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MATHEMATICAL FUZZY LOGICS

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Abstract. The last decade has seen an enormous development in infinite-valued systems and in particular in such systems which have become known as mathematical fuzzy logics.

The paper discusses the mathematical background for the interest in such systems of mathematical fuzzy logics, as well as the most important ones of them. It concentrates on the propositional cases, and mentions the first-order systems more superficially. The main ideas, however, become clear already in this restricted setting.

§1. Prehistory. Generalized sets with a graded notion of membership degree have been introduced in 1965 by the US-American system scientist Lotfi Zadeh [120] under the name *fuzzy sets* and by the German mathematician Dieter Klaua [87, 88, 89] under the name *mehrwertige Mengen* (many-valued sets).

There was a slight formal difference between both approaches: Zadeh used the whole real unit interval as the set of membership degrees, but Klaua used "only" a finite set of equidistant points from this interval. However, there was a big difference in their motivations. Zadeh's main problem was that he needed such generalized sets, and the vague notions they represent, to model complex systems. In more detail this background is explained in [115]. Klaua, on the other hand, got his inspiration from a discussion following a colloquium talk Karl Menger had given in Berlin.

Accordingly, and influenced by their respective personalities, there was also a big difference in the influence both these scientists achieved. Zadeh gave for more than 40 years inspiring talks at many conferences offering and supporting his ideas for innovative applications whereas Klaua, bound to stay in relative isolation in East Germany after the Berlin wall had been built, only offered his ideas in German language publications.

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It was part of Klaua's approach from the very beginning, and soon obvious for logically minded people also with respect to Zadeh's approach, that the membership degrees could well be understood as truth degrees for a membership predicate which was treated in the realm of some suitable many-valued logic. As such a background logic Klaua had chosen Łukasiewicz's finitely-valued systems.[1]

As we will focus our attention on the infinitely-valued case, we shall further on only use the term "fuzzy set" to refer to these generalized sets.

The whole story of approaches toward set theoretic foundations for fuzzy set theory has been surveyed in [58, 59] and shall not be the topic of the present considerations. Instead, here the formal logics related to fuzzy set theory shall be considered.

The usual relationships between set algebra and—some suitable, propositional—logic remain valid also in the generalized setting of fuzzy sets. In the first years of the development of fuzzy set theory there was only rarely a reference to the background logic, and even if such a reference happened this logic was used mainly in a naive way, but not to give a fully formalized set theory based upon such a suitable many-valued logic.

Only few papers initially have been devoted to the underlying logic(s) for fuzzy set theory. Important early contributions had been provided by Joseph Goguen [52] and Robin Giles [50]. Also the present author has always emphasized this relationship, e.g., in [53] and particularly in the monograph [55]. Essentially these early papers have been related solely to applications, or only to generalizations—usually called "fuzzifications"—of well known elementary mathematical notions: and then they often had only a l'art pour l'art character. The more set theoretically oriented early papers mainly discussed set algebraic aspects.

These discussions proved to be rather important also from a logical point of view. One of their essential outcomes was the agreement that it would be welcome for applications to have in the realm of fuzzy sets not only an intersection operation which is, as usual in classical set theory, an idempotent operation, but to have additionally a non-idempotent intersection operation available. First examples of both types of intersection operations had already been given in [120]: there Zadeh defined the "standard" intersection of two fuzzy sets by taking pointwise the minimum of the respective membership degrees, and he mentioned as two further possibilities to take pointwise either the (arithmetic) product of the respective membership degrees or a combination of them which corresponded to the strong, i.e., arithmetic conjunction of the Łukasiewicz systems.

[1]Usually references to notions for many-valued logics will be omitted. The interested reader may always consult [56].

§**2. Mathematical preliminaries.** A fuzzy set A is usually a fuzzy subset of a given set $\mathbb{X}$ and characterized by its membership function $\mu_A\colon \mathbb{X} \mapsto [0,1]$. The set $\mathbb{X}$ is often called the *universe of discourse*. This notation derives from [120]. So these fuzzy sets are (possibly) first-level objects of a cumulative hierarchy, with the elements of $\mathbb{X}$ as urelements. But the usual applications do not need higher level fuzzy sets. And also for our discussion of the background logic such higher level fuzzy sets do not matter.

2.1. Set algebra for fuzzy sets. Mathematically it is customary to identify such a fuzzy subset of $\mathbb{X}$ with its membership function. Accordingly, $\mu_A(a)$ and $A(a)$ both are used to denote the membership degree of the object $a \in \mathbb{X}$ w.r.t. the fuzzy set A. For any binary operation $*$ between membership degrees the pointwise approach means to define from it a binary operation $\circledast$ for fuzzy sets such that the fuzzy set $A \circledast B$ is characterized by

$$A \circledast B(x) = A(x) * B(x) \quad \text{for all } x \in \mathbb{X}. \tag{1}$$

Hence, Zadeh's "standard" *intersection* $A \cap B$ and *union* $A \cup B$ are characterized by

$$A \cap B(x) = \min\{A(x), B(x)\}, \quad A \cup B(x) = \max\{A(x), B(x)\}. \tag{2}$$

Additionally, again following the first proposal from [120], one usually also defines the *complement* $\complement_{\mathbb{X}}A$ of a fuzzy set A by the condition

$$\complement_{\mathbb{X}}A(x) = 1 - A(x) \quad \text{for all } x \in \mathbb{X}. \tag{3}$$

2.2. T-norms and t-conorms. For the previously mentioned non-idempotent intersection operation the mathematically oriented part of the fuzzy community reached, mainly in the first half of the 1980s, a consensus that it should be defined via (1) from a triangular norm $*$. Such *triangular norms—t-norms* for short—had first been considered in the context of probabilistic metric spaces to get a suitable version of a triangle inequality, cf. e.g., [114], and found since independent interest in different contexts, cf. [5, 90]. They are isotonic, associative, and commutative binary operations in the unit interval which have 1 as their neutral element. This means that they make the unit interval an ordered monoid.

The class of all t-norms is, however, very large and not yet really well understood. So the question appears to restrict to suitable subclasses, e.g., to the continuous t-norms or to the left-continuous ones.[2] Standard examples for continuous t-norms are the min-operation in [0, 1], also called Gödel t-norm T_{G}, the arithmetic product in [0, 1], also called product t-norm T_{P}, and the Łukasiewicz t-norm $T_{\mathsf{L}}\colon (u, v) \mapsto \max\{u + v - 1, 0\}$ which is the truth degree function of the strong conjunction in the Łukasiewicz many-valued

[2]For a t-norm T left-continuity means that all the unary functions T_a with $T_a(x) = T(a, x)$ for each $a \in [0, 1]$ are left-continuous. For continuity the conditions (i) that T is continuous as a binary function and (ii) that all the T_a are continuous coincide; cf. [90] or [56].

systems. And a standard example for a left-continuous t-norm which is not continuous is the nilpotent minimum T_{NM} defined as

$$T_{\mathsf{NM}}(u,v) = \begin{cases} \min\{u,v\}, & \text{if } u+v>1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

These examples for continuous t-norms are even characteristic in the sense that each continuous t-norm is an ordinal sum of isomorphic copies of $T_{\mathsf{L}}, T_{\mathsf{P}}, T_{\mathsf{G}}$, cf. [90] and also [56].

Parallel with t-norms one often considers also *t-conorms*: these are isotonic, associative, and commutative binary operations in the unit interval which have 0 as their neutral element. For the set algebra of fuzzy sets they define (possibly) non-idempotent unions, and for the background logics they constitute (possibly) non-idempotent disjunctions.

There is a natural 1-1 duality between t-norms and t-conorms. By

$$1 - S(u,v) = T(1-u, 1-v) \tag{5}$$

one determines a t-conorm S for any t-norm T, and conversely determines a t-norm T for any t-conorm S. This relationship connects e.g., the truth degree function $(u,v) \mapsto \max\{u+v-1, 0\}$ of the Łukasiewicz strong conjunction with that one of the corresponding strong disjunction $(u,v) \mapsto \min\{u+v, 1\}$.

Obviously (5) constitutes, for the background logic, a de Morgan connection between suitably chosen conjunctions and disjunctions—as long as the function $u \mapsto 1-u$ acts as the truth degree function of a negation. And indeed this is the truth degree function of the negation of the Łukasiewicz systems, which was already used in the definition (3) of the complement of a fuzzy set.

Summing up, one has for the background logic idempotent "weak" connectives for conjunction and disjunction, determined by the minimum and the maximum operation in $[0,1]$. Furthermore one is interested to have (possibly) non-idempotent "strong" connectives for conjunction and disjunction, determined by a t-norm and a t-conorm, usually one the dual of the other according to (5).

