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ON THE COMPUTATIONAL COMPLEXITY OF THE NUMERICALLY DEFINITE SYLLOGISTIC AND RELATED LOGICS

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Abstract. The *numerically definite syllogistic* is the fragment of English obtained by extending the language of the classical syllogism with numerical quantifiers. The *numerically definite relational syllogistic* is the fragment of English obtained by extending the numerically definite syllogistic with predicates involving transitive verbs. This paper investigates the computational complexity of the satisfiability problem for these fragments. We show that the satisfiability problem (= finite satisfiability problem) for the numerically definite syllogistic is strongly NP-complete, and that the satisfiability problem (= finite satisfiability problem) for the numerically definite relational syllogistic is NEXPTIME-complete, but perhaps not strongly so. We discuss the related problem of probabilistic (propositional) satisfiability, and thereby demonstrate the incompleteness of some proof-systems that have been proposed for the numerically definite syllogistic.

§1. Introduction.

Inspection of the argument

$$\begin{array}{l} \textsf{At least 13 artists are beekeepers} \\ \textsf{At most 3 beekeepers are carpenters} \\ \underline{\textsf{At most 4 dentists are not carpenters}} \\ \textsf{At least 6 artists are not dentists} \end{array} \tag{1}$$

shows it to be *valid*: any circumstance in which all the premises are true is one in which the conclusion is true. For suppose the premises are true. Take any collection of thirteen artists who are beekeepers; since at most three of these may be carpenters, at least ten must be non-carpenters; and since, of these ten, at most four may be dentists, at least six must be non-dentists.

Considerably more thought shows the argument

$$\begin{array}{l} \textsf{At most 1 artist admires at most 7 beekeepers} \\ \textsf{At most 2 carpenters admire at most 8 dentists} \\ \textsf{At most 3 artists admire at least 7 electricians} \\ \textsf{At most 4 beekeepers are not electricians} \\ \textsf{At most 5 dentists are not electricians} \\ \underline{\textsf{At most 1 beekeeper is a dentist}} \\ \textsf{At most 6 artists are carpenters} \end{array} \tag{2}$$

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to be likewise valid—assuming, that is, that the quantified subjects in these sentences scope over their respective objects. For suppose to the contrary that the premises are true, but the conclusion false. By the negation of the conclusion, take any collection of seven artists who are carpenters; by the first two premises, at least four of these seven must admire eight or more beekeepers and nine or more dentists; and by the third premise, at least one of these four satisfies the additional property of admiring at most six electricians. Let A be such an artist, then, and consider any set of eight beekeepers and any set of nine dentists admired by A: by the fourth and fifth premises, respectively, at least four of these beekeepers and four of these dentists must be electricians. But since A admires only six electricians altogether, these sets of four beekeepers and four dentists must overlap by at least two, which contradicts the final premise.

Argument (1) is couched in a fragment of English obtained by extending the language of the classical syllogism with numerical quantifiers. Such arguments were treated at length by de Morgan [4] (Ch. VIII), who presented a list of what he took to be the valid numerical syllogisms. Adapting de Morgan's terminology slightly, we shall call this fragment the *numerically definite syllogistic*. Argument (2) is couched in a fragment of English obtained by extending the numerically definite syllogistic with predicates involving transitive verbs. We might call this fragment the *numerically definite relational syllogistic*. (Precise definitions are given below.)

Surprisingly, de Morgan seems to have been the first to investigate the numerically definite syllogistic, though his account was followed by Boole [1] (reprinted as [2], Sec. IV) and Jevons [9], (reprinted as [10], Part I, Sec. IV). Boole's rather peculiar methodology of trying to apply familiar algebraic techniques directly to logical problems resulted in an opaque and more or less unusable account; Jevons' approach, by contrast, which was based on the transformation of logical problems into equivalent systems of equations, is recognizably related the analysis of the present paper. For a brief historical overview, see [7], pp. 25–63.

