THE BULLETIN OF SYMBOLIC LOGIC
Volume 14, Number 3, Sept. 2008

RECONSIDERING ORDERED PAIRS

DANA SCOTT AND DOMINIC MCCARTY

Abstract. The well known Wiener–Kuratowski explicit definition of the ordered pair, which sets $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$, works well in many set theories but fails for those with classes which cannot be members of singletons. With the aid of the Axiom of Foundation, we propose a *recursive* definition of ordered pair which addresses this shortcoming and also naturally generalizes to ordered tuples of greater length. There are many advantages to the new definition, for it allows for uniform definitions working equally well in a wide range of models for set theories. In ZFC and closely related theories, the rank of an ordered pair of two *infinite sets* under the new definition turns out to be equal to the maximum of the ranks of the sets.

"I shall not pursue these considerations here, but only emphasize that it is still a problem how the ordered n-tuple can be defined in the most suitable way."—Thoralf Skolem [16], p. 46.

§1. The history of ordered pairs. In 1914, Wiener [17] showed how to reduce the theory of relations to the theory of classes by presenting relations as classes of ordered pairs and by defining ordered pairs in terms of classes. He set

$$\langle x, y\rangle = \{\{\{x\}, 0\}, \{\{y\}\}\}.$$

At around that same time, Hausdorff [9] reduced the levels of curly brackets with this definition:

$$\langle x, y\rangle = \{\{x, 0\}, \{y, 1\}\}.$$

Here 0 is the empty set and $1 = \{0\}$. Later Kuratowski [13] improved both definitions to:

$$\langle x, y\rangle = \{\{x\}, \{x, y\}\}.$$

The nice motivation here is that any (partially) ordered set can be given its ordering by using the class of all initial segments in the ordering. Note that in terms of Russell's theory of types, $\langle x, y\rangle$ is two types higher than x and y. Indeed, in modern terms, the rank of the ordered pair of two sets x and y is two levels higher than the maximum of the ranks of the two sets. In the

Received February 27, 2007.

1079-8986/08/1403-0004/$2.90

Zermelo–Fraenkel universe of sets this makes little difference as the ranks of sets are unbounded. In other set theories this can be a difficulty.

In 1941 Quine, in a conversation with Goodman, proposed a definition of the ordered pair which reduced the type difference to 1, presenting $\langle x, y\rangle$ as the class of unit subclasses of x union the class of complements of unit subclasses of y. Goodman [8] then presented a more complicated version of Quine's definition, which allows immediate generalizations to ordered tuples of any finite length.

In 1966 Quine (see [14]) refined these ideas again, presenting a new definition which reduces the type difference to *zero*. He proceeded under the assumptions that all objects are classes and that the basic notions of integer arithmetic are available. First he defined this operation on sets:

$$\theta(a) = \{b \in a \mid b \notin \mathbb{N}\} \cup \{b + 1 \mid b \in a \cap \mathbb{N}\}\,.$$

Note that $0 \notin \theta(x)$ always holds. He then proposed the definition

$$\langle x, y\rangle = \{\theta(a) \mid a \in x\} \cup \{\theta(b) \cup \{0\} \mid b \in y\}\,.$$

It is easily seen that, given $\langle x, y\rangle$, one can extract x, y, and the knowledge of which is first and which is second. Boolos in his 1970 paper [5] used a slight generalization of Quine's method in order to have closure of the constructible levels L_α under finite sequences. Quine's definition has the advantage that every object is an ordered pair, because

$$w = \langle\{z \mid \theta(z) \in w\}, \{z \mid \theta(z) \cup \{0\} \in w\}\rangle.$$

The new definition we shall propose also expands the class of ordered pairs, but not this drastically. Also, when we have need of a tagging method such as Quine used, we find a more general method has additional advantages.

Quine in [15], pp. 36–37, suggests defining ordered tuples in a different way:

$$\langle a_0, a_1, \ldots, a_{n-1}\rangle = \{\{\{a_i\}, i\} \mid i < n\}\,.$$

(Actually he uses the range of indices $\{1, 2, \ldots, n\}$, but starting with 0 does not hurt.) His point is that only one von Neumann ordinal is a singleton, namely $1 = \{0\}$. But the possible ambiguity of having $\{\{0\}\}$ as a member of a tuple does not cause any confusion in working out what the various terms of a tuple are. This definition can be taken as seeing tuples as direct generalizations of this definition of ordered pairs:

$$\langle a, b\rangle = \{\{\{a\}, 0\}, \{\{b\}, 1\}\}.$$

Moreover, the idea could be extended to transfinite sequences indexed by arbitrary ordinals. However, from the point of view of this paper, the idea does not work for other index sets, and, as we explain later, such tuples cannot be regarded as functions. Also, the definition is rank (type) increasing.

For more historical background, the reader should consult the extensive review article Kanamori [10].

§2. A brief review of set theories.

§2. A brief review of set theories. In this section we briefly state some relevant facts about the well known set theories ZF, NGB, KM, and KPU.

2.1. ZFC. First, in ZF, or Zermelo–Fraenkel set theory, recall that all variables range over sets; classes have no formal existence and are in some presentations merely used schematically as abbreviations for formulae. The axioms of this theory (of course formulated in first-order logic) are familiar. By name we have: Extensionality, Foundation, Pairing, Union, Powerset, and Infinity. These need to be augmented by two schemata:

Comprehension Schema: For each formula φ with free variables among $x, y, z_1, \dots, z_n$,

$$\forall y\, \forall z_0, \dots, z_{n-1}\, \exists w\, \forall x\, [x \in w \leftrightarrow x \in y \wedge \varphi[x, z_0, \dots, z_{n-1}]].$$

Replacement Schema: For each formula φ with free variables among $x, y, z_0, \dots, z_{n-1}$,

$$\forall w\, \forall z_0, \dots, z_{n-1} [\forall x \in w \exists! y \,.\, \varphi[x, y, z_0, \dots, z_{n-1}] \\ \rightarrow \exists u\, \forall x \in w \exists y \in u \,.\, \varphi[x, y, z_0, \dots, z_{n-1}]].$$

It is easy to combine these into one schema, but this economy is of no importance here.

The preceeding axioms imply the existence of a unique set with no elements, designated by 0 (sometimes written as Ø). In addition, for any set x there is a unique "successor" set characterized by:

$$\forall y\, [y \in x \cup \{x\} \leftrightarrow y \in x \vee y = x].$$

However Infinity is formulated, it is necessary to prove the existence of a least set satisfying:

$$0 \in z \wedge \forall x \in z \,.\, x \cup \{x\} \in z.$$

This set is commonly denoted by ω, the least infinite von Neumann ordinal. It of course models the natural numbers. We give the formal introduction of ordinals from a different point of view in the next section, where we also present the arithmetic operations on ordinals.

