THE BULLETIN OF SYMBOLIC LOGIC
Volume 14, Number 1, March 2008

INDEPENDENCE PROPERTY AND HYPERBOLIC GROUPS

ERIC JALIGOT, ALEXEY MURANOV, AND AZADEH NEMAN

Abstract. In continuation of [JOH04, OH07], we prove that existentially closed *CSA*-groups have the independence property. This is done by showing that there exist words having the independence property relative to the class of torsion-free hyperbolic groups.

§1. Introduction. A *CSA-group* is a group G in which every maximal abelian subgroup A is malnormal, i.e., satisfies $A \cap A^g = \{1\}$ for each element g of G which is not in A. The interest in these groups is motivated on the one hand by the question of existence of new groups with well behaved first-order theories, and ultimately of so-called *bad groups* of finite Morley rank which may be *CSA*-groups under certain circumstances [Jal01]. On the other hand, it is motivated by the fact that *CSA*-groups are usually more suitable than other groups for solving equations, an important step towards understanding their first-order theory.

The torsion of *CSA*-groups under consideration can be restricted as follows. If f is a function from the set of prime integers into $\mathbb{N} \sqcup \{\infty\}$, we call a *$CSA_f$-group* any *CSA*-group which contains no elementary abelian p-subgroup of rank $f(p)+1$ for any prime p such that $f(p)$ is finite. Once such a function f is fixed, it is easily seen that the first-order class of CSA_f-groups is inductive, and thus contains *existentially closed CSA_f*-groups. If a *CSA*-group contains an involution, then it must be abelian, a case of low interest from the group-theoretic point of view and from the model-theoretic point of view as well, since all abelian groups are known to be stable. Hence we work with *CSA*-groups without involutions, i.e., with CSA_f-groups where $f(2) = 0$. It is known, when $f(2) = 0$, that existentially closed CSA_f-groups are simple, in the strong sense that simplicity is provided by a first-order formula. Moreover, their maximal abelian subgroups are conjugate, divisible, and of Prüfer p-rank $f(p)$ for each prime p. We refer to [JOH04, Sect. 2, 5] for details.

Received July 11, 2007.
2000 *Mathematics Subject Classification.* Primary 03C45; Secondary 20F06, 20F67.

1079-8986/08/1401-0004/$2.10

Concerning the model theory of existentially closed CSA_f-groups, always assuming $f(2) = 0$, it has also been proved in [JOH04] that such groups are not ω-stable. This was done by counting the number of types, more precisely by showing the existence of $2^{\aleph_0}$ types over the empty set. The same method of counting types has later been used in [OH07] to prove that the first-order theory of existentially closed CSA_f-groups $(f(2) = 0)$ is not superstable. We will show here by a much more elementary argument that their first-order theory is indeed far from being stable. We shall prove that they have the independence property using a standard small cancellation argument from combinatorial group theory.

§2. Relative independence property.

§2. Relative independence property. The main result of the present paper leading to the independence property of existentially closed CSA-groups is the following.

THEOREM 2.1. *There exists a group word $w(x, y)$ in two variables such that the formula $\ulcorner w(x, y) = 1 \urcorner$ has the independence property relative to the class of torsion-free hyperbolic groups.*

For the moment we postpone the proof of Theorem 2.1, and rather give the definition used in this theorem and derive the corollary concerning the first-order theories of existentially closed CSA-groups which particularly interests us.

Hyperbolic groups have been introduced by Gromov in [Gro87] and have been intensively studied in geometric group theory since [CDP90, GdlH90]. They are finitely presented groups for which, by definition, the Cayley graph "looks like" a tree. In this sense they are finitely generated groups with only "few" relations between the generators, and hence can be seen as a generalization of finitely generated free groups. For example, by Fact 2.4 below, torsion-free hyperbolic groups are CSA-groups, an elementary property of free groups.