From a modern perspective, we may formalize Argument (1) in the language of first-order logic, extended with counting quantifiers, as

$$\begin{array}{l} \exists_{\geq 13}x(\text{artist}(x) \wedge \text{beekeeper}(x)) \\ \exists_{\leq 3}x(\text{beekeeper}(x) \wedge \text{carpenter}(x)) \\ \exists_{\leq 4}x(\text{dentist}(x) \wedge \neg\text{carpenter}(x)) \\ \hline \exists_{\geq 6}x(\text{artist}(x) \wedge \neg\text{dentist}(x)). \end{array} \tag{3}$$

The notion of validity for such arguments may then be reconstructed in terms of the standard apparatus of validity for first-order sequents, with the counting quantifiers $\exists_{\geq C}$ and $\exists_{\leq C}$ given the obvious semantics. Note that the use of counting quantifiers enables us to restrict attention to formulas featuring only a single variable x. That is to say: the formulas in question lie in the *one-variable fragment of first-order logic with counting quantifiers*. Argument (2) may be formalized along similar lines. For example, its first premise would be rendered as

$$\exists_{\leq 1} x(\text{artist}(x) \wedge \exists_{\leq 7} y(\text{beekeeper}(y) \wedge \text{admire}(x, y)));$$

and similarly for the other propositions involved. As the reader may readily verify, all of the resulting formulas feature at most two distinct variables—say, x and y. That is to say: they lie in the *two-variable fragment of first-order logic with counting quantifiers*. The satisfiability problem for this latter fragment is known to be decidable. From this perspective, then, the most salient question that confronts us regarding the numerically definite syllogistic and its relational counterpart is the computational complexity of the problem of determining the validity of arguments within them. That question forms the subject of this paper.

Technically, the main results reported below are: (i) the satisfiability problem (= finite satisfiability problem) for any logic between the numerically definite syllogistic and the one-variable fragment of first-order logic with counting quantifiers is strongly NP-complete; and (ii) the satisfiability problem and finite satisfiability problem for any logic between the numerically definite relational syllogistic and the two-variable fragment of first-order logic with counting quantifiers are both NEXPTIME-complete, but perhaps not strongly so. We discuss the related problem of *probabilistic* (propositional) satisfiability, and use the results of this investigation to demonstrate the incompleteness of some proof-systems that have been proposed for the numerically definite syllogistic.

§2. Preliminaries. In the sequel, we employ first-order logic extended with the *counting quantifiers* $\exists_{\leq C}$, $\exists_{\geq C}$ and $\exists_{=C}$, for any $C \geq 0$, under the obvious semantics. Note that, in this language, $\exists x\phi$ is logically equivalent to $\exists_{\geq 1} x\phi$, and $\forall x\phi$ is logically equivalent to $\exists_{\leq 0} x \neg\phi$. The *one-variable fragment with counting quantifiers*, here denoted $\mathcal{C}^1$, is the set of function-free first-order formulas featuring at most one variable, but with counting quantifiers allowed. We assume for simplicity that all predicates in $\mathcal{C}^1$ have arity at most 1.

We define the fragment $\mathcal{N}^1$ to be the set of $\mathcal{C}^1$-formulas of the forms

$$\begin{array}{ll} \exists_{\geq C} x(p(x) \wedge q(x)) & \exists_{\geq C} x(p(x) \wedge \neg q(x)) \\ \exists_{\leq C} x(p(x) \wedge q(x)) & \exists_{\leq C} x(p(x) \wedge \neg q(x)), \end{array} \tag{4}$$

where p and q are unary predicates. Linguistically, we think of unary predicates as corresponding to common nouns, and the formulas (4) to English sentences of the forms

$$\begin{array}{ll} \textsf{At least } C \ p \textsf{ are } q & \textsf{At least } C \ p \textsf{ are not } q \\ \textsf{At most } C \ p \textsf{ are } q & \textsf{At most } C \ p \textsf{ are not } q, \end{array} \tag{5}$$

respectively. (We have simplified the presentation here by ignoring the issue of singular/plural agreement; this has no logical or computational significance, and in the sequel, we silently correct any resulting grammatical infelicities.) We call the fragment of English defined by these sentence-forms the *numerically definite syllogistic*.