2.3. Inclusion relations and implications. Parallel to the set algebraic operations for fuzzy sets one has definitions of an *inclusion* relation. Zadeh [120] defined a *binary* relation $\subseteq$ as

$$A \subseteq B \quad \text{iff} \quad A(x) \leq B(x) \quad \text{for all } x \in \mathbb{X}, \tag{6}$$

whereas Klaua [87, 88] introduced *graded* inclusion relations $\sqsubseteq$, defined in [88] as

$$[\![A \sqsubseteq_{\mathsf{L}} B]\!] = [\![\forall x (A(x) \rightarrow_{\mathsf{L}} B(x))]\!] = \inf_{x \in \mathbb{X}} \min\{1 - A(x) + B(x), 1\}. \tag{7}$$

Here $\rightarrow_{\mathsf{L}}$ is the implication of the Łukasiewicz systems, and $[\![H]\!]$ is shorthand for the truth degree of the formula H.

Obviously one has $A \subseteq B$ iff $[\![A \sqsubseteq_{\mathsf{L}} B]\!] = 1$, such that (7) is a natural generalization of (6).

Therefore one should also care about implication connectives for the background logic.

In this case, however, there is among the fuzzy set people not as much agreement as in the case of e.g., the non-idempotent conjunction connectives. Besides approaches which started from abstract properties generalized implication connectives should have, like [34, 116, 118], there are mainly two competing approaches.

1. One approach generalizes the characterizations of the classical implication connective using disjunction and negation, or using conjunction and negation.
2. A second approach generalizes the characterization of the intuitionistic implication connective in Heyting algebras as

$$u \wedge v \leq w \quad \text{iff} \quad u \leq v \rightarrowtail w, \tag{8}$$

with lattice meet $\wedge$ and relative pseudo-complement $\rightarrowtail$, in the sense that it uses a t-norm $\otimes$ and instead of (8) the *adjointness condition*

$$u \otimes v \leq w \quad \text{iff} \quad u \leq v \rightarrowtail w. \tag{9}$$

Usually implications formed according to (1) are called *S-implications*, and such ones formed according to (2) are called *residuated implications* or simply *R-implications*.

Technically, the adjointness condition (9) forces the t-norm $\otimes$ to be left continuous. Additionally one has

$$u \rightarrowtail v = \sup\{z \mid u \otimes z \leq v\} \tag{10}$$

as an equivalent characterization of the R-implication operation. (Indeed, the supremum is always a maximum here.)

It is interesting to mention that also the continuity of t-norms has an algebraic equivalent: a (left-continuous) t-norm $\otimes$ is continuous iff its residuation operation satisfies the *divisibility condition*

$$u \wedge v = u \otimes (u \rightarrowtail v) \tag{11}$$

with $\wedge$ as the minimum operation.

§3. Residuated implications versus S-Implications. The basic properties of classical logic, particularly the logical equivalence of

$$\varphi \rightarrow \psi \quad \text{and} \quad \neg\varphi \vee \psi \quad \text{and} \quad \neg(\varphi \wedge \neg\psi), \tag{12}$$

allows a definitional introduction of the implication connective. Hence, the introduction of the implication connective as residuation via (10) or via the adjointness condition (9) seems to be quite sophisticated, and perhaps unnecessarily complicated.

If one has in mind, for the present case, a definitional approach toward implication similar to (12), one would have to start for a definition of an implication connective either with a generalized disjunction or with a generalized conjunction, and had to add in any case a generalized negation. This has been done e.g., in [20, 83], but this approach does not really become simpler as the other one via (9) or (10), because one needs to fix either, besides the basic t-norm, an additional negation, or one has to fix a negation together with a disjunction.

However, the main disadvantage of the use of an S-implication is that one looses a quite natural strong soundness property one has for the rule of detachment in the residuated t-norm based approach. From the adjointness condition (9) one has always

$$u \otimes (u \rightarrowtail v) \leq v \tag{13}$$

simply because of

$$u \otimes (u \rightarrowtail v) \leq v \quad \text{iff} \quad (u \rightarrowtail v) \otimes u \leq v \quad \text{iff} \quad u \rightarrowtail v \leq u \rightarrowtail v.$$

Therefore one has, in the t-norm based approach with implication defined as residuation, a natural lower bound for the truth degree of a formula ψ which has been derived from $\varphi \to \psi$ and φ via the rule of detachment: the truth degree of the formula $\varphi \& (\varphi \to \psi)$.

A similar property is lacking for the approach via S-implications. In this case, say starting from a t-norm $\otimes$ and a negation $\rightharpoondown$, the corresponding property to (13) would be

$$u \otimes \rightharpoondown (u \otimes \rightharpoondown v) \leq v. \tag{14}$$

But this fails already in the case that, independent of the choice of the t-norm $\otimes$, the negation $\rightharpoondown$ is the Gödel negation $\sim$, i.e., the negation of the Gödel systems. For this negation and any $v > 0$ one has $\sim v = 0$, hence $u \otimes \sim v = 0$, which means $\sim (u \otimes \sim v) = 1$ and $u \otimes \sim (u \otimes \sim v) = u$. Now one may choose $u > v$ to see that (14) fails.

Hence, for the use of logical calculi for modeling reasoning an approach via R-implications is the more suitable one. Therefore we shall concentrate here on this type of approach.

It is characteristic for this approach that he disregards t-conorms, i.e., that he does not discuss disjunction operations primarily, and that he defines an "internal" negation always as "implies falsum". This has the consequence that often—i.e., always if the basic t-norm does not have zero divisors[3]—this internal negation is the (non-involutive) negation of the Gödel system G and not the involutive negation $x \mapsto 1 - x$ of the Łukasiewicz system L.

[3]Zero divisors of a t-norm T are reals $0 < u, v < 1$ with $T(u, v) = 0$.

This has some disadvantages which one tries to overcome by extending these fuzzy logics with an additional involutive negation, cf. [43].

§4. Algebraic properties of the truth degree sets. In theoretical considerations the fuzzy people usually do not specify the t-norm which determines their non-idempotent intersection operation. So also the background logic should refer to some t-norm, however not to a particular one.

From this point of view, the algebraic structure of the set of membership degrees, i.e., the algebra of truth degrees of the underlying logic, becomes the core of the matter. And following this type of approach, one may even be interested to determine the underlying logic by an algebraic semantics.

To reach such a goal, it is of interest to find some suitable variety of algebraic structures including the t-norm based structures over the real unit interval. The following conditions seem to be structurally important for t-norms $\otimes$:

- $\langle [0, 1], \otimes, 1\rangle$ is an abelian monoid,
- the usual ordering $\leq$ of the reals is a (lattice) ordering in $[0, 1]$ which has a universal lower bound and a universal upper bound,
- both structures "fit together": $\otimes$ is non-decreasing with respect to this lattice ordering,
- and there is a further residuation operation $\rightarrowtail$ such that $\otimes$ and $\rightarrowtail$ form an adjoint pair, i.e., satisfy (9).

Algebraic structures $[0, 1]_\otimes = \langle [0, 1], \otimes, \rightarrowtail, 0, 1\rangle$ with these properties often are called *t-norm algebras*.

Leaving the support undetermined, residuated abelian lattice-ordered monoids may be suitable truth degree structures for such t-norm-related logics.

In general, however, abelian lattice-ordered monoids may have different elements as their universal upper bound of the lattice and as their neutral element of the monoid. This is not the case for the t-norm-based systems, they make $[0, 1]$ into an *integral* abelian lattice-ordered monoid as truth degree structure, i.e., one in which the universal upper bound of the lattice ordering and the neutral element of the monoidal structure coincide.

Summing up, hence, we are going to consider *bounded integral commutative residuated lattices*, i.e., algebraic structures $\mathbf{L} = \langle L, \cap, \cup, *, \rightarrowtail, \mathbf{0}, \mathbf{1}\rangle$ such that L is a lattice under $\cap$, $\cup$ with universal lower bound $\mathbf{0}$ and universal upper bound $\mathbf{1}$, and an abelian lattice-ordered monoid under $*$ with neutral element $\mathbf{1}$, and such that the operations $*$ and $\rightarrowtail$ form an adjoint pair, i.e., satisfy

$$x * z \leqslant y \quad \Longleftrightarrow \quad z \leqslant (x \rightarrowtail y). \tag{15}$$

(In the fuzzy logic community, *residuated lattices* is the short name for these structures.) In this framework one additionally introduces, following the understanding of the negation connective e.g., in intuitionistic logic, a further operation $-$ by $-x =_{\mathrm{df}} x \rightarrowtail \mathbf{0}$. But usually one does not discuss a strong disjunction separately.