If $\phi(\overline{x}, \overline{y})$ is a formula in a given language $\mathcal{L}$, and $\mathcal{C}$ is any (not necessarily elementary) class of $\mathcal{L}$-structures, we say that ϕ has the *independence property relative to* $\mathcal{C}$ if for any positive integer n, there exists a structure M_n in $\mathcal{C}$ with sequences of tuples $\overline{x}_1, \dots, \overline{x}_i, \dots, \overline{x}_n$, and $\overline{y}_\sigma$, where there are 2^n indices σ varying over the set of subsets of $\{1, \dots, n\}$, such that the following holds in M_n:

$$M_n \models \phi(\overline{x}_i, \overline{y}_\sigma) \quad \text{if and only if} \quad i \in \sigma.$$

When the class $\mathcal{C}$ consists of all models of a complete first-order theory T, this definition corresponds to the usual notion of independence property of the formula ϕ relative to the first-order theory T, as defined in [She90]. As we are dealing with groups here, the language will be that of groups, and our formula will typically be a group equation in two variables, i.e., of the form

$\ulcorner w(x, y) = 1 \urcorner$ with $w(x, y)$ a *group word* in two variables x and y (i.e., a word in $x^{\pm 1}$ and $y^{\pm 1}$).

Before the proof of Theorem 2.1, we derive the corollary concerning the model theory of existentially closed CSA-groups.

COROLLARY 2.2. *Assume $f(2) = 0$ and let G be an existentially closed CSA_f-group, or more generally a group having the same universal theory as an existentially closed CSA_f-group. Then the first-order theory of G has the independence property.*

For deriving Corollary 2.2 from Theorem 2.1, we will use the following fact.

FACT 2.3. *Assume $f(2) = 0$ and let G be a group having the same universal theory as an existentially closed CSA_f-group. Then any CSA_f-group embeds into a model of the first-order theory of G.*

PROOF. It was explicitly seen in the proof of [JOH04, Corollary 8.2] that the universal theory of an existentially closed CSA_f-group is true in any CSA_f-group. Now the fact that the models of the universal part $T_\forall$ of a first-order theory T are precisely the substructures of models of T [Hod93, Corollary 6.5.3] yields our claim. $\dashv$

We will also use the following fact, according to which CSA-groups can be seen as a generalization of torsion-free hyperbolic groups.

FACT 2.4. [MR96, Proposition 12] *Every torsion-free hyperbolic group is CSA.*

We shall give here a short proof for reader's convenience. By [CDP90, Corollaire 7.2] or [GdlH90, Théorème 38], the centralizer of any element of infinite order in a hyperbolic group is virtually cyclic, i.e., has a cyclic subgroup of finite index. Hence Fact 2.4 is indeed a special case of the following purely group theoretic lemma.

LEMMA 2.5. *Let G be a torsion-free group.*

a. *If G is virtually cyclic, then G is cyclic.*

b. *If centralizers of nontrivial elements of G are cyclic, then G is CSA.*

PROOF. a. Assume G is virtually cyclic. First, it is easy to verify that in every torsion-free virtually cyclic group the center has finite index. (This is true even for virtually cyclic groups without involutions.) Second, there is a theorem, often attributed to Issai Schur, which states that if the index of the center of a group is finite, then the derived subgroup of the group is also finite. In particular, if in a torsion-free group the center has finite index, then the group is abelian. An elementary proof of this theorem of Schur may be found in [Ros62]: the idea is to show that in a group whose center has index n, the number of commutators is at most $(n-1)^2 + 1$, and that the $(n+1)$st power of each commutator is the product of n commutators. Hence G is abelian. Now it follows that G is cyclic from the classification of finitely generated abelian groups, or from the simpler fact that in a torsion-free abelian group, each two cyclic subgroups either intersect trivially, or are both contained in some cyclic subgroup.