The sentence Some p are q may be equivalently written At least 1 p is a q, and the sentence All p are q may be equivalently—if somewhat unidiomatically—written At most 0 p are not q. Thus, the numerically definite syllogistic generalizes the ordinary syllogistic familiar from logic textbooks. Furthermore, the sentence There are at least C p may be equivalently written At least C p are p; and similarly for There are at most C p. There is a convention in the classical tradition of taking the sentences Every p is a q and No p is a q to imply that there exists some p. We do not adopt this convention.

We will have occasion below to extend $\mathcal{N}^1$ slightly. Let $\mathcal{N}^{1+}$ consist of $\mathcal{N}^1$ together with the set of $\mathcal{C}^1$-formulas of the forms

$$\begin{array}{ll} \exists_{\geq C} x(\neg p(x) \wedge q(x)) & \exists_{\leq C} x(\neg p(x) \wedge q(x)) \\ \exists_{\geq C} x(\neg p(x) \wedge \neg q(x)) & \exists_{\leq C} x(\neg p(x) \wedge \neg q(x)). \end{array} \tag{6}$$

The formulas in the upper row are of course logically equivalent to $\mathcal{N}^1$-formulas, but those in the lower row are not, and correspond to slightly less natural English sentences with negated subjects and predicates, thus:

At least C non-p are not q At most C non-p are not q.

Turning now to Argument (2), we take the *two-variable fragment with counting quantifiers*, here denoted $\mathcal{C}^2$, to be the set of function-free first-order formulas featuring at most two variables, but with counting quantifiers allowed. We assume for simplicity that all predicates in $\mathcal{C}^2$ have arity at most 2. And we define the fragment $\mathcal{N}^2$ to be the set of $\mathcal{C}^2$-formulas consisting of $\mathcal{N}^1$ together with all formulas of the forms

$$\begin{array}{ll} \exists_{\geq C} x(p(x) \wedge \exists_{\geq D} y(q(y) \wedge r(x,y))) & \exists_{\geq C} x(p(x) \wedge \exists_{\leq D} y(q(y) \wedge r(x,y))) \\ \exists_{\leq C} x(p(x) \wedge \exists_{\geq D} y(q(y) \wedge r(x,y))) & \exists_{\leq C} x(p(x) \wedge \exists_{\leq D} y(q(y) \wedge r(x,y))), \end{array}$$

where p and q are unary predicates, and r is a binary predicate. (It is not permitted to reverse the order of variables in the binary atoms $r(x, y)$.) Linguistically, we think of binary predicates as corresponding to transitive verbs, and the above formulas to English sentences of the forms

$$\begin{array}{ll} \text{At least } C\ p\ r \text{ at least } D\ q & \text{At least } C\ p\ r \text{ at most } D\ q \\ \text{At most } C\ p\ r \text{ at least } D\ q & \text{At most } C\ p\ r \text{ at most } D\ q, \end{array} \tag{7}$$

respectively. (Again, we ignore the issue of singular and plural phrases.) Note that the sentence-forms in (7) may exhibit scope ambiguities; we have resolved these by stipulating that subjects always scope over objects. With this stipulation, we call the fragment of English defined by the sentence-forms (5) and (7) the *numerically definite relational syllogistic*.

We take it as uncontentious that the correspondence between (4) and (5) provides a rational reconstruction of the notion of validity for arguments in the numerically definite syllogistic: such an argument is valid just in case the corresponding $\mathcal{N}^1$-sequent is valid according to the usual semantics of first-order logic with counting quantifiers. Moreover, for every $\mathcal{N}^1$-formula, there is another $\mathcal{N}^1$-formula logically equivalent to its negation. Hence, the notion of validity for $\mathcal{N}^1$-sequents (with single consequent) is dual to the notion of satisfiability for sets of $\mathcal{N}^1$-formulas in the standard way. Similar remarks apply to $\mathcal{N}^2$ and the numerically definite relational syllogistic. We note in passing that the fragments $\mathcal{N}^1$ and $\mathcal{N}^2$ are 'flat', in the sense that they do not allow for nesting of the key logical constructions.