§**5. Fuzzy propositional logics.** Languages for propositional logics with an algebraic semantic given by a class $\mathcal{K}$ of residuated lattices should therefore have connectives $\wedge, \vee, \&, \rightarrow$ corresponding to the operations $\cap, \cup, *, \rightarrowtail$, and either a negation connective $\neg$ corresponding to $-$ or a truth degree constant $\overline{0}$ to denote $\mathbf{0}$. *Validity* of a formula φ in a fixed residuated lattice $\mathbf{L}$ means that φ gets the value $\mathbf{1}$ for all $\mathbf{L}$-evaluations.

The class $\mathcal{K}$ of residuated lattices which defines such a logic $\mathfrak{L}_{\mathcal{K}}$ should preferably be a variety. If this variety is generated by a class of t-norm algebras, the logic $\mathfrak{L}$ will be called *t-norm based.* If this variety is even generated by a single t-norm algebra $[0,1]_{\otimes}$ this logic should be denoted $\mathfrak{L}(\otimes)$.

The whole class $\mathcal{K}$ determines the general semantics, and the t-norm algebras from $\mathcal{K}$ determine the standard semantics for $\mathfrak{L}_{\mathcal{K}}$

Particular cases of t-norm based fuzzy propositional logics are the infinite valued Łukasiewicz logic L, the infinite valued Gödel logic G, and the product logic Π introduced in [76]. They have algebraic semantics given by the classes of all MV-algebras, of all pre-linear Heyting algebras (often called Gödel algebras), and of all product algebras, respectively. Hence one has also $\mathsf{L} = \mathfrak{L}(T_{\mathsf{L}})$, $\mathsf{G} = \mathfrak{L}(T_{\mathsf{G}})$, and $\Pi = \mathfrak{L}(T_{\mathsf{P}})$.

§**6. The logic of continuous t-norms.** The class of t-norm algebras (with a continuous t-norm or not) is not a variety: it is not closed under direct products because each t-norm algebra is linearly ordered. Hence one may expect that it would be helpful for the development of a logic of continuous t-norms to extend the class of all divisible t-norm algebras in a moderate way to get a variety. And indeed this idea works: it was developed by Petr Hájek and in detail explained in [64].

The core points are that one considers instead of the divisible t-norm algebras, which are linearly ordered integral monoids as mentioned previously, now lattice ordered integral monoids which are divisible, which have an additional residuation operation connected with the semigroup operation via an adjointness condition like (9), and which satisfy the pre-linearity condition

$$(x \rightarrowtail y) \vee (y \rightarrowtail x) = \mathbf{1}. \tag{16}$$

These structures have been called BL-algebras; they are completely defined in the following way.

DEFINITION 6.1. $\mathbf{L} = \langle L, \vee, \wedge, *, \rightarrowtail, \mathbf{0}, \mathbf{1} \rangle$ *is a* BL-algebra *iff*

(i) $(L, \vee, \wedge, \mathbf{0}, \mathbf{1})$ *is a bounded lattice,*
(ii) $(L, *, \mathbf{1})$ *is an abelian monoid,*

(iii) *the operations* $*$ *and* $\rightarrowtail$ *satisfy the* adjointness *condition* (15),
(iv) *the* pre-linearity *condition* (16) *is satisfied,*
(v) *the* divisibility *condition is satisfied, i.e., one has always*[4]

$$x * (x \rightarrowtail y) = x \wedge y. \tag{17}$$

The pre-linearity condition (16) is equivalent to the condition that one always has

$$((x \rightarrowtail y) \rightarrowtail z) \rightarrowtail (((y \rightarrowtail x) \rightarrowtail z) \rightarrowtail z) = \mathbf{1}. \tag{18}$$

The axiomatization of Hájek [64] for the *basic t-norm logic* BL (in [56] denoted BTL), i.e., for the class of all well-formed formulas which are valid in all BL-algebras, is given in a language $\mathcal{L}_T$ which has as basic vocabulary the connectives $\rightarrow, \&$ and the truth degree constant $\overline{0}$, taken in each BL-algebra $\langle L, \cap, \cup, *, \rightarrowtail, \mathbf{0}, \mathbf{1}\rangle$ as the operations $\rightarrowtail, *$ and the element $\mathbf{0}$. Then this t-norm based logic BL has the following axiom schemata:

($\mathsf{Ax_{BL}}$1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$,
($\mathsf{Ax_{BL}}$2) $\varphi \& \psi \rightarrow \varphi$,
($\mathsf{Ax_{BL}}$3) $\varphi \& \psi \rightarrow \psi \& \varphi$,
($\mathsf{Ax_{BL}}$4) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi \& \psi \rightarrow \chi)$,
($\mathsf{Ax_{BL}}$5) $(\varphi \& \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$,
($\mathsf{Ax_{BL}}$6) $\varphi \& (\varphi \rightarrow \psi) \rightarrow \psi \& (\psi \rightarrow \varphi)$,
($\mathsf{Ax_{BL}}$7) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$,
($\mathsf{Ax_{BL}}$8) $\overline{0} \rightarrow \varphi$,

and has as its (only) inference rule the rule of detachment.

Starting from the primitive connectives $\rightarrow, \&$ and the truth degree constant $\overline{0}$, the language $\mathcal{L}_T$ of BL is extended by definitions of additional connectives $\wedge, \vee, \neg$:

$$\varphi \wedge \psi =_{\mathrm{df}} \varphi \& (\varphi \rightarrow \psi), \tag{19}$$

$$\varphi \vee \psi =_{\mathrm{df}} ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi), \tag{20}$$

$$\neg \varphi =_{\mathrm{df}} \varphi \rightarrow \overline{0}, \tag{21}$$

where φ, ψ are formulas of the language of that system.

Calculations (in BL-algebras) show that the additional connectives $\wedge, \vee$ just have the lattice operations $\cap, \cup$ as their truth degree functions.

The system BL is an implicative logic in the sense of Rasiowa [112]. So one gets a general soundness and completeness result.

THEOREM 6.1 (General Completeness). *A formula φ of the language $\mathcal{L}_T$ is derivable within the axiomatic system* BL *iff φ is valid in all BL-algebras.*

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[4]This condition is equivalent, in the linearly ordered case, to the standard algebraic definition of divisibility as explained e.g., in [56, 84].

However, it is shown in [64] that already the class of all BL-chains, i.e., of all linearly ordered BL-algebras, provides an adequate algebraic semantics.

THEOREM 6.2 (General Chain Completeness). *A formula φ of $\mathcal{L}_T$ is derivable within the axiomatic system* BL *iff φ is valid in all BL-chains.*

But even more is provable and leads back to the starting point of the whole approach: the theorems of BL are just those formulas which hold true w.r.t. all divisible t-norm algebras. This was, extending preliminary results from [63], finally proved in [24].

THEOREM 6.3 (Standard Completeness). *The class of all formulas which are provable in the system* BL *coincides with the class of all formulas which are logically valid in all t-norm algebras with a continuous t-norm.*

The main steps in the proof are to show (i) that each BL-algebra is a subdirect product of subdirectly irreducible BL-chains, i.e., of linearly ordered BL-algebras which are not subdirect products of other BL-chains, and (ii) that each subdirectly irreducible BL-chain can be embedded into the ordinal sum of some BL-chains which are either trivial one-element BL-chains, or linearly ordered MV-algebras, or linearly ordered product algebras, such that (iii) each such ordinal summand is locally embedable into a t-norm based residuated lattice with a continuous t-norm, cf. [24, 63] and again [56].

This is a lot more of algebraic machinery as necessary for the proof of the General Completeness Theorem 6.1 and thus offers a further indication that the extension of the class of divisible t-norm algebras to the class of BL-algebras made the development of the intended logical system easier. But even more can be seen from this proof: *the class of BL-algebras is the smallest variety which contains all the divisible t-norm algebras*, i.e., all the t-norm algebras determined by a continuous t-norm. And the algebraic reason for this is that each variety may be generated from its subdirectly irreducible elements, cf. again [19, 32].

But the algebraic machinery allows even deeper insights. After some particular results e.g., in [80, 82], the study of such subvarieties of the variety of all BL-algebras which are generated by single t-norm algebras $\langle [0,1], \wedge, \vee, \otimes, \rightarrowtail, 0, 1\rangle$ with a continuous t-norm $\otimes$ led to (finite) axiomatizations of those t-norm based logics which have a standard semantics determined just by this continuous t-norm algebra, cf. [45].

