b \overset{\mathrm{df}}{\iff} a, b \in \mathbb{B} \wedge a \sqsubseteq b \\ \wedge \forall_{x,y,z \in \mathbb{B}} (x \wr b \wedge y \,\mathbb{ES}\, ax \wedge y \sqsubseteq b \wedge z \wr y \wedge a \sqsubseteq z \Rightarrow x \wr z).$$

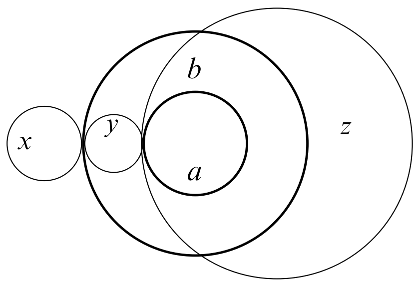


FIGURE 10. a is internally concentric to b.

Jaśkowski points to the fact that

FACT 18.9. *In three-dimensional Euclidean space for any open* (*resp. closed*) *balls* A, B

$$A \,\mathbb{IC}\, B \iff A \subseteq B \wedge \mathrm{m}_A = \mathrm{m}_B\,.$$

Moreover, we obtain that

FACT 18.10. *If the relation* $\sqsubseteq$ *is reflexive, then so is the relation* $\mathbb{IC}$.

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[19] In [11, Theorem 1] Jaśkowski only asserts that for any open (resp. closed) balls A, B, C: if $B \cap C = \emptyset$, then $A \,\mathbb{ES}\, BC$ iff $A \,\mathbb{ET}\, B$ and $A \,\mathbb{ET}\, C$.

PROOF. By the definition of $\mathbb{IC}$ for any $a \in \mathbb{B}$: $a \,\mathbb{IC}\, a$ iff $a \sqsubseteq a$ and $\forall_{x,y,z \in \mathbb{B}}(x \wr a \wedge y \,\mathbb{ES}\, ax \wedge y \sqsubseteq a \wedge z \wr y \wedge a \sqsubseteq z \Rightarrow x \wr z)$. But, by the definition of $\mathbb{ES}$, if $y \,\mathbb{ES}\, ax$ then $y \wr a$; so $y \not\sqsubseteq a$, by $(\mathrm{r}_{\sqsubseteq})$ and $(\mathrm{df}\wr)$. ⊣

Now we can define in $\mathbb{B}$ the concentricity relation as follows

$$a \circledcirc_{\mathrm{J}} b \stackrel{\mathrm{df}}{\iff} a \,\mathbb{IC}\, b \vee b \,\mathbb{IC}\, a.$$

REMARK 18.1. In [11] Jaśkowski defined not the concentricity relation but *the middle point of a given ball b* as the set $\{x \in \mathbb{B} \mid x \,\mathbb{IC}\, b \vee b \,\mathbb{IC}\, x\}$. By a *point* he means the middle point of some ball.

From a technical point of view the difference is irrelevant. We chose this definition since it is a better one from a point of view of analyzing properties of definitions of concentricity in T-structures. ⊣

Immediately from the above definition we have that

$$\forall_{b \in \mathbb{B}}\, b \circledcirc_{\mathrm{J}} b, \tag{$\mathrm{r}_{\circledcirc_{\mathrm{J}}}$}$$

$$\forall_{a,b \in \mathbb{B}}(a \circledcirc_{\mathrm{J}} b \iff b \circledcirc_{J} a), \tag{$\mathrm{s}_{\circledcirc_{\mathrm{J}}}$}$$

$$\forall_{a,b \in \mathbb{B}}(a \circledcirc_{\mathrm{J}} b \Longrightarrow a \sqsubseteq b \vee b \sqsubseteq a), \tag{$++_{\mathrm{J}}$}$$

which are those properties of the concentricity relation that are essential to develop the theory of T-structures (resp. $\mathrm{T}_{\circledcirc}$-structures).

Now we can define the set of all points in a standard way by means of $(\mathrm{df}\,\Pi)$ and define the equidistance relation (as Jaśkowski actually did) by $(\mathrm{df}'\,\Delta^{\Pi})$. So we can introduce all the axioms for T-, T′- and T*-structures, respectively.

Of course in three-dimensional Euclidean space for open (resp. closed) balls the relations $\mathbb{IC}$ and $\circledcirc_{\mathrm{J}}$ have proper interpretations. It is routinely verified that

FACT 18.11. *For any A, B from $\mathbf{BO}_{\mathbb{R}^3}$ (resp. $\mathbf{BC}_{\mathbb{R}^3}$)*

$$A \circledcirc_{\mathrm{J}} B \iff A \copyright B.$$

The above shows that Jaśkowski's definition is proper one and we infer that (cf. Fact 18.6)

FACT 18.12. *In any* T*-structure (resp.* $\mathrm{T}_{\circledcirc}$*-structure) for any a, b from* $\mathbb{B}$

$$a \circledcirc_{\mathrm{J}} b \iff a \circledcirc b.$$

Appendix A. Mereology. By a *mereological structure* we mean a relational structure $\langle M, \sqsubseteq\rangle$, where M is a non-empty set and $\sqsubseteq$ is a binary relation in $M \times M$. We will call the relation $\sqsubseteq$ —after Leśniewski [13]—*ingrediens relation*, i.e., in case if $x \sqsubseteq y$, we say that x *is an ingrediens of* y. Any mereological structure satisfies the following axioms (see [21])

$$\forall_{x,y,z \in M}(x \sqsubseteq y \wedge y \sqsubseteq z \Longrightarrow x \sqsubseteq z), \tag{$\mathrm{t}_{\sqsubseteq}$}$$

$$\forall_{X \in 2^M \setminus \{\emptyset\}} \exists!_{z \in M}\; z \;\mathsf{sum}\; X, \tag{$\exists!\,\mathsf{sum}$}$$

where sum is a binary relation in $M \times 2^M$ such that

$$z \;\mathsf{sum}\; X \overset{\text{df}}{\iff} \forall_{x \in X}\, x \sqsubseteq z \\ \wedge \forall_{y \in M}\big(y \sqsubseteq z \Rightarrow \exists_{x \in X}\, \exists_{u \in M}(u \sqsubseteq y \wedge u \sqsubseteq x)\big). \quad (\text{df}\,\mathsf{sum})$$

If $z \;\mathsf{sum}\; X$, then we say that z *is the mereological sum of elements of* X. The axiom ($\mathrm{t}_{\sqsubseteq}$) says that the relation $\sqsubseteq$ is transitive, ($\exists!\,\mathsf{sum}$) is the axiom of sum existence and says that the relation sum is a function of the second argument.