Let $\mathcal{L}$ be any logic. The *satisfiability problem* for $\mathcal{L}$ is the problem of determining whether a given finite set of $\mathcal{L}$-formulas is satisfiable (has a model); likewise, the *finite satisfiability problem* for $\mathcal{L}$ is the problem of determining whether a given finite set of $\mathcal{L}$-formulas is finitely satisfiable (has a finite model). A logic $\mathcal{L}$ is said to have the *finite model property* if every finite, satisfiable set of $\mathcal{L}$-formulas is finitely satisfiable. Thus, $\mathcal{L}$ has the finite model property just in case the satisfiability and finite satisfiability problems for $\mathcal{L}$ coincide. As usual, we take the *size* of any set Φ of $\mathcal{L}$-formulas to be the number of symbols in Φ, counting each occurrence of a logical connective or non-logical symbol as 1. (Technically, one is supposed to take into account *how many* non-logical symbols occur in Φ; but for the logics considered here, this would make no difference.) The computational complexity of the satisfiability problem and the finite satisfiability problem for $\mathcal{L}$ can then be understood in the normal way. Care is required, however, when the formulas of $\mathcal{L}$ contain numerical constituents, as is the case with the logics considered here. Under *unary coding*, a positive numerical constituent C is taken to have size C; under *binary coding*, by contrast, the same constituent is taken to have size $\lfloor \log_2 C \rfloor + 1$, in recognition of the fact that C can be encoded as a bit string without leading zeros. When giving upper complexity bounds, binary coding is the more stringent accounting method; when giving lower complexity bounds, unary coding is. In the sequel, binary coding will be assumed, unless it is explicitly stated to the contrary. A problem is sometimes said to be *strongly* NP-complete if it is NP-complete (under binary coding), and remains NP-hard even under unary coding; and similarly for other complexity classes.

In a logic with negation, a *literal* is an atomic formula or the negation of an atomic formula; in a logic with negation and disjunction, a *clause* is a disjunction of literals.

Henceforth, all logarithms have base 2.

§3. Complexity of logics between $\mathcal{N}^1$ and $\mathcal{C}^1$. In this section, we consider logics containing $\mathcal{N}^1$ but contained in $\mathcal{C}^1$.

Lemma 1. *The satisfiability problem for $\mathcal{N}^1$ is NP-hard, even under unary coding.*

Proof. If G is an undirected graph (no loops or multiple edges), a 3-*colouring* of G is a function t mapping the nodes of G to the set $\{0, 1, 2\}$ such that no edge of G joins two nodes mapped to the same value. We say that G is 3-*colourable* if a 3-colouring of G exists. The problem of deciding whether a given graph G is 3-colourable is well-known to be NP-hard. We reduce it to $\mathcal{N}^1$-satisfiability.

Let the nodes of G be $\{1, \ldots, n\}$. For all i $(1 \leq i \leq n)$ and k $(0 \leq k < 3)$, let p_i^k be a fresh unary predicate. Think of $p_i^k(x)$ as saying: "x is a colouring of G in which node i has colour k". Let Φ_G be the set of $\mathcal{N}^1$-formulas consisting of

$$\exists_{\leq 3} x (p(x) \wedge p(x)) \tag{8}$$

$$\{\exists_{\leq 0} x (p_i^j(x) \wedge p_i^k(x)) \mid 1 \leq i \leq n, 0 \leq j < k < 3\} \tag{9}$$

$$\{\exists_{\geq 1} x (p_i^k(x) \wedge p(x)) \mid 1 \leq i \leq n, 0 \leq k < 3\} \tag{10}$$

$$\{\exists_{\leq 0} x (p_i^k(x) \wedge p_j^k(x)) \mid (i, j) \text{ is an edge of } G, 0 \leq k < 3\}. \tag{11}$$

We prove that Φ_G is satisfiable if and only if G is 3-colourable.

