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GENTZEN'S PROOF OF NORMALIZATION FOR NATURAL DEDUCTION

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Gentzen writes in the published version of his doctoral thesis *Untersuchungen über das logische Schliessen* (Investigations into logical reasoning) that he was able to prove the normalization theorem only for intuitionistic natural deduction, but not for classical. To cover the latter, he developed classical sequent calculus and proved a corresponding theorem, the famous cut elimination result. Its proof was organized so that a cut elimination result for an intuitionistic sequent calculus came out as a special case, namely the one in which the sequents have at most one formula in the right, succedent part. Thus, there was no need for a direct proof of normalization for intuitionistic natural deduction. The only traces of such a proof in the published thesis are some convertibilities, such as when an implication introduction is followed by an implication elimination [1934–35, II.5.13]. It remained to Dag Prawitz in 1965 to work out a proof of normalization. Another, less known proof was given also in 1965 by Andres Raggio.

We found in February 2005 an early handwritten version of Gentzen's thesis, with exactly the above title, but with rather different contents: Most remarkably, it contains a detailed proof of normalization for what became the standard system of natural deduction. The manuscript is located in the Paul Bernays collection at the ETH-Zurich with the signum *Hs.* 974: 271. Bernays must have gotten it well before the time of his being expelled from Göttingen on the basis of the racial laws in April 1933. He seems to have never mentioned the existence of a proof of normalization for natural deduction by Gentzen, even if he discussed Gentzen's work extensively [Bernays 1965, 1970] and was also fully aware of the published proof of normalization by Prawitz [1965]. This is even the more strange if we consider that Bernays had presented Gentzen's calculus of natural deduction at The Institute for Advanced Study in his lectures from 1935–36, titled *Logical Calculus*. These lectures give no indication of Gentzen's aims in setting up the system of natural deduction, such as a subformula property of normal derivations to be used in proofs of consistency.

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Remarkably, most of page 2 of Gentzen's manuscript is reproduced photographically among the frontmatter of Gentzen's *Collected Papers* of 1969. The editor indicates that the source for the manuscript is Bernays. It is astonishing that he did not mention in this connection, say, the complete set of detour convertibilities that stand very prominently on top of page 9 of the manuscript, or the announcement of the proof of normalization in Gentzen's summary of his further results at the end of the first introductory chapter, and its realization in chapter three.

Gentzen's original plan for the thesis seems to have been the following: First set up the calculus of natural deduction, a rule of induction included (chapter I). Next show that the calculus is equivalent to an axiomatic calculus (II). Then show normalization for intuitionistic natural deduction (III). In the next step, show that classical arithmetic, and especially the question of its consistency, reduces to intuitionistic arithmetic (IV). Finally, extend the subformula property of natural deduction to the system of intuitionistic arithmetic, to prove the consistency of the latter (V).

Gentzen's grand plan for a doctoral thesis failed, as he found out some time around the turn of the new year 1933, because the system of arithmetic does not have, and cannot have, a subformula property. He took out the chapter on the translation of classical into intuitionistic arithmetic, sent it first to Heyting, then to *Mathematische Annalen* in mid-March 1933, and developed classical and intuitionistic sequent calculi for pure logic to be presented as a doctoral thesis. This latter cannot have taken more than a few months, because the thesis was accepted in early June 1933. The finished thesis has not been preserved in Göttingen or elsewhere, but the published version should be the same, save for possible minor corrections.

In addition to the first introductory chapter that presents the system of natural deduction and the third on normalization, the handwritten manuscript contains the very first sketches towards an intuitionistic sequent calculus, a brief section on a multisuccedent classical sequent calculus, and translations between natural deduction and an axiomatic system of logic, the topic of the originally planned chapter II. These translations are different from those in the published thesis in which Gentzen started from an axiomatic calculus, the one of Hilbert [1931]. Axioms are given there separately for each connective, so that an intuitionistic calculus can be obtained by leaving out the law of double negation. Derivations in that calculus are first translated into intuitionistic natural deduction, then to sequent calculus, then to the axiomatic calculus, to make a full circle. The last of these translations is rather complicated. Afterwards the analogous translations for the corresponding classical calculi are outlined. In the manuscript, the axiomatic calculus is that of Hilbert and Ackermann [1928] that contains only disjunction, negation, and universal quantification. Gentzen gives a classical system of natural deduction, called *N2*, that has rules for conjunction, negation, and universal quantification. Contrary to the published thesis, a direct translation of derivations in *N2* to derivations in the Hilbert–Ackermann calculus is given. This translation seems to be at the origin of the discovery of sequent calculus, but no trace of it can be seen in the published thesis.

Letters from Gentzen to Heyting from the winter and spring of 1933 show that the latter had received a manuscript with the translation from classical to intuitionistic arithmetic in January or early February 1933 (see Menzler-Trott [2007]). Heyting was corresponding at the same time also with Gödel, and a letter shows that Gödel reacted rather strongly to the news that someone else had done the thing (see Gödel's *Collected Works*, vol. IV). In the end, Gentzen withdrew his paper from publication, but Bernays kept a set of galley proofs through which the work was preserved and finally published in English translation in 1969 and in the German original in 1974. A page of corrections in the manuscript of the thesis is left where there once was a planned fourth chapter, all corrections done in the proofs. It can be gathered that the text of the article came directly from that chapter.