b. Assume now centralizers of nontrivial elements of G are cyclic. In particular, commuting is an equivalence relation on $G \setminus \{1\}$. Consider an arbitrary maximal abelian subgroup A of G. Then A is infinite cyclic. Suppose A is not malnormal. Let a be a generator of A, and let b be an element of $G \setminus A$ such that $A \cap A^b \neq \{1\}$. Since commutativity is transitive and A is maximal, $A = A^b$. Therefore, $a^b \in \{a^{\pm 1}\}$, and hence $(a^b)^b = a$. Since a commutes with b^2, and b^2 commutes with b, we have that a commutes with b, and therefore $b \in A$, which gives a contradiction. Thus A is malnormal. Hence G is CSA. ⊣

We remark that, in addition to Fact 2.4, Lemma 2.5 implies that all maximal abelian subgroups of a torsion-free hyperbolic group are cyclic.

Proof of Corollary 2.2. By Theorem 2.1, there exists a group word $w(x, y)$ in two variables x and y such that the formula $\ulcorner w(x, y) = 1 \urcorner$ has the independence property relative to the class of torsion-free hyperbolic groups. This means that for any positive integer n, there exists a torsion-free hyperbolic group G_n with sequences of elements $x_1, \dots, x_i, \dots, x_n$, and y_σ, with $\sigma \subseteq \{1, \dots, n\}$, such that

$$G_n \models w(x_i, y_\sigma) = 1 \quad \text{if and only if} \quad i \in \sigma.$$

By Fact 2.4, any torsion-free hyperbolic group G_n is a CSA-group, and even, as it is torsion-free, a CSA_f-group for an arbitrary function f.

Let now $\mathrm{Th}(G)$ denote the first-order theory of a group G having the same universal theory as an existentially closed CSA_f-group. By Fact 2.3, each group G_n embeds into a model of $\mathrm{Th}(G)$. In particular, as the truth of the formula $\ulcorner w(x, y) = 1 \urcorner$ is preserved under embeddings, $\mathrm{Th}(G)$ contains the formula

$$(\exists_{1 \leq i \leq n}\, x_i)\,(\exists_{\sigma \in 2^n}\, y_\sigma)\,\Big[\big(\bigwedge_{i \in \sigma} w(x_i, y_\sigma) = 1\big) \wedge \big(\bigwedge_{i \notin \sigma} w(x_i, y_\sigma) \neq 1\big)\Big].$$

Since this is true for any positive n, the formula $\ulcorner w(x, y) = 1 \urcorner$ has the independence property relative to $\mathrm{Th}(G)$. Hence $\mathrm{Th}(G)$ has the independence property. ⊣

§3. Small-cancellation groups. In this section we consider groups together with their presentations by generators and defining relations. We take most definitions and results from [Ol91], which is our main source of references for the combinatorial treatment of group presentations. Some terms are also borrowed from [CCH81].

Recall that a *group presentation* is an ordered pair $\langle A \parallel \mathcal{R} \rangle$, also written as $\langle A \parallel R = 1; R \in \mathcal{R} \rangle$, where A is an arbitrary set, called the *alphabet*, and $\mathcal{R}$ is a set of *group words* over A, called *relators* or *defining relators*. Every relator from $\mathcal{R}$ is a word in the alphabet $A \sqcup A^{-1}$, composed of elements of A and their formal inverses. Relators can also be viewed as a simplified form of *terms* in the language of groups augmented with constants from A.

By abuse of notation, the same symbol $\langle A \parallel \mathcal{R}\rangle$ shall be used to denote a certain group given by this presentation, that is a group G where all constants from A are interpreted in such a way that G is generated by their interpretations, all the relations $\ulcorner R = 1\urcorner$, $R \in \mathcal{R}$, hold in G, and all other relations of this form that hold in G are just their group-theoretic consequences. A presentation of a group G is any presentation $\langle A \parallel \mathcal{R}\rangle$ such that $G \cong \langle A \parallel \mathcal{R}\rangle$. Every group has a presentation (by the multiplication table, for example).

