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THE COMPLEXITY OF ORBITS OF COMPUTABLY ENUMERABLE SETS

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Abstract. The goal of this paper is to announce there is a single orbit of the c.e. sets with inclusion, $\mathcal{E}$, such that the question of membership in this orbit is Σ^1_1-complete. This result and proof have a number of nice corollaries: the Scott rank of $\mathcal{E}$ is $\omega_1^{\mathrm{CK}} + 1$; not all orbits are elementarily definable; there is no arithmetic description of all orbits of $\mathcal{E}$; for all finite $\alpha \geq 9$, there is a properly Δ^0_α orbit (from the proof).

§1. Introduction. In the classic paper [32], Post suggested that the study of the lattice of computably (recursively) enumerable (c.e.) sets was fundamental in computability theory. Post observed that, at the time, all known undecidability proofs worked by coding some "noncomputability," (coded by a certain kind of set) into the theory at hand thereby arguing that the relevant structures could emulate computation. He argued that for such proofs the central object was the notion of a computably enumerable set, being one which is effectively generated as the range of some effective process. The basic example is the set of consequences of a computably enumerable set of axioms for a formal system. Of course, the other key concept in computability was that of Turing [36] who introduced the notion of reducibility. Reducibilities are pre-orderings used to measure relative computational complexity.

The interplay of these two basic objects, (Turing) reducibility and effectively enumerated (c.e.) sets has a long and rich history. Clearly the computably enumerable sets under union and intersection form a lattice, denoted by $\mathcal{E}$. Their Turing degrees form an upper semilattice, denoted by $\mathcal{R}$. Ever since the groundbreaking paper of Post, there has been a persistent intuition that structural properties of computably enumerable sets have reflections in their degrees, and conversely. In particular, *definability* in $\mathcal{E}$ should be linked with *information content* as measured by $\mathcal{R}$.

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The simplest possible illustration of this is the fact that the complemented members of $\mathcal{E}$ are exactly the members of $\mathbf{0}$ the degree of the computable sets. An excellent and deep example is Martin's result that the Turing degrees of maximal sets are exactly the high computably enumerable Turing degrees,[1] (that is, their jumps are as complex as possible, $\mathbf{a}' = \mathbf{0}''$), where a co-infinite set A is a maximal c.e. set iff for all c.e. sets B, if $A \subseteq B$ then either $A =^* B$ or $B =^* \omega$, where $=^*$ denotes equivalence modulo the filter of finite sets. Since a set A is finite iff every subset is complements in $\mathcal{E}$, it is natural to consider $\mathcal{E}^*$, $\mathcal{E}$ modulo the filter of finite sets. Thus A is a maximal set iff it represents a co-atom in $\mathcal{E}^*$.

An original impetus for work on $\mathcal{E}^*$ was *Post's Problem*: Post observed that the coding inherent in all known undecidability proofs of the time were so faithful that all computably enumerable sets seemed to be either the $\mathbf{0}'$ the degree of the halting problem, or $\mathbf{0}$ the degree of the computable sets. He asked if this was always the case. Post's problem is the following: *Are there c.e. sets of intermediate Turing degree*? *Is there a computably enumerable degree* $\mathbf{a}$ *with* $\mathbf{0} < \mathbf{a} < \mathbf{0}'$? Post even suggested a program to answer this question. Since complemented members of $\mathcal{E}$ are computable, but $\emptyset' = \{x : \varphi_x(x)\downarrow\}$, the halting problem, had many infinite computably enumerable sets disjoint from it, perhaps a very "noncomplemented" member of $\mathcal{E}$ would turn out to be incomplete. Toward this goal, Post defined a computably enumerable set A to be *simple* if $\omega - A$ was infinite and for all infinite computably enumerable W, $W \cap A \neq \emptyset$. Post constructed a simple set and then proved that a simple set A is not of the same m-degree as $\mathbf{0}'$. Post's program was to come up with a thinness property of the complement of a c.e. set which would guarantee Turing incompleteness. Interpreted more liberally, we can generalize Post's program and ask whether there is any definable property of a c.e. set in $\mathcal{E}^*$ which would guarantee Turing incompleteness.

While Post's problem was eventually solved by the development of the priority method independently by Friedberg [18] and Muchnik [31] out of the work of Kleene and Post [24], whether Post's Program could be realized successfully remained open for much longer. Myhill observed that if Post's original program were to succeed then *maximal sets*, first constructed by Friedberg, should be Turing incomplete. On the other hand, rather than being Turing incomplete, Tennenbaum suggested that all maximal sets would be Turing *complete*. As we have seen above, following earlier work of Tennenbaum, Sacks, and Yates, and others, Martin classified the degrees of maximal sets as precisely the high computably enumerable degrees. Martin's theorems can be seen as a partial realization of Tennenbaum's intuition. That is, in spite of the fact that maximal sets may not necessarily be complete, they do have high information content in the sense that they have the same jump

[1]Indeed the reader should recall that, more generally, a set A is low$_n$ iff $A^{(n)} = \emptyset^{(n)}$ iff $\Delta^0_{n+1} = \Delta^A_{n+1}$, and A is high$_n$ iff $A^{(n)} = \emptyset^{(n+1)}$ iff $\Delta^0_{n+2} = \Delta^A_{n+1}$.

as the halting problem. That is, as far as the jump operator is concerned, they are indistinguishable from the halting problem. Martin had discovered the first *invariant class* in $\mathcal{R}$ in the sense that the high degrees are precisely those realized by the (definable class of) maximal sets.

Definability goes hand in hand with automorphisms of structures. Thus, a class $\mathcal{C}$ of computably enumerable degrees is *invariant* if there is a collection $\mathcal{C}'$ of computably enumerable sets closed under automorphisms of $\mathcal{E}$ such that $\mathcal{C} = \{\deg(A)\colon A \in \mathcal{C}'\}$. The following definition will be important for our subsequent discussion.

Definition 1.1. $A \approx \hat{A}$ iff there is a map, Φ, from the c.e. sets to the c.e. sets preserving inclusion, $\subseteq$, (so $\Phi \in \mathrm{Aut}(\mathcal{E})$) such that $\Phi(A) = \hat{A}$.

By [35], $\mathcal{E}$ can be replaced with $\mathcal{E}^*$, since Soare showed that every automorphism of $\mathcal{E}^*$ is equivalent to one on $\mathcal{E}$ and conversely (as long we focus on infinite and co-infinite sets). Early work by Lachlan and others showed that both of the automorphism groups $\mathrm{Aut}(\mathcal{E})$ and $\mathrm{Aut}(\mathcal{E}^*)$ were large since each had $2^{\aleph_0}$ automorphisms. Martin [30] used a priority construction to show that a certain construction of Post (hypersimplicity) was not invariant under automorphisms of $\mathcal{E}$.

Post's original program was to look at thinness properties of the lattice of the complement of a c.e. set A. We will see shortly this original program cannot be solved. But there are several solutions to various modified versions of this program.

The earliest solution to a modified Post's Program was due to Marchenkov [29] who showed that a certain type of maximal set in a related quotient structure gave a solution. Specifically, if you change the game and replace the integers by computably enumerable equivalence classes η, you can get a structure consisting of the c.e. sets factored out by this equivalence relation. Then for a suitable choice of η certain η-maximal sets provide a solution to Post's program, at least in a generalized sense.