From the axioms of mereological structures we obtain that the relation $\sqsubseteq$ is also reflexive, antisymmetric and separative, and that being the mereological sum of elements of a given non-empty set coincides with being the supremum of elements of this set (see [16, 17, 21])

$$\forall_{x \in M}\, x \sqsubseteq x, \qquad (\mathrm{r}_{\sqsubseteq})$$

$$\forall_{x,y \in M}(x \sqsubseteq y \wedge y \sqsubseteq x \Longrightarrow x = y), \qquad (\mathrm{as}_{\sqsubseteq})$$

$$x \not\sqsubseteq y \Longrightarrow \exists_{z \in M}\big(z \sqsubseteq x \wedge \neg\exists_{u \in M}(u \sqsubseteq z \wedge u \sqsubseteq y)\big), \qquad (\mathrm{sep}_{\sqsubseteq})$$

$$\forall_{z \in M}\, \forall_{X \subseteq M}(z \;\mathsf{sum}\; X \iff X \neq \emptyset \wedge z = \sup\nolimits_{\sqsubseteq} X), \qquad (\mathrm{A.1})$$

$$|M| > 1 \iff \forall_{z \in M}\, \forall_{X \subseteq M}(z \;\mathsf{sum}\; X \Leftrightarrow z = \sup\nolimits_{\sqsubseteq} X), \qquad (\mathrm{A.2})$$

where '$|M|$' denotes the cardinal number of M.

Notice that from ($\mathrm{t}_{\sqsubseteq}$) and ($\mathrm{sep}_{\sqsubseteq}$) we obtain ($\mathrm{r}_{\sqsubseteq}$). Moreover, from ($\mathrm{t}_{\sqsubseteq}$), ($\mathrm{as}_{\sqsubseteq}$), ($\mathrm{sep}_{\sqsubseteq}$) and (df sum) we obtain (see [16, 17])

$$\forall_{X \in 2^M}\, \forall_{z,y \in M}(z \;\mathsf{sum}\; X \wedge y \;\mathsf{sum}\; X \Rightarrow z = y). \qquad (\text{fun}\,\mathsf{sum})$$

Thus, for mereological structures we may use axioms: ($\mathrm{t}_{\sqsubseteq}$), ($\mathrm{as}_{\sqsubseteq}$), ($\mathrm{sep}_{\sqsubseteq}$) and

$$\forall_{X \in 2^M \setminus \{\emptyset\}} \exists_{z \in M}\, z \;\mathsf{sum}\; X, \qquad (\exists\,\mathsf{sum})$$

with the definition (df sum).

We introduce the following definitions for any $x, y \in M$

$$x \sqsubset y \overset{\text{df}}{\iff} x \sqsubseteq y \wedge x \neq y, \qquad (\text{df}\,\sqsubset)$$

$$x \bigcirc y \overset{\text{df}}{\iff} \exists_{u \in M}(u \sqsubseteq x \wedge u \sqsubseteq y), \qquad (\text{df}\,\bigcirc)$$

$$x \wr y \overset{\text{df}}{\iff} \neg\exists_{u \in M}(u \sqsubseteq x \wedge u \sqsubseteq y), \qquad (\text{df}\,\wr)$$

$$\mathbb{1} := (\iota\, z)\; z \;\mathsf{sum}\; M, \qquad (\text{df}\,\mathbb{1})$$

$$x \sqcup y := (\iota\, z)\; z \;\mathsf{sum}\; \{x, y\} = (\iota\, z)\; z \;\mathsf{sum}\; \{u \in M \mid u \sqsubseteq x \vee u \sqsubseteq y\}, \qquad (\text{df}\,\sqcup)$$

$$x \bigcirc y \Longrightarrow x \sqcap y := (\iota\, z)\; z \;\mathsf{sum}\; \{u \in M \mid u \sqsubseteq x \wedge u \sqsubseteq y\}, \qquad (\text{df}\,\sqcap)$$

$$x \neq \mathbb{1} \Longrightarrow -x := (\iota\, z)\; z \;\mathsf{sum}\; \{y \in M \mid y \wr x\}. \qquad (\text{df}_1\, -)$$

If $x \sqsubset y$ (resp. $x \bigcirc y$; $x \wr y$) we will say that x *is a part of* y (resp. x *overlaps* y; x *is exterior to* y). The following are properties of mereological structures (see [16, 17])

$$\forall_{x\in M\setminus\{\mathbb{1}\}}\, x = - - x, \tag{A.3}$$

$$\forall_{x\in M\setminus\{\mathbb{1}\}}\, x \wr -x, \tag{A.4}$$

$$\forall_{x,y\in M}(x \wr y \iff y \neq \mathbb{1} \wedge x \sqsubseteq -y), \tag{A.5}$$

$$\forall_{x,y\in M}(x \not\sqsubseteq y \iff y \neq \mathbb{1} \wedge x \bigcirc -y), \tag{A.6}$$

$$\forall_{x,y\in M\setminus\{\mathbb{1}\}}(x \sqsubseteq y \iff -y \sqsubseteq -x), \tag{A.7}$$