A presentation $\langle A \parallel \mathcal{R}\rangle$ is called *finite* if both sets A and $\mathcal{R}$ are finite. A group is *finitely presented* if it has a finite presentation.

Since *cyclic reduction* of relators, i.e., *cyclic shifts* and cancellation of pairs of adjacent mutually inverse letters, does not change the group given by the presentation, every group has a presentation with all relators *cyclically reduced*. Presentations, or sets of relators, in which all relators are cyclically reduced shall be called *cyclically reduced* themselves.

A cyclically reduced presentation $\langle A \parallel \mathcal{R}\rangle$, or a set of relators $\mathcal{R}$, shall be called *symmetrized* if $\mathcal{R}$ contains the *visual inverse* R^{-1} and every *cyclic shift* YX of every $R \equiv XY \in \mathcal{R}$. (The visual inverse of $ab^{-1}c$, for example, is $c^{-1}ba^{-1}$; XY denotes the concatenation of the words X and Y.) The *symmetrized closure*, or *symmetrization*, of $\mathcal{R}$ is the minimal symmetrized set of relators containing $\mathcal{R}$.

The opposite of "symmetrized" is "concise": following [CCH81], we call a cyclically reduced presentation $\langle A \parallel \mathcal{R}\rangle$, or a set of relators $\mathcal{R}$, *concise* if no two distinct elements of $\mathcal{R}$ are cyclic shifts of each other or of each other's visual inverses.

In what follows, all group presentations shall be assumed cyclically reduced. The length of a word X shall be denoted by $|X|$.

If $R_1 \equiv XY_1$ and $R_2 \equiv XY_2$ are two distinct words in a symmetrized set of relators $\mathcal{R}$, then X is called a *piece* (of R_1) relative to $\mathcal{R}$.

DEFINITION 3.1. Let λ be a number in $[0, 1]$. Let $\langle A \parallel \mathcal{R}\rangle$ be a group presentation, and $\tilde{\mathcal{R}}$ be the symmetrization of $\mathcal{R}$. Then the set $\mathcal{R}$, or the presentation $\langle A \parallel \mathcal{R}\rangle$, is said to satisfy the *condition* $C'(\lambda)$ if $|X| < \lambda|R|$ for any $R \in \tilde{\mathcal{R}}$ and for any subword X of R which is a piece relative to $\tilde{\mathcal{R}}$.

The condition $C'(\lambda)$ with "small" λ is a standard example of a so-called "small cancellation condition". There are various such small cancellation conditions and generally such conditions allow one to prove that large traces of relators remain in all their consequences.

The following fact about the condition $C'(\lambda)$ shows that it can be used for constructing hyperbolic groups.

FACT 3.2. [GdlH90, Théorème 33] *A finitely presented group which has a presentation satisfying $C'(1/6)$ is hyperbolic.*

As an intermediate step of our proof of Theorem 2.1, we are going to use *asphericity* of a group presentation. Various forms of asphericity and their mutual implications are studied in [CCH81]. What we will actually need is the asphericity defined in [Ol91, §13], though it is defined there in a more general setting of *graded* presentations. To avoid confusion with other forms of asphericity, however, we will borrow the term *diagrammatic asphericity* from [CCH81]. (It follows from [Ol91, Theorem 32.2] that the asphericity defined in [Ol91, §13] in the case of a non-graded cyclically reduced presentation is indeed the same as diagrammatic asphericity in [CCH81].) Diagrammatic asphericity of a group presentation means that there exists no reduced spherical van Kampen diagram over that presentation. As we do not use the definition of asphericity in our argument, we shall not explain it any further.

Taking small values of λ in the condition $C'(\lambda)$ results in diagrammatic asphericity. Indeed, the $C'(1/5)$ condition implies a related small cancellation condition $C(6)$ [Ol91, p. 127], which yields diagrammatic asphericity by [Ol91, Theorem 13.3]. Hence we have the following fact.