The chapter on normalization is a later addition to the manuscript, distinguished by a standard handwriting style that can be read without greater difficulties. The other parts are written in the "deutsche Schrift," an obsolete form of handwriting that is quite hard to read. There are also notational and terminological changes as compared to the first chapter, such as the use of horseshoe implication in place of the earlier arrow. The following rules of the calculus of natural deduction for intuitionistic logic come from the introductory chapter, with the notation changed into that of chapter III:

$$\begin{array}{cccc}
AI & AE & OI & OE \\
\dfrac{A \quad B}{A\&B} & \dfrac{A\&B}{A} \quad \dfrac{A\&B}{B} & \dfrac{A}{A \vee B} \quad \dfrac{B}{A \vee B} & \dfrac{A \vee B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C}
\end{array}$$

$$\begin{array}{cccc}
UI & UE & EI & EE \\
\dfrac{Dy}{(x)Dx} & \dfrac{(x)Dx}{Da} & \dfrac{Da}{(Ex)Dx} & \dfrac{(Ex)Dx \quad \begin{matrix}[Dy]\\ C\end{matrix}}{C}
\end{array}$$

$$\begin{array}{ccc}
FI & FE & V \\
\dfrac{\begin{matrix}[A]\\ B\end{matrix}}{A \supset B} & \dfrac{A \quad A \supset B}{B} & \dfrac{\curlywedge}{A}
\end{array}$$

Rule V is from a marginal remark that indicates a change with the negation rules: In chapter III, negation is defined through *falsum* and only rule V is kept of the earlier negation rules. Rules *UI* and *EE* have the standard variable restrictions. Square brackets indicate any number $\geqslant 0$ of discharged assumption formulas. Derivations are such that implication introductions discharge in the end any remaining open assumptions. The same is true of Gentzen's printed thesis, even if his system of natural deduction is usually presented as one in which completed derivations can have open assumptions. Thus, in Gentzen's calculus all formulas in a normal derivation are subformulas of the endformula that is also the longest formula in the derivation.

Contrary to the other extant parts of Gentzen's thesis manuscript, the detailed proof of normalization for intuitionistic natural deduction is what he would have described as "publikationsreif verfasst," ready for publication. The text is, even if ready for publication, not easy to read. Gentzen had developed his own style of writing in which the proof of normalization is presented by laborious verbal arguments, with a minimum of formal notation. The same style can be seen, even though to a lesser extent, in the published thesis and in, say, the proof of a crucial lemma 3.4.3 in Gentzen's [1938] proof of the consistency of arithmetic.

It is seen from the stenographic manuscripts that Gentzen left behind that he had arrived at the system of natural deduction by September 1932. (See also our [2008] for details on these Gentzen papers.) The idea of a normal form appears also at this time. A lost series of notes with the signum *HÜG*, for "Hügel" or "hillock" in English, seems to have contained the working-out of details of normalization. The proof in the present manuscript is very detailed, especially in questions such as the change of variables, called "purification of derivations," to the effect that steps of normalization of derivations can be made without violating variable restrictions. The main idea appears already in the $\vee, \exists$-free fragment: An introduction followed by the corresponding elimination gives rise to what is called today a detour convertibility, and its elimination removes the "hillock formula," a local peak of maximal length, from the derivation. The said fragment was of interest because it arose from the study of a classical system of natural deduction. Thus, Prawitz [1965] proves first a normalization theorem for classical logic without $\vee, \exists$. Gentzen gives a proof directly for the full language of intuitionistic logic. The main obstacle is when a disjunction or existence elimination separates a step of introduction from the corresponding elimination. Gentzen gives what are now called permutative conversions by which the order of elimination rules can be permuted until a detour convertibility is reached. The resulting proof is the same as the standard proof today, say in Troelstra and Schwichtenberg's text book *Basic Proof Theory*.

As said, the proof of normalization is done in great detail, with lots of cases and nested subcases. Gentzen's papers usually contain an elaborate numbering system for complicated proofs. It seems that in the normalization proof, these numberings have been added later, because they are found in the margin. We have adjusted them so that they appear in the text in the way in which similar numberings appear in Gentzen's publications of the time.

Further terminological changes, as compared to the other chapters, include that the calculus is called *NI* instead of *N1I*, what were called proofs are now derivations, and propositions are formulas. The German originals are, first, *NJ*, in which the letter J is just a variant writing of the upper case I that stands for *intuitionistisch*, and *Herleitung* instead of *Beweis*, *Formel* instead of *Aussage*. More details on Gentzen's manuscript can be found in our [2008]. We have added a descriptive title to the chapter.

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THE NORMALIZATION OF DERIVATIONS[1]

Dissertation. III. Chapter

G. Gentzen

Overview: We shall show that each *NI*-derivation can be brought into a rather simple form. It will have, e.g., the following property: Each formula in the derivation is a subformula of the endformula, eventually with different individual variables. The basic idea of the transformation is: In an arbitrary derivation, any formulas can appear in the first place. One considers now, among those formulas that are not subformulas of the endformula, a longest one. It will be, in general, the lower formula of an "introduction" of its outermost sign and the main upper formula of an "elimination" of the same sign. But then it can be subsequently eliminated through a reduction. (Comp. I Chapter §5.)

Certain exceptions (with the inferences *OE, EE, V*) require a special treatment, but the situation is not essentially different.