Yet another generalization of Theorem 6.1 deserves to be mentioned. To state it, let us call *schematic extension* of BL every extension which consists in an addition of axiom schemata to the axiom schemata of BL. And let us denote such an extension by BL($\mathcal{C}$). And call BL($\mathcal{C}$)-algebra each BL-algebra **A** which makes **A**-valid all formulas of $\mathcal{C}$.

Then one can prove, as done in [64], an even more general completeness result.

THEOREM 6.4 (Strong General Completeness). *For each set $\mathcal{C}$ of axiom schemata and any formula φ of $\mathcal{L}_T$ there are equivalent*:

(i) φ *is derivable within* $\mathsf{BL}(\mathcal{C})$;
(ii) φ *is valid in all* $BL(\mathcal{C})$*-algebras*;
(iii) φ *is valid in all* $BL(\mathcal{C})$*-chains*.

For the standard semantics this result holds true only in a restricted form: one has to restrict the consideration to *finite* sets $\mathcal{C}$ of axiom schemata, i.e., to finite theories. For the Łukasiewicz logic L, which is the extension of BL by the schema $\neg\neg\varphi \to \varphi$ of double negation, this has already been shown in [64]. And for arbitrary continuous t-norms this follows from results of Zuzana Hanikova [80, 81].

§7. The logic of left continuous t-norms. The guess of Francesc Esteva and Lluís Godo [40] has been that one should arrive at the logic of left continuous t-norms if one starts from the logic of continuous t-norms and deletes the continuity condition, i.e., the divisibility condition (11).

The algebraic approach needs only a small modification: in the definition of the BL-algebras one has simply to delete the divisibility condition. The resulting algebraic structures have been called *MTL-algebras*. They again form a variety.

Following this idea, one has to modify the previous axiom system in a suitable way. And one has to delete the definition (19) of the connective $\wedge$, because this definition (together with suitable axioms) essentially codes the divisibility condition. The definition (20) of the connective $\vee$ remains unchanged.

As a result one now considers a new system MTL of mathematical fuzzy logic, characterized semantically by the class of all MTL-algebras. It is connected with the axiom system

($\mathsf{Ax_{MTL}}1$) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
($\mathsf{Ax_{MTL}}2$) $\varphi \& \psi \to \varphi$,
($\mathsf{Ax_{MTL}}3$) $\varphi \& \psi \to \psi \& \varphi$,
($\mathsf{Ax_{MTL}}4$) $(\varphi \to (\psi \to \chi)) \to (\varphi \& \psi \to \chi)$,
($\mathsf{Ax_{MTL}}5$) $(\varphi \& \psi \to \chi) \to (\varphi \to (\psi \to \chi))$,
($\mathsf{Ax_{MTL}}6$) $\varphi \wedge \psi \to \varphi$,
($\mathsf{Ax_{MTL}}7$) $\varphi \wedge \psi \to \psi \wedge \varphi$,
($\mathsf{Ax_{MTL}}8$) $\varphi \& (\varphi \to \psi) \to \varphi \wedge \psi$,
($\mathsf{Ax_{MTL}}9$) $\overline{0} \to \varphi$,
($\mathsf{Ax_{MTL}}10$) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$,

together with the rule of detachment (w.r.t. the implication connective $\to$) as (the only) inference rule.

Again, the system MTL is an implicative logic in the sense of Rasiowa [112], giving a general soundness and completeness result as for the previous system BL. Proofs of these results were given in [40].

THEOREM 7.1 (General Completeness). *A formula φ of the language $\mathcal{L}_T$ is derivable within the system* MTL *iff φ is valid in all MTL-algebras.*

Furthermore it is shown, in [40], that again already the class of all MTL-chains provides an adequate algebraic semantics.

THEOREM 7.2 (General Chain Completeness). *A formula φ of $\mathcal{L}_T$ is derivable within the axiomatic system* MTL *iff φ is valid in all MTL-chains.*

And again, similar as for the BL-case, even more is provable: the system MTL characterizes just these formulas which hold true w.r.t. all those t-norm based logics which are determined by a left continuous t-norm, cf. [85].

THEOREM 7.3 (Standard Completeness). *The class of formulas which are provable in the system* MTL *coincides with the class of formulas which are logically valid in all t-norm algebras with a left continuous t-norm.*

This result again means, as the similar one for the logic of continuous t-norms, that the variety of all MTL-algebras is the smallest variety which contains all t-norm algebras with a left continuous t-norm.

Because of the fact that the BL-algebras are the divisible MTL-algebras, one gets another adequate axiomatization of the basic t-norm logic BL if one extends the axiom system of MTL with the additional axiom schema

$$\varphi \land \psi \to \varphi \mathbin{\&} (\varphi \to \psi). \tag{22}$$

The simplest way to prove that this implication is sufficient is to show that the inequality $x * (x \rightarrowtail y) \leq x \cap y$, which corresponds to the converse implication, holds true in each MTL-algebra. Similar remarks apply to further extensions of MTL.

Also for MTL an extended completeness theorem similar to Theorem 6.4 holds true. (The notions MTL($\mathcal{C}$) and MTL($\mathcal{C}$)-algebra are used similar to the BL case.)

THEOREM 7.4 (Strong General Completeness). *For each set $\mathcal{C}$ of axiom schemata and any formula φ of $\mathcal{L}_T$ the following are equivalent*:

(i) *φ is derivable within the system* MTL($\mathcal{C}$);
(ii) *φ is valid in all MTL($\mathcal{C}$)-algebras*;
(iii) *φ is valid in all MTL($\mathcal{C}$)-chains.*

For much more information upon completeness matters for different systems of fuzzy logic the reader may consult [29].

§8. Extensions to first-order logics. The extensions of these propositional logics to first-order ones follows the standard lines of approach: one has to start from a first-order language[5] $\mathcal{L}$ and a suitable residuated lattice $\mathbf{A}$, and has to define $\mathbf{A}$-interpretations $\mathbf{M}$ by fixing a nonempty domain $M = |\mathbf{M}|$

[5]With the two standard quantifiers $\forall, \exists$, but without function symbols for the present considerations.

and by assigning to each predicate symbol of $\mathcal{L}$ an $\mathbf{A}$-valued relation in M (of suitable arity) and to each constant an element from (the support of) $\mathbf{A}$.

The satisfaction relation is also defined in the standard way. The quantifiers $\forall$ and $\exists$ are interpreted as taking the infimum or supremum, respectively, of all the values of the relevant instances.

That this approach works well one has either to suppose that the underlying lattices of the interpretations are complete lattices, or at least that all the necessary infima and suprema do exist in these lattices. Interpretations over lattices which satisfy this last mentioned condition are called *safe* by Hájek [64].

For the logic BL of continuous t-norms, Hájek [64] added the axioms

($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$, where t is substitutable for x in φ,
($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$, where t is substitutable for x in φ,
($\forall$2) $(\forall x)(\chi \to \varphi) \to (\chi \to (\forall x)\varphi)$, where x is not free in χ,
($\exists$2) $(\forall x)(\varphi \to \chi) \to ((\exists x)\varphi \to \chi)$, where x is not free in χ,
($\forall$3) $(\forall x)(\chi \vee \varphi) \to \chi \vee (\forall x)\varphi$, where x is not free in χ.

and the rule of generalization to the propositional calculus yielding the system $\mathsf{BL}\forall$.

Then he was able to prove the following completeness theorem.

THEOREM 8.1 (General Chain Completeness). *A first-order formula φ is $\mathsf{BL}\forall$-provable iff it is valid in all safe interpretations over BL-chains.*

This result can be extended to a lot of other first-order fuzzy logics, e.g., to $\mathsf{MTL}\forall$.

We will not discuss further completeness results here but refer to the extended survey [74]. But it should be mentioned that, as suprema are not always maxima and infima not always minima, the truth degree of an existentially/universally quantified formula may not be the maximum/minimum of the truth degrees of the instances. It is, however, interesting to have conditions which characterize models in which the truth degrees of each existentially/universally quantified formula is witnessed as the truth degree of an instance. Cintula/Hájek [75] study this problem.

The topic of first-order fuzzy logics with identity deserves some attention. The core problem is, as in any many-valued logic, whether the identity symbol should be interpreted by the standard, i.e., "two-valued" identity relation, or whether one should allow for graded identity relations inside the interpretations.