FACT 3.3. *A presentation satisfying $C'(1/5)$ is diagrammatically aspherical.*

Diagrammatic asphericity of a group presentation implies many algebraic properties of the group. An important consequence is the following.

FACT 3.4. [Ol91, Theorem 13.4] *If $\langle A \parallel \mathcal{R}\rangle$ is a concise diagrammatically aspherical group presentation, then the defining relators are independent, i.e., no relation in the set $\{\ulcorner R = 1 \urcorner \mid R \in \mathcal{R}\}$ is a consequence of the others.*

Following [CCH81] and [Mur07, Sect. 6], we say that a group presentation $\langle A \parallel \mathcal{R}\rangle$ is *singularly aspherical* if it is diagrammatically aspherical, concise, and no element of $\mathcal{R}$ can be decomposed as a concatenation of several copies of the same subword (i.e., does not represent a proper power in the free group $\langle A \parallel \varnothing\rangle$).

If $\langle A \parallel \mathcal{R}\rangle$ is a singularly aspherical presentation, then the relation module of $\langle A \parallel \mathcal{R}\rangle$ is a free G-module by [Ol91, Corollary 32.1]. One can see then, using the fact that all odd-dimensional homology groups of nontrivial finite cyclic groups are nontrivial [Bro94], that this prevents the presence of torsion.

FACT 3.5. [Mur07, Lemma 64] *Every group with singularly aspherical presentation is torsion-free.*

Note that Fact 3.5 can also be deduced from [Hue79, Theorem 3], where the finite-order elements are classified in the more general case of *combinatorially aspherical* groups (defined in [Hue79, CCH81]).

We have now all the tools from combinatorial group theory needed to build families of finitely presented groups with the desired properties.

PROOF OF THEOREM 2.1. Consider, for example, the following word $w(x, y)$ in two letters x and y:

$$w(x, y) = xy^7x^2y^6x^3y^5x^4y^4x^5y^3x^6y^2x^7y.$$

For an integer $n \geq 1$, let A be a set of $n + 2^n$ elements $a_1, \dots, a_i, \dots, a_n$, and b_σ, with 2^n indices σ varying in the set of all subsets of $\{1, \dots, n\}$. Define two sets of relators:

$$\mathcal{R} = \{w(a_i, b_\sigma) \mid 1 \leq i \leq n;\ \sigma \in 2^n;\ i \in \sigma\}$$

and

$$\mathcal{S} = \{w(a_i, b_\sigma) \mid 1 \leq i \leq n;\ \sigma \in 2^n\}.$$

Denote by $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{S}}$ their respective symmetrizations. Clearly both $\mathcal{R}$ and $\mathcal{S}$ are concise.

Consider now the finitely presented group

$$G = \langle A \parallel \mathcal{R} \rangle = \langle A \parallel \tilde{\mathcal{R}} \rangle,$$

and denote by F the free group $\langle A \parallel \varnothing \rangle$, of rank $n + 2^n$. Then G is the quotient of F by the normal closure of $\mathcal{R}$, $\mathcal{R}$ being viewed as a subset of F.

One can check directly that any piece relative to $\tilde{\mathcal{S}}$ of any relator from $\tilde{\mathcal{S}}$ is either of the form $(a_i^m b_\sigma^n)^{\pm 1}$, or of the form $(b_\sigma^n a_i^m)^{\pm 1}$. Then we deduce that the length of such a piece is at most 8. As the length of every relator is 7×8, we obtain

$$|X| \leq \frac{1}{7}|R| < \frac{1}{6}|R|$$

for any $R \in \tilde{\mathcal{S}}$ and any piece X of R relative to $\tilde{\mathcal{S}}$. Hence the presentation $\langle A \parallel \mathcal{S} \rangle$ satisfies $C'(1/6)$. In particular, the subpresentation $\langle A \parallel \mathcal{R} \rangle$ satisfies $C'(1/6)$.