§**1.** *Some new notations.*

As concerns an *NI*-derivation we define:

1. 1. For the inference figures except V:

1. 1 1. One formula is the *main formula*, namely $A\&B$ with *UI, UE*, $A \vee B$ with *OI, OE*, ... $(Ex)Dx$ with *EI, EE*.

1. 1 2. There appear, further, subformulas of these, possibly with different variables, these are A, B, Dx in the schemes.

We shall call each such formula a *neighbor* of the main formula (and conversely, symmetrically.)

1. 1 3. There appear further, with *OE* and *EE*, formulas C. We shall say that the lower formula C is *bonded* with both of the two upper formulas C (with *OE*), resp. with the one in *EE*. (A symmetric relation.) The two upper formulas in *OE* are, then, not bonded with each other.

1. 2. A *bond* is a sequence of (formally equal) proposition formulas that stand one under another in a derivation thread, such that each formula is bonded with the one standing under it (and therefore also with the one above it) but so that the bond cannot be continued in the same way either upwards or downwards.

[1][From the handwritten German manuscript Hs. 974: 271, *Untersuchungen über das logische Schliessen*, pp. 13–27, in the Paul Bernays collection of the ETH-Zurich. Edited and translated by Jan von Plato. The title has been added by the editor.]

1. 2 1. (The concept of a "bond" serves as a generalization of the concept of "a formula in a derivation." Namely, such sets of C's can be considered in practice as one formula.)

We let a bond consist of just one formula. Then we have: Each proposition formula of a derivation belongs to a bond.

We shall apply also to bonds certain modes speech such as subformula, outermost sign, and mean then an arbitrary formula of the bond.

We shall designate the uppermost formula of a bond as its *representant*. The latter determines the bond uniquely. (The downmost would not, because of *OE*.)

Two bonds are neighbors if one formula of one bond is a neighbor of one formula of the other bond.

Of two bonds that are neighbors, one is a subformula of the other, possibly with different variables. (For this holds in general for formulas that are neighbors.)

1. 3. A *hillock*[2] is a bond the neighboring bonds of which are subformulas of it (eventually with different variables). The *grade* of a formula is the number of logical signs in it.

A hillock can also be characterized in the following way: Its neighbors have a lesser grade.

A bond that has the greatest grade found in a derivation is always a hillock.

Therefore there is no derivation without a hillock.

An *inner hillock* is a hillock to which the endformula does not belong.

1. 4. The main theorem of this chapter is the following **hillock theorem:** An arbitrary *NI*-derivation, the endformula of which does not contain any free variables, can be transformed into an *NI*-derivation of the same formula *without inner hillocks*.

(In this case the bond of the endformula is the only hillock.) All further results of this chapter, e.g., the more intuitive subformula theorem (that says less) become easy consequences of the main theorem.

1. 5. The concept of a "main hillock."[3]

We shall define a relation of "higher" between certain formulas of a derivation.:

When two formulas stand in the same thread, the one that is upper is higher. If in an inference figure the main formula is an upper formula, and if there are further upper formulas (*OE, FE, EE*), the latter, as well as everything that stands above them, are higher than the main formula as well as everything that stands above it.

[2]Corresponds more or less to the notion of a "peak," the latter is more specific.

[3]The hillock to be reduced first is one.

We carry over the relation higher into bonds by calling a bond higher than another one if its representant is higher than the representant of the other bond.

A *main hillock* is an inner hillock such that there is (in the derivation) no inner hillock of a greater grade, and no higher one of the same grade.

Each derivation that has an inner hillock has also a main hillock. (Because the relation of higher has no cycles.)

§**2.** *Proof of the hillock theorem.*

It is sufficient to show:

An *NI*-derivation with an inner hillock can be so transformed, with the endformula maintained, that a main hillock is eliminated and thereby the number of inner hillocks of the greatest grade decreased, without the appearance of inner hillocks of a greater grade.

Let us, then, choose a specific main hillock for reduction. The transformation of the derivation happens in four steps.

The fourth step gives the reduction proper. The others are preparatory to it.

2. 1. *The first step.* Renaming of individual variables. (Purification of the derivation.)

2. 1 1. The following shall be achieved:

2. 1 1 1. Each individual variable shall occur as only bound or only free.

2. 1 1 2. Each individual variable shall be the eigenvariable[4] (x) of at most one inference figure (*UI* or *EE*).

2. 1 1 3. The eigenvariable of a *UI* resp. *EE* must appear only above the inference line of the *UI* resp. only above the upper formula C of the *EE*. (We call this part of the derivation an "upper part" of the *UI* resp. *EE*.)

2. 1 1. All of this is achieved through the following renaming:

2. 1 2 1. If an individual variable occurs in the derivation both as free and as bound, it will be substituted everywhere where it occurs free by one and the same variable that does not occur yet anywhere.

2. 1 2 2. The eigenvariable of a *UI* resp. *EE* will be substituted in the whole upper part of this inference by a variable that does not occur yet anywhere.

2. 1 3. We have to convince ourselves now that

2. 1 3 1. The derivation has turned into a derivation with the same endformula.

2. 1 3 2. The number of inner hillocks of the greatest grade has not been multiplied and no new inner hillock of the greatest grade has appeared.

[4]Old notation a, new notation x.