However, we want to focus on the structures $\mathcal{E}$ and $\mathcal{E}^*$. In these structures, there is a positive solution to another modified version of Post's Program. This solution will be discussed shortly but for now we want to focus on the failure of the original program. This leads us to a definition which will be important:

Definition 1.2. $\mathcal{L}^*(A)$ is $\{W \cup A\colon W \text{ an c.e. set}\}$ under $\subseteq$ modulo the ideal of finite sets $(\mathcal{F})$. (The outside of a set.)

The final blow to Post's original program was the marvelous paper of Soare [35], who showed that maximal sets form an orbit in $\mathrm{Aut}(\mathcal{E})$. In particular, no "extra" property together with maximality could guarantee incompleteness. Moreover, the paper Cholak, Downey, and Stob [9] showed that no property of $\mathcal{L}^*(A)$ *alone* could guarantee Turing incompleteness for a computably enumerable set A. That is, Cholak, Downey, and Stob proved that if for any computably enumerable set A there is a set computably enumerable Turing complete B with the same lattice of supersets as A, $\mathcal{L}^*(A) \cong \mathcal{L}^*(B)$.

Soare's [35] was highly influential. The methods introduced constructed Δ^0_3 automorphisms of $\mathcal{E}^*$. Here we will classify automorphisms according to the complexity of a presentation of them. A presentation will be a function $f : \omega \to \omega$ such that $W_e \mapsto W_{f(e)}$ induces the automorphism. If $e \mapsto f(e)$ a Δ^0_3 function, we would call the automorphism Δ^0_3, and $A \approx_{\Delta^0_3} \hat{A}$. While later papers presented Soare's automorphism machinery argument as a more thematic and flexible tree argument (beginning with Cholak [2] and [4] and Harrington and Soare [22]) most of the key underlying ideas for constructing automorphisms of $(\mathcal{E})$ are in Soare's original paper.

The principal tool used is called the (or, in view of recent work, *an*) *Extension Lemma*. Roughly speaking, constructing an automorphism works as follows. We wish to show maximal $A \approx \hat{A}$. We are given two versions of the universe ω, called ω and $\hat{\omega}$ with $A \subset \omega$ and $\hat{A} \subset \hat{\omega}$, with enumerations of c.e. $\{W_e : e \in \omega\}$ subsets of ω and $\{V_e : e \in \omega\}$ subsets of $\hat{\omega}$. We must define some sort of mapping as follows

$$\begin{array}{lll} \omega & \mapsto & \hat{\omega} \\ A & \mapsto & \hat{A} \\ W_e & \mapsto & \hat{W}_{f(e)} \\ \hat{V}_{g(e)} & \leftarrow\!\shortmid & V_e. \end{array}$$

We must have the $\hat{W}_{f(e)}$ and $\hat{V}_{g(e)}$ so that we can argue that the mapping induces an automorphism of $\mathcal{E}^*$ by a back and forth argument. Here we are thinking of *building* the hatted sets $\hat{W}$ and $\hat{V}$. At the very least, intersections should be respected. That is, if $\overline{A} \cap W_e$ is infinite, then $\overline{\hat{A}} \cap \hat{W}_{f(e)}$ would need to be infinite. Similarly, if $W_e - \hat{V}_{g(q)}$ is infinite then so too must be $\hat{W}_{f(e)} - V_q$, etc. Evidently, any possible diagram we can think of denoting intersections and difference would need to be respected. We represent these intersections and differences by *states*, which are strings measuring which W_es and $\hat{V}_k$s on the ω side an element is in, and which $\hat{W}_j$s and V_qs a hatted element is in on the $\hat{\omega}$ side. We would write this basic requirement as

$$\mathcal{R}_\sigma : \exists^\infty x \in \text{state } \sigma \text{ iff } \exists^\infty \hat{x} \in \text{state } \hat{\sigma}.$$

Soare's original idea is to begin on the $\mathcal{L}^*(A) \cong \mathcal{L}^*(\hat{A})$. He would make this an isomorphism and then extend this outside isomorphism to an automorphism by an isomorphism of the lattice of computably enumerable subsets of A to those of $\hat{A}$.

Concentrating on the $\mathcal{L}^*(A) \cong \mathcal{L}^*(\hat{A})$ part, as we go along, elements appear to be in this region (that is, in $\overline{A}_s$ or $\overline{\hat{A}}_s$), and we build corresponding sets to match the states measuring intersections. This would seem not too hard in the case of a maximal set since for any sets W either $W \cap \overline{A}$ is finite or W almost contains $\overline{A}$. On the hatted side, all we would need to do is either have the corresponding $\hat{W}$ empty, or containing $\overline{\hat{A}}$, and similarly for the mappings from the hatted side back. The information as to which is correct is Σ^0_3 information and can be handled by a priority argument.

However, the *heart* of Soare's method is the following. As we go along enumerating hatted sets as elements stream into $\overline{A}_s$ many of these will be based on *wrong* information (such as the fact that at stage s they might appear in the complement of A yet might be in A), and will later enter A, and hence be in $W_e \searrow A$. They will enter A_t and $\hat{A}_u$ in various *entry* states some caused by the W_e and V_e played by the opponent, and the hatted sets played by *us*. The key problem is how to handle these wrongly enumerated elements and be able to *extend* the $\mathcal{L}^*(A) \cong \mathcal{L}^*(\hat{A})$ correspondence to an automorphism. For the automorphism machinery to succeed, it is *necessary* to ensure that *for all* entry *states* σ *(and dually for* $\hat{\sigma}$*), if infinitely many elements enter A in state* σ*, then there is some covering entry state* $\hat{\tau}$ *(*τ*, respectively).* Here *covering* means that it is within *our* power to add elements into sets under *our* control to be able to match states.

Soare's Extension Lemma shows that this necessary condition is sufficient. Soare showed that we have not already killed that automorphism, meaning that the necessary condition is satisfied, then there is a strategy which enables us to extend the partial matching into a full automorphism. On the inside, that is the lattice of subsets of A to those of $\hat{A}$ the map is Δ^0_3, and in the case of maximal sets, Soare's original result shows that if A and $\hat{A}$ are maximal, then $A \approx_{\Delta^0_3} \hat{A}$.

There is a lot of subsequent work on automorphisms and invariance in the lattice of c.e. sets. Almost all of it either uses Soare's original Extension Lemma as a black box, or modified it, to prove various results on the lattice of c.e. sets. Examples include the work of Maass [27], Maass and Stob [28], and Downey and Stob [17].

Early on, the methods seemed so powerful that anything seemed possible. Perhaps all sets were automorphic to complete sets, as suggested by Soare [34]. Certainly Harrington and Soare, and Cholak, independently, showed that all sets were automorphic to high sets.

Hand in hand with this work constructing automorphisms was another line of investigation, where *failures* of the automorphism machinery could be exploited to provide *definability* results in $\mathcal{E}^*$. A classic example of this is the following theorem of Harrington and Soare [20] [21] who showed that a more general form of Post's Program indeed has a positive solution.