It follows now from Fact 3.2 that G is hyperbolic. By Fact 3.3, the presentation $\langle A \parallel \mathcal{S} \rangle$ is diagrammatically aspherical. As clearly no element of $\mathcal{S}$ is a proper power in the free group F, the presentation $\langle A \parallel \mathcal{S} \rangle$ is in fact singularly aspherical, and such is the subpresentation $\langle A \parallel \mathcal{R} \rangle$. Fact 3.5 implies now that G is torsion-free.

Finally, applying Fact 3.4 to $\langle A \parallel \mathcal{S} \rangle$, we obtain that for all $i \in \{1, \dots, n\}$ and $\sigma \subseteq \{1, \dots, n\}$,

$$G \models w(a_i, b_\sigma) = 1 \quad \text{if and only if} \quad i \in \sigma.$$

As n can be any positive integer, the formula $\ulcorner w(x, y) = 1 \urcorner$ has the independence property relative to the class of torsion-free hyperbolic groups. The proof of Theorem 2.1 is now complete. $\dashv$

§4. **Concluding remarks.** We conclude with a few remarks.

4.1. The group word $w(x, y)$ built in the proof of Theorem 2.1 is in two variables and without additional constants. Hence in Corollary 2.2 one finds directly a formula with the independence property having two single variables only (see [Poi85, Théorème 12.18] for general reductions of this kind), and without parameters.

4.2. As announced in [Sel07], a torsion-free hyperbolic group is stable, and hence cannot have the independence property. Hence one cannot imagine a version of Theorem 2.1 where the class of groups would consist of groups elementarily equivalent to a given torsion-free hyperbolic group, or more generally to a fixed *finite* set of torsion-free hyperbolic groups. Our proof provides however a countable set of torsion-free hyperbolic groups.

4.3. The proof of Theorem 2.1 actually provides a group word $w(x, y)$ such that the formula $\ulcorner w(x, y) = 1 \urcorner$ has the independence property relative to the class of torsion-free finitely presented $C'(1/6)$-groups, a class significantly smaller than that of all torsion-free hyperbolic groups. Choosing w long enough, one can similarly produce formulas having the independence property relative to the class of torsion-free finitely presented $C'(\lambda)$-groups with λ arbitrarily small. Indeed, if one denotes by P_n the "probability" that a cyclically reduced group word $w(x, y)$ of length $n \geq 1$ gives the independence property relative to the class of torsion-free $C'(\lambda)$-groups (with the formula $\ulcorner w(x, y) = 1 \urcorner$), then one can check in a rather straightforward manner the rough estimate

$$P_n \geq 1 - \frac{n^2}{2^{\lambda n}} - \frac{4n}{2^{\lambda n}},$$

and hence P_n tends rapidly to 1 as n tends to the infinity. This estimate can be found by verifying that the "probability" of occurrence of a subword of a cyclic shift of w of length $\lceil \lambda n \rceil$ in two given distinct "positions" with respect to w is not greater than $1/2^{\lambda n}$, and that the "probability" that a cyclic shift of w contains a syllable of length $\lceil \lambda n \rceil$ is less than $4n/2^{\lambda n}$. For example, the probability that in a cyclically reduced group word $w(x, y)$ of length at least 6 the same word of length 4 occurs as a subword starting from the first letter and also starting from third letter, is at most $1/(3 \cdot 3 \cdot 3 \cdot 2) = 1/54$; indeed, once all the letters of w except the first 4 are fixed, there is at most one way to complete w so as to obtain a word whose initial subword of length 4 occurs again starting from the third letter, but there are at least $3 \cdot 3 \cdot 3 \cdot 2$ ways to complete it to obtain a cyclically reduced group word.

Roughly speaking, for any fixed $\lambda > 0$, by choosing an arbitrary but sufficiently long group word w, one will "most probably" obtain the formula $\ulcorner w(x, y) = 1 \urcorner$ with the independence property relative to the class of torsion-free finitely presented $C'(\lambda)$-groups.