So we introduce the following definition for any $x, y \in M$

$$x \not\sqsubseteq y \Longrightarrow x - y := x \sqcap -y. \tag{df$_2$ $-$}$$

Moreover, for any $x, y, z \in M$ and any $X, Y \subseteq M$ we have (see [16, 17])

$$x \bigcirc x, \tag{r$_\bigcirc$}$$

$$x \bigcirc y \iff \neg\, x \wr y, \tag{A.8}$$

$$x \sqsubset y \wedge y \sqsubset z \Longrightarrow x \sqsubset z, \tag{t$_\sqsubset$}$$

$$x \sqsubset y \wedge y \sqsubseteq z \Longrightarrow x \sqsubset z, \tag{A.9}$$

$$x \sqsubseteq y \wedge y \sqsubset z \Longrightarrow x \sqsubset z, \tag{A.10}$$

$$x \sqsubset y \Longrightarrow \exists_{z\in M}(z \sqsubset y \wedge z \wr x), \tag{A.11}$$

$$x \sqsubseteq y \iff \forall_{z\in M}(z \wr y \Rightarrow z \wr x), \tag{A.12}$$

$$x \sqcup y = \sup_{\sqsubseteq}\{x, y\}, \tag{A.13}$$

$$x \bigcirc y \Longrightarrow x \sqcap y = \inf_{\sqsubseteq}\{x, y\}, \tag{A.14}$$

$$x \bigcirc y \Longrightarrow \forall_{z\in M}(z \sqsubseteq x \sqcap y \Leftrightarrow z \sqsubseteq x \wedge z \sqsubseteq y), \tag{A.15}$$

$$x \sqsubseteq y \iff y = x \sqcup y \iff x = x \sqcap y, \tag{A.16}$$

$$y \sqsubset x \Longrightarrow y = x - (x - y), \tag{A.17}$$

$$x \;\mathsf{sum}\; \{x\}, \tag{A.18}$$

$$x \;\mathsf{sum}\; \{y \in M \mid y \sqsubseteq x\}, \tag{A.19}$$

$$x \;\mathsf{sum}\; X \wedge y \;\mathsf{sum}\; Y \wedge X \subseteq Y \Longrightarrow x \sqsubseteq y, \tag{A.20}$$

$$x \in X \Longrightarrow x \;\mathsf{sum}\; \{y \in X \mid y \sqsubseteq x\}, \tag{A.21}$$

$$z \;\mathsf{sum}\; X \iff \forall_{y\in M}\big(y \bigcirc z \Leftrightarrow \exists_{x\in X}\, x \bigcirc y\big), \tag{A.22}$$

$$z \;\mathsf{sum}\; X \Longrightarrow \forall_{y\in M}\big(y \sqsubseteq z \Leftrightarrow \forall_{u\in M}(u \sqsubseteq y \Rightarrow \exists_{x\in X}\, x \bigcirc u)\big), \tag{A.23}$$

$$|M| > 1 \iff \exists_{x,y\in M}\, x \wr y, \tag{$\exists\wr$}$$

$$|M| > 1 \iff \neg\exists_{x\in M}\,\forall_{y\in M}\, x \sqsubseteq y. \tag{$\nexists 0$}$$

Two following theorems[20] are crucial in showing that some structures we make use of are mereological structures. Let us remind here that a structure $\langle A, \leq, 0, 1\rangle$ is a *Boolean lattice* iff it is bounded, distributive and complemented lattice.

THEOREM A.1. [22, 16, 17] *If $\langle A, \leq, \mathbf{0}, \mathbf{1}\rangle$ is a non-degenerate complete Boolean lattice, $M := A \setminus \{\mathbf{0}\}$ and $\sqsubseteq := \leq|_M := \leq \cap (M \times M)$, then $\langle M, \sqsubseteq\rangle$ is a mereological structure and for any $x \in M$, $X \subseteq M$*

$$x \mathrel{\mathsf{sum}} X \iff X \neq \emptyset \wedge x = \sup\nolimits_{\leq} X.$$

It is known that for a Boolean algebra $\langle A, +, \cdot, -, \mathbf{0}, \mathbf{1}\rangle$, the structure $\langle A, \leq, \mathbf{0}, \mathbf{1}\rangle$ with $\leq$ defined as below

$$x \leq y \overset{\text{df}}{\iff} y = x + y \iff x = x \cdot y$$

is a Boolean lattice. Thus Theorem A.1 holds also for any non-degenerate complete Boolean algebra $\langle A, +, \cdot, -, \mathbf{0}, \mathbf{1}\rangle$ with the standard relation $\leq$.

THEOREM A.2. [22, 16, 17] *Let $\langle M, \sqsubseteq\rangle$ be a mereological structure, o be such that $\mathsf{o} \notin M$, $\mathbb{1}$ be the unity of $\langle M, \sqsubseteq\rangle$ and $M^{\circ} := M \cup \{\mathsf{o}\}$.*

(i) *Let $\leq := \sqsubseteq \cup (\{\mathsf{o}\} \times M^{\circ})$, i.e., for any $x, y \in M^{\circ}$*

$$x \leq y \overset{\text{df}}{\iff} x \sqsubseteq y \vee x = 0.$$

Then $\langle M^{\circ}, \leq, \mathsf{o}, \mathbb{1}\rangle$ is a non-degenerate complete Boolean lattice and for $x \in M$, $X \subseteq M$

$$x \mathrel{\mathsf{sum}} X \iff X \neq \emptyset \wedge x = \sup\nolimits_{\leq} X.$$

(ii) *Let for any $x, y \in M^{\circ}$*

$$x + y := \begin{cases} x \sqcup y & \text{if } x, y \in M, \\ x & \text{if } y = \mathsf{o}, \\ y & \text{if } x = \mathsf{o}, \end{cases} \qquad -x := \begin{cases} -x & \text{if } x \in M \setminus \{\mathbb{1}\}, \\ \mathsf{o} & \text{if } x = \mathbb{1}, \\ \mathbb{1} & \text{if } x = \mathsf{o}, \end{cases}$$

$$x \cdot y := \begin{cases} x \sqcap y & \text{if } x \bigcirc y, \\ \mathsf{o} & \text{if } x \wr y \text{ or } x = \mathsf{o} \text{ or } y = \mathsf{o}. \end{cases}$$