As is common, we have used the notation for *bounded quantifiers*, where for example $\forall x \in y \,.\, \varphi[x]$ abbreviates $\forall x\, [x \in y \rightarrow \varphi[x]]$, and $\exists x \in y \,.\, \varphi[x]$ abbreviates $\exists x\, [x \in y \wedge \varphi[x]]$.

The Axiom of Choice (AC, for short) has many familiar formulations, and the system ZFC is ZF + AC.

2.2. Set theories with classes. Russell's paradox ushered in the "limitation of size doctrine" as a way of avoiding antinomies; collections defined by formulas too large to be sets and not break the system of axioms could be referred to as classes. *Proper classes*, i.e., classes which are not sets, are usually thought of as being "too large" to be sets. In ZFC, classes are given no formal existence, and their use in proofs is always a time-saving way of talking about formulae. Von Neumann–Gödel–Bernays set theory, or NGB, is the most prominent set theory which gives classes a formal existence. NGB can be thought of as "ZF with classes". The language is richer in that we have variables which range over sets (denoted by lowercase letters) and variables which range over classes (denoted by uppercase letters). NGB adds a new axiom schema which says that any condition in this expanded language which does not include quantification over classes defines a class; this is called the Axiom Schema of Predicative Comprehension. NGB also adds an extensionality axiom for classes. The axiom schemas of ZF are then replaced by single axioms involving classes. NGB has the advantage that it is finitely axiomatizable; see [6]. All statements in the language of NGB which involve only sets are provable in NGB if and only if they are provable in ZF. This remains true if we add AC (for sets) to both theories. Producing a finitely axiomatizable theory with this property was one of Gödel's motivations for his development of set theory with classes, see [7].

In von Neumann's version of the theory, one extension of the "limitation of size" doctrine is even made an axiom: a class is a set if, and only if, there is no (class) mapping of it onto the class of all sets. This statement is equivalent to the conjunction of the Axiom of Union, the Axiom of Replacement, and the Axiom of Global Choice:

Global Choice: There exists a function F whose domain is the class of all nonempty sets such that for every nonempty set x, $F(x) \in x$.

Of course, "function" here means function in the traditional sense. There is nothing stopping us from using a new kind of function; more on this later. For more on the axiom of global choice, the reader should consult [6].

Kelley–Morse set theory, or KM, replaces the Axiom Schema of Predicative Comprehension in NGB with the Axiom Schema of Impredicative Comprehension: any condition in the language of NGB (which is also the language of KM) defines a class. Thus, KM is basically second order set-theory. KM is not equivalent to ZF in the sense mentioned above. However, all theorems of KM which mention only sets *are* derivable from a theory obtained by adding to ZF an axiom schema which implies the existence of many inaccessible cardinals, but this extension of ZF gives more theorems about sets than KM does. The appendix of Kelley's book [11] contains an exposition of KM.

2.3. Set theories with atoms. Certain models of ZF and KM are often given with reference to inaccessible cardinals, which are vastly larger than cardinals such as $\aleph_0, \aleph_1, \aleph_\omega, \aleph_{\omega_1}$, or $2^{\aleph_0}$. In the investigations of Gödel's constructible universe, L, it was found that many initial segments of L have nice closure properties making them into useful set theories in their own right. Kripke–Platek set theory, KP, axiomatizes what holds in such restricted universes; see [3] for an exposition.

KPU (KP with *urelemente*) set theory admits the existence of objects called *atoms*. These are objects which are never in the range of the $\in$-relation, but which are distinct from one another and from the empty set (which is not an atom). We now present the axioms of KPU. Let the variables $p, q, r, \ldots$ range over atoms, $a, b, c, \ldots$ range over sets, and $x, y, z, x_0, \ldots$ range over both. A Δ_0 *formula* is a logical formula in which all the quantifiers are bounded.

AXIOM 2.1. **Extensionality:**

$$\forall x\, \forall a\, \forall b\, [x \in a \leftrightarrow x \in b] \rightarrow a = b.$$

AXIOM 2.2. **Foundation:** For all formulas $\varphi[x]$ in which y does not occur free,

$$\exists x \,.\, \varphi[x] \rightarrow \exists x\, [\varphi[x] \wedge \forall y \in x \,.\, \neg\varphi[y]].$$

AXIOM 2.3. **Pairing:**

$$\forall x\, \forall y\, \exists a\, [x \in a \wedge y \in a].$$

AXIOM 2.4. **Union:**

$$\forall a\, \exists b\, \forall y \in a\, \forall x \in y [x \in b].$$

AXIOM 2.5. Δ_0**-Separation Schema:** For all Δ_0 formulas φ in which b does not occur free,

$$\forall a\, \exists b\, \forall x\, [x \in b \leftrightarrow x \in a \wedge \varphi[x]].$$

AXIOM 2.6. Δ_0**-Collection Schema:** For all Δ_0 formulas φ in which b does not occur free,

$$\forall a\, \forall x \in a\, \exists y \,.\, \varphi[x, y] \rightarrow \exists b\, \forall x \in a\, \exists y \in b \,.\, \varphi[x, y].$$

KPU^+ is KPU plus the axiom:

$$\exists a\, \forall x\, [x \in a \leftrightarrow \exists p \,.\, x = p],$$

which says that there is a set of all atoms.

KP is KPU plus the axiom:

$$\forall x\, \exists a \,.\, x = a,$$

which says that there are no atoms.

For a study of the consequences and uses of these set theories, consult Barwise [3].

2.4. A quick look at non-well-founded set theories. In [1], Aczel proposes a new kind of set theory which gives non-well-founded sets mathematical uses.

In set theories with an Axiom of Foundation where all sets are built up "from below," Extensionality provides the means of determining whether two sets are equal. Extensionality tells us that for any two sets a and b, $a = b$ if, and only if, $a \subseteq b$ and $b \subseteq a$. However, Extensionality is less enlightening without Foundation. If $a = \{a\}$ and $b = \{b\}$, then all Extensionality tells us is that $a = b$ if, and only if, $a = b$! Aczel's set theory has a different conception of set, but also has a tool for determining when two sets are equal.

Consider a directed graph $G = \langle V, E \rangle$, where the vertices are taken to be sets. A *pointed graph* is a graph with a distinguished node called its *point*. A pointed graph is called *accessible* if for every node in the graph there is a path from the point to that node. Call an accessible pointed graph an *apg*. For the sake of our discussion, we assume that apg's are all directed trees; this is not necessary, but only serves to simplify matters. A *decoration* of a graph is an assignment of a set to each node in the graph such that the elements of a set assigned to a node are the sets assigned to the children of that node. A *picture* of a set is an apg in which the set is assigned to the point. Even without the Axiom of Foundation, we can see that every set has a picture: given a set x, consider every set y that occurs in an $\in$-chain

$$\cdots y \in x_n \in x_{n-1} \in \cdots \in x_1 \in x_0 \in x.$$

Make a graph with the vertices the set of all such y, and the edges the set of all (Wiener–Kuratowski) pairs $\langle a, b \rangle$ such that $a \in b$. Then that apg is a picture of x.