2. 1 3 3. The above three [one in 1 and two in 2] requirements are satisfied in the new derivation.

2. 1 3 4. About 2. 1 3 1.: We divide this question into three parts and shall later always lay the same division as a basis:
The following three requirements are to be satisfied for a figure to represent a derivation with a prescribed endformula.

1. The figure has tree form and has the right endformula, the "*inference figures*" are correct, i.e., producible from the inference schemes through a permissible instantiation. (We shall not count the assumption formulas that are associated to inference figures as parts of the figures.)

2. The uppermost formulas of a figure can be associated to the inference figures as *assumption formulas* in such a way that each uppermost formula belongs to exactly one inference figure and has the form and position that is permitted by the schemes.

3. The *variable condition* is satisfied in the figure. The variable condition is now (different from earlier): The eigenvariable of a *UI* or *EE* must appear among the formulas of an inference figure only in the formula Dx, and in no assumption formula in the upper part (2.113) of the inference figure on which the lower formula of the inference figure depends.

2. 1 3. Working-out of 2. 1 3 1–2. 1 3 3:

2. 1 4 1. (2. 1 3 1.) Division as already said:

2. 1 4 1. 1. The endformula is obviously unchanged, because it could not contain any free variables. The inference figures are changed, repeatedly, in at most the way that a variable is substituted through a new variable all over (in the inference figure), or all over in places in which it occurred free. (This holds because of the variable condition also for a *UI* resp. *EE* the eigenvariable of which was renamed.) All inference figures remain correct here.

2. 1 4 1. 2. The association of assumption formulas to the inference figures is left as it was. All things are now in order, save that it is established that the same substitution finds place in an assumption formula as in its inference figure. This could be doubted in renaming 2. 1 2 2 in the following case: The assumption formula contains an eigenvariable of a *UI* resp. *EE* and stands in its upper part, whereas the inference figure concerned is neither in this part, nor is it the *UI* resp. *EE* itself. But then it has to stand below the inference line of the *UI* resp. *EE*, i.e., its lower formula depends on the said assumption and we have a contradiction with the variable conditions, 2. part.

2. 1 4 1. 3. The variable condition is trivially satisfied in the new inference figure, because the new eigenvariable of a *UI* resp. *EE* occurs only in such places in which the old one already occurred.

2. 1 4 2. (2. 1 3 2) Neighboring bonded formulas are always affected by the same substitution, each bond has become a bond, each hillock a hillock of the same grade, each non-hillock a non-hillock.

2. 1 4 3. (2. 1 3 3) Conditions 2. 1 1 1 and 2. 1 1 3 are obviously satisfied in the new derivation. 2. 1 1 2 remains:
Let there be a variable in a formula that is the eigenvariable of two different inference figures. Both inference lines stand below this formula because of 2. 1 1 3. We consider the lower of the two inference figures. If it were a *UI*, the eigenvariable would appear in its upper formula, contrary to 2. 1 1 3 applied to the upper inference figure. If it were an *EE*, the eigenvariable would appear in an associated assumption formula, contrary to either 2. 1 1 3 or to the variable condition applied in both cases to the upper inference figure.

2. 2. *Second step.* This vanishes in the case that the main hillock is already a bond that consists of a single formula (which is the normal case). In the case that it is not, the second step serves to diminish the main hillock by one formula.

We consider the two lowest formulas of the hillock, and let both have the form C. Thus, the derivation looks as follows: (We keep an eye on the arrangement of the upper formulas of an inference figure, not necessarily on the inference schemes.)

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Gamma\\ |C\end{array}\quad \Delta}{|C}{\scriptstyle I\ EE}\qquad \Theta}{D}{\scriptstyle II}\\ \Lambda\end{array} \qquad \text{resp.:} \qquad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Gamma\\ |C\end{array}\quad \Delta\quad \begin{array}{c}\Pi\\ C\end{array}}{|C}{\scriptstyle I\ OE}\qquad \Theta}{D}{\scriptstyle II}\\ \Lambda\end{array}$$

In these, the two formulas C with a stroke denote the said formulas, D the lower formula of the inference figure the upper formula of which is the lower of the two. $\Gamma, \Delta, \Theta, \Lambda, \Pi$ represent the remaining parts of the derivations. (These can be empty.) For example, $\begin{array}{c}D\\ \Lambda\end{array}$ denotes the part that consists of all formulas that do not stand above D, Γ the part that consists of all formulas above the C in question, Δ and Θ denote possible further upper formulas that belong to the inference figure, together with whatever stands above them. The derivation is now transformed into the following form:

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Gamma\\ |C\end{array}\quad \Theta}{D}{\scriptstyle (II)}\qquad \Delta}{D}{\scriptstyle \begin{array}{c}I\\ (EE)\end{array}}\\ \Lambda\end{array} \qquad \text{resp.:} \qquad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Gamma\\ |C\end{array}\quad \Theta}{D}{\scriptstyle (II)}\qquad \Delta\qquad \dfrac{\begin{array}{c}\Pi\\ C\end{array}\quad \Theta}{D}{\scriptstyle (II)}}{D}{\scriptstyle \begin{array}{c}I\\ (OE)\end{array}}\\ \Lambda\end{array}$$

2. 2 1. We have to show again that (comp. 2. 1 3)

2. 2 1 1. the derivation turns into a derivation with the same endformula.