Then $\langle M^{\circ}, +, \cdot, -, \mathsf{o}, \mathbb{1}\rangle$ is a non-degenerate complete Boolean algebra and for $x, y \in M^{\circ}$

$$x \leq y \iff x \sqsubseteq y \vee x = \mathsf{o},$$

where $\leq$ is introduced in a standard way, i.e., for any $x, y \in M^{\circ}$: $x \leq y \iff y = x + y$. Moreover for any $x \in M$ and $X \subseteq M$

$$x \mathrel{\mathsf{sum}} X \iff X \neq \emptyset \wedge x = \sup\nolimits_{\leq} X.$$

[20] Concerning these theorems see Footnote 1 on pp. 333–334 in [22].

LEMMA A.3. [12] *Let* $\langle A, \leq, \mathbf{0}, \mathbf{1}\rangle$ *be a non-degenerate Boolean lattice,* $A_\times := A\times A$ *and* $\leq_\times := \{\langle\langle x, y\rangle, \langle u, z\rangle\rangle \mid x \leq u \wedge y \leq z\}$. *Then the Cartesian product* $\langle A_\times, \leq_\times, \langle \mathbf{0}, \mathbf{0}\rangle, \langle \mathbf{1}, \mathbf{1}\rangle\rangle$ *is also a non-degenerate Boolean lattice* (*the* product lattice).

If $\langle A, \leq, \mathbf{0}, \mathbf{1}\rangle$ *is complete, then* $\langle A_\times, \leq_\times, \langle \mathbf{0}, \mathbf{0}\rangle, \langle \mathbf{1}, \mathbf{1}\rangle\rangle$ *is also complete. Moreover, for any subset* $\mathcal{X}$ *of* $A_\times$

$$\sup_{\leq_\times} \mathcal{X} = \langle \sup_\leq \mathrm{pr}_1[\mathcal{X}], \sup_\leq \mathrm{pr}_2[\mathcal{X}]\rangle,$$

where $\mathrm{pr}_i \colon A_\times \to A$ *is the projection map defined by* $\mathrm{pr}_i(\langle x_1, x_2\rangle) := x_i$, *for* $i = 1, 2$.

LEMMA A.4. *Let* $\langle M, \sqsubseteq\rangle$ *be a mereological structure,* $\mathbb{1}$ *be its unity,* o *be such that* $\mathsf{o} \notin M$ *and* $M^\circ := M \cup \{\mathsf{o}\}$. *Let* $\langle M^\circ, \leq, \mathsf{o}, \mathbb{1}\rangle$ *be the non-degenerate complete Boolean lattice which we obtain by Theorem A.*2.

For $M_\times := M^\circ \times M^\circ$ *let* $\langle M_\times, \leq_\times, \langle \mathsf{o}, \mathsf{o}\rangle, \langle \mathbb{1}, \mathbb{1}\rangle\rangle$ *be the non-degenerate complete product lattice which we obtain by Lemma A.*3.

Then for $M^+_\times := M_\times \setminus \{\langle \mathsf{o}, \mathsf{o}\rangle\}$ *and* $\sqsubseteq_\times := \leq_\times|_{\mathbb{R}^+_\times}$, *by Theorem A.*1, $\langle M^+_\times, \sqsubseteq_\times\rangle$ *is a mereological structure. Moreover, by Lemma A.*3, *for any* $\mathcal{X} \subseteq M_\times$ *and* $\langle x, y\rangle \in M_\times$ *the condition* $\langle x, y\rangle \,\mathsf{sum}_{\sqsubseteq_\times}\, \mathcal{X}$ *holds iff one of the following conditions is the case*

(a) $x \,\mathsf{sum}\, \mathrm{pr}_1[\mathcal{X}]$ *and* $y \,\mathsf{sum}\, \mathrm{pr}_2[\mathcal{X}]$,
(b) $x = \mathsf{o}$, $\mathrm{pr}_1[\mathcal{X}] = \{\mathsf{o}\}$ *and* $y \,\mathsf{sum}\, \mathrm{pr}_2[\mathcal{X}]$,
(c) $x \,\mathsf{sum}\, \mathrm{pr}_1[\mathcal{X}]$, $y = \mathsf{o}$ *and* $\mathrm{pr}_2[\mathcal{X}] = \{\mathsf{o}\}$.

Appendix B. Some special family of sets. Let $X \neq \emptyset$ and $\mathcal{F}$ be a family of subsets of X such that $\emptyset, X \in \mathcal{F}$ and $\langle \mathcal{F}, \subseteq, \emptyset, X\rangle$ is a complete Boolean lattice.[21] Let a be such that $\mathsf{a} \notin X$. For any $A \subseteq X \cup \{\mathsf{a}\}$ we put

$$A^{\mathsf{a}} := A \cup \{\mathsf{a}\},$$
$$A^{-\mathsf{a}} := A \setminus \{\mathsf{a}\}.$$

We define the family $\mathcal{F}_{\mathsf{a}}$ as follows

$$\mathcal{F}_{\mathsf{a}} := \mathcal{F} \cup \{A^{\mathsf{a}} \in 2^{X\cup\{\mathsf{a}\}} \mid A \in \mathcal{F}\}. \tag{df $\mathcal{F}_{\mathsf{a}}$}$$