In ordinary ZFC it is an easy application of what is often called *Mostowski's Collapsing Lemma* that each well-founded apg (i.e., one with no infinite downward paths) has a unique decoration, and that every well-founded apg is a picture of a unique set. There are also formulations not depending on the Axiom of Choice. The axiom Aczel introduces treats non-well-founded apg's.

AXIOM 2.7. **Anti-Foundation Axiom (AFA):** Every apg has a unique decoration.

Thus, though we have now jettisoned the Axiom of Foundation as a way to understand what sets look like, we still have some idea: each set can be visualized as an apg. This also helps solves the problem from above: if $a = \{a\}$ and $b = \{b\}$, then a and b are both decorations of the same apg, so they are the same set.

The reader may wonder about the relative consistency of ZF with Foundation and ZF with Anti-Foundation; the consistency of one indeed implies the consistency of the other; for a nice discussion of Anti-Foundation and some competing non-well-founded set theories, consult [1]. The reader interested in non-well-founded set theory is encouraged to read the fascinating [2].

§**3. Examples of recursive definitions.** The technique of definition by recursion is an extremely useful one in mathematics, and one which we use repeatedly here. First we state the KPU $\in$-Recursion Theorem. (For a proof of this theorem, the reader should consult [3].) We recall that a formula φ is a a Σ_1 formula if and only if it is in the smallest set E containing the Δ_0 formulas and closed under conjunction, disjunction, bounded quantification, and having the property that if φ is in E and u is any variable, then $\exists u \,.\, \varphi[u]$ is in E. We assume the following statement as an axiom for now, and comment on its justification later. Function-class should be interpreted as usual, i.e., as a class of Wiener–Kuratowski pairs.

AXIOM 3.1. **$\in$-Recursion:** Let F be an $n+2$-ary function-class defined by a Σ_1 formula. Then there exists a unique function-class G, also defined by a Σ_1 formula, such that

$$G(x_0, \ldots, x_n, y) = F(x_0, \ldots, x_n, y, \{G(x_0, \ldots, x_n, z) \mid z \in y\}).$$

The analogue of this theorem for ZF does not have any restrictions on the formula defining G. (For more on this topic, the reader may also consult Kunen [12].)

We now give some examples of definitions by recursion. We recall that a set x is *transitive* if it includes its union, in symbols, $\bigcup x \subseteq x$, in words, *every member of a member is a member.* It is easy to prove that any set x which is included in a transitive set is included in a least one, called its *transitive closure* and denoted by $\mathsf{TC}(x)$. In the system ZF as outlined here, it is not among the assumptions that this closure exists, and this essential fact must be proved. The usual proof uses ω-recursion, a special case of (3.1). For, given any set x, define by ω-recursion the sequence of sets x_n, where $x_0 = x$, and $x_{n+1} = \bigcup x_n$, for all $n \in \omega$. It is then an easy exercise to verify that

$$\mathsf{TC}(x) = \bigcup_{n=0}^{\infty} x_n.$$

It should be noted that this procedure works even in ZF^- (ZF without the Axiom of Foundation), but the more general recursion in (3.1) we need later essentially rests on the idea of foundation.

We recall (with no appeal to Foundation) that a von Neumann ordinal is a transitive set well-ordered by the membership relation. Moreover, recursion on ordinals can be justified without using Foundation.

The *powerset* of a set x is symbolized and defined as follows:

$$\mathcal{P}(x) = \{y \mid y \subseteq x\}.$$

By iterating the powerset operation throughout all the ordinals, we obtain a covering of the well-founded part of the universe by partial universes indexed by the ordinals.

DEFINITION 3.2. **The Cumulative Hierarchy:**

$$V_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}(V_\beta),$$

where α is any ordinal.

Assuming this definition by ordinal recursion has been justified, it is easy to check that $V_0 = 0$; that, for successor ordinals $\alpha = \beta + 1$, the partial universe V_α is the powerset of V_β; that, for limit ordinals α,

$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta;$$

and that each V_α is transitive.

So far ZF^- is sufficient, but the Axiom of Foundation is equivalent to asserting that every set is in some V_α. We may define, then, for all sets x, the *rank* of x as the least α such that $x \in V_{\alpha+1}$. A formal definition might read:

$$\rho(x) = \{\beta \mid \forall\alpha\,[x \in V_{\alpha+1} \to \beta < \alpha]\}\,.$$

It would then follow that

$$\forall x\,.\,x \subseteq V_{\rho(x)}.$$

We note that with the aid of the Axiom of Foundation, we can simplify the definition of ordinal to *a transitive set of transitive sets*. We could also say an ordinal is a transitive set of ordinals. As a formal definition we could write the following.

DEFINITION 3.3. **Ordinal Numbers:**

$$\forall x\,[\mathsf{Ord}(x) \leftrightarrow \forall y \in x[\mathsf{Ord}(y) \wedge y \subseteq x]].$$

But in what sense is this a definition? It seems *circular*. The answer is that this is a definition of a predicate by $\in$-recursion justified by the Axiom of Foundation as was (3.1). Informally one can say that if you know the meaning of Ord on all members of a set x, then you know whether the predicate applies to x (definitions of predicates can also be phrased as definitions of $\{0, 1\}$-valued functions).

For some others examples, consider these next definitions which were first formulated by Tarski.

DEFINITION 3.4. **Rank:**

$$\rho(x) = \mathsf{TC}(\{\rho(y) \mid y \in x\}).$$

We can then link this definition to the definition of ordinals as follows:

$$\forall x\,[\mathsf{Ord}(x) \leftrightarrow \rho(x) = x \leftrightarrow \exists z \,.\, \rho(z) = x].$$

And here are the arithmetic operations:

DEFINITION 3.5. **Ordinal Operations:**

(i) $A + B = A \cup \{A + b \mid b \in B\}$,
(ii) $A \cdot B = \{A \cdot b + a \mid b \in B \wedge a \in A\}$,
(iii) $B^A = \{0 \mid B \neq 0 \vee A = 0\} \cup \{B^a \cdot b + c \mid a \in A \wedge b \in B \wedge c \in B^a\}$.

The idea of (3.5) when adding two ordinals α and β is that $\alpha+\beta$ consists of the union of two ranges of ordinals. First we have $\alpha = \{\xi \mid \xi < \alpha\}$ followed by an order-isomorphic copy of β, namely $\{\alpha + \eta \mid \eta < \beta\}$. If we already knew what ordinal addition meant, this is an entirely correct description of $\{\zeta \mid \zeta < \alpha + \beta\}$. But, having realized this, we turn the observation into a recursive definition, and then argue by transfinite induction (which for von Neumann ordinals is nothing more than an application of the Axiom of Foundation) that the definition maps ordinals to ordinals and gives the correct results (say, in comparison to other well known definitions). What is remarkable, however, is that by replacing $<$ by $\in$ the definition makes sense for defining $A + B$ for an *arbitrary pair* of sets.