THEOREM 1.3 (Harrington and Soare [20]). *There is a definable property* $Q(A)$ *such that, if a c.e. set* A *satisfies* $Q(A)$*, then* A *is Turing incomplete.*

There were precursers to the Harrington-Soare result. Harrington used the idea of exploiting the failure of the machinery to get a definition of being a halting problem in the lattice of c.e. sets. Similarly, Lerman and Soare [26] showed that there are low simple sets that are elementarily inequivalent, in that one has a property called d-simplicity and one has not, where d-simplicity is an elementary property implying certain facts about entry states. Another example of this can be found in Downey and Harrington [14] where the "no fat orbit" theorem is proven. The simplest form of the Downey-Harrington result below says that no c.e. set has an orbit hitting all nonzero degrees.

THEOREM 1.4 (Downey and Harrington—No fat orbit). *There is a property $S(A)$, a prompt low degree $\mathbf{d_1}$, a prompt high$_2$ degree $\mathbf{d_2}$ greater than $\mathbf{d_1}$, and tardy high$_2$ degree $\mathbf{e}$ such that for all $E \leq_T \mathbf{e}$, $\neg S(E)$ and if $\mathbf{d_1} \leq_T D \leq_T \mathbf{d_2}$ then $S(D)$.*

We remark that this "failure" methodology has yielded similar definability results in other structures such as the lattice of Π^0_1 classes, as witnessed by Weber [37] and [38], Cholak and Downey [6], and Downey and Montalbán [15]. Perhaps the best example of the methodology is the following proof of the definability of the double jump classes, the proof using "patterns" which are more or less direct reflections of blockages to the automorphism machinery.

THEOREM 1.5 (Cholak and Harrington [12]). *Let $\mathcal{C} = \{\mathbf{a}\colon \mathbf{a}$ is the Turing degree of a Σ_3 set greater than $\mathbf{0}''\}$. Let $\mathcal{D} \subseteq \mathcal{C}$ such that $\mathcal{D}$ is upward closed. Then there is a nonelementary $(\mathcal{L}_{\omega_1,\omega})$ $\mathcal{L}(A)$ property $\varphi_{\mathcal{D}}(A)$ such that $D'' \in \mathcal{D}$ iff there is an A where $A \equiv D$ and $\varphi_{\mathcal{D}}(A)$.*

COROLLARY 1.6. *If $\mathbf{a}'' > \mathbf{b}''$ then there is a $A \in \mathbf{a}$ such that for all $B \in \mathbf{b}$, A is not automorphic to B (in fact, $\mathcal{L}^*(A) \ncong \mathcal{L}^*(B)$).*

Related here is the following conjecture of Harrington.

CONJECTURE 1.7 (Harrington). *For all A and degrees $\mathbf{d}$ if $A' \leq_T \mathbf{d}'$ is there $\hat{A} \in \mathbf{d}$ such that $\mathcal{L}^*(A) \cong \mathcal{L}^*(\hat{A})$?*

For more of these results one can see the paper [11].

§**2. New results.** The present work is motivated by basic questions about the automorphism group of $\mathcal{E}^*$. How complicated is it? If $A \approx \hat{A}$ is $A \approx \hat{A}$ witnessed by an *arithmetical* automorphism? How complicated is $\{W_e\colon W_e \approx A\}$ for a fixed A? The following conjecture was made by Ted Slaman and Hugh Woodin in 1989.

CONJECTURE 2.1. [33] *The set $\{\langle i, j\rangle\colon W_i \approx W_j)\}$ is Σ^1_1-complete.*

This conjecture was claimed to be true by the authors in the mid 1990s, but no proof appeared. One of the roles of this announcement and the full paper [7] is to correct that omission. The proof we will present is far simpler than all previous (and hence unpublishable) proofs. Indeed, much of the material reported in this paper due to Cholak and Harrington was developed toward making the proof of the conjecture accessible. The other important role of this communcation and the full paper is to prove a stronger result.

THEOREM 2.2 (The Main Theorem). *There is a c.e. set A such that the index set* $\{i : W_i \approx A\}$ *is* Σ^1_1*-complete.*

As mentioned in the abstract this theorem does have a number of nice corollaries.

COROLLARY 2.3. *Not all orbits are elementarily definable; there is no arithmetic description of all orbits of* $\mathcal{E}$.

COROLLARY 2.4. *The Scott rank of* $\mathcal{E}$ *is* $\omega_1^{\mathrm{CK}} + 1$.

PROOF. Our definition that a structure has Scott rank $\omega_1^{\mathrm{CK}} + 1$ is that there is an orbit such that membership in that orbit is Σ^1_1-complete. There are other equivalent definitions of a structure having Scott Rank $\omega_1^{\mathrm{CK}} + 1$ and we refer the readers to [1]. ⊣

A consequence of the *method* of the proof (and some further effort to preserve quantifiers) is the following.

THEOREM 2.5. *For all finite* $\alpha > 8$ *there is a properly* Δ^0_α *orbit.*

Prior to this paper [7] all known orbits were Δ^0_3 with the single exception of the orbit of Cholak and Harrington [10] which constructed a pair of sets Δ^0_5 automorphic but not Δ^0_3.

Before we turn to the proof of Theorem 2.2, we will discuss the background to the Slaman-Woodin Conjecture. Certainly the set $\{\langle i, j\rangle : W_i \approx W_j)\}$ is Σ^1_1. Why would we believe it to be Σ^1_1-complete? The following result is from the folklore.[2]

THEOREM 2.6 (Folklore[3]). *There is a computable listing,* $\mathcal{B}_i$, *of computable Boolean algebras such that the set* $\{\langle i, j\rangle : \mathcal{B}_i \cong \mathcal{B}_j\}$ *is* Σ^1_1*-complete.*

Note that $\mathcal{L}^*(A)$ is a definable structure in $\mathcal{E}$ with a parameter for A. The following result says that the full complexity of the isomorphism problem for Boolean algebras of Theorem 2.6 is present in the supersets of a c.e. set.

THEOREM 2.7. [25] *Effectively in* i *there is a c.e. set* H_i *such that* $\mathcal{L}^*(H_i) \cong \mathcal{B}_i$.

COROLLARY 2.8. *The set* $\{\langle i, j\rangle : \mathcal{L}^*(H_i) \cong \mathcal{L}^*(H_j)\}$ *is* Σ^1_1*-complete.*

Slaman and Woodin's idea was to replace "$\mathcal{L}^*(H_i) \cong \mathcal{L}^*(H_j)$" with "$H_i \approx H_j$". Unfortunately, this very attractive idea is doomed, as we now see.

[2] We think it is well known that the isomorphism problems for Boolean Algebras and Trees are Σ^1_1-complete, at least in the form stated in Theorems 2.6 and 5.1. We have searched for a reference to a proof for these theorems without success. It seems very likely that these theorems were known to Kleene. There are a number of places where something very close to what we want appears; for example, see the example at the end of Section 5 of [19] and surely there are earlier examples (for example, [39]). All of these constructions work by coding the Harrison ordering. In the full paper we give self-contained proofs of the folklore theorems we use.

[3] See Section 5 of the full paper [7] for more information and a proof.

DEFINITION 2.9 (The sets disjoint from A).

$$\mathcal{D}(A) = (\{B : \exists W(B \subseteq A \cup W \text{ and } W \cap A =^* \emptyset)\}, \subseteq).$$

Let $\mathcal{E}_{\mathcal{D}(A)}$ be $\mathcal{E}$ modulo $\mathcal{D}(A)$. A is *$\mathcal{D}$-hhsimple* iff $\mathcal{E}_{\mathcal{D}(A)}$ is a Boolean Algebra. A is *$\mathcal{D}$-maximal* iff $\mathcal{E}_{\mathcal{D}(A)}$ is the trivial Boolean Algebra.