For any sets $A, B \in \mathcal{F}_{\mathsf{a}}$

$$\sup^{\mathsf{a}}_{\subseteq}\{A, B\} = \begin{cases} \sup_\subseteq\{A, B\} & \text{if } A, B \in \mathcal{F}, \\ (\sup_\subseteq\{A^{-\mathsf{a}}, B^{-\mathsf{a}}\})^{\mathsf{a}} & \text{otherwise,} \end{cases}$$

$$\inf^{\mathsf{a}}_{\subseteq}\{A, B\} := \begin{cases} (\inf_\subseteq\{A^{-\mathsf{a}}, B^{-\mathsf{a}}\})^{\mathsf{a}} & \text{if } A, B \in \mathcal{F}_{\mathsf{a}} \setminus \mathcal{F}, \\ \inf_\subseteq\{A^{-\mathsf{a}}, B^{-\mathsf{a}}\} & \text{otherwise,} \end{cases}$$

[21] It is possible that for some $\mathcal{G} \subsetneq \mathcal{F}$ we obtain: $\sup_\subseteq \mathcal{G} \neq \bigcup \mathcal{G}$. Boolean lattice of regular open (resp. closed) sets in a topological space can serve as an example, see Appendix C.

$$-^{\mathsf{a}}A := \begin{cases} (-A)^{\mathsf{a}} & \text{if } A \in \mathcal{F}, \\ -(A^{-\mathsf{a}}) & \text{otherwise.} \end{cases}$$

Moreover, for any $\mathcal{A} \subseteq \mathcal{F}_{\mathsf{a}}$

$$\sup\nolimits^{\mathsf{a}}_{\subseteq} \mathcal{A} = \begin{cases} \sup_{\subseteq} \mathcal{A} & \text{if } \mathcal{A} \subseteq \mathcal{F}, \\ (\sup_{\subseteq}\{A^{-\mathsf{a}} \mid A \in \mathcal{A}\})^{\mathsf{a}} & \text{otherwise,} \end{cases}$$

$$\inf\nolimits^{\mathsf{a}}_{\subseteq} \mathcal{A} = \begin{cases} (\inf_{\subseteq}\{A^{-\mathsf{a}} \mid A \in \mathcal{A}\})^{\mathsf{a}} & \text{if } \mathcal{A} \subseteq \mathcal{F}_{\mathsf{a}} \setminus \mathcal{F}, \\ \inf_{\subseteq}\{A^{-\mathsf{a}} \mid A \in \mathcal{A}\} & \text{otherwise.} \end{cases}$$

Thus, $\langle \mathcal{F}_{\mathsf{a}}, \subseteq, \emptyset, X^{\mathsf{a}} \rangle$ is a Boolean lattice.

THEOREM B.1. *Let $\mathcal{F}$ be a family of subsets of X such that $\emptyset, X \in \mathcal{F}$ and $\langle \mathcal{F}, \subseteq, \emptyset, X \rangle$ is a complete Boolean lattice. Let $\mathsf{a} \notin X$. Then*

(i) *$\langle \mathcal{F}_{\mathsf{a}}, \subseteq, \emptyset, X^{\mathsf{a}} \rangle$ is a non-degenerate complete Boolean lattice;*
(ii) *$\langle \mathcal{F}^{+}_{\mathsf{a}}, \subseteq \rangle$ is a mereological structure, where $\mathcal{F}^{+}_{\mathsf{a}} := \mathcal{F}_{\mathsf{a}} \setminus \{\emptyset\}$;*
(iii) *$\forall_{A \in \mathcal{F}^{+}} \forall_{B \in \mathcal{F}^{+}_{\mathsf{a}} \setminus \mathcal{F}^{+}}\, B \nsubseteq A$.*

Appendix C. Regular sets in topological spaces. Let $\langle X, \mathscr{O} \rangle$ be an arbitrary topological space, Int and Cl being, respectively, the operations of interior and closure in $\langle X, \mathscr{O} \rangle$, $\mathrm{Int}, \mathrm{Cl}\colon 2^X \to 2^X$. We call a subset A of X *regular open* iff $A = \mathrm{Int}\,\mathrm{Cl}\,A$. Let $\mathbf{RO}_{\mathscr{O}}$ be the set of all regular open sets in $\langle X, \mathscr{O} \rangle$. Notice that $\emptyset, X \in \mathbf{RO}_{\mathscr{O}}$ and for any $A \subseteq X$, $\mathrm{Int}\,\mathrm{Cl}\,A \in \mathbf{RO}_{\mathscr{O}}$. The following properties of open domains are widely known.

FACT C.1. *Let $\langle X, \mathscr{O} \rangle$ be a topological space.*

(i) *The structure $\langle \mathbf{RO}_{\mathscr{O}}, \cup^{\circ}, \cap^{\circ}, -^{\circ}, \emptyset, X \rangle$ with operations*

$$U \cup^{\circ} V := \mathrm{Int}\,\mathrm{Cl}(U \cup V), \tag{df$\cup^{\circ}$}$$

$$U \cap^{\circ} V := U \cap V, \tag{df$\cap^{\circ}$}$$

$$-^{\circ} U := \mathrm{Int}(X \setminus U) \tag{df$-^{\circ}$}$$

is a non-degenerate Boolean algebra.

(ii) *For any $U, V \in \mathbf{RO}_{\mathscr{O}}$: $U \leq V \overset{\text{df}}{\iff} U \cup^{\circ} V = V \iff U \subseteq V$.*
(iii) *The structure $\langle \mathbf{RO}_{\mathscr{O}}, \cup^{\circ}, \cap^{\circ}, -^{\circ}, \emptyset, X \rangle$ is a complete Boolean algebra, since for any $\mathscr{U} \subseteq \mathbf{RO}_{\mathscr{O}}$: $\inf_{\leq} \mathscr{U} = \mathrm{Int} \bigcap \mathscr{U} \in \mathbf{RO}_{\mathscr{O}}$ and $\sup_{\leq} \mathscr{U} = \mathrm{Int}\,\mathrm{Cl} \bigcup \mathscr{U}$ in $\mathbf{RO}_{\mathscr{O}}$.*
(iv) *The structure $\langle \mathbf{RO}_{\mathscr{O}}, \subseteq, \emptyset, X \rangle$ is a non-degenerate complete Boolean lattice.*

From Fact C.1 and Theorem A.1 we immediately obtain the following corollary.