One proves using $\in$-induction (which is equivalent to using Foundation) that 0 is the neutral element for this $+$, and that the operation is associative. Moreover, the sets A and $\{A + b \mid b \in B\}$ are disjoint. Thus the cancellation law, in the form

$$A + B = A + C \rightarrow B = C,$$

therefore follows, again by induction. Thus, the set $A + B$ also has the right *cardinality* for the sum of two sets. As for rank, we find that $\rho(A + B) = \rho(A) + \rho(B)$.

Remembering that $1 = \{0\}$, it is interesting to note that $A + 1 = A \cup \{A\}$. On the other hand, understanding $1 + A$ requires a recursive trace:

$$1 + A = \{0\} \cup \{1 + a \mid a \in A\}\,.$$

So, $1 + A$ is always a non-empty set, and aside from the element 0 it contains a one-one copy of A by non-empty sets. We can also recapture A from $1 + A$, since we have

$$A = \{a \mid 1 + a \in 1 + A\}\,.$$

Noting that $1 + \omega$ consists of the *positive* integers, we can guess that we can use the construction of $1 + A$ instead of the Quine trick mentioned at the end of Section 1. We explain what we mean by this in the next section.

The idea of multiplication for ordinals is based on the *Division Algorithm*. One must take care, however, inasmuch as the operations of addition and multiplication are often *non-commutative* when transfinite ordinals are involved. Nevertheless, it is the case that all ordinals less than a product $\alpha \cdot \beta$ can be divided uniquely by α on the left, producing a *quotient* less than β and a *remainder* less than α. Again, we turn this observation around into a $\in$-recursive definition applying to all sets. What we achieve is an associative operation with 0 as a null element, 1 as the neutral element, and which is distributive across addition in the right-hand position; specifically,

$$A \cdot (B + C) = (A \cdot B) + (A \cdot C).$$

There are also uniqueness results, so that $A \cdot B$ consists of B-tagged disjoint copies of A. Hence, the ordinal product of sets always has the correct cardinal size for a binary product. As for rank, we find that $\rho(A \cdot B) = \rho(A) \cdot \rho(B)$.

When we come to exponentiation, we have to remember that ordinal exponentiation is quite different from the cardinal operation. For example, ω^ω is still a (rather small) *countable* ordinal. The general idea of computing β^α is to consider that any given smaller ordinal can be divided by a *largest* power α^η with $\eta < \beta$. The quotient then has to be less than α and the remainder is then less than α^η. Hence, a recursive definition is found, since these steps always produce smaller ordinals. And note that continuing in a recursive fashion requires stopping after a finite number of such resolutions, since we cannot generate an infinite decreasing sequence of ordinals. As regards rank, we will find that $\rho(B^A) = \rho(B)^{\rho(A)}$.

By the way, in the special case when $B = 0$ and $A \neq 0$, we have to set $B^A = 0$, which explains why the definition looks a little odd. We conclude this brief review by noting that these algebraic equations for exponents hold for all sets:

$$C^0 = 1,\ C^{A+B} = (C^A) \cdot (C^B),\ \text{and}\ (C^A)^B = C^{A \cdot B}.$$

A slightly more complicated recursive definition is that of *Boolean-valued universes*. Let $\mathbb{B}$ be a complete Boolean algebra. For the moment, let $f : A \rightharpoonup B$ mean that f is a function mapping a subset of A into B (that is, a *partial* function). Define

$$V_\alpha^{\mathbb{B}} = \bigcup_{\beta<\alpha} \left\{ f \mid f : V_\beta^{\mathbb{B}} \rightharpoonup \mathbb{B} \right\},$$

and

$$V^{\mathbb{B}} = \bigcup_{\alpha \in \mathsf{Ord}} V_\alpha^{\mathbb{B}}.$$

(For more on Boolean-Valued universes, consult [4].) It is clearly possible to define a "rank within $V^{\mathbb{B}}$". It is perhaps interesting to note that under the new definition of ordered pair of the next section, this notion of rank and the usual notion of rank "eventually" coincide.

Given a model $\langle M, E\rangle$ of the axioms of Foundation and Extensionality (where it is also impossible, from "outside the model", to construct an infinite descending sequence $\ldots E\ x_n\ E\ x_{n-1} \ldots E\ x$), we can construct by $\in$-recursion a transitive model $\langle N, \in\rangle$ and an isomorphism from $\langle M, E\rangle$ onto $\langle N, \in\rangle$. The model N is called the *transitive collapse* of N. This is *Mostowski's Collapsing Lemma*.

§**4. Defining ordered tuples.** Once a notion of order pair has been defined in one's set theory, the common next step is to construe relations and functions as *sets of ordered pairs*. The usual way to represent an n-tuple, then, is as this kind of set:

$$\{\langle 0, a_0\rangle, \langle 1, a_1\rangle, \langle 2, a_2\rangle, \ldots, \langle n-1, a_{n-1}\rangle\}.$$

In other words, *first pairs, then tuples*.

And anyone with even just basic experience in axiomatic set theory knows that this approach is fine—provided the terms of the tuples can be elements of collections. But it is also clear that there is a sense of *duplication*: 2-tuples are defined in terms of pairs, but a 2-tuple is *not* a pair. Or is it? What is wrong with the equation:

$$\langle a, b\rangle = \{\langle 0, a\rangle, \langle 1, b\rangle\}?$$

Well, a lot of things are wrong. For example, both of the consequent statements $\langle 0, 0\rangle \in \langle 0, 0\rangle$ and $\langle 1, 0\rangle \in \langle 1, 0\rangle$ violate the Axiom of Foundation no matter what definition of ordered pair is being used. Can this be fixed? Here is our proposal:

DEFINITION 4.1. **Ordered Pairs:**

$$\langle x, y\rangle = \{\langle 0, t\rangle \mid t \in x\} \cup \{\langle 1, u\rangle \mid u \in y\}.$$

One sees that the use of pairs on the right-hand side has been pushed down one level over what appeared in the faulty definition. This change has two effects: first, if in many cases the rank of a pair $\langle 0, t\rangle$ is the same as the rank of t, and rank of $\langle 1, u\rangle$ is the same as the rank of u, then the rank of $\langle x, y\rangle$ will just be the rank of $x \cup y$ (that is, ranks are mostly not being raised); and second, we are defining paring of sets in terms of pairings involving elements, which sounds like a recursive definition. Is it?