LEMMA 2.10. *If A is simple then $\mathcal{E}_{\mathcal{D}(A)} \cong_{\Delta^0_3} \mathcal{L}^*(A)$.*

It is an old result of Lachlan [25] that A is hhsimple iff $\mathcal{E}_{\mathcal{D}(A)}$ is a Boolean algebra. Except for the creative sets, until recently, all known orbits were orbits of $\mathcal{D}$-hhsimple sets. We direct the reader to [10] for a further discussion of this claim and for an orbit of $\mathcal{E}$ which does not contain any $\mathcal{D}$-hhsimple sets. The following are relevant theorems from [10].

THEOREM 2.11. *If A is $\mathcal{D}$-hhsimple and A and $\hat{A}$ are in the same orbit then $\mathcal{E}_{\mathcal{D}(A)} \cong_{\Delta^0_3} \mathcal{E}_{\mathcal{D}(\hat{A})}$.*

THEOREM 2.12 (using [27]). *If A is $\mathcal{D}$-hhsimple and simple (i.e., hhsimple) then $A \approx \hat{A}$ iff $\mathcal{L}^*(A) \cong_{\Delta^0_3} \mathcal{L}^*(\hat{A})$.*

Hence the Slaman-Woodin plan of attack fails. In fact even more is true.

THEOREM 2.13. *If A and $\hat{A}$ are automorphic then $\mathcal{E}_{\mathcal{D}(A)}$ and $\mathcal{E}_{\mathcal{D}(\hat{A})}$ are Δ^0_6-isomorphic.*

Hence in order to prove Theorem 2.2 we must code everything into $\mathcal{D}(A)$. This is completely contrary to all approaches used to try to prove the Slaman-Woodin Conjecture over the years. We will point out two more theorems from [10] to show how far the sets we use for the proof must be from simple sets, in order to prove Theorem 2.2.

THEOREM 2.14. *If A is simple then $A \approx \hat{A}$ iff $A \approx_{\Delta^0_6} \hat{A}$.*

THEOREM 2.15. *If A and $\hat{A}$ are both promptly simple then $A \approx \hat{A}$ iff $A \approx_{\Delta^0_3} \hat{A}$.*

§3. Future work and the degrees of the constructed orbits. While this work does answer many open questions about the orbits of c.e. sets, there are many questions left open. But perhaps these open questions are of a more degree-theoretic flavor. We will list a number of questions here.

QUESTION 3.1 (Completeness). *Which c.e. sets are automorphic to complete sets?*

Of course, by [20], we know that not every c.e. set is automorphic to a complete set, and partial classifications of precisely which sets can be found in [17] and Harrington and Soare [22, 23].

QUESTION 3.2 (Cone Avoidance). *Given an incomplete c.e. degree* $\mathbf{d}$ *and an incomplete c.e. set A, is there an $\hat{A}$ automorphic to A such that $\mathbf{d} \not\leq_T \hat{A}$?*

QUESTION 3.3 (Can single jumps be coded into $\mathcal{E}$?). *Let J be C.E.A. in $\mathbf{0}'$ but not of degree $\mathbf{0}''$. Is there a degree $\mathbf{a}$ such that $\mathbf{a}' \equiv_T J$ and, for all $A \in \mathbf{a}$, there is an $\hat{A}$ with A automorphic to $\hat{A}$ and $\hat{A}' <_T \mathbf{a}'$ or $\hat{A}'|_T\mathbf{a}'$?*

QUESTION 3.4 (Can a single Turing degree be coded into $\mathcal{E}$?). *Is there a degree $\mathbf{d}$ and an incomplete set A such that, for all $\hat{A}$ automorphic to A, $\mathbf{d} \leq \hat{A}$? $A \in \mathbf{d}$?*

In a technical sense, these may not have a "reasonable" answer. Thus the following seems a reasonable question.

QUESTION 3.5. *Are these arithmetical questions*?

In this paper we do not have the space to discuss the import of these questions. Furthermore, it not clear how this current work impacts possible approaches to these questions. At this point we will just direct the reader to slides of a presentation of Cholak [5]; perhaps a paper reflecting on these issues will appear later.

One of the issues that will impact all of these questions is which degrees can be realized in the orbits that we construct in Theorem 2.2 and 2.5. A set is *hemimaximal* iff it is the nontrivial split of a maximal set. A degree is *hemimaximal* iff it contains a hemimaximal set. [17] proved that the hemimaximal sets form an orbit, and in some sense, this orbit is very large degree-theoretically. While it is known by [14] that there is no orbit containing sets of all nonzero degrees, the orbit of hemimaximal sets contain representatives of all jump classes ([16]).

We are able to also show that we can construct our orbits to contain at least a fixed hemimaximal degree (possibly along others) or contain all hemimaximal degrees (again possibly along others). However, what is open is if every such orbit must contain a representative of every hemimaximal degree or only hemimaximal degrees. For the proofs of these claims, we direct the reader to Section 5.3.

§4. Past work and other connections. The paper [7] is a fourth paper in a series of loosely connected papers, the previous three being by Harrington and Cholak [12], [13], and [10]. We have seen above that results from [10] determine the direction one must take to prove Theorem 2.2. The above results from [10] depend heavily on the main result in [13] whose proof depends on special $\mathcal{L}$-patterns and several theorems about them which can be found in [12]. It is not necessary to understand any of the above-mentioned theorems from any of these papers to understand the proof of Theorem 2.2.

But the proof of Theorem 2.2 does depend on Theorems 2.16, 2.17, and 5.10 of [10]; see Section 5. The proof of Theorem 2.5 also needs Theorem 6.3 of [10]. The first two theorems are straightforward but the third and fourth require work. The third is another modified "Extension Theorem." The fourth is what we might call a "Restriction Theorem"; it restricts the possibilities for automorphisms.

Fortunately, we are able to use these four theorems from [10] as *black boxes*. These four theorems provide a clean interface between the two papers. If one wants to understand the *proofs* of these four theorems one must go to [10]; otherwise, the paper [7] is completely independent from its three predecessors. In the next section we will explore the statements of Theorems 5.10 and 6.3 of [10] in more detail.

4.1. An algebraic framework. Crucial to Theorem 2.2 is the following theorem of the second two authors which demonstrates that Δ^0_3 Extension Lemmas are central to our understanding of the automorphism group of $\mathcal{E}^*$.

THEOREM 4.1 (Theorem 6.3 of [10]). *Assume D and $\hat{D}$ are automorphic via Ψ. Then D and $\hat{D}$ are automorphic via Θ where $\Theta \restriction \mathcal{E}(D)$ is Δ^0_3.*

Theorem 4.1 says that *inside* any automorphism can be thought of as Δ^0_3. The proof of this result and others we will need relies heavily on the framework of the second two authors who have recast the idea of an Extension Lemma *algebraically* so that the dynamic notions of entry states and matching are replaced by *extendible Boolean algebras* and *supports*. In particular, these proofs rely on Theorem 4.5. We will briefly discuss these methods of Cholak and Harrington.