4.4. Another property of a formula witnessing the instability of a first-order theory, orthogonal in some sense to the independence property, is the *strict order property*. One may wonder whether the first-order theory $\mathrm{Th}(G)$ of an existentially closed CSA-group G has this property. We provide here some speculations on this question, again for formulas $\phi(x, y)$ of the form $\ulcorner w(x, y) = 1 \urcorner$ for some group word w, and we concentrate on weaker properties such as the *n-strong order property* SOP_n defined in [She96, Definition 2.5] for $n \geq 3$. We refer to [She96, Sect. 2] for a general discussion of these properties, and we just recall the implications

$$\begin{array}{rll} \text{strict order property} & \Longrightarrow \cdots & \\ & \Longrightarrow \mathrm{SOP}_{n+1} & \\ & \Longrightarrow \mathrm{SOP}_n & \\ & \Longrightarrow \cdots & \\ & \Longrightarrow \mathrm{SOP}_3 & \Longrightarrow \text{non-simplicity} \end{array}$$

If w is a "long" and "arbitrarily chosen" cyclically reduced group word in two letters x and y (see above), then "most probably" the formula $\ulcorner w(x, y) = 1 \urcorner$ will not exemplify SOP_n. The reason is that essentially the same kind of arguments used in the proof of Theorem 2.1 provide torsion-free $C'(1/6)$-groups generated by elements $a_1, \ldots, a_n$ in which $w(a_i, a_j) = 1$ if and only if $j = i + 1$ modulo n, and thus the graph defined on an existentially closed CSA_f-group by the formula $\ulcorner w(x, y) = 1 \urcorner$ contains cycles of size n, contrary to one of the two requirements for the condition SOP_n (the other one being the existence in some model of an infinite chain in this graph). Hence one may be tempted to look at "short" words. As usual in our context of commutative transitive groups, the very short word $[x, y]$ witnessing the commutation of x and y boils down to an equivalence relation, and hence is useless. Incidentally, the formula $\phi(x, y)$ used in [SU06, Proposition 4.1] to prove that Group Theory "in general" has SOP_3 is

$$(xyx^{-1} = y^2) \wedge (x \neq y).$$

In our context of CSA-groups, it implies $y = 1$. Hence the absence of certain triangles (a_1, a_2, a_3) satisfying

$$\phi(a_1, a_2) \wedge \phi(a_2, a_3) \wedge \phi(a_3, a_1)$$

(provided by [Sta91, p. 493] in the context of arbitrary groups) is immediate in our case, but one cannot hope to find an infinite chain in the graph associated to $\phi(x, y)$. Hence this formula could not exemplify the SOP_3 in our context of CSA-groups. This means that a formula examplifying the SOP_n of an existentially closed CSA-group, if it exists, cannot involve only an "arbitrarily chosen long" equation, and it does not seem to involve only "short" equations.

Acknowledgments. We thank Abderezak Ould Houcine who pointed out fallacies in preliminary attempts towards the present work.

REFERENCES

[Bro94] K. S. Brown, ***Cohomology of groups***, Graduate Texts in Mathematics, vol. 87, Springer, New York, 1994, corrected reprint of the 1982 original.

[CCH81] I. M. Chiswell, D. J. Collins, and J. Huebschmann, *Aspherical group presentations*, ***Mathematische Zeitschrift***, vol. 178 (1981), no. 1, pp. 1–36.

[CDP90] M. Coornaert, T. Delzant, and A. Papadopoulos, ***Géométrie et théorie des groupes***, Lecture Notes in Mathematics, vol. 1441, Springer, Berlin, 1990, Les groupes hyperboliques de Gromov. [Gromov hyperbolic groups], with an English summary.

[GdlH90] É. Ghys and P. de la Harpe (editors), ***Sur les groupes hyperboliques d'après Mikhael Gromov***, Progress in Mathematics, vol. 83, Birkhäuser Boston Inc., Boston, MA, 1990, papers from the Swiss seminar on Hyperbolic Groups held in Bern, 1988.