To check that a proper $\in$-recursion is being carried out, note that by this definition we have $\langle 0, 0\rangle = 0$, hence

$$\langle 0, y\rangle = \{\langle 1, u\rangle \mid u \in y\}, \text{and}$$
$$\langle 1, y\rangle = \{0\} \cup \{\langle 1, u\rangle \mid u \in y\}.$$

In other words, we can see that the function $\langle 1, y\rangle$ has a proper $\in$-recursive definition, and that the function $\langle 0, y\rangle$ has an *explicit definition* in terms of $\langle 1, y\rangle$. Call these function $\tau_0(y)$ and $\tau_1(y)$ for short, and note that only the second had been introduced recursively. Now lay down the explicit definition

$$\langle x, y\rangle = \{\tau_0(t) \mid t \in x\} \cup \{\tau_1(u) \mid u \in y\}.$$

By the uniqueness of recursive definitions it is now an elementary exercise to prove (4.1) as a theorem. Therefore the original definition is justified.

As a matter of fact, strange as it seems, $\langle 1, y\rangle = 1 + y$ and $\langle 0, y\rangle = (1 + y) - \{0\}$. Hence, our new definition is very much in the spirit of Quine's idea mentioned in section 1. However, our $1 + y$ is more recursive than his. The proof that if $\langle x, y\rangle = \langle w, z\rangle$ then $x = w$ and $y = z$ follows from what we already know about the function $1 + y$.

With this new definition of ordered pair in hand we can now easily formulate a new definition of ordered n-tuple for any $n < \omega$. But first we need the definition of cartesian product.

DEFINITION 4.2. **Cartesian Product:**

$$A \times B = \{\langle a, b\rangle \mid a \in A \wedge b \in B\}.$$

DEFINITION 4.3. **Ordered n-tuple:**

$$\langle a_i \mid i \leq n\rangle = \bigcup_{i=0}^{n-1} \{i\} \times a_i.$$

Then an ordered n-tuple is a set of ordered pairs, so the uniqueness property we have already shown for pairs implies that ordered n-tuples uniquely determine their terms. Hence, this definition *unifies the notions of ordered pair and* 2*-tuple*.

More graphically, but less exactly, we can write the definition as:

$$\langle a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}\rangle = \{0\} \times a_0 \cup \{1\} \times a_1 \cup \{2\} \times a_2 \cup \cdots \\ \cup \{n-2\} \times a_{n-2} \cup \{n-1\} \times a_{n-1}.$$

We already pointed out that under our definition, an ordered pair is a kind of function. More generally, and speaking informally, let f be any "function" which we wish to represent in the universe as a *set*, which we call f_{set}. We assume we know what function values $f(x)$ are. Then we have:

DEFINITION 4.4. **Functions as Sets:**

$$f_{\mathsf{set}} = \bigcup_{f(x)=y} \{x\} \times y.$$

But perhaps a more rigorous approach would be to define *functional abstraction* using the familiar lambda notation with any well formed term τ.

DEFINITION 4.5. **Functional Abstraction:**

$$\lambda x \in A \,.\, \tau[x] = \bigcup \{\{x\} \times \tau[x] \mid x \in A\}.$$

And we have to follow this with an official:

DEFINITION 4.6. **Function Value:**

$$f(x) = \{t \mid \langle x, t\rangle \in f\}\,.$$

And finally we see this is an adequate combination of definitions, because we can prove the following useful properties:

THEOREM 4.7. ***Properties of Abstraction:***

(i) $\lambda x \in A \,.\, \tau = \lambda x \in A \,.\, \sigma \leftrightarrow \forall x \in A \,.\, \tau = \sigma$,
(ii) $\lambda x \in A \,.\, \tau \subseteq \lambda x \in A \,.\, \sigma \leftrightarrow \forall x \in A \,.\, \tau \subseteq \sigma$,
(iii) $\lambda x \in A \,.\, \tau \cup \sigma = \lambda x \in A \,.\, \tau \cup \lambda x \in A \,.\, \sigma$,
(iv) $\forall x \in A \,.\, (\lambda x \in A \,.\, \tau)(x) = \tau$,
(v) $\forall x \notin A \,.\, (\lambda x \in A \,.\, \tau)(x) = 0$.

The observant reader may have already noticed that with this definition of function (indeed, with the new definition of ordered n-tuple), it is impossible to extract the *intended domain of definition of a function*; for the places where the function vanishes never show up in its set representation. In fact, we have:

THEOREM 4.8. **Further Properties of Abstraction:**

(i) $\lambda x \in A \,.\, \tau \subseteq \lambda x \in B \,.\, \sigma \leftrightarrow \forall x \in A \cap B \,.\, \tau \subseteq \sigma \wedge \forall x \in A - B \,.\, \tau = 0$,
(ii) $\lambda x \in A \,.\, \tau = \lambda x \in B \,.\, \sigma \leftrightarrow$

$$\forall x \in A \cap B \,.\, \tau = \sigma \wedge \forall x \in A - B \,.\, \tau = 0 \wedge \forall x \in B - A \,.\, \sigma = 0.$$

In order to have a set to represent a function and to determine its full domain, we have to remedy the situation by "tagging" every value in the range.

THEOREM 4.9. **Tagged Abstraction:**

(i) $\lambda x \in A \,.\, 1 + \tau = \lambda x \in B \,.\, 1 + \sigma \leftrightarrow A = B \wedge \forall x \in A \,.\, \tau = \sigma$,
(ii) $\lambda x \in A \,.\, 1 + \tau \subseteq \lambda x \in B \,.\, 1 + \sigma \leftrightarrow A \subseteq B \wedge \forall x \in A \,.\, \tau \subseteq \sigma$.

The reason this alteration works is that a set of the form $1 + x$ is never empty; however, to allow the alternative, another definition is required for function values.

DEFINITION 4.10. **Untagged Function Value:**

$$f(\!(x)\!) = \{t \mid \langle x, 1 + t\rangle \in f\}\,.$$

THEOREM 4.11. **Properties of Tagged Abstraction:**

(i) $\forall x \in A \,.\, (\lambda x \in A \,.\, 1 + \tau)(\!(x)\!) = \tau$,
(ii) $\forall x \notin A \,.\, (\lambda x \in A \,.\, 1 + \tau)(\!(x)\!) = 0$.

We can also employ tags for tuples.

DEFINITION 4.12. **Tagged Ordered Pairs and n-Tuples:**

(i) $\langle\!\langle a, b\rangle\!\rangle = \langle 1 + a, 1 + b\rangle$,
(ii) $\langle\!\langle a_0, a_1, \ldots, a_{n-1}\rangle\!\rangle = \langle 1 + a_0, 1 + a_1, \ldots, 1 + a_{n-1}\rangle$,
(iii) $\langle\!\langle a_i \mid i < n\rangle\!\rangle = \langle 1 + a_i \mid i < n\rangle$.