Fix a c.e. set A. Then the structure $\mathcal{S}(A) = \{B : \exists C(C \sqcup B = A)\}$, the Boolean algebra of (c.e.) *splits* of A. Let $\mathcal{R}(A) = \{R : R \subseteq A$ and R computable$\}$, with $\mathcal{S}_R(A)$ the quotient of $\mathcal{S}(A)$ by $\mathcal{R}(A)$, and $=^R, \subseteq^R$ the corresponding quotient relations. It is proven in Cholak and Harrington [10] that $\mathcal{S}_R(A)$ is always a Σ^0_3 Boolean algebra. Ones that have representations of low complexity are especially important to us. A uniformly computable listing $\mathcal{S} = \{S_i : i \in \omega\}$ of splits of A is called an *effective listing of splits* of A iff there exists another uniformly computable listing of splits of A, $\{\hat{S}_i : i \in \omega\}$, with $S_i \sqcup \hat{S}_i = A$ for all i. The idea here is based around the fact that those elements x in W_e and then later enter A give rise to effective listings of splits.

DEFINITION 4.2 (Cholak and Harrington [10]). A Σ^0_3 subalgebra $\mathcal{B}$ of $\mathcal{S}_R(A)$ is called *extendible* iff there exists a representation $\mathcal{S}$ and B of $\mathcal{B}$ such that $\mathcal{S}$ is an effective listing of splits of A and B is a Δ^0_3 set.

Again following Cholak and Harrington [10], we consider a partial map Θ between splits of A and splits of $\hat{A}$ (for general sets A and $\hat{A}$) to be an isomorphism between a substructure $\mathcal{B}$ of $\mathcal{S}_R(A)$ and a substructure $\hat{B}$ of $\mathcal{S}_R(\hat{A})$, iff Θ preserves $\subseteq^R$, for each equivalence class S_R of $\mathcal{B}$, if $S \in S_R$ then $\Theta(S)$ exists, and for each equivalence class $\hat{S}_R$ of $\hat{B}$, $\Theta^{-1}(\hat{S})$ exists for all $\hat{S} \in \hat{S}_R$. Then two extendible algebras $\mathcal{B}$ and $\hat{B}$ are *extendibly isomorphic* via Θ iff

1. There is an effective listing $\mathcal{S}$ and B witnessing that $\mathcal{B}$ is an extendible algebra.
2. There is an effective listing $\hat{S}$ and $\hat{B}$ witnessing that $\hat{\mathcal{B}}$ is an extendible algebra.

3. For all $i \in B$, there is a $j \in \hat{B}$, with $\Theta(S_i) = \hat{S}_j$, and
4. For each $j \in \hat{B}$, there is an $i \in B$ with $\Theta(S_i) = \hat{S}_j$.
5. The partial map Θ' indiced by Θ describes an isomorphism between $\mathcal{B}$ and $\hat{\mathcal{B}}$, as above.

The first algebraic version of the Extension Lemma is the following.

THEOREM 4.3 (Cholak and Harrington [10]). *Let $\mathcal{B} \subseteq \mathcal{S}_R(A)$ and $\hat{\mathcal{B}}_R \subseteq \mathcal{S}_R(\hat{A})$ be two extendible Boolean algebras, which are Δ^0_3 extendibly isomorphic via Θ. Then there is a Φ which is a Δ^0_3 isomorphism between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$ such that for all $i \in B$, $\Phi(S_i) =_R \Theta(S_i)$, and for all $i \in \hat{B}$, $\Phi^1(\hat{S}_i) =_R \Theta^{-1}(\hat{S}_i)$.*

The key idea here is that it is possible to extend the extendible isomorphism between $\mathcal{B}$ and $\hat{\mathcal{B}}$ to an isomorphism between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$. Of course, nothing comes without price, and the *proof* of this result (and similar ones) relies on dynamic extension lemmas, of one type or another. In the paper Cholak and Harrington [10], Theorem 4.3 is proven using a modification of Cholak's *Translation Theorem*, Cholak [3].

While it is not directly pertinent to the present paper, we point out how Cholak and Harrington applied theorems like Theorem 4.3 using the idea of supports. This notion is related to the relationship between $\mathcal{L}^*(A)$ and $\mathcal{B}$, in some sense focusing on the relationship between the outside and the inside. We say a c.e. set S *supports* X if $S \subseteq X$ and $(X - A) \sqcup S$ is c.e. For example, $W_e \searrow A$, the elements of W_e which begin outside of A_s and then later enter A support W_e. More generally, an extendible algebra $\mathcal{B}$ supports a substructure $\mathcal{L}$ of $\mathcal{L}^*(A)$ (a subcollection of $\{W_e \cup A, \cup\}$ modulo finite sets), if for all $W \in \mathcal{L}$ there is an $i \in B$ with S_i supporting W.

DEFINITION 4.4 (Cholak and Harrington [10]). Assume that $\mathcal{L}^*(A)$ and $\mathcal{L}^*(\hat{A})$ are isomorphic via Ψ, $\mathcal{B}$ and $\hat{\mathcal{B}}$ are extendible algebras isomorphic via Θ, $\mathcal{B}$ supports $\mathcal{L}$ and $\hat{\mathcal{B}}$ supports $\hat{\mathcal{L}}$. Then we say that Ψ and Θ *preserve the supports* if for $W \in \mathcal{L}$, there is an $i \in B$ such that S_i supports W and $(\Psi(W \cup A) - \hat{A}) \sqcup \Theta(S_i)$ is c.e., and for all $\hat{W} \in \hat{\mathcal{L}}$, there is an $i \in \hat{\mathcal{B}}$ with $\hat{S}_i$ supporting $\hat{W}$ and $\Psi^{-1}(\hat{W} \cup \hat{A}) - A) \sqcup \Theta^{-1}(\hat{S}_i)$ is c.e.

Then one algebraic version of the Extension Lemma is the following.

THEOREM 4.5 (Cholak and Harrington [10]). *Assume that $\mathcal{L}^*(A)$ and $\mathcal{L}^*(\hat{A})$ are isomorphic via Ψ, $\mathcal{B}$ and $\hat{\mathcal{B}}$ are extendible algebras isomorphic via Θ, $\mathcal{B}$ supports $\mathcal{L}^*(A)$ and $\hat{\mathcal{B}}$ supports $\mathcal{L}^*(\hat{A})$, with Ψ and Θ preserving supports.*

Then there is an automorphism Λ of $\mathcal{E}^*$ with $\Lambda(A) = \hat{A}$, $\Lambda \restriction \mathcal{L}^*(A) = \Psi$, and such that $\Lambda \restriction \mathcal{E}^*(A)$ is Δ^0_3.

For reasons which become clear later, one final result from Cholak and Harrington's paper that we will need concerns extendible algebras of computable sets. An extendible algebra $\mathcal{B}$ of $\mathcal{S}_R(\omega)$ is called an *extendible algebra of computable sets*, as the splits of ω are computable sets.