2. 2 1 2. the number of inner hillocks of the greatest grade does not increase and none of a greater grade appear.

2. 2 1 3. the bond of C's remains a main hillock and is diminished by one formula.

2. 2 2 1. (2. 2 1 1) Division as in 2. 1 3:

2. 2 2 1. 1. The new figure has tree form and has the old endformula. How is it with the inference figures? *II* is not changed by the transformation, it just appears possibly twice. In *I*, D has taken the place of C, together with an instance of *EE* resp. *OE*. All of the other inference figures have remained completely unaltered, though those in Θ can appear twice after the transformation.

2. 2 2 1. 2. The uppermost formulas remain *associated* to the same inference figures as assumption formulas as before. In the case that the part Θ was doubled, the same associations that held earlier within Θ are naturally kept within each part.

The *form* of assumption formulas is, now as before, suitable to their inference figures, because neither has been changed. (The change of C into D in *I* changes obviously nothing here.)

It must be studied carefully that their *position* relative to each other satisfied the requirements.

If the inference figure belongs to $\overset{D}{\Lambda}$, its assumption formulas that stand above D can have changed places and been multiplied (Θ), but this doesn't harm anything because they stand now as before above D and are uppermost formulas, and nothing is prescribed of their number and further arrangement.

If the inference figure belongs to $\overset{\Gamma}{C}$, $\overset{\Pi}{C}$, Δ, Θ, the position of the assumption formulas has remained unchanged. There remain the inference figures *I* and *II* to be considered. Assumption formulas that belong to inference figure *I* stand in Γ and Π, therefore clearly also after the transformation in a correct relation to the inference figure. Possible assumption formulas associated to *II* can, as we shall soon show, stand only in Θ. Therefore also these stand in the right place after the transformation. The following shows that there cannot be any associated assumption formulas above the upper formula C of *II*: There can be assumption formulas, on the basis of the inference schemes, over the upper formula C of the inference figures *OE, EE* as well as over the C of a *FI*. C cannot be of the first kind, because it is the lowest formula of a bond. (Otherwise the bond would be extensible further down.) C cannot be of the second kind, because it belongs to a hillock, whereas the B of a *FI* is a neighbor of the $A \supset B$ and $A \supset B$ is not a subformula of B.

2. 2 2 1. 3. The variable condition is manifestly satisfied for every *UI* resp. *EE* that belong to one of the parts $\overset{\Gamma}{C}$, $\overset{\Pi}{C}$, Δ, Θ. For it states then something about only the conditions within this part, and these have remained completely unchanged. The variable condition remains satisfied also for each *UI* resp. *EE* that belongs to the part D. (For D depends only on assumption formulas of the same form as before.) Inference figure *II* cannot be a *UI*, because C would then be a neighboring subformula (possibly with a different variable) of D, against the definition of a hillock. It can be, instead, an *EE*. Then Θ is its upper part (because D and C are not permitted to be bonded). The variable condition is now obviously satisfied, because inference figure *II* together with its upper part Θ appears in the new figure completely unchanged.

Also inference figure *I* requires some attention in case it is an *EE*. True, its upper part Δ remains unchanged, but it would be thinkable that the eigenvariable of the *EE* appeared in D. This is, however, impossible by 2. 1 1 3.

2. 2 2 2. (2. 2 1 2) Is it possible that there appear inner hillocks that have the same or greater grade than C?

As of next we consider those formulas of the above schemes that are designated by C, D. The C of the part $\overset{\Gamma}{C}$ belongs to the main hillock. (see 2. 2 2 3.) If the C of $\overset{\Pi}{C}$ belongs to a hillock after the change, this was the case also before. (Because the neighborhood relations in $\overset{\Pi}{C}$ and Π have not been changed, and there were none for C in I.) Thus, there is no new hillock here.

How is is with D? C is, as the lowest formula of a hillock in Π, the main formula. C as a lower formula can be either[5] a neighbor of B, and consequently a subformula of C, so of a lesser grade, or else (if *II* is one of *OE, EE*) bonded with the lowest formula of Θ. Now, Θ is just the part that was possibly doubled in the change. I.e.: If there appeared, through a transformation, at all a hillock of the same or greater grade than that of C, its representative formula (1. 2 1) must belong to Θ. (The parts $\Gamma, \Delta, \Pi, \Theta$ contain after the change the same hillocks that were there before.) And there must have been a hillock of the same grade in the same place in Θ already before the change. (It is seen similarly as for C that this holds also for D.)

We must now take into consideration the fact that the bond C was a main hillock. By the definition of "higher," all formulas of Θ were (before the transformation) higher than C and the formulas above C, for C was, as

[5][Margin has in a clumsy hand: Case *FE*?]

noticed, the main formula of inference figure II. Thus, had there occurred in Θ before the change a representative formula of a hillock of the same or greater grade as that of C, the bond of C would not have been a main hillock and we would have a contradiction.

2. 2 2 3. (2. 2 1 3) We have already established that there is after the change no hillock of a greater grade than that of C. Before the change, the entire part Θ as well as parts of Γ and Λ were higher than C. (though not Δ and Π.) After the change they are exactly the same parts. (With the doubling of Θ only one part Θ.) And no formula can be a representative of a hillock in these parts if it was not such already earlier. (comp. 2. 2 2 2.)