The reason that these definitions might be preferable is that with the original definitions we suggested here, we would have:

$$\langle a, b, c\rangle = \langle a, b, c, 0, 0\rangle.$$

But, with the alternative definition, we have:

$$\langle\!\langle a, b, c\rangle\!\rangle \neq \langle\!\langle a, b, c, d, e\rangle\!\rangle,$$

though it is the case that

$$\langle\!\langle a, b, c\rangle\!\rangle \subseteq \langle\!\langle a, b, c, d, e\rangle\!\rangle.$$

The point here is that the tagged tuple $\langle\!\langle a_0, a_1, \ldots, a_{n-1}\rangle\!\rangle$ uniquely determines the length n, while a tuple $\langle a_0, a_1, \ldots, a_{n-1}\rangle$ only determines a subset of n (namely, $\{i < n \mid a_i \neq 0\}$). Both conventions have advantages.

An important new use of the new ordered pairs is that it is now possible to define ordered pairs and, more generally, sequences of *classes*.

§**5. Adjusting for atoms.** For the proceeding section, we amend the definition of the cumulative hierarchy as follows: let V_0 be the class of all atoms, let $V_{\alpha+1}$ be the class of sets that are either in V_α, or a subset of V_α, for all α, and at limit ordinals θ let V_θ be the class of all sets contained in V_ρ for some $\rho < \theta$. We can make this a formal definition by recursion as follows. For any set of atoms A, we let $V_0[A] = A$, let $V_{\alpha+1}[A]$ be the set of sets that are either subsets of $V_\alpha[A]$ or members of $V_\alpha[A]$, for all α, and at limit ordinals θ let $V_\theta[A]$ be the set of sets contained in $V_\gamma[A]$ for some $\gamma < \theta$. Then for all ordinals α, let V_α be the class of all sets in some $V_\alpha[A]$. For any set x,

$$x' = \{y \in \mathsf{TC}(x) \mid \exists a \,.\, a = y\}$$

is a set, so it follows that the two definitions produce the same classes.

Now consider a theory like KPU. We first define $\langle x, y\rangle$ in four cases.

DEFINITION 5.1. Let p and q be atoms, and a and b be sets. Then

(i) $\langle a, b\rangle = \{\langle 0, t\rangle \mid t \in a\} \cup \{\langle 1, u\rangle \mid u \in b\}$,
(ii) $\langle a, q\rangle = \{\langle 0, t\rangle \mid t \in a\} \cup \{\{0, q\}\}$,
(iii) $\langle p, b\rangle = \{\{p\}\} \cup \{\langle 1, u\rangle \mid u \in b\}$,
(iv) $\langle p, q\rangle = \{\{p\}\} \cup \{\{0, q\}\}$.

We want prove that $\forall x\,[0 \in \langle 1, x\rangle \wedge 0 \notin \langle 0, x\rangle]$. Note that the first clause is immediate, because $\forall x \,.\, \langle 0, 0\rangle = 0 \in \langle 1, x\rangle$. Now for the second. If p is an atom, $\langle 0, p\rangle = 0 \cup \{\{0, p\}\} \not\ni 0$. Also $0 \notin 0 = \langle 0, 0\rangle$. For x a set, $\langle 0, x\rangle = 0 \cup \{\langle 1, u\rangle \mid u \in x\}$, and $0 \notin \langle 0, x\rangle$ as no set $\langle 1, u\rangle$ is empty by the first clause.

Now we show, similarly as before, that our new definition arises from a valid recursively defined function. Let $\pi\colon V \to V$ be $\pi(y) = \{0\} \cup \{\pi(u) \mid u \in y\}$ if y is a set, and $\pi(y) = \{0\} \cup \{\{0, y\}\}$ if y is an atom.

Then by the $\in$-Recursion Theorem (which is a theorem of KPU) π is a well-defined function. But by the uniqueness part of that theorem we must have that $\pi(y) = \langle 1, y\rangle$ for all sets y, because for all y,

$$\langle 1, y\rangle = \{0\} \cup \{\langle 1, y\rangle \mid u \in y\}\,.$$

If y is an atom then it is trivial that $\pi(y) = \langle 1, y\rangle$. Similarly,

$$\langle x, y\rangle = \{\{\pi(v) \mid v \in t\} \mid t \in x\} \cup \{\pi(u) \mid u \in y\}$$

when x and y are both sets;

$$\langle x, y\rangle = \{\{\pi(v) \mid v \in t\} \mid t \in x\} \cup \{\{0, y\}\}$$

when x is a set and y is an atom;

$$\langle x, y\rangle = \{\{x\}\} \cup \{\pi(u) \mid u \in y\}$$

when x is an atom and y is a set; and

$$\langle x, y\rangle = \{\{x\}\} \cup \{\{0, y\}\}$$

when x and y are both atoms, so the new pairing notion is well-defined.

We show next that for all $\alpha \geq \omega$, V_α is closed under the ordered pair operation.

THEOREM 5.2. If $\alpha \geq \omega$ and $x, y \in V_\alpha$, then $\langle x, y\rangle \in V_\alpha$.

We give the proof when x and y are both sets. If $\alpha = \omega$, one shows by induction on the maximum of the ranks that $\langle x, y\rangle$ is a finite subset of V_ω. If $\alpha > \omega$, first assume that α is a limit ordinal, and let $\omega \leq \gamma < \alpha$ be such that $x, y \in V_\gamma$. So $x, y \subseteq V_\gamma$. By the inductive hypothesis,

$$\{\langle 0, t\rangle \mid t \in x\}, \{\langle 1, u\rangle \mid u \in y\} \subseteq V_\gamma.$$

Then

$$\{\langle 0, t\rangle \mid t \in x\} \cup \{\langle 1, u\rangle \mid u \in y\} \subseteq V_\gamma,$$

so

$$\{\langle 0, t\rangle \mid t \in x\} \cup \{\langle 1, u\rangle \mid u \in y\} \in V_{\gamma+1} \subseteq V_\alpha.$$

If α is a successor ordinal, let θ be its predecessor. Then $x, y \subseteq V_\theta$, and by the inductive hypothesis

$$\{\langle 0, t\rangle \mid t \in x\}, \{\langle 1, u\rangle \mid u \in y\} \subseteq V_\theta.$$

Then

$$\{\langle 0, t\rangle \mid t \in x\} \cup \{\langle 1, u\rangle \mid u \in y\} \subseteq V_\theta,$$

so

$$\{\langle 0, t\rangle \mid t \in x\} \cup \{\langle 1, u\rangle \mid u \in y\} \in V_{\theta+1} = V_\alpha.$$

We note that if p is an atom, then $\{\{p\}\}, \{\{0, p\}\} \in V_2$, so the proof can be easily adapted to this case.

So our new notion of ordered pairs does not increase infinite rank.