THEOREM 4.6 (Theorem 5.10 of [10]). *Let $\mathcal{B}$ be an extendible algebra of computable sets and similarly for $\hat{\mathcal{B}}$. Assume the two are extendibly isomorphic via Π. Then there is a Φ such that Φ is a Δ^0_3 isomorphism between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$, Φ maps computable subsets to computable subsets, and, for all $R \in \mathcal{B}$, $(\Pi(R) - \hat{A}) \sqcup \Phi(R \cap A)$ is computable (and dually).*

4.2. Some algebraic orbits. In [10], Cholak and Harrington use Theorem 4.5 to give algebraic proofs of many known theorems from the literature, such as the maximal and hemimaximal results. (Additionally they use the algebraic methods for new results such as new orbits.) We remark that the algebraic view does go back to Herrmann's proof that a certain class of sets (now called *Hermann sets*) were automorphic. This result was proven using a hitherto unobserved *algebraic* consequence of the original Soare paper about the preservation of computable sets under Soare's construction. Given that is not well understood and is a critical proof of Theorem 2.2, we would like to explore these algebraic proofs with some more detail.

DEFINITION 4.7. $\mathcal{C}(A)$ is the set of W_e such that either $\overline{A} \subseteq W_e$ or $W_e \subseteq^* A$.

THEOREM 4.8 (Soare's Automorphism Theorem [35]). *Let A and $\hat{A}$ be two noncomputable computably enumerable sets.*

1. *Then there is a Δ^0_3 isomorphism Λ between $\mathcal{E}(A) \cup \mathcal{C}(A)$ and $\mathcal{E}(\hat{A}) \cup \mathcal{C}(\hat{A})$. Furthermore, a Δ^0_3-index for Λ can be found uniformly from indexes for A and $\hat{A}$.*
2. *In addition, Λ preserves the computable subsets of A.*

[35] explicitly stated Theorem 4.8.1. Theorem 4.8.2 was observed, in unpublished work, by Herrmann. Assume that R is a computable subset of A. Herrmann's observation was that $\overline{R} \in \mathcal{C}(A)$ and hence $\Lambda(R) \sqcup \Lambda(\overline{R}) =^* \hat{\omega}$ and therefore Λ maps R to a computable subset of $\hat{A}$. This observation of Herrmann was never published and is one of the key facts he used in showing that the Herrmann sets form an orbit; see [8].

THEOREM 4.9. [35] *The maximal sets form an orbit.*

PROOF. Assume that A and $\hat{A}$ are maximal. Then $\mathcal{C}(A) = \mathcal{E}$. If $W \subseteq A$ then let $\Psi(W) = \Lambda(W)$. If $W \cup A =^* \omega$ there is a computable set R_W such that $R_w \subseteq^* A$ and $\overline{R_W} \subseteq^* A$ and then let $\Psi(W) = \Lambda(W \cup R_W) \sqcup \overline{\Lambda(R_W)}$. It is not difficult to show Ψ is an automorphism. ⊣

Recall that set is *hemimaximal* iff it is the nontrivial split of a maximal set.

THEOREM 4.10. [17] *The hemimaximal sets form an orbit.*

PROOF. Assume $A_1 \sqcup A_2 = A$ where the A_is are not computable and A is maximal. Dually for $\hat{A}$. Assume that Θ_i is an isomorphism from $\mathcal{E}^*(A_i)$ to $\mathcal{E}^*(\hat{A}_i)$ that preserves the computable subsets (from Theorem 4.8).

As with the maximal sets, it is enough to define an isomorphism Λ between $\mathcal{E}^*(A)$ and $\mathcal{E}^*(\hat{A})$ preserving the computable subsets. If $X \subseteq^* A$ then let $\Lambda(X) = \Theta_1(X \cap A_1) \sqcup \Theta_2(X \cap A_2)$. Let $R \in \mathcal{R}(A)$. Then $R \cap A_i$ is computable. So $\Theta_i(R \cap A_i)$ is computable. Hence $\Theta_1(R \cap A_1) \sqcup \Theta_2(R \cap A_2)$ is computable. The complexity of the resulting automorphism is Δ^0_3. ⊣

Definition 4.11. We say that a c.e. set H is *strongly r-separable* if, for all c.e. sets W disjoint from H, there is a computable set R such that $W \subset R$, $H \subset \overline{R}$, and $R - W$ is infinite. We say that a set H is *Herrmann* if it is both $\mathcal{D}$-maximal and strongly r-separable.

Theorem 4.12 (Herrmann, see [8]). *The Herrmann sets form an orbit (under Δ^0_3 automorphisms).*

Proof. Let H be a Herrmann set. Since H is $\mathcal{D}$-maximal for all W there is a W_W such that either $W \subseteq^* H \sqcup W_W$ or $\overline{W} \subseteq^* H \sqcup W_W$. Furthermore, since H is Herrmann, for all W, there is a computable R_W such that either $W \subseteq^* H \sqcup R_W$ or $\overline{W} \subseteq^* H \sqcup R_W$. Note that finding R_W and determining which case holds can be done using an oracle computable in $\mathbf{0}''$.

Assume $\overline{W} \subseteq^* H \sqcup R_W$. Then $W \cup H \sqcup R_W =^* \omega$. Therefore $(W \cap \overline{R_W}) \cup H =^* \overline{R}_W$. Recall that $X \backslash Y = \{x | \exists s(x \in X_s - Y_s)\}$. Hence $((W \cap \overline{R}_W)\backslash H) \sqcup (H\backslash(W \cap \overline{R}_W)) =^* \overline{R}_W$. Thus there is a computable subset $R_{H,W} = H\backslash(W \cap \overline{R}_W)$ of H such that

$$W =^* (W \cap R_{H,W}) \sqcup (\overline{R}_{H,W} \cap \overline{R}_W) \sqcup (W \cap R_W). \tag{4.12.1}$$

Again note that finding $R_{H,W}$ can be done using an oracle computable in $\mathbf{0}''$.

Now using $\mathbf{0}''$ find a pairwise disjoint collection of R_i such that $R_i \cap H = \emptyset$ and, for all e, $R_{W_e} \subseteq \bigsqcup_{i \leq e} R_e$. Since H is Herrmann it is possible to find such a collection. Do the same for $\hat{H}$.

Use Theorem 4.8 to get Λ mapping $\mathcal{E}(H)$ to $\mathcal{E}(\hat{H})$. Let p_i be a computable one-to-one onto map from R_i to $\hat{R}_i$ and, for $W \subseteq^* R_i$, let $\Lambda_i(W) = p_i(W)$. All of these maps take computable subsets to computable subsets.

Now we will work on defining our automorphism Φ. First assume $W \subseteq^* H \sqcup \bigsqcup R_i$. Let $\Phi(W) = \Lambda(H \cap W) \sqcup \bigsqcup \Lambda_i(R_i \cap W)$. It is not hard to show for Φ, as defined so far, is order-preserving, for all W, $\Phi(W)$ is an r.e. set, and if W is computable so is $\Phi(W)$.

Now consider the case when $W \not\subseteq^* H \sqcup \bigsqcup R_i$. In that case, Equation 4.12.1 holds, and we can use that to define $\Phi(W)$ in terms of subsets of $H \sqcup \bigsqcup R_i$:

$$\Phi(W) =^* \Phi(W \cap R_{H,W}) \sqcup (\overline{\Phi(R_{H,W})} \cap \overline{\Phi(R_W)}) \sqcup \Phi(W \cap R_W).$$

As defined $\Phi(W)$ is an r.e. set. It is not difficult to show Φ is order preserving and hence well-defined. Thus Φ is an automorphism taking H to $\hat{H}$. ⊣

For more on preserving the computable sets and an algebraic proof of Theorem 4.8 we refer the reader to [10].