Suppose $\mathfrak{A} \models \Phi_G$. By (8), $|p^{\mathfrak{A}}| \leq 3$. Fix any i $(1 \leq i \leq n)$. No $a \in p^{\mathfrak{A}}$ satisfies any two of the predicates p_i^0, p_i^1, p_i^2, by (9); on the other hand, each of these predicates is satisfied by at least one element of $p^{\mathfrak{A}}$, by (10); therefore, $|p^{\mathfrak{A}}| = 3$, and each element a of $p^{\mathfrak{A}}$ satisfies exactly one of the predicates p_i^0, p_i^1, p_i^2. Now fix any $a \in p^{\mathfrak{A}}$, and, for all i $(1 \leq i \leq n)$, define $t_a(i)$ to be the unique k $(0 \leq k < 3)$ such that $\mathfrak{A} \models p_i^k[a]$, by the above argument. The formulas (11) then ensure that t_a defines a colouring of G. Conversely, suppose that $t \colon \{1, \ldots, n\} \to \{0, 1, 2\}$ defines a colouring of G. Let $\mathfrak{A}$ be a structure with domain $A = \{0, 1, 2\}$; let all three elements satisfy p; and, for all $k \in A$, let p_i^k be satisfied by the single element $k + t(i)$ (where the addition is modulo 3). It is routine to verify that $\mathfrak{A} \models \Phi_G$. We note that all numerical subscripts in the formulas of Φ are bounded by 3. Thus, NP-hardness remains however those numerical subscripts are coded. ⊣

So much for the lower complexity bound for $\mathcal{N}^1$. We now proceed to establish a matching upper bound for the whole of $\mathcal{C}^1$, thus establishing the NP-completeness of any fragment between $\mathcal{N}^1$ and $\mathcal{C}^1$. To understand better the nature of the problem, consider first any $\mathcal{C}^1$-formula ϕ having the form

$$\bigwedge_{1 \leq i \leq m} \exists_{\bowtie_i C_i} x \phi_i, \tag{12}$$

where the symbols $\bowtie_i$ are any of $\{\leq, \geq, =\}$, and the formulas ϕ_i involve no quantifiers, no individual constants, no proposition letters, and no occurrences of equality. (Thus, the ϕ_i are Boolean combinations of atoms of the form $p(x)$.) We will see presently that such formulas are as general as we need.

Let the unary predicates occurring in ϕ, enumerated in some order, be $p_1, \ldots, p_l$. Call any formula of the form $\pi = \pm p_1(x) \wedge \cdots \wedge \pm p_l(x)$ a 1-*type*; and let these 1-types, enumerated in some order, be $\pi_1, \ldots, \pi_L$, where $L = 2^l$. Evidently, any structure $\mathfrak{A}$ interpreting the $p_1, \ldots, p_l$ can be characterized, up to isomorphism, by the sequence of cardinal numbers $(\alpha_1, \ldots, \alpha_L)$, where α_j is the cardinality of the set $\{a \in A \colon \mathfrak{A} \models \pi_j[a]\}$ for all j $(1 \leq j \leq L)$. Denote this sequence by $\alpha(\mathfrak{A})$. Now, for all i $(1 \leq i \leq m)$ and j $(1 \leq j \leq L)$, define

$$a_{i,j} = \begin{cases} 1 & \text{if} \models \pi_j \to \phi_i, \\ 0 & \text{otherwise.} \end{cases}$$

Interpreting any arithmetic operations involving infinite cardinals in the expected way, we see that, if $\mathfrak{A} \models \phi$, then $\alpha(\mathfrak{A})$ is a non-degenerate solution of

$$\begin{array}{cccc} a_{1,1}x_1 + & \cdots + & a_{1,L}x_L & \bowtie_1 & C_1 \\ \vdots & & \vdots & \vdots & \vdots \\ a_{m,1}x_1 + & \cdots + & a_{m,L}x_L & \bowtie_m & C_m. \end{array} \tag{13}$$

(*Non-degenerate* means that at least one value is non-zero.) Conversely, given any non-degenerate solution $\alpha_1, \ldots, \alpha_L$ of (13), we can easily construct a model $\mathfrak{A}$ of ϕ such that $\alpha(\mathfrak{A}) = (\alpha_1, \ldots, \alpha_L)$. Setting $C = \max\{C_i \mid 1 \leq i \leq m\}$, we also see that, if $\alpha_1, \ldots, \alpha_L$ is a solution of (13), then so is $\beta_1, \ldots, \beta_L$, where $\beta_j = \min(\alpha_j, C)$ for all j $(1 \leq j \leq L)$. Thus, ϕ is satisfiable if and only if it is finitely satisfiable; moreover, to determine whether ϕ is satisfiable, it suffices to determine whether the system (13) of equations and inequalities has a non-degenerate solution over $\mathbb{N}$.