§6. Applications. In this section, let us assume that V_0, the class of atoms, is a set. For $\alpha \geq \omega$, as we have just shown, the new definition of ordered pairs allows relations over V_α to occur as sets in $V_{\alpha+1}$. We will use the new ordered pairs to define syntax and semantics for $\langle V_\alpha, \in\rangle$ in such a way that the set of all logical formulae is a subset of V_ω, and for $\alpha \geq \omega$, the satisfaction relation on $\langle V_\alpha, \in\rangle$ is definable in $\langle V_{\alpha+1}, \in\rangle$. In the following paragraphs, α is always meant to denote an ordinal $\geq \omega$, and any ordered pairs, ordered n-tuples, or functions are meant to be *of the untagged new variety*.

6.1. Syntax. Let $x_0, x_1, x_2, \ldots$ be the variables in the language of set theory. We define a map from the set of formulas into V_ω by recursion. We note that many authors choose to make negation a basic notion in the process of constructing the set of all logical formulae; we demonstrate an equivalent approach. Our "tokenization" process will be such that, for all logical formulae φ, an operation of negation can be defined so that the token for $\neg\neg\varphi$ is the same as the token for φ. Note that the tokens we are using are analogous to Gödel numbers.

Definition 6.1. **Tokens for formulae:**

(i) $\#(x_i \in x_j) = \langle 0, \langle i, j\rangle\rangle$,
(ii) $\#(x_i \notin x_j) = \langle 1, \langle i, j\rangle\rangle$,
(iii) $\#(\varphi \wedge \psi) = \langle 2, \langle \#\varphi, \#\psi\rangle\rangle$,
(iv) $\#(\varphi \vee \psi) = \langle 3, \langle \#\varphi, \#\psi\rangle\rangle$,
(v) $\#(\forall x_i\, \varphi) = \langle 4, \langle i, \#\varphi\rangle\rangle$,
(vi) $\#(\exists x_i\, \varphi) = \langle 5, \langle i, \#\varphi\rangle\rangle$.

Let F be the set of all tokens of all formulae. Then $F \subseteq V_\omega$. Note also that

$$F = \langle \omega \times \omega,\ \omega \times \omega,\ F \times F,\ F \times F,\ \omega \times F,\ \omega \times F\rangle.$$

Indeed, it is easily seen that the map

$$A \mapsto \langle \omega \times \omega,\ \omega \times \omega,\ A \times A,\ A \times A,\ \omega \times A,\ \omega \times A\rangle$$

is continuous, and so must have a least fixed point. We could have just as easily defined F to be that least fixed point. Moreover, there is a more general fixed-point theorem we will need to invoke. A map $f : PA \to PA$, is *monotone* if for all $a_1, a_2 \subseteq A$, $a_1 \subseteq a_2 \Rightarrow f(a_1) \subseteq f(a_2)$. Such a map has a least fixed point.

Theorem 6.2 (Knaster–Tarski). A monotone map $F\colon PA \to PA$ has a least fixed point.

By way of proof, let $B = \bigcap\{Y \in PA \mid F(Y) \subseteq Y\}$. We claim that $F(B) \subseteq B$. If not, let $x \in F(B) - B$. $x \notin B$ so there is some C such that $F(C) \subseteq C$ and $x \notin C$. $B \subseteq C$ so $F(B) \subseteq F(C) \subseteq C$, contradiction.

Also, $F(B) \subseteq B$ so $F(F(B)) \subseteq F(B)$, so $B \subseteq F(B)$. So $B = F(B)$. Clearly B is the least fixed point of F.

As of now, when trying to evaluate the token for a given formula, it is necessary to convert the formula to a (classically) logically equivalent one where the negations are "as far inside as possible." We can resolve this difficulty by defining a map $\neg\colon F \to F$ by induction.

DEFINITION 6.3. **The negation map:**

(i) $\neg(\langle 0, \langle i, j\rangle\rangle) = \langle 1, \langle i, j\rangle\rangle$,
(ii) $\neg(\langle 1, \langle i, j\rangle\rangle) = \langle 0, \langle i, j\rangle\rangle$,
(iii) $\neg(\langle 2, \langle n, m\rangle\rangle) = \langle 3, \langle \neg n, \neg m\rangle\rangle$,
(iv) $\neg(\langle 3, \langle n, m\rangle\rangle) = \langle 2, \langle \neg n, \neg m\rangle\rangle$,
(v) $\neg(\langle 4, \langle i, m\rangle\rangle) = \langle 5, \langle i, \neg m\rangle\rangle$,
(vi) $\neg(\langle 5, \langle i, m\rangle\rangle) = \langle 4, \langle i, \neg m\rangle\rangle$.

It is easily seen that for all $n \in F$, $\neg\neg n = n$.

6.2. Semantics. If v is a function from ω to V_α (all this means, under the new definition of ordered pair, is that $\forall x, y\, [\{x\}\times y \subseteq v] \to x \in \omega \wedge y \in V_\alpha$), let $v_i = \{x \mid \langle i, x\rangle\} \in v$. Then

$$V_\alpha^{<\omega} = \{v \mid v \text{ is a function from } \omega \text{ to } V_\alpha \text{ and } \exists j \in \omega \forall i \in \omega [i \geq j \to v_i = 0]\}.$$

Informally, let us consider the usual notion of satisfaction in a model, which we can notate as

$$\langle V_\alpha, \in\rangle \vDash \varphi[v].$$

We intend here that the values of $v_0, v_1, v_2, \ldots$ are being assigned to the free variables $x_0, x_1, x_2, \ldots$. The formal definition of satisfaction should then give us a relation $S \subseteq V_\alpha^{<\omega} \times F$ such that

$$S = \left\{\langle v, n\rangle \mid v \in V_\alpha^{<\omega} \wedge \exists\phi \,\centerdot\, n = \#\phi \wedge \langle V_\alpha, \in\rangle \vDash \phi[v]\right\}.$$

Inasmuch as $V_\alpha^{<\omega}$ and F are subsets of V_α, we have $V_\alpha^{<\omega} \times F \subseteq V_\alpha$. Hence, $S \in V_{\alpha+1}$.

If $v \in V_\alpha^{<\omega}$, let $v[a/i] = (v - \{i\} \times v_i) \cup \{i\} \times a$. We can now formally define S to be the least subset of V_α satisfying the following conditions:

DEFINITION 6.4. **The satisfaction relation:**

(i) $\forall v \in V_\alpha^{<\omega}\, \forall i, j \in \omega \left[v_i \in v_j \to v\ S\ \langle 0, \langle i, j\rangle\rangle\right]$,
(ii) $\forall v \in V_\alpha^{<\omega}\, \forall i, j \in \omega \left[v_i \notin v_j \to v\ S\ \langle 1, \langle i, j\rangle\rangle\right]$,
(iii) $\forall v \in V_\alpha^{<\omega}\, \forall n, m \in F\, [v\ S\ n \wedge v\ S\ m \to v\ S\ \langle 2, \langle n, m\rangle\rangle]$,
(iv) $\forall v \in V_\alpha^{<\omega}\, \forall n, m \in F\, [v\ S\ n \vee v\ S\ m \to v\ S\ \langle 3, \langle n, m\rangle\rangle]$,
(v) $\forall v \in V_\alpha^{<\omega}\, \forall i \in \omega\, \forall n \in F\, [\forall a \in V_\alpha \,\centerdot\, v[a/i]\ S\ n \to v\ S\ \langle 4, \langle i, n\rangle\rangle]$,
(iv) $\forall v \in V_\alpha^{<\omega}\, \forall i \in \omega\, \forall n \in F\, [\exists a \in V_\alpha \,\centerdot\, v[a/i]\ S\ n \to v\ S\ \langle 5, \langle i, n\rangle\rangle]$.