§5. **A sketch of the proof of Theorem 2.2.** The proof of Theorem 2.2 is quite complex and involves several ingredients. The proof will be easiest to understand if we introduce each of the relevant ingredients in context.

The following theorem will prove to be useful.

THEOREM 5.1 (Folklore[4]). *There is a computable listing T_i of computable infinite branching trees and a computable infinite branching tree $T_{\Sigma^1_1}$ such that the set $\{i : T_{\Sigma^1_1} \cong T_i\}$ is Σ^1_1-complete.*

The idea for the proof of Theorem 2.2 is to code each of the above T_is into the orbit of A_{T_i}. Informally let $\mathcal{T}(A_T)$ denote this encoding; $\mathcal{T}(A_T)$ will not be defined in this announcement. But we will discuss in some details some of the ingredients and resulting complexity. The game plan is as follows:

1. *Coding*: For each T build an A_T such that $T \cong \mathcal{T}(A_T)$ via an isomorphism $\Lambda \leq_T \mathbf{0}^{(2)}$.
2. *Coding is preserved under automorphic images*: If $\hat{A} \approx A_T$ via an automorphism Φ then $\mathcal{T}(\hat{A})$ exists and $\mathcal{T}(\hat{A}) \cong T$ via an isomorphism Λ_Φ, where $\Lambda_\Phi \leq_T \Phi \oplus \mathbf{0}^{(2)}$.
3. *Sets coding isomorphic trees belong to the same orbit*: If $T \cong \hat{T}$ via isomorphism Λ then $A_T \cong A_{\hat{T}}$ via an automorphism Φ_Λ where $\Phi_\Lambda \leq_T \Lambda \oplus \mathbf{0}^{(2)}$.

Thus $A_{T_{\Sigma^1_1}}$ and A_{T_i} are in the same orbit iff $T_{\Sigma^1_1}$ and T_i are isomorphic. Since the latter question is Σ^1_1-complete so is the former question.

We will build a pairwise disjoint collection of D_χs to code the tree T. $A_T = D_\lambda$ will code the empty node in T. The basic module to construct a D involves the construction of a computable set R and a subset M of R. There will be infinitely many pairwise disjoint Rs. Inside Rs the Ds will be Friedberg splits of M. It is well understood how to split an r.e. set M into Friedberg splits.

Depending on the construction either $M =^* R$ or M is maximal inside R; i.e., $M \cup \overline{R}$ is maximal. If M is maximal inside R then D is hemimaximal inside R and we say D *lives* in R. If $M =^* R$ then, inside R, D is a computable set and D does not live in R. Note that D living in R is a definable property as is D is computable in R.

The big issue of the construction will be to decide when D lives in R and when not. It is well understood how to construct a maximal set inside R. Furthermore, it is also well understood how to use a dumping argument to alter the maximal set construction to force $M =^* R$. We will use a dumping construction to alter the construction of the desired M. The decision whether D lives in R will be handled by a tree argument. Here we will not discuss the tree argument but discuss issues that go into deciding whether D lives in R.

Each node χ of T will have infinitely many pairwise disjoint $R_{\chi,i}$ associated with χ. For these $R_{\chi,i}$ the corresponding set is $M_{\chi,i}$. D_χ will always be a

[4]See Section 5 of the full paper for more information and a proof.

Friedberg split of $M_{\chi,i}$. If χ^+ is a successor of χ in T then, for almost all $R_{\chi,i}$, D_{χ^+} will be a Friedberg split of $M_{\chi,i}$. The collection of all $R_{\chi,i}$ will be pairwise disjoint. When constructed in this fashion the D_χ code T and hence we have part 1 of the game plan under control.

We have to work on part 2 and 3 of the game plan. For part 3 we are going to set things up so that if we know where the D_χ and $R_{\chi,i}$ go then we will be able to construct the desired automorphism.

Let's look at the hatted side of the construction briefly to work on Part 2. We are told A_T goes to $\hat{A}$. We can gather together a collection of pairwise disjoint computable sets $\hat{R}_{\lambda,i}$ such that either $\hat{A}$ lives in $\hat{R}_{\lambda,i}$ or $\hat{A}$ is computable inside $\hat{R}_{\lambda,i}$. We can assume that this list is maximal; i.e., if $\hat{A}$ lives in $\hat{R}$ or is computable in $\hat{R}$, then $\hat{R} \subseteq \bigsqcup \hat{R}_{\lambda,i}$. There may be many such lists. But we can show modulo a computable set each $R_{\lambda,i}$ must be sent to some $\hat{R}_{\lambda,j}$.

Now we are in position to pick out the successors of $\hat{A}$ on the hatted side. They are the sets $\hat{D}_{\lambda^+}$ which live into almost all of $\hat{R}_{\lambda,i}$ that $\hat{A}$ lives in. More or less these sets must be the automorphic images of the $D_{\lambda^+,i}$s. We better ensure that these sets behave like a good successor. The only way to do this is to control how the $D_{\lambda^+,i}$s behave.

Any set D which looks like a D_χ must be a split of one of the D_χs. If, for infinitely many χ and i, D lives in $R_{\chi,i}$ then there must be exactly one χ such that, for all i, D lives in $R_{\chi,i}$ iff D_χ lives in $R_{\chi,i}$ and for almost all i, D lives in $R_{\chi^-,i}$ iff D_{χ^-} lives in $R_{\chi^-,i}$. In this case D will be a Friedberg split of D_χ.

If we can get the D to behave properly (as discussed above) then their automorphic images must also behave properly. So the $\hat{D}_{\lambda^+}$ which we have found above in fact code the successors of λ in $\hat{T}$. Once we have this we can find their successors.

For each $\lambda^+ = \hat{\chi}$ we can find a collection of pairwise disjoint computable sets $\hat{R}_{\hat{\chi},i}$ such that either $\hat{D}_{\hat{\chi}}$ lives in $\hat{R}_{\hat{\chi},i}$ or $\hat{D}_{\hat{\chi}}$ is computable inside $\hat{R}_{\hat{\chi},i}$. We can assume that this list is maximal; i.e., if $\hat{D}_{\hat{\chi}}$ lives in $\hat{R}$ or is computable in $\hat{R}$, then $\hat{R} \subseteq \bigsqcup \hat{R}_{\hat{\chi},i} \sqcup \bigsqcup \hat{R}_{\lambda,i}$. There may be many such lists. Given that $\hat{D}_{\hat{\chi}}$ does code a successor of $\hat{A}$, there is some node χ of length 1 such that modulo a computable set each $R_{\chi,i}$ must be sent to some $\hat{R}_{\hat{\chi},j}$. Now we can bootstrap our way to find a successor of $\hat{D}_{\hat{\chi}}$ and so on. Hence part 2 is now under control.

So, for each possible D, we will make sure that either D is hemimaximal inside finite many $R_{\chi,i}$ or D behaves as above; i.e., D will be a Friedberg split of some D_χ. To do this we will use a coherence/state argument not unlike the argument used to construct maximal sets. The state of $R_{\chi,i}$ will be those D which D is hemimaximal in $R_{\chi,i}$. Determining the state of a set is Σ^0_3 rather than Σ^0_1. Hence this is another reason we must do this whole construction on a tree. If $R_{\chi,i}$ is in a low e-state then we must dump it. We dump $R_{\chi,i}$ by making $M_{\chi,i} =^* R_{\chi,i}$ as discussed above.