[Gro87] M. Gromov, *Hyperbolic groups*, ***Essays in group theory***, Mathematical Sciences Research Institute Publications, vol. 8, Springer, New York, 1987, pp. 75–263.

[Hod93] W. Hodges, ***Model theory***, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993.

[Hue79] J. Huebschmann, *Cohomology theory of aspherical groups and of small cancellation groups*, ***Journal of Pure and Applied Algebra***, vol. 14 (1979), no. 2, pp. 137–143.

[Jal01] E. Jaligot, *Full Frobenius groups of finite Morley rank and the Feit-Thompson theorem*, this Bulletin, vol. 7 (2001), no. 3, pp. 315–328.

[JOH04] E. Jaligot and A. Ould Houcine, *Existentially closed CSA-groups*, ***Journal of Algebra***, vol. 280 (2004), no. 2, pp. 772–796.

[Mur07] A. Yu. Muranov, *Finitely generated infinite simple groups of infinite commutator width*, ***International Journal of Algebra and Compututation***, vol. 17 (2007), no. 3, pp. 607–659, arXiv.org preprint: `math.GR/0608688`.

[MR96] A. G. Myasnikov and V. N. Remeslennikov, *Exponential groups. II. Extensions of centralizers and tensor completion of CSA-groups*, ***International Journal of Algebra and Computation***, vol. 6 (1996), no. 6, pp. 687–711.

[Ol91] A. Yu. Ol′shanskiĭ, ***Geometry of defining relations in groups***, Mathematics and its Applications (Soviet Series), vol. 70, Kluwer, Dordrecht, 1991, translated from the 1989 Russian original by Yu. A. Bakhturin.

[OH07] A. Ould Houcine, *On superstable CSA-groups*, to appear in ***Annals of Pure and Applied Logic***, `http://math.univ-lyon1.fr/~ould/`, 2007.

[Poi85] B. Poizat, ***Cours de théorie des modèles***, Bruno Poizat, Villeurbanne, 1985, Une introduction à la logique mathématique contemporaine. [An introduction to contemporary mathematical logic].

[Ros62] M. Rosenlicht, *On a result of Baer*, ***Proceedings of the American Mathematical Society***, vol. 13 (1962), no. 1, pp. 99–101.

[Sel07] Z. Sela, *Diophantine geometry over groups VIII: Stability*, preprint: `http://www.ma.huji.ac.il/~zlil/`, 2007.

[She90] S. Shelah, ***Classification theory and the number of nonisomorphic models***, second ed., North-Holland, Amsterdam, 1990.

[She96] ———, *Toward classifying unstable theories*, ***Annals of Pure and Applied Logic***, vol. 80 (1996), no. 3, pp. 229–255.

[SU06] S. Shelah and A. Usvyatsov, *Banach spaces and groups—order properties and universal models*, ***Israel Journal of Mathematics***, vol. 152 (2006), pp. 245–270.

[Sta91] J. R. Stallings, *Non-positively curved triangles of groups*, ***Group theory from a geometrical viewpoint*** (***Trieste,*** 1990), World Scientific Publishing, River Edge, NJ, 1991, pp. 491–503.

UNIVERSITÉ DE LYON
CNRS AND UNIVERSITÉ LYON 1
INSTITUT CAMILLE JORDAN CNRS UMR 5208
43, BOULEVARD DU 11 NOVEMBRE 1918
F-69622 VILLEURBANNE CEDEX, FRANCE
E-mail: jaligot@math.univ-lyon1.fr

UNIVERSITÉ DE LYON
UNIVERSITÉ LYON 1
INSTITUT CAMILLE JORDAN CNRS UMR 5208
43, BOULEVARD DU 11 NOVEMBRE 1918
F-69622 VILLEURBANNE CEDEX, FRANCE
E-mail: muranov@math.univ-lyon1.fr
E-mail: azadeh@math.univ-lyon1.fr