Direct translations of the identity axioms of classical first-order logic into, e.g., the language of the Łukasiewicz systems force that the interpretation of the identity symbol has to be the standard identity relation, cf. [117]. Similarly, for a wide class of first-order fuzzy logics the addition of the axioms

$\mathsf{Id1}$: $x \approx y \vee \neg x \approx y$
$\mathsf{Id2}$: $x \approx x$
$\mathsf{Id3}$: $x \approx y \to (\varphi(x, \vec{z}) \to \varphi(y, \vec{z}))$, for y substitutable for x in φ

forces that the identity symbol $\approx$ can only be understood as meaning standard identity. A general completeness theorem like Theorem 8.1 remains valid in this case too, cf. [74].

For the case of the Łukasiewicz logics, however, a slight modification of the standard identity axioms – particularly of the Leibniz schema, as given in [54], allows for graded identity relations, cf. also [56]. For fuzzy logics in general similarity relations, i.e., graded equivalence relations offer such an approach, cf. e.g., [64] and also [107]. For the restricted case of Horn formulas an approach is offered by Radim Bělohlávek and Vilém Vychodil [15, 16]. They consider a first-order language with function symbols and the identity symbol $\approx$ as the only predicate symbol. Their models for sets of Horn formulas[6] therefore have to be algebraic structures with graded identity relations. However, the aim of these authors is not to develop an identity logic, they mainly are interested to use the approach to characterize classes of algebraic structures with graded identity relations and to find "fuzzified" versions of results from universal algebra.

§9. Some generalizations. The standard approach toward t-norm based logics, as explained in Sections 6 and 7, has been modified in various ways. The main background ideas are the extension or the modification of the expressive power of these logical systems.

A first extension is provided by the addition of truth degree constants besides $\overline{0}$. This approach mainly is motivated by a mirroring of a generalization, explained later on in Section 10, which discusses graded notions of entailment and provability, but is not restricted to this particular purpose. A nice recent survey is [38].

Another, quite fundamental addition to the standard vocabulary of the languages of t-norm based systems was proposed in [7]: a unary propositional operator $\triangle$ which has for t-norm algebras the semantics

$$\triangle(x) = 1 \quad \text{for } x = 1, \quad \triangle(x) = 0 \quad \text{for } x \neq 1. \tag{23}$$

This unary connective can be added to the systems BL and MTL via the additional axioms

$$\begin{array}{ll} \triangle\varphi \vee \neg\triangle\varphi, & \triangle(\varphi \vee \varphi) \to (\triangle\varphi \vee \triangle\psi), \\ \triangle\varphi \to \varphi, & \triangle\varphi \to \triangle\triangle\varphi, \\ \multicolumn{2}{c}{\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi).} \end{array}$$

[6]These authors even consider fuzzy sets of Horn formulas, i.e., they work in a Pavelka-style fuzzy logic as explained later on in Section 10. But because this type of approach can be mirrored in standard fuzzy logics with sufficiently many truth degree constants, cf. also Section 10, this approach is already discussed here.

This addition leaves all the essential theoretical results, like correctness and completeness theorems, valid: of course w.r.t. suitably expanded algebraic structures.

A further stream of papers discusses the addition of an involutive negation, i.e., a negation which satisfies the double negation law, for those cases where the standard negation of the t-norm based system is not involutive. This is e.g., the case for the product logic which has the Gödel negation as its standard negation. By the way, it should be noticed that this non-involutive Gödel negation is the standard negation of all those t-norm algebras with a t-norm $\otimes$ which does not have zero-divisors.[7] A very general approach is given in [43], and a more particular axiomatization problem discussed in [62].

There is a whole family of systems which span a kind of web between monoidal t-norm logic MTL and classical logic. A nice survey of this class of logics is given in [41]. These authors do also relate these t-norm-based logics to substructural logics, in particular to Hiroakira Ono's hierarchy [108, 48, 91] of extensions of the full Lambek calculus. So corresponds MTL to Ono's system FL_{ew}, i.e., to the full Lambek calculus with exchange and weakening, extended by pre-linearity.[8]

Furthermore do the kinds of residuated lattices which appear as algebraic semantics for these logics also contain particular cases of structures which are of importance as quantum structures [37]. But the problem of a possible relationship between quantum logic and fuzzy logics remains an open one [60].

Another area of extensions, partly related to the previously mentioned one, is devoted to the problem of a unified treatment of different, usually two, t-norms and their related connectives within one logical system. Here the focus is on the join of the systems based upon the Łukasiewicz t-norm and upon the product t-norm. The great advantage of this unification is that the Łukasiewicz t-norm essentially allows to treat the addition, as may be seen from the truth degree function of the Łukasiewicz (arithmetical) disjunction, and that the product t-norm adds the treatment of the usual product: and this means that the elementary arithmetic (in the unit interval) can be discussed in this combined system. This combined system has been considered in two strongly related forms, denoted $\mathsf{L\Pi}$ and $\mathsf{L\Pi}\frac{1}{2}$. The distinction between both systems is that $\mathsf{L\Pi}$ has both t-norms T_{L} and T_{P} and their related (residual) implications and negations among their basic connectives, and that $\mathsf{L\Pi}\frac{1}{2}$ adds a truth degree constant for the truth degree $\frac{1}{2}$. These two systems are discussed in detail e.g., in [26, 25, 27, 39, 44].

[7] Zero-divisors of a t-norm T are such reals $0 < u, v < 1$ for which $T(u, v) = 0$ holds.

[8] The property which is called pre-linearity in the fuzzy logic community is called linearity by Ono.

Moreover, there is a series of papers which is devoted to weakening the systems BL and MTL in such a way that one deletes the explicit reference to the truth degree constant $\overline{0}$ and considers the *falsity free* fragments of the previous systems. From the algebraic point of view, in case of the system BL the characteristic structures become the *hoops* which are algebraic structures $\mathbf{H} = \langle H, *, \Rightarrow, \mathbf{1}\rangle$ such that $\langle H, *, \mathbf{1}\rangle$ is an abelian monoid and that the further binary operation $\Rightarrow$ satisfies the equations

$$\begin{aligned} x \Rightarrow x &= \mathbf{1}, \\ x * (x \Rightarrow y) &= y * (y \Rightarrow x), \\ (x * y) \Rightarrow z &= x \Rightarrow (y \Rightarrow z). \end{aligned}$$

The definition

$$x \sqsubseteq y =_{\text{def}} (x \Rightarrow y = \mathbf{1})$$

provides an ordering $\sqsubseteq$ with universal upper bound $\mathbf{1}$ which makes $\langle H, *, \mathbf{1}\rangle$ an ordered monoid, and which has the additional property that the operations $*, \Rightarrow$ become an adjoint pair w.r.t. this ordering.

Hoops with the additional property

$$x \Rightarrow (y \Rightarrow z) \sqsubseteq (y \Rightarrow (x \Rightarrow z)) \Rightarrow z$$

can in a natural way be generated from t-norm algebras with continuous t-norms, as has been shown in [1]. So one has a kind of competing generalization of t-norm algebras. And for this kind of algebraic semantics one can find adequate axiomatizations for corresponding hoop logics quite similar to the approaches of Sections 6 and 7. The details have been developed in [42].

Another group of papers discusses the generalization of the algebraic semantics from the case of abelian lattice ordered monoids with residuation to the case of non-commutative lattice ordered semigroups. In this context one tries to define non-commutative BL-algebras or non-commutative MTL-algebras, and similarly defines non-commutative t-norms, also called pseudo-t-norms. For these cases one has also adequate axiomatization, and similar results as in Sections 6 and 7. The most important ones of these papers are [33, 47, 67, 68, 69, 86, 92].

Hájek [72] even gives a common generalization of all of these generalized fuzzy logics, thus giving up divisibility, the falsity constant, and commutativity. The corresponding algebras he calls *fleas* (or *flea algebras*), and the logic is the *flea logic* FlL. There are examples of fleas on $(0, 1]$ not satisfying divisibility, nor commutativity, and having no least element.

These systems delete, in the terminology of Ono's hierarchy of extensions of the full Lambek calculus, the exchange property. Similarly there have been discussed systems without the weakening property, e.g., in [94, 93]. These systems are related to the generalization of t-norms to uninorms, i.e., in algebraic terms to the deletion of the integrality condition.

It shall be sufficient to mention these generalizations here. More details offer e.g., the survey papers [41, 61, 74].

§10. Extensions with graded notions of inference. Having in mind that fuzzy logics, also in their form as formalized logical systems, should be a (mathematical) tool for approximate reasoning makes it desirable that they should be able to deal with graded inferences.

The systems of t-norm based logics discussed up to now have been designed to formalize the logical background for fuzzy sets, and they allow themselves for degrees of truth of their formulas. But they all have crisp notions of consequence, i.e., of entailment and of provability.