Therefore the formula C has in fact remained a main hillock and further manifestly diminished by one formula.

2. 2 3. There is at the end of step 2 [2. 2] still a repetition of step 1 (2. 1). Observe that what was achieved by step 2 (2. 2 1 3) remains (on the basis of 2. 1 4 2). (2. 2 1 1 and 2. 2 1 2 remain naturally, because of 2. 1 3 1 and 2. 1 3 2.)

2. 3. *Third step.*

If after performing step 2 the main hillock still consists of more than one formula, step 2 is repeated until it consists of one single formula.

It is essential for this that the properties (2. 1 1) of the derivation, achieved through step 1 on the basis of 2. 2 1, are maintained each time.

2. 4. *Fourth step.* The proper reduction of the main hillock.

Now the main hillock consists of *one* formula in the derivation. It is the lower formula of an inference figure and the upper formula of another inference figure. For: It cannot be the endformula (as an inner hillock) and it cannot be an assumption formula, because each assumption formula is a neighbor of the main formula of the associated inference figure, and a subformula of it (possibly with a different variable).

We have further: The hillock is a main formula in both of the inference figures to which it belongs, provided that the inference figure is not V, for which there is not even any definition of a concept of main formula. (Because there appear in an inference figure, except for the main formula and except for V, only subformulas that are neighbors of the main formula, and formulas (C in *OE, EE*) that are bonded with others.)

We shall next distinguish the two cases:

1. The main hillock does not belong to any inference figure V.
2. The main hillock does belong to an inference figure V.

2. 4 1. First case. By what was said before, the main hillock is the lower formula and upper formula of two inference figures, and in both cases the main formula.

Then obviously exactly the following five cases are possible:

The two inference figures are *AI, AE*; *OI, OE*; *FI, FE*; *UI, UE*; or *EI, EE*.

Thus, the outermost sign of the hillock is eliminated right after its introduction. It is for this reason that the hillock can be removed, and this happens after the following five reduction schemes, corresponding to the five cases: (Only one possibility of the two in *AE* and *OI* is treated, the other one is cleared in a completely analogous way)

We lay down for each case the form of the given derivation and next to it the new form that appears through reduction.[6]

$\&$

$$\dfrac{\dfrac{\begin{matrix}\Gamma & \Delta\\ A & B\end{matrix}}{|A\&B}}{\begin{matrix}A\\ \Omega\end{matrix}} \qquad \begin{matrix}\Gamma\\ A\\ \Omega\end{matrix}$$

$\vee$

$$\dfrac{\dfrac{\Gamma}{A}\atop |A\vee B \quad \begin{matrix}[A]\\ \Theta\\ C\end{matrix} \quad \begin{matrix}[B]\\ \Lambda\\ C\end{matrix}}{\begin{matrix}C\\ \Omega\end{matrix}} \qquad \begin{matrix}\begin{bmatrix}\Gamma\\ A\end{bmatrix}\\ \Theta\\ C\\ \Omega\end{matrix}$$

$\supset$

$$\dfrac{\begin{matrix}\Gamma\\ A\end{matrix} \quad \dfrac{\begin{matrix}[A]\\ \Theta\\ B\end{matrix}}{|A\supset B}}{\begin{matrix}B\\ \Omega\end{matrix}} \qquad \begin{matrix}\begin{bmatrix}\Gamma\\ A\end{bmatrix}\\ \Theta\\ B\\ \Omega\end{matrix}$$

$(\ )$

$$\dfrac{\dfrac{\begin{matrix}\Gamma(x)\\ Dx\end{matrix}}{|(y)Dy}}{\begin{matrix}Dz\\ \Omega\end{matrix}} \qquad \begin{matrix}\Gamma(z)\\ Dz\\ \Omega\end{matrix}$$

(E)

$$\dfrac{\dfrac{\Gamma}{Dz}\atop |(Ey)Dy \quad \begin{matrix}[Dx]\\ \Theta(x)\\ C\end{matrix}}{\begin{matrix}C\\ \Omega\end{matrix}} \qquad \begin{matrix}\begin{bmatrix}\Gamma\\ Dz\end{bmatrix}\\ \Theta(z)\\ C\\ \Omega\end{matrix}$$

The formula with a stroke denotes the hillock. $\Gamma, \Delta, \Theta, \lambda, \Omega$ denote the other parts of the derivations. They can be empty. $\begin{matrix}[A]\\ \Theta\\ C\end{matrix}$, e.g., means: C together with all the formulas that stand above it, $[A]$ denotes assumption formulas of the form A that belong to the inference figure the upper formula of which is C. (it is possible that C itself is such a formula, then Θ is empty and C identical with $[A]$.)

In a scheme for a modified derivation, $\begin{bmatrix}\Gamma\\ A\end{bmatrix}$ means that the part Γ is to be written above each of the mentioned assumption formulas A, so that Γ appears as many times as the assumption formula appeared (possibly not at all).

Where there stands $\Gamma(x)$ in a scheme of the old derivation, and $\Gamma(z)$ in the new one, the following is meant (and the same with $\Theta(x), \Theta(z)$): $\Gamma(z)$ denotes that part of derivation that is obtained from $\Gamma(x)$ when the variable x is substituted by z everywhere in $\Gamma(x)$.

[6]These correspond to precisely the schemes in the I. chapter, §5.