S can clearly be defined in $V_{\alpha+1}$ in this manner. It is also possible to define it as a least fixed point of a monotone map.

Consider the map $f : V_{\alpha+1} \to V_{\alpha+1}$ defined by

$$\begin{aligned} f(X) = \ & \big\{\langle v, \langle 0, \langle i, j\rangle\rangle\rangle \mid v \in V_\alpha^{<\omega} \wedge v_i \in v_j \wedge i, j \in \omega\big\} \\ & \cup \big\{\langle v, \langle 1, \langle i, j\rangle\rangle\rangle \mid v \in V_\alpha^{<\omega} \wedge v_i \notin v_j \wedge i, j \in \omega\big\} \\ & \cup \big\{\langle v, \langle 2, \langle n, m\rangle\rangle\rangle \\ & \qquad \mid v \in V_\alpha^{<\omega} \wedge \langle v, n\rangle \in X \wedge \langle v, m\rangle \in X \wedge n, m \in F\big\} \\ & \cup \big\{\langle v, \langle 3, \langle n, m\rangle\rangle\rangle \\ & \qquad \mid v \in V_\alpha^{<\omega} \wedge [\langle v, n\rangle \in X \vee \langle v, m\rangle \in X] \wedge n, m \in F\big\} \\ & \cup \big\{\langle v, \langle 4, \langle i, n\rangle\rangle\rangle \in V_\alpha^{<\omega} \times (\{4\} \times (\omega \times F)) \\ & \qquad \mid \forall a \in V_\alpha \langle v[a/i], n\rangle \in X\big\} \\ & \cup \big\{\langle v, \langle 5, \langle i, n\rangle\rangle\rangle \in V_\alpha^{<\omega} \times (\{5\} \times (\omega \times F)) \\ & \qquad \mid \exists a \in V_\alpha \langle v[a/i], n\rangle \in X\big\}. \end{aligned}$$

Then f is monotone so it has a least fixed point, namely S. We can also talk about definability in V_α.

DEFINITION 6.5. **Definable subsets of V_α:** Given $v_0, \ldots, v_{n-1} \in V_\alpha$, let $v = \bigcup_{0 \leq i < n}\{i\} \times v_i$, then we define

$$\mathsf{Def}_\alpha[v_0, \ldots, v_{n-1}] = \{x \in V_{\alpha+1} \mid \exists m \in F[\forall z \,.\, v[z/n] \ S \ m \leftrightarrow z \in x]\}.$$

§7. Ordered pairs under AFA. As we remarked, the $\in$-Recursion Theorem is equivalent to the Axiom of Foundation. But if we examine our introduction of new ordered pairs carefully, we see that the $x \mapsto 1 + x$ operator can still be made to work under Anti-Foundation. So can the definition of new ordered pairs; for, given x, look at the canonical picture p of x. This is an apg (so, a tree). All we have to do next, at each node in p, is to add a child with no children. Now let $1 + x$ be the unique decoration of that set. Then our operator $x \mapsto 1 + x$ has all the properties that it did before, and we can use it to define new ordered pairs with the same definition as before.

REFERENCES

[1] PETER ACZEL, ***Non-well-founded sets***, Center for the Study of Language and Information, 1988.

[2] ALEXANDRU BALTAG, *Sts*: *A structural theory of sets*, ***Advances in modal logic***, vol. 2, CSLI Publications, 2000.

[3] JON BARWISE, ***Admissible sets and structures***, Perspectives in Mathematical Logic, Springer-Verlag, 1975.

[4] JOHN L. BELL, ***Set theory: Boolean-valued models and independence proofs***, third ed., Oxford Logic Guides, vol. 47, Oxford University Press, 2005.

[5] GEORGE BOOLOS, *On the semantics of the constructible levels*, ***Zeitschrift für mathematische Logik und Grundlagen der Mathematik***, (1970), pp. 139–148.

[6] ABRAHAM A. FRAENKEL, YEHOSHUA BAR-HILLEL, and AZRIEL LEVY, ***Foundations of set theory***, Studies in Logic and the Foundations of Mathematics, vol. 67, North-Holland Publishing Company, 1973.

[7] KURT GÖDEL, ***The consistency of the axiom of choice and of the generalised continuum hypothesis***, Annals of Mathematical Studies, 1940.

[8] NELSON GOODMAN, *Sequences*, ***The Journal of Symbolic Logic***, vol. 6 (1941), pp. 150–153.

[9] FELIX HAUSDORFF, ***Grundzüge der Mengenlehre***, de Gruyter, 1914.

[10] AKIHIRO KANAMORI, *The empty set, the singleton, and the ordered pair*, this BULLETIN, vol. 9 (2003), no. 3, pp. 273–298.

[11] JOHN L. KELLEY, ***General topology***, D. Van Nostrand, 1955.

[12] KENNETH KUNEN, ***Set theory: An introduction to independence proofs***, Studies in Logic and the Foundations of Mathematics, vol. 102, Elsevier, 1980.

[13] KAZIMIERZ KURATOWSKI, *Sur la notion de l'ordre dans la théorie des ensembles*, ***Fundamenta Mathematicae***, vol. 2 (1921), pp. 161–171.

[14] WILLARD V. QUINE, *On ordered pairs and relations*, ***Selected logic papers***, Random House, 1966.

[15] ———, ***Philosophy of logic***, 2nd ed., Harvard University Press, 1986.

[16] THORALF SKOLEM, *Two remarks on set theory*, ***Mathematica Scandinavica***, vol. 5 (1957), pp. 40–46.

[17] NORBERT WIENER, *A simplification of the logic of relations*, ***From Frege to Gödel: A source book in mathematical logic 1879–1931*** (van Heijenoort, editor), Harvard University Press, 1967, pp. 224–227.

CARNEGIE MELLON UNIVERSITY, USA
Current address: 1149 Shattuck Avenue, Berkeley, CA 94707, USA
E-mail: dana.scott@cs.cmu.edu

MATHEMATICS DEPARTMENT, UC BERKELEY, USA
Current address: 1401 Red Hawk Circle Apt. K-115, Fremont, CA 94538, USA
E-mail: dominic@math.berkeley.edu