It is natural to ask whether it is possible to generalize these considerations to the case that one starts from *fuzzy sets of formulas*, and that one gets from them as consequence hulls again fuzzy sets of formulas. This problem was first treated by Jan Pavelka [109]. The basic monograph elaborating this approach is [107]. Accordingly such approaches are sometimes called *Pavelka-style*, but they have—with emphasis on the syntactic side of the matter—also been coined approaches with *evaluated syntax*. Here I'll call them *GI-approaches*.

Such an approach with graded inferences has to deal with fuzzy sets $\Sigma^\sim$ of formulas, i.e., besides formulas φ also their membership degrees $\Sigma^\sim(\varphi)$ in $\Sigma^\sim$. And these membership degrees are just the truth degrees. We may assume that these degrees again form a residuated lattice $\mathbf{L} = \langle L, \cap, \cup, *, \rightarrowtail, 0, 1\rangle$. Thus we (slightly) generalize the standard notion of fuzzy set (with membership degrees from the real unit interval). Therefore the appropriate language has the same logical connectives as in the previous considerations.

A GI-approach is an easy matter as long as the entailment relationship is considered. An evaluation e is a *model* of a fuzzy set $\Sigma^\sim$ of formulas iff

$$\Sigma^\sim(\varphi) \leqslant e(\varphi) \tag{24}$$

holds for each formula φ. This immediately yields that the semantic consequence hull of $\Sigma^\sim$ should be characterized by the membership degrees

$$\mathcal{C}^{\mathsf{sem}}(\Sigma^\sim)(\psi) = \bigwedge\{e(\psi) \mid e \text{ model of } \Sigma^\sim\} \tag{25}$$

for each formula ψ.

For a syntactic characterization of this entailment relation it is necessary to have some calculus $\mathbb{K}$ which treats formulas of the language together with truth degrees. So the language of this calculus has to extend the language of the basic logical system by having also symbols for the truth degrees. We indicate these symbols by overlined letters like $\overline{a}, \overline{c}$, and realize the common treatment of formulas and truth degrees by considering *evaluated formulas*, i.e., ordered pairs $(\overline{a}, \varphi)$ consisting of a truth degree symbol and a formula.

This transforms each fuzzy set $\Sigma^{\sim}$ of formulas into a (usual) set of evaluated formulas, again denoted by $\Sigma^{\sim}$.

So $\mathbb{K}$ has to allow to derive evaluated formulas from sets of evaluated formulas, using suitable axioms and rules of inference. These axioms are usually only formulas φ which, however, are used in the derivations as the corresponding evaluated formulas $(\overline{1}, \varphi)$. The rules of inference have to deal with evaluated formulas.

Each $\mathbb{K}$-derivation of an evaluated formula $(\overline{a}, \varphi)$ counts as a derivation of φ *to the degree* $a \in L$. The *provability degree* of φ from $\Sigma^{\sim}$ in $\mathbb{K}$ is the supremum over all these degrees. The syntactic consequence hull of $\Sigma^{\sim}$ is the fuzzy set $\mathcal{C}^{\mathsf{syn}}_{\mathbb{K}}$ of formulas characterized by the membership function

$$\mathcal{C}^{\mathsf{syn}}_{\mathbb{K}}(\Sigma^{\sim})(\psi) = \bigvee \{a \in L \mid \mathbb{K} \text{ derives } (\overline{a}, \psi) \text{ out of } \Sigma^{\sim}\} \tag{26}$$

for each formula ψ.

Despite the fact that $\mathbb{K}$ is a standard calculus for evaluated formulas, this is—for infinite truth degree structures—an *infinitary* notion of provability for usual formulas.

For the infinite-valued Łukasiewicz logic L this machinery works particularly well because it needs in an essential way the continuity of the residuation operation. The corresponding calculus $\mathbb{K}_{\mathsf{L}}$ has as axioms any axiom system of the infinite-valued Łukasiewicz logic L which provides together with the rule of detachment an adequate axiomatization of L, but $\mathbb{K}_{\mathsf{L}}$ replaces this standard rule of detachment by the generalized form

$$\frac{(\overline{a}, \varphi) \quad (\overline{c}, \varphi \to \psi)}{(\overline{a * c}, \psi)} \tag{27}$$

for evaluated formulas.

The soundness result then says that the $\mathbb{K}_{\mathsf{L}}$-provability of an evaluated formula $(\overline{a}, \varphi)$ means that $a \leq e(\varphi)$ holds for every valuation e. And this just means that the formula $\overline{a} \to \varphi$ is valid—however as a formula of an *extended* propositional language which has all the truth degree constants among its vocabulary. Of course, for this extended language the evaluations e have to satisfy $e(\overline{a}) = a$ for each $a \in [0, 1]$.

The soundness and completeness results for $\mathbb{K}_{\mathsf{L}}$ say that a *strong completeness theorem* holds true giving

$$\mathcal{C}^{\mathsf{sem}}(\Sigma^{\sim})(\psi) = \mathcal{C}^{\mathsf{syn}}_{\mathbb{K}_{\mathsf{L}}}(\Sigma^{\sim})(\psi) \tag{28}$$

for each formula ψ and each fuzzy set $\Sigma^{\sim}$ of formulas.

If one takes the previously mentioned turn and extends the standard language of propositional L by truth degree constants for all degrees $a \in [0, 1]$, and if one reads each evaluated formula $(\overline{a}, \varphi)$ as the formula $\overline{a} \to \varphi$, then a slight modification $\mathbb{K}^{+}_{\mathsf{L}}$ of the former calculus $\mathbb{K}_{\mathsf{L}}$ again provides an adequate axiomatization: one has to add the *bookkeeping axioms*

$$(\overline{a} \,\&\, \overline{c}) \equiv \overline{a * c},$$
$$(\overline{a} \to \overline{c}) \equiv \overline{a \rightarrowtail_{\mathsf{L}} c},$$

as explained e.g., in [107]. And if one is interested to have evaluated formulas together with the extension of the language by truth degree constants, one has also to add the *logical constant introduction rule*

$$\frac{(\overline{a}, \varphi)}{\overline{a} \to \varphi}.$$

However, even a stronger result is available which refers only to a notion of derivability over a countable language. The completeness result (28), for $\mathbb{K}_{\mathsf{L}}^{+}$ instead of $\mathbb{K}_{\mathsf{L}}$, becomes already provable if one adds truth degree constants only for all the *rationals* in [0, 1], as was shown in [64]. And this extension of L is even a conservative one, cf. [79], i.e., $\mathbb{K}_{\mathsf{L}}^{+}$ proves only such *constant-free* formulas of the language with rational constants which are already provable in the standard infinite-valued Łukasiewicz logic L.

So the GI-approach with graded notion of provability and entailment can suitably be mirrored inside standard fuzzy logics with sufficiently many truth degree constants.

For more details the reader may also consult e.g., [64, 107, 119].

§11. Complexity issues. Each (left-continuous) t-norm T determines four important sets of formulas:

$\mathsf{1TAUT}(T)$	the set of all 1-tautologies
$\mathsf{posTAUT}(T)$	the set of all positive tautologies
$\mathsf{1SAT}(T)$	the set of 1-satisfiable formulas
$\mathsf{posSAT}(T)$	the set of all positively satisfiable formulas.

Here a 1-tautology is a formula valid in $[0,1]_T$, i.e., having for each evaluation of propositional variables by elements of [0, 1] the value 1 in $[0,1]_T$. And a positive tautology is a formula which has for each evaluation a positive value in $[0,1]_T$. Similarly 1-satisfiability means to have the $[0,1]_T$ value 1 for some evaluation, and positive satisfiability means to have for some evaluation a positive $[0,1]_T$-value.

In the same way one defines analogous sets corresponding to sets of t-norms; in particular, with BL referring to the set of all continuous t-norms, one defines

$$\mathsf{1TAUT}(\mathsf{BL}) = \bigcap\{\mathsf{1TAUT}(T) \mid T \text{ a continuous } T\text{-norm}\},$$
$$\mathsf{posTAUT}(\mathsf{BL}) = \bigcap\{\mathsf{posTAUT}(T) \mid T \text{ a continuous } T\text{-norm}\},$$

and similarly for the satisfiability cases.

There are interesting results on the computational complexity of these sets. So it was, already in [64], shown that if the t-norm T is T_{L}, or T_{G}, or T_{P}, then $\mathsf{1TAUT}(T)$ and $\mathsf{posTAUT}(T)$ are co-NP-complete, and $\mathsf{1SAT}(T)$ and $\mathsf{posSAT}(T)$ are NP-complete. This result was partly strengthened in [81] yielding that $\mathsf{1TAUT}(T)$ is co-NP-complete for each continuous t-norm T.