COROLLARY C.2. *A structure $\langle \mathbf{RO}_{\mathscr{O}} \setminus \{\emptyset\}, \subseteq \rangle$ is a mereological structure.*

FACT C.3. *For any $U, V \in \mathbf{RO}_{\mathscr{O}}$, $U \subseteq V$ iff $\mathrm{Cl}\,U \subseteq \mathrm{Cl}\,V$.*

We call a subset A of X *regular closed* iff $A = \mathrm{Cl}\,\mathrm{Int}\,A$. Let $\mathbf{RC}_{\mathscr{O}}$ be the set of all regular closed sets in $\langle X, \mathscr{O}\rangle$. Notice that $\emptyset, X \in \mathbf{RC}_{\mathscr{O}}$, for any $A \subseteq X$ we have $\mathrm{Cl}\,\mathrm{Int}\,A \in \mathbf{RC}_{\mathscr{O}}$, and $A \in \mathbf{RC}_{\mathscr{O}}$ iff $X \setminus A \in \mathbf{RO}_{\mathscr{O}}$.

FACT C.4. *Let $\langle X, \mathscr{O}\rangle$ be a topological space.*

1. *The structure $\langle \mathbf{RC}_{\mathscr{O}}, \cup^{\bullet}, \cap^{\bullet}, -^{\bullet}, \emptyset, X\rangle$ with operations*

$$F \cup^{\bullet} G := F \cup G, \tag{df$\cup^{\bullet}$}$$

$$F \cap^{\bullet} G := \mathrm{Cl}\,\mathrm{Int}(F \cap G), \tag{df$\cap^{\bullet}$}$$

$$-^{\bullet} F := \mathrm{Cl}(X \setminus F) \tag{df$-^{\bullet}$}$$

   is a non-degenerate Boolean algebra.
2. *For any $F, G \in \mathbf{RC}_{\mathscr{O}}$: $F \leq G \overset{\text{df}}{\iff} F \cup^{\bullet} G = G \iff F \subseteq G$.*
3. *$\langle \mathbf{RC}_{\mathscr{O}}, \cup^{\bullet}, \cap^{\bullet}, -^{\bullet}, \emptyset, X\rangle$ is a complete Boolean algebra, since for any $\mathscr{F} \subseteq \mathbf{RC}_{\mathscr{O}}$: $\inf_{\leq} \mathscr{F} = \mathrm{Cl}\,\mathrm{Int} \bigcap \mathscr{F} \in \mathbf{RC}_{\mathscr{O}}$ and $\sup_{\leq} \mathscr{F} = \mathrm{Cl} \bigcup \mathscr{F}$ in $\mathbf{RC}_{\mathscr{O}}$.*
4. *$\langle \mathbf{RC}_{\mathscr{O}}, \subseteq, \emptyset, X\rangle$ is a non-degenerate complete Boolean lattice.*

From Fact C.4 and Theorem A.1 we immediately obtain the following corollary.

COROLLARY C.5. *$\langle \mathbf{RC}_{\mathscr{O}} \setminus \{\emptyset\}, \subseteq\rangle$ is a mereological structure.*

FACT C.6. *For any $F, G \in \mathbf{RC}_{\mathscr{O}}$, $F \subseteq G$ iff $\mathrm{Int}\,F \subseteq \mathrm{Int}\,G$.*

From facts C.3 and C.6 we obtain

FACT C.7. (i) Cl *is an isomorphism from $\langle \mathbf{RO}_{\mathscr{O}} \setminus \{\emptyset\}, \subseteq\rangle$ onto $\langle \mathbf{RC}_{\mathscr{O}} \setminus \{\emptyset\}, \subseteq\rangle$.*

(ii) Int *is an isomorphism from $\langle \mathbf{RC}_{\mathscr{O}} \setminus \{\emptyset\}, \subseteq\rangle$ onto $\langle \mathbf{RO}_{\mathscr{O}} \setminus \{\emptyset\}, \subseteq\rangle$.*

(iii) $\mathrm{Int} \circ \mathrm{Cl} = \mathrm{id}_{\mathbf{RO}_{\mathscr{O}}}$ *and* $\mathrm{Cl} \circ \mathrm{Int} = \mathrm{id}_{\mathbf{RC}_{\mathscr{O}}}$.

Appendix D. Pieri's geometry. Let X a be non-empty set and Δ be a ternary relation in X, i.e., $\Delta \subseteq X \times X \times X$. Members of X are called *points*. The relation Δ is a relation of *equidistance* in X and the condition '$\langle p, q, r\rangle \in \Delta$' is read as follows: "points p and q are equidistant from point r". We write '$pq\,\Delta\,r$' instead of '$\langle p, q, r\rangle \in \Delta$'. *A Pieri's structure* is any pair $\mathcal{P} = \langle X, \Delta\rangle$ that satisfies all axioms of Pieri's geometry expressed in set theoretical language (see [15], the English translation can be found in [14], pp. 160–270).

By means of Δ we define, for any points p and q, *the sphere whose center point is p and such that q is its element*

$$\mathrm{S}_p^q := \{r \in X \mid rq\,\Delta\,p\}. \tag{df$\,\mathrm{S}_p^q$}$$

We let that $p = q$ and it follows from the axioms of Pieri that $\mathrm{S}_p^p = \{p\}$.

We define sets $\mathbf{S} := \{\mathrm{S}_p^q \mid p, q \in X\}$ and

$$\mathbf{S}^{\circ} := \{\mathrm{S}_p^q \mid p, q \in X \wedge p \neq q\}. \tag{df$\,\mathbf{S}^{\circ}$}$$

Elements of $\mathbf{S}^{\circ}$ will be called *proper spheres*. Clearly $\mathbf{S}^{\circ} \subsetneq \mathbf{S}$. Moreover, let $\mathbf{S}_p := \{\mathrm{S}_p^q \mid q \in X\}$ and $\mathbf{S}_p^{\circ} := \mathbf{S}^{\circ} \cap \mathbf{S}_p$.