As for the collection of all $R_{\chi,i}$ we want them to have the property that for all W, either W is a split of the D_χs or there is a finite set F of χ and i such that either $W \subseteq^* \bigsqcup_{(\chi,i)\in F} R_{\chi,i}$ or $\overline{W} \subseteq^* \bigsqcup_{(\chi,i)\in F} R_{\chi,i}$. This cannot be achieved via an effective construction but is achievable on a tree. This has the side effect that there will be more sets $R_{\chi,i}$ where $M_{\chi,i} =^* R_{\chi,i}$.

We will also construct that $R_{\chi,i}$ such that for all χ, the $R_{\chi^-,i}$ and $R_{\chi,i}$ form an extendible algebra of computable sets, $\mathcal{B}_\chi$. We wish to use Theorem 4.6. For each χ, the map $R_{\chi,i}$ to $\hat{R}_{\hat{\chi},i}$ is an extendible isomorphism. We want to claim that we can use these pieces and an isomorphism between T and $\hat{T}$ to construct an automorphism not unlike what we did in Section 4.2 and, in particular, with the proof of Theorem 4.12. With one caveat this is the case.

The caveat is that if χ and $\hat{\chi}$ have the same length we want to ensure that D_χ lives in $R_{\chi,i}$ iff $\hat{D}_{\hat{\chi}}$ lives in $\hat{R}_{\hat{\chi},i}$. This requires two more additions to the above description. First we must construct all A_Ts using the same tree construction. Second the coherence/state argument above must be extended so that the above homogeneity is preserved. So if $M_{\chi,i} =^* R_{\chi,i}$ then for all $\hat{\chi}$ of the same length as χ, $M_{\hat{\chi},i} =^* R_{\hat{\chi},i}$.

5.1. Invariants. It might appear that $\mathcal{T}(A)$ is an invariant which determines the orbit of A. But there is no reason to believe for an arbitrary A that $\mathcal{T}(A)$ is well defined. The following theorem shows that $\mathcal{T}(\hat{A})$ is an invariant as far as the orbits of the A_Ts are concerned. In the full paper, we prove a more technical version of the following theorem.

THEOREM 5.2. *If $\hat{A}$ and A_T are automorphic via Ψ and $T \cong \mathcal{T}(\hat{A})$ via Λ then $A_T \approx \hat{A}$ via Φ_Λ where $\Phi_\Lambda \leq_T \Lambda \oplus \mathbf{0}^{(8)}$.*

SKETCH. For A_T the above construction gives us a $\mathbf{0}''$ (they are constructed on the true path) listing of the sets D_χ, $R_{\chi,i}$, and $M_{\chi,i}$. So they are available for us to use here. The idea is to recover images of these sets on the hatted side. This recovery relies on Theorem 6.3 of [10] that if D and $\hat{D}$ are automorphic via Ψ, then D and $\hat{D}$ are automorphic via Θ where $\Theta \restriction \mathcal{E}(D)$ is Δ^0_3, as well as more intricate material on extendible algebras (in particular, careful application of Theorem 5.10 of [10]). Then using these recovered sets we construct the desired automorphism as hinted to above. It takes $\Lambda \oplus \mathbf{0}^{(8)}$ to recover the needed sets. The construction of the automorphism needs the recovered sets and an oracle for $\mathbf{0}^{(2)}$. ⊣

5.2. Properly Δ^0_α orbits.

THEOREM 5.3 (Folklore[5]). *For all finite α there is a computable tree T_{i_α} from the list in Theorem 2.6 such that, for all computable trees T, T and T_{i_α} are isomorphic iff T and T_{i_α} are isomorphic via an isomorphism computable in deg$(T) \oplus 0^{(\alpha)}$. But, for all $\beta < \alpha$ there is an i^*_β such that $T_{i^*_\beta}$ and T_{i_α} are isomorphic but are not isomorphic via an isomorphism computable in $0^{(\beta)}$.*

[5]See Section 5 of the full paper [7] for more information and a proof.

It is open if the above theorem holds for all α such that $\omega \geq \alpha < \omega_1^{\mathrm{CK}}$. But if it does then so does the theorem below.

THEOREM 5.4. *For all finite $\alpha > 8$ there is a properly Δ^0_α orbit.*

PROOF. Assume that $A_{T_{i_\alpha}}$ and $\hat{A}$ are automorphic via an automorphism Φ. Hence, by part 2 of the game plan, $\mathcal{T}(\hat{A})$ and T_{i_α} are isomorphic. Since $\mathcal{T}(\hat{A})$ is computable in $0^{(8)}$, $\alpha > 8$, and by Theorem 5.3, $\mathcal{T}(\hat{A})$ and T_{i_α} via a $\Lambda \leq_T 0^{(\alpha)}$. By Theorem 5.2, $\hat{A}$ and $A_{T_{i_\alpha}}$ are automorphic via an automorphism computable in $0^{(\alpha)}$.

Fix β such that $8 \geq \beta < \alpha$. By part 3 of the game plan and the above paragraph, $A_{T_{i_\alpha}}$ and $A_{T_{i^*_\beta}}$ are automorphic via an automorphism computable in $0^{(\alpha)}$. Now assume $A_{T_{i^*_\beta}} \approx A_{T_{i_\alpha}}$ via Φ. By part 2 of the game plan, $\mathcal{T}(A_{T_{i^*_\beta}}) \cong T_{i_\alpha}$ via Λ_Φ, where $\Lambda_\Phi \leq_T \Phi \oplus \mathbf{0^{(2)}}$. Since $\mathcal{T}(A_{T_{i^*_\beta}})$ is computable in $0^{(8)}$ and $\mathcal{T}(A_{T_{i^*_\beta}})$ is isomorphic to $T_{i^*_\beta}$ via an isomorphism computable in $0^{(\beta)}$ (part 1 of the game plan), by Theorem 5.3, $\Lambda_\Phi >_T 0^{(\beta)}$. Hence $\Phi >_T 0^{(\beta)}$. ⊣

5.3. Our orbits and hemimaximal degrees. Recall that a set is *hemimaximal* iff it is the nontrivial split of a maximal set. A degree is *hemimaximal* iff it contains a hemimaximal set.

Let T be given. Construct A_T as above. For all i, either A_T is hemimaximal in R_i or $A_T \cap R_i$ is computable. If A_T is hemimaximal in R_i then $A_T \cap R_i$ is a split of maximal set $M \sqcup \overline{R}_i$ and hence $A_T = (A_T \cap R_i)$ is a hemimaximal set. $A_T = \bigsqcup_{i \in \omega}(A_T \cap R_i)$ where $A_T \cap R_i$ is either hemimaximal or computable. So the degree of A_T is the infinite join of hemimaximal degrees. It is not known if the (infinite) join of hemimaximal degrees is hemimaximal. Moreover, this is not an effective infinite join. But if we control the degrees of $A_T \cap R_i$ we can control the degree of A_T. By modifying our proofs we can achieve the following degree controls.

THEOREM 5.5. *Let H be hemimaximal. We can construct A_T such that $A_T \equiv_T H$.*

Indeed, we can show the following.

THEOREM 5.6. *There is an A_T whose orbits contain a representative of every hemimaximal degree, and hence of all jump classes.*

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