Corresponding results have been proved in [9, 66] for the logic BL of continuous t-norms.

THEOREM 11.1. $\mathsf{1TAUT}(\mathsf{BL})$ *and* $\mathsf{posTAUT}(\mathsf{BL})$ *are co-NP-complete, and* $\mathsf{1SAT}(\mathsf{BL})$ *as well as* $\mathsf{posSAT}(\mathsf{BL})$ *are NP-complete.*

Furthermore there are several results on equality or inequality among the sets involved (see [64, 66]). So one has e.g.,

$$\mathsf{1SAT}(\mathsf{G}) = \mathsf{posSAT}(\mathsf{G}) = \mathsf{1SAT}(\Pi) = \mathsf{posSAT}(\Pi),$$

but also $\mathsf{1SAT}(\mathsf{L}) \neq \mathsf{posSAT}(\mathsf{L})$ and $\mathsf{posSAT}(\mathsf{BL}) = \mathsf{posSAT}(\mathsf{L})$.

For the 1-tautologicity the papers [2, 3, 4] contain interesting results relating the property of a formula being a 1-tautology of one of the logics $\mathsf{L}, \mathsf{G}, \Pi$ to their being a 1-tautology of finitely may finite valued logics of estimated complexity, and similar results for $\mathsf{1TAUT}(\mathsf{BL})$. For example, φ is in $\mathsf{1TAUT}(\mathsf{L})$ if and only if it is a 1-tautology of the finitely valued Łukasiewicz logic L_m for m being $2^{\#(\varphi)}$ where $\#(\varphi)$ is the number of occurrences of variables in φ.

Remind that for predicate logics $\mathfrak{L}_{\mathcal{K}}\forall$, the general models are safe interpretations over any *linearly ordered* $\mathfrak{L}_{\mathcal{K}}$-algebra, and the standard models—for t-norm based logics—are interpretations over any t-norm algebra which is also an $\mathfrak{L}_{\mathcal{K}}$-algebra.

DEFINITION 11.1. *Let φ be a closed formula of the language of $\mathfrak{L}_{\mathcal{K}}\forall$.*

(1) *φ is a* general $\mathfrak{L}_{\mathcal{K}}\forall$-tautology *if φ is valid in each safe interpretation over any* linearly ordered *$\mathfrak{L}_{\mathcal{K}}$-algebra*; $\mathsf{genTAUT}(\mathfrak{L}_{\mathcal{K}}\forall)$ *is the set of all general $\mathfrak{L}_{\mathcal{K}}\forall$-tautologies.*

(2) *φ is a* standard $\mathfrak{L}_{\mathcal{K}}\forall$-tautology *if φ is valid in each safe interpretation over any* standard *$\mathfrak{L}_{\mathcal{K}}$-algebra*; $\mathsf{stTAUT}(\mathfrak{L}_{\mathcal{K}}\forall)$ *is the set of all standard $\mathfrak{L}_{\mathcal{K}}\forall$-tautologies.*

(3) *φ is* $\mathfrak{L}_{\mathcal{K}}\forall$-satisfiable *if φ is valid in some safe interpretation over some* linearly ordered *$\mathfrak{L}_{\mathcal{K}}$-algebra*; $\mathsf{genSAT}(\mathfrak{L}_{\mathcal{K}}\forall)$ *is the set of all $\mathfrak{L}_{\mathcal{K}}\forall$-satisfiable sentences.*

(4) *φ is* standardly $\mathfrak{L}_{\mathcal{K}}\forall$-satisfiable *if φ is valid in some interpretation over some* standard *$\mathfrak{L}_{\mathcal{K}}$-algebra*; $\mathsf{stSAT}(\mathfrak{L}_{\mathcal{K}}\forall)$ *is the set of all standardly $\mathfrak{L}_{\mathcal{K}}\forall$-satisfiable formulas.*

It was already shown in [64] that if $\mathfrak{L}$ is the logic BL or one of its specifications $\mathsf{L}, \mathsf{G}, \Pi$, then $\mathsf{genTAUT}(\mathfrak{L}\forall)$ is Σ_1-complete and $\mathsf{genSAT}(\mathfrak{L}\forall)$ is Π_1-complete. And this result has been extended in [71] to any t-norm based logic $\mathfrak{L}(T)\forall$ for a continuous t-norm T.

For standard semantics the situation is different. Already Matthias Ragaz [110, 111] proved that the set $\mathsf{stTAUT}(\mathsf{L}\forall)$ of standard tautologies of the infinite valued Łukasiewicz logic is Π_2-complete. Generalizing this, Hájek [64] showed also that $\mathsf{stTAUT}(\mathsf{G}\forall) = \mathsf{genTAUT}(\mathsf{G}\forall)$ and that therefore the set $\mathsf{stTAUT}(\mathsf{G}\forall)$ of standard tautologies of the infinite valued Gödel logic is Σ_1-complete.

These results have been considerably extended by Franco Montagna [100] yielding the following facts.

THEOREM 11.2. (1) *For each set $\mathcal{K}$ of continuous t-norms containing a t-norm different from T_{G}, the set* $\mathsf{stTAUT}(\mathfrak{L}_{\mathcal{K}}\forall)$ *is Π_2-hard.*

(2) *If $\mathcal{K}$ is a non-empty set of continuous t-norms containing a t-norm which has, in its ordinal sum representation, a product component or a non-extremal*[9] *Łukasiewicz component then* $\mathsf{stTAUT}(\mathfrak{L}_{\mathcal{K}}\forall)$ *is not arithmetical.*

It should be mentioned that the arithmetical complexity of the set $\mathsf{stTAUT}(\mathfrak{L}(T)\forall)$ remains undetermined if T is e.g., one of the t-norms $\mathsf{L} \oplus \mathsf{L}$, $\mathsf{G} \oplus \mathsf{L}$, $\mathsf{L} \oplus \mathsf{G}$, or $\mathsf{L} \oplus \mathsf{G} \oplus \mathsf{L}$, with $\oplus$ denoting the ordinal sum operation.

For standard satisfiability Hájek [64] proved that the sets $\mathsf{stSAT}(\mathsf{G}\forall)$ and $\mathsf{stSAT}(\mathsf{L}\forall)$ are Π_1-complete. He also proved, in [65], that the set $\mathsf{stSAT}(\Pi\forall)$ is not arithmetical, and gave in [70] also the following results.

THEOREM 11.3. *If $\otimes$ is a continuous t-norm whose first component is* G, *or* L, *then* $\mathsf{stSAT}(\mathfrak{L}(\otimes)\forall)$ *is Π_1-complete.*

The reason is that one has, under these assumptions, $\mathsf{stSAT}(\mathfrak{L}(\otimes)\forall) = \mathsf{stSAT}(\mathsf{G}\forall)$, as well as $\mathsf{stSAT}(\mathfrak{L}(\otimes)\forall) = \mathsf{stSAT}(\mathsf{L}\forall)$.

For the product logic Π Montagna [99] added the more general

THEOREM 11.4. *If $\mathcal{K}$ is a non-empty set of continuous t-norms containing T_{P}, or a t-norm whose first ordinal summand is T_{P}, then* $\mathsf{stSAT}(\mathcal{L}_{\mathcal{K}}\forall)$ *is not arithmetical.*

The complexity of $\mathsf{stSAT}(\mathfrak{L}(T)\forall)$ for continuous t-norms T which do not have a first component in their ordinal sum representation is a open problem.

More complexity results are surveyed in [71].

§12. Toward more general approaches. The previously discussed main approaches toward mathematical fuzzy logics are subject to some further generalizations. Besides the non-commutative extensions which have already been mentioned, two approaches deserve to be considered here.

The first one has a strong algebraic touch with its reference to generalized closure operators. The second one aims to give a unified view from a position which considers the use of the implication connective and the subdirect representation property as the core of the matter.

12.1. Approaches via generalized closure operators. The approach of Section 10 was historically the first one toward "fuzzified", i.e., graded notions of inference. The general background for such generalizations is that in classical logic the syntactic as well as the semantic consequence relations, i.e.,

[9]This means being neither first nor last summand.

the provability as well as the entailment relations, are equivalent to closure operators within the set of formulas. And the graded operators $\mathcal{C}^{\mathsf{sem}}$ and $\mathcal{C}^{\mathsf{syn}}$ are generalized closure operators—in a suitable sense.