It is known that Mario Pieri's system of geometry is definitionally equivalent to David Hilbert's system and it is provable that

FACT D.1. [7] *Every Pieri's structure* $\langle X, \Delta\rangle$ *is isomorphic to* $\langle \mathbb{R}^3, \Delta^{\mathbb{R}^3}\rangle$, *where for any* $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^3$

$$\bar{x}\,\bar{y}\,\Delta^{\mathbb{R}^3}\,\bar{z} \overset{\text{df}}{\iff} \varrho(\bar{x}, \bar{z}) = \varrho(\bar{y}, \bar{z}). \tag{df $\Delta^{\mathbb{R}^3}$}$$

FACT D.2. $\langle \mathbb{R}^3, \Delta^{\mathbb{R}^3}\rangle$ *is a Pieri's structure.*

Detailed descriptions of the constructions that are necessary to prove the above result can be found in [7]. Some metamathematical results concerning Pieri's system of geometry can also be found in [19].

D.1. Topology in Pieri's structures. Let $\mathcal{P} = \langle X, \Delta\rangle$ be an arbitrary Pieri's structure. Constructing a topological space founded upon $\mathcal{P}$ we will refer to the notion of *the set of interior points of a given sphere*, introduced as follows for any $S \in \mathbf{S}$

$$\mathrm{Ptint}(S) := \{r \in X \mid \exists_{p,q\in S}(p \neq q \wedge r\,\mathbf{B}\,pq\}. \tag{df Ptint}$$

where $\mathbf{B}$ is the ternary *betweenness* relation defined by means of Δ. In light of the preceding subsection of this paper we can say that the set of interior points of any non-degenerate sphere whose center is p takes the role of a three-dimensional Euclidean open ball whose center is p. Let us accept that if $S \in \mathbf{S}^{\circ}_p$, then the set $\mathrm{Ptint}(S)$ will be denoted by 'B_p'. Moreover, let $\mathbf{BO}_p$ be the family of all open balls such that p is their center, i.e.,

$$\mathbf{BO}_p := \{\mathrm{Ptint}(S) \mid S \in \mathbf{S}^{\circ}_p\}. \tag{df $\mathbf{BO}_p$}$$

The family $\{\mathbf{BO}_p\}_{p\in X}$ consisted of all Euclidean balls satisfies the following three conditions.

(a) For every $p \in X$, $\mathbf{BO}_p \neq \emptyset$. For every $B \in \mathbf{BO}_p$: $p \in B$.
(b) If $p \in B \in \mathbf{BO}_q$, then there exists $B' \in \mathbf{BO}_p$ such that $B' \subseteq B$.
(c) For all $B_1, B_2 \in \mathbf{BO}_p$ there exists $B_3 \in \mathbf{BO}_p$ such that $B_3 \subseteq B_1 \cap B_2$.

Therefore three operations $\mathrm{Int}_{\mathcal{P}}, \mathrm{Cl}_{\mathcal{P}}, \mathrm{Fr}_{\mathcal{P}} \colon 2^X \to 2^X$ such that

$$\mathrm{Int}_{\mathcal{P}}\,A := \{p \in Y \mid \exists_{B\in\mathbf{BO}_p}\,B \subseteq A\}, \tag{df $\mathrm{Int}_{\mathcal{P}}$}$$

$$\mathrm{Cl}_{\mathcal{P}}\,A := X \setminus \mathrm{Int}_{\mathcal{P}}(X \setminus A), \tag{df $\mathrm{Cl}_{\mathcal{P}}$}$$

$$\mathrm{Fr}_{\mathcal{P}}\,A := \mathrm{Cl}_{\mathcal{P}}\,A \setminus \mathrm{Int}_{\mathcal{P}}\,A \tag{df $\mathrm{Fr}_{\mathcal{P}}$}$$

satisfy standard conditions put on topological interior, closure and fringe operations. Hence we may introduce a topology in the set X, that is in 2^X we can distinguish a family of open sets, i.e., of all those sets that are equal to their interior. We define

$$\mathscr{O}_{\mathcal{P}} := \{A \in 2^X \mid A = \mathrm{Int}_{\mathcal{P}}\,A\}.$$

$\langle X, \mathscr{O}_{\mathcal{P}}\rangle$ is a topological space and we say that it is *based upon* $\mathcal{P}$ or that it is the topological space *for* $\mathcal{P}$.

Let $\mathbf{BO}_{\mathcal{P}} := \bigcup_{p\in X} \mathbf{BO}_p$. Clearly $\mathbf{BO}_{\mathcal{P}} \subseteq \mathscr{O}_{\mathcal{P}}$, the family $\mathbf{BO}_{\mathcal{P}}$ is a base for $\langle X, \mathscr{O}_{\mathcal{P}}\rangle$, so for every $U \in \mathscr{O}_{\mathcal{P}}$ there exists a family $\mathscr{B} \subseteq \mathbf{BO}_{\mathcal{P}}$ such that $U = \bigcup \mathscr{B}$. Thus for any $\mathscr{B} \subseteq \mathbf{BO}_{\mathcal{P}}$ the set $\bigcup \mathscr{B}$ is open. In consequence

$$\mathscr{O}_{\mathcal{P}} = \{U \in 2^X \mid \exists_{\mathscr{B}\subseteq\mathbf{BO}_{\mathcal{P}}}\, U = \bigcup \mathscr{B}\}.$$