We have to show that

1. the new figure represents again a derivation, with the same endformula.

2. the number of hillocks of the greatest grade diminishes in the reduction and no hillocks of a greater grade appear.

2. 4 1 1. Division as in 2. 1 3 4.

2. 4 1 1. 1. We have a tree form. The endformula is unchanged (also if Ω is empty).

The following holds for the inference figures:

They are in general taken over from the old derivation. An exception is made, though, by the inference figures in $\overset{\Gamma(z)}{Dz}$ with universal reduction and in $\begin{matrix}[Dx]\\ \Theta(x)\\ C\end{matrix}$ with existence reduction. These have come from the inference figures of the old derivation through a substitution of z for x. It was specified to be so for Γ, Θ, it holds for Dz on the following basis: By the rule for substitution in the inference schemes, I. chapter, both of Dy and Dz come from Dx through the substitution of y resp. z for x. By the variable condition, y does not occur in Dx, (z can occur there). It follows that one obtains Dz through substitution of z for y in Dy. — Our claim holds as well for C, because y does not occur in C on the basis of the variable condition.

An inference figure remains in general correct if a variable is substituted by another one that did not occur bound in it previously. (This holds because of 2. 1 1 1.)

2. 4 1 1. 2. The assumption formulas.

2. 4 1 1. 2 1). $\overset{\Delta}{B}$ is left out in &-reduction, as well as $\begin{matrix}[B]\\ \Lambda\\ C\end{matrix}$ in $\vee$-reduction, and as a consequence certain assumption formulas are left out. This, however, has no effect.

2. 4 1 1. 2 2). The assumption formulas in $\overset{\Gamma}{A}$, $\overset{\Gamma(y)}{Dy}$, $\overset{\Gamma}{Dz}$, belong (in all reductions in which there is a part denoted by Γ) to inference figures that stand in this part itself or in $\overset{A}{\Omega}$ resp. $\overset{C}{\Omega}$, $\overset{B}{\Omega}$, $\overset{Dz}{\Omega}$. They are associated after the reduction to "the same" inference figures. This is clear and in order if we have case 2. (Except for universal reduction in which $\overset{\Gamma(y)}{Dy}$ is modified through the substitution of z. But we have, because of the variable condition: An assumption formula in $\overset{\Gamma(y)}{Dy}$, the inference figure of which belongs to $\overset{Dz}{\Omega}$, cannot contain y. Therefore it is not at all affected by the substitution by z.)

In case 1, that is, when the inference figure belongs to $\overset{\Gamma}{A}$, $\overset{\Gamma(y)}{Dy}$, $\overset{\Gamma}{Dz}$ itself, the matter is in order with &- and universal reduction. (A substitution of z for y in an assumption formula and its inference figure does no harm, because z did not occur bound (2. 1 1 1).)

With &, $\vee$, and existence reduction, a multiplication (or also a disappearance) of Γ can occur. The association of assumptions is fixed in each part in itself in the way it was done earlier in the single part.

2. 4 1 1. 2 3). We consider now assumption formulas in the parts $\begin{matrix}[A]\\ \Theta\\ C\end{matrix}$, $\begin{matrix}[A]\\ \Theta\\ B\end{matrix}$, $\begin{matrix}[Dy]\\ \Theta(y)\\ C\end{matrix}$ of the old derivation.

These are, in the first place, those assumption formulas that belong to the inference figures *OE, FI, EE* removed by the reduction. These were denoted by $[A], [D]$ in the schemes. The parts Γ are attached to these in the reduction, so they have not become, say, assumption formulas without associated inference figures.

Further assumption formulas in the parts considered can belong to inference figures in the parts themselves or to $\overset{C}{\Omega}$ resp. $\overset{B}{\Omega}$. The association can in both cases be carried over without further ado to the new figure. Existence reduction requires some attention, because y is here substituted by z. But it suffices to present the same considerations as with universal reduction under 2. 4 1 1. 22.

2. 4 1 1. 2 4). Now there remain still those assumption formulas to be considered that belong to Ω. Also their inference figures must belong to Ω, and the associations are obviously kept under the reduction.

2. 4 1 1. 3. The variable conditions.

2. 4 1 1. 3 1). Let us consider next *A, O* and *F*-reductions.

There appear in the new figure only such inference figures as there appeared already in the old figure. Therefore the first part of the variable conditions remains satisfied. The lower formulas of the new figure can, instead, depend on assumption formulas of which the corresponding ones did not depend in the old proof figure, in case of a *UI* resp. *EE*. This is obviously possible only if the *UI* resp. *EE* stands in $\begin{matrix}[A]\\ \Theta\\ C\end{matrix}$, $\begin{matrix}[A]\\ \Theta\\ B\end{matrix}$ and the new assumption formulas stand in Γ. The variable condition could be violated if the eigenvariable of the *UI* resp. *EE* appeared in such an assumption. This is, however, not possible, because it cannot appear in Γ at all, in accordance with 2. 1 1 2.

2. 4 1 1. 3 2). We shall now onsider the *U*- and *E*-reductions.

If in inference figures *UI, EE* of the new figure the eigenvariable is not z, essentially the above considerations are valid.

If an inference figure has z as an eigenvariable, it must have had z as an eigenvariable already in the old proof figure. (Because y could, by 2. 1 1 2, be the eigenvariable of exactly that one *UI* resp. *EE* that was removed by the reduction.)