The context is that of L-fuzzy subsets of the set $\textsc{For}$ of all formulas of some language $\mathcal{L}$. Following [109], a (*graded*) *closure operator* is a mapping $C^{\sim} : \mathcal{F}(\textsc{For}) \to \mathcal{F}(\textsc{For})$ satisfying for all $X, Y \in \mathcal{F}(\textsc{For})$ the conditions

$(\mathbf{C}1)$	$X \subseteq C^{\sim}(X)$,	(increasingness)
$(\mathbf{C}2)$	$X \subseteq Y \Rightarrow C^{\sim}(X) \subseteq C^{\sim}(Y)$,	(isotony)
$(\mathbf{C}3)$	$C^{\sim}(C^{\sim}(X) \subseteq C^{\sim}(X)$	(idempotency)

w.r.t. the binary fuzzy set inclusion $\subseteq$ of (6).

In a more algebraic setting this type of approach was considerably generalized by Giangiacomo Gerla [49]. The main ideas are sketched e.g., in [57]. However this generalization does not really pay off: a general completeness theorem like (28) is not available. What one needs instead, in search for a completeness result, that are specifications which restrict the full generality of this approach, and lead back to a satisfactory situation mainly in case of the Łukasiewicz logic.

Additionally there have been diverse papers, e.g., [17, 18, 21, 22, 35, 49], which treated generalized inferences as graded inference relations, often intending applications in the field of approximate reasoning. However, there was not always a nice correspondence between graded inference relations and graded closure operators.

Unifying generalizations which offer such a correspondence are given in [13, 14] and [113]. These authors modify the previous definition of a graded closure operator and understand by an implicative closure operator a mapping $C^{*}\colon \mathcal{F}(\textsc{For}) \to \mathcal{F}(\textsc{For})$ satisfying

$(\mathbf{C}^{*}1)$	$\models X \sqsubseteq C^{*}(X)$,	(increasingness)
$(\mathbf{C}^{*}2)$	$\models X \sqsubseteq Y \to C^{*}(X) \sqsubseteq C^{*}(Y)$,	(isotony)
$(\mathbf{C}^{*}3)$	$\models C^{*}(C^{*}(X) \sqsubseteq C^{*}(X)$	(idempotency)

w.r.t. the graded fuzzy set inclusion $\sqsubseteq$ of (7). This graded relation is here supposed to be defined w.r.t. some residuated implication $\to$ instead of only the Łukasiewicz implication $\to_{\mathsf{L}}$; and just this implication is also used in condition $(\mathbf{C}^{*}2)$.

Of course, the conditions $(\mathbf{C}1)$ and $(\mathbf{C}^{*}1)$ coincide, as do $(\mathbf{C}3)$ and $(\mathbf{C}^{*}3)$. But this more general case allows, as shown in [113], an equivalent transformation into conditions for a graded inference relation, i.e., for an L-fuzzy relation $Cn^{*} \in \mathcal{F}(\mathcal{F}(\textsc{For}) \times \textsc{For})$.

To correspond to such an implicative closure operator such a relation Cn^{*} has to satisfy the conditions

$(\mathbf{Cn}^{*}1)$	$\models A(\varphi) \to Cn^{*}(A, \varphi)$,	(reflexivity)
$(\mathbf{Cn}^{*}2)$	$B \subseteq A \;\; \Rightarrow \;\; \models Cn^{*}(B, \varphi) \to Cn^{*}(A, \varphi)$,	(isotony)
$(\mathbf{Cn}^{*}3)$	$\models Cn^{*}(A \cup B, \varphi) \to (B \sqsubseteq C^{*}(A) \to Cn^{*}(A, \varphi))$	(graded cut)

for arbitrary $A, B \in \mathcal{F}(\text{For})$, $\varphi \in \text{For}$, and the fuzzy set $C^*(A) \in \mathcal{F}(\text{For})$ characterized by $C^*(A)(\varphi) = Cn^*(A, \varphi)$.

This generalization covers all the approaches mentioned above, and also the entailment relation of the Gödel logic G. But, unfortunately, it covers neither the case of the Łukasiewicz logic L, nor of the product logic Π.

12.2. An implication based approach. Historically, mathematical fuzzy logics did arise out of the problem to find suitable background logics for fuzzy set theory. This fact causes their t-norm relatedness, and also their particular algebraic semantics. What is lacking, however, is a more general point of view which allows one to distinguish what is a fuzzy logic and what is not a fuzzy logic.

One such idea is the approach via generalized, "graded" closure operators as explained in the previous subsection. A second idea is to coin only such logics as fuzzy logics which have algebraic semantics determined by suitable classes of t-norm algebras.

It is interesting that recently another such approach has been offered: an approach which looks at fuzzy logics as logics of comparable truth degrees. This is a proposal of Petr Cintula [28].

The starting point is a generalization of the implicative logics of Helena Rasiowa [112]. A propositional logic $\mathfrak{L}$ is called *weakly implicative* iff it has in its language a (definable) connective $\rightarrow$ which satisfies the derivability conditions

$$\vdash \varphi \rightarrow \varphi, \qquad \varphi, \varphi \rightarrow \psi \vdash \psi,$$
$$\varphi \rightarrow \psi, \psi \rightarrow \chi \vdash \varphi \rightarrow \chi,$$

and for each n-ary connective c additionally

$$\varphi \rightarrow \psi, \psi \rightarrow \varphi \vdash c(\dots, \varphi, \dots) \rightarrow c(\dots, \psi, \dots).$$

It is characteristic for these weakly implicative logics that they have an adequate semantics determined by a class of ordered matrices with an upset D of designated truth degrees and an ordering $\preceq$ characterized by

$$x \preceq y \quad \text{iff} \quad (x \rightarrow y) \in D.$$

These matrices are just the prestandard matrices of [36].

The *fuzzy logics* are those ones among the weakly implicative logics that have an adequate semantics formed by those matrices for which $\preceq$ is a linear ordering, i.e., by the standard matrices in the sense of [36].

This seems to be a very promising notion which deserves further study. These fuzzy logics can e.g., be characterized as those weakly implicative logics which have the *linear extension* property, i.e., for which each formula unprovable in a theory T is already unprovable in a linear extension T' of T, which means in a theory T' satisfying $T' \vdash \varphi \rightarrow \psi$ or $T' \vdash \psi \rightarrow \varphi$ for any formulas φ, ψ.

If one additionally restricts oneself to finitary weakly implicative logics, then the fuzzy logics among them can also be characterized as those ones which have the *prelinearity* property, as well as those ones which have the *subdirect representation* property. Here the prelinearity property for a logic $\mathfrak{L}$ means that one gets $T \vdash \chi$ from $T, \varphi \to \psi \vdash \chi$ and $T, \psi \to \varphi \vdash \chi$ for each $\mathfrak{L}$-theory T, and the subdirect representation property means that each ordered $\mathfrak{L}$-matrix is a subdirect product of linearly ordered $\mathfrak{L}$-matrices.

§**13. Concluding remarks.** There is a reasonable amount of further ideas which could not be included here. Only some of them shall at least be mentioned.

For investigation of *compactness* of the logics in question (in different senses of the word compactness) see [31]; for *interpolation* in fuzzy logics see [10]. Higher order fuzzy logics have been discussed in [11], and a Church-Henkin style fuzzy type theory is given in [104]. Weaker systems are considered e.g., in [101, 102], fragments in [30].

There are also proof theoretic studies toward fuzzy logics. Sequent as well as hypersequent calculi for fuzzy logics are discussed e.g., in [95, 97, 96] and in [6].

Also a game theoretic semantics via dialogue games has been generalized [46, 8, 23], and Kripke type semantics considered [98].

What also has not been discussed here, besides a lot of further technical results, are different ideas for applications as e.g., to the sorites paradox [78], to methods of approximate reasoning [51], or to natural language modeling [103, 106], but also to give axiomatic versions of a set theory for fuzzy sets [58, 77] or to develop fuzzified versions of particular mathematical theories, e.g., of arithmetic [73]. Some of these ideas are also mentioned in [74], others are not yet surveyed.

The whole field has seen a rapid development in the last years which is still ongoing.

Whether there is something like a "genuine" fuzzy logic inside the large family of systems discussed here is a matter of occasional debates, cf. e.g., [12, 105], however unsolved at present and very probably not solvable at all with formal means.

For completeness it shall also be mentioned that there is another understanding of the terminus "fuzzy logic" (FL). To distinguish both, the one discussed here is also called "FL in narrow sense" (FLn), and the other "FL in wider sense" (FLw). This FLw mainly is a conglomeration partly of methodological ways to apply fuzzy sets in knowledge engineering, and partly of heuristic approaches toward commonsense reasoning using vaguely determined notions.

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