From the above constructions it follows that

FACT D.3. *If $\mathcal{P} = \langle X, \Delta\rangle$ is a Pieri's structure, then the topological space $\langle X, \mathscr{O}_{\mathcal{P}}\rangle$ has all the topological properties of three-dimensional Euclidean space. So $\langle X, \mathscr{O}_{\mathcal{P}}\rangle$ is connected.*[22]

In Euclidean spaces for any pair of distinct points p and q there exists exactly one ball whose center is p and such that q lies on its surface. For the sake of considerations carried out in this paper it is useful to introduce partial operator in $X \times X$ defined only for pairs of distinct points. Thus for $p \neq q$ we accept

$$\mathrm{B}_p^q := (\iota\, B)\,(B \in \mathbf{BO}_{\mathcal{P}} \wedge \mathrm{Fr}_{\mathcal{P}}\, B = \mathrm{S}_p^q)$$

where $\mathrm{Fr}_{\mathcal{P}}$ is topological fringe operation defined in a standard way in $\langle X, \mathscr{O}_{\mathcal{P}}\rangle$. Clearly

$$\mathrm{B}_p^q = \{r \in X \mid \exists_{z\in\mathrm{S}_p^q}\, r\, \mathbf{B}_{\mathcal{P}}\, zp\} \cup \{p\}.$$

Directly from the definition of an open ball in $\langle X, \Delta\rangle$ it follows that for an arbitrary ball B from $\mathbf{BO}_{\mathcal{P}}$ there exist distinct points p and q such that $B = \mathrm{B}_p^q$. Indeed, if $B \in \mathbf{BO}_{\mathcal{P}}$, then there exists $p \in X$ such that $B \in \mathbf{BO}_p$. Hence in $\mathbf{S}_p^\circ$ there is a sphere S such that $B = \mathrm{Ptint}(S)$. Since S is non-degenerate we have that there is a point q such that $q \in S$ and $q \neq p$. Thus $S = \mathrm{S}_p^q$ and $B = \mathrm{Ptint}(\mathrm{S}_p^q)$. Therefore $B = \mathrm{B}_p^q$.

The following lemmas, corollaries and facts are easy consequences of the fact that there exists only one, up to isomorphism, Pieri's structure. We omit their proofs, since these pose no serious difficulties. Let $\mathcal{P}_1 = \langle X_1, \Delta_1\rangle$ and $\mathcal{P}_2 = \langle X_2, \Delta_2\rangle$ be Pieri's structures and I be their isomorphism.

LEMMA D.4. *Let $p, q \in X_1$ and $p \neq q$. Then $I[\mathrm{B}_p^q] = \mathrm{B}_{I(p)}^{I(q)}$, where $I[\mathrm{B}_p^q]$ is the image of the ball B_p^q under I.*

Since every ball in a given Pieri's structure may be treated as the value of B_p^q we can strengthen the above lemma to the following corollaries for any $A \in 2^{X_1}$.

$$A \in \mathbf{BO}_{\mathcal{P}_1} \quad \text{iff} \quad I[A] \in \mathbf{BO}_{\mathcal{P}_2}. \tag{D.1}$$

Moreover, for any $A \in 2^{X_1}$

$$I[\mathrm{Int}_{\mathcal{P}_1} A] = \mathrm{Int}_{\mathcal{P}_2}(I[A]), \tag{D.2}$$

$$I^{-1}[\mathrm{Int}_{\mathcal{P}_2} A] = \mathrm{Int}_{\mathcal{P}_1}(I^{-1}[A]), \tag{D.3}$$

[22] See Footnote 14.

$$I[\mathrm{Cl}_{\mathcal{P}_1} A] = \mathrm{Cl}_{\mathcal{P}_2}(I[A]), \tag{D.4}$$

$$I^{-1}[\mathrm{Cl}_{\mathcal{P}_2} A] = \mathrm{Cl}_{\mathcal{P}_1}(I^{-1}[A]). \tag{D.5}$$

For any Pieri's structure $\mathcal{P} = \langle X, \Delta \rangle$ let $\mathbf{RO}_{\mathcal{P}}$ (resp. $\mathbf{RC}_{\mathcal{P}}$) be the set of all *regular open* (*closed*) *sets* in the topological space $\langle X, \mathscr{O}_{\mathcal{P}} \rangle$.

Let $\mathcal{P}_1 = \langle X_1, \Delta_1 \rangle$ and $\mathcal{P}_2 = \langle X_2, \Delta_2 \rangle$ be Pieri's structures and I be their isomorphism. Then for any $A \in 2^{X_1}$

$$A \in \mathbf{RO}_{\mathcal{P}_1} \iff I[A] \in \mathbf{RO}_{\mathcal{P}_2}, \tag{D.6}$$

$$A \in \mathbf{RC}_{\mathcal{P}_1} \iff I[A] \in \mathbf{RC}_{\mathcal{P}_2}. \tag{D.7}$$

and moreover, I is a homeomorphism of $\langle X_1, \mathscr{O}_{\mathcal{P}_1} \rangle$ and $\langle X_2, \mathscr{O}_{\mathcal{P}_2} \rangle$.

FACT D.5. *Let $\mathcal{P} = \langle X, \Delta \rangle$ be a Pieri's structure and let I be an isomorphism of $\mathcal{P}$ to $\langle \mathbb{R}^3, \Delta^{\mathbb{R}^3} \rangle$. Then I is a homeomorphism of $\langle X, \mathscr{O}_{\mathcal{P}} \rangle$ and $\langle \mathbb{R}^3, \varrho \rangle$ with a standard topology.*

In the set $\mathbb{R}^3$ we introduce a topology defined in a usual way. The following theorem is an easy consequence of the fact that there are no atoms in the Boolean algebra consisted of open non-empty domains in $\mathbb{R}^3$.

FACT D.6. *The structure $\langle \mathbf{RO}^+_{\mathbb{R}^3}, \subseteq \rangle$ is atomless, i.e.*, $\min_{\subsetneq}(\mathbf{RO}^+_{\mathbb{R}^3}) = \emptyset$.

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