An inference figure with the eigenvariable z should, now, belong to $\overset{Dz}{\Omega}$ resp. $\overset{C}{\Omega}$ (by 2. 1 1 3).

So it remains itself unchanged in the reduction. The first part of the variable conditions is secured by this. Now, instead, the lower formulas of the *UI* resp. *EE* can depend on assumption formulas. These would belong to inference figures standing further down, i.e., to those in Ω. However, such assumption formulas have remained completely unchanged in the reduction, as was already established in 2. 4 1 1. 2 2–4.

2. 4 1 1 2. (The hillock.)

The substitution of z for y has with U- and E-reduction no effect on the number and grade of the hillock. (Because the concepts of neighborhood, boundedness, bond, subformula with possibly different variables, hillock, grade, are independent of such substitution.)

Therefore we do not need to pay attention to substitution in considering these reductions.

The formulas A, B, Dz in the new derivation have all a lesser grade than the reduced main hillock, for they are its subformulas (with possibly other variables).

Thus, there remain to be considered only those hillocks of the new derivation that belong entirely to one of the parts Γ, Θ, Ω, as well as those to which a C belongs.

Such a hillock must have occurred already before the reduction at the corresponding place. (Observe that the loss of assumption formulas that can take place through the reduction cannot make a bond into a hillcock if it wasn't one before. For an assumption formula is always a subformula of its neighboring formulas (possibly with a different variable.)

The hillocks in Θ and Ω, as well as in Γ with A- and U-reduction, further the (possible) hillocks that contain C, correspond one-to-one to hillocks of the same grade in the old derivation.

With the parts Γ in *O, F*- and E-reduction, one can say only: To each new hillock there is an old one of the same grade; But there is no one-to-one correspondence, because Γ can have been multiplied by the reduction.

We have in any case already:

No hillocks of a greater grade than the one the main hillock had have appeared.

We shall be finished if we succeed in showing that there appeared in the case of *O*-, *F*- and *E*-reductions in part Γ of the old derivation at most hillocks of a lesser grade than the main hillock. To this end, we have to turn back to the definition of a main hillock: Namely, the formulas in Γ are all higher (in the old derivation) than the main hillock. This can be read off at once from the definition of higher and the reduction schemes. —

The main hillock becomes removed and no new hillocks of its grade appear. Therefore the number of hillocks of the greatest grade is diminished.

2. 4 2. Second case. The main hillock belongs to a *V*-inference figure. It is then the lower formula of the inference. (For if it is the upper formula ⅄, it is in addition a lower formula of a *V*-inference figure, because ⅄ is not the main formula of any inference figure.)

We distinguish six cases according to whether the hillock is an upper formula of *AE, OE, FE, UE, EE*, or *V*. (Further ones are not possible, because the main formula in the *I*-inference figures is a lower formula.)

The six reduction schemes:

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Delta\\ \text{⅄}\end{array}}{|A\&B}}{A}\\ \Omega\end{array}\qquad \begin{array}{c}\dfrac{\begin{array}{c}\Delta\\ \text{⅄}\end{array}}{A}\\ \Omega\end{array}\qquad\qquad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Delta\\ \text{⅄}\end{array}}{|A\vee B}\quad \begin{array}{c}[A]\\ \Theta\\ C\end{array}\quad \begin{array}{c}[B]\\ \Lambda\\ C\end{array}}{C}\\ \Omega\end{array}\qquad \begin{array}{c}\left[\dfrac{\begin{array}{c}\Delta\\ \text{⅄}\end{array}}{A}\right]\\ \Theta\\ C\\ \Omega\end{array}\qquad\qquad \begin{array}{c}\dfrac{\begin{array}{c}\Gamma\\ A\end{array}\quad \dfrac{\begin{array}{c}\Delta\\ \text{⅄}\end{array}}{|A\supset B}}{B}\\ \Omega\end{array}\qquad \begin{array}{c}\dfrac{\begin{array}{c}\Delta\\ \text{⅄}\end{array}}{B}\\ \Omega\end{array}$$

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Delta\\ \text{⅄}\end{array}}{|(y)Dy}}{Dz}\\ \Omega\end{array}\qquad \begin{array}{c}\dfrac{\begin{array}{c}\Delta\\ \text{⅄}\end{array}}{Dz}\\ \Omega\end{array}\qquad\qquad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Delta\\ \text{⅄}\end{array}}{|(Ex)Dx}\quad \begin{array}{c}[Dy]\\ \Theta\\ C\end{array}}{C}\\ \Omega\end{array}\qquad \begin{array}{c}\left[\dfrac{\begin{array}{c}\Delta\\ \text{⅄}\end{array}}{Dy}\right]\\ \Theta\\ C\\ \Omega\end{array}\qquad\qquad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Delta\\ \text{⅄}\end{array}}{|\text{⅄}}}{A}\\ \Omega\end{array}\qquad \begin{array}{c}\dfrac{\begin{array}{c}\Delta\\ \text{⅄}\end{array}}{A}\\ \Omega\end{array}$$

[Written in the middle of the next manuscript page:]

Subformula theorem from the hillock theorem:

There is, among the formulas that are not subformulas of the endformula, one of a greatest grade. This is an inner hillock. Now one applies the hillock theorem.