It was shown in Borosh and Treybig [3] that, if $\mathcal{E}$ is a system of linear equations and inequalities with integer coefficients, then $\mathcal{E}$ has a solution over $\mathbb{N}$ if and only if $\mathcal{E}$ has such a solution in which all values are bounded by a fixed exponential function of the size of $\mathcal{E}$, measured in the obvious way. It follows easily that the problem of determining the satisfiability of such systems is in NP. The difficulty in the present context, however, is that the number of variables L in (13) is at best bounded by an exponential function of the size of ϕ. We need to show that, if there is a non-degenerate solution of (13) over $\mathbb{N}$, then there is a non-degenerate solution over $\mathbb{N}$ in which only a small number of values are non-zero.

To set the scene, we recall another well-known result (see, e.g., Paris [16], Chapter 10). Denote the set of *non-negative rationals* by $\mathbb{Q}^+$. The following

Lemma shows that, when seeking solutions to (13) over $\mathbb{Q}^+$, we can indeed limit attention to solutions with few non-zero values.

LEMMA 2. *Let $\mathcal{E}$ be a system of m linear equations with rational coefficients. If $\mathcal{E}$ has a solution over $\mathbb{Q}^+$, then $\mathcal{E}$ has a solution over $\mathbb{Q}^+$ with at most m non-zero entries.*

PROOF. We can write $\mathcal{E}$ as $\mathbf{Ax} = \mathbf{c}$, where $\mathbf{A}$ is a rational matrix with m rows and, say, L columns, and $\mathbf{c}$ is a rational column vector of length m. If $\mathbf{b}$ is any solution of $\mathcal{E}$ in $\mathbb{Q}^+$ with $k > m$ non-zero entries, the k columns of $\mathbf{A}$ corresponding to the non-zero entries of $\mathbf{b}$ must be linearly dependent. Thus, there exists a non-zero rational vector $\mathbf{b}'$ with zero-entries wherever $\mathbf{b}$ has zero-entries, such that $\mathbf{Ab}' = \mathbf{0}$. But then it is easy to find a rational number ε such that $\mathbf{b}+\varepsilon\mathbf{b}'$ is a solution of $\mathcal{E}$ in $\mathbb{Q}^+$ with fewer than k non-zero entries. ⊣

The question we face is whether the corresponding bound holds when solutions are sought in $\mathbb{N}$, rather than $\mathbb{Q}^+$. Here, the bound of m given in Lemma 2 no longer holds (see Lemma 4). Nevertheless, a slightly weaker bound can be obtained.

DEFINITION 1. A *Boolean equation* is any equation of the form $a_1x_1 + \cdots + a_nx_n = c$, where each a_i $(1 \leq i \leq n)$ is either 0 or 1, and c is a natural number.

The proof of the following two Lemmas uses an argument due to Eisenbrand and Shmonina [5]. We repeat it here in full for completeness.

LEMMA 3. *Let $\mathcal{E}$ be a system of m Boolean equations in L variables. If $\mathcal{E}$ has a non-degenerate solution over $\mathbb{N}$, then $\mathcal{E}$ has a non-degenerate solution over $\mathbb{N}$ with at most $m\log(L+1)$ non-zero entries.*

PROOF. We write $\mathcal{E}$ as $\mathbf{Ax} = \mathbf{c}$, where $\mathbf{A}$ is a matrix of 0s and 1s with m rows and L columns, $\mathbf{c}$ is a column vector over $\mathbb{N}$ of length m, and $\mathbf{x} = (x_1, \ldots, x_L)^T$. If $\mathcal{E}$ has a non-degenerate solution over $\mathbb{N}$, let $\mathbf{b} = (b_1, \ldots, b_L)^T$ be such a solution with a minimal number $k > 0$ of non-zero entries. We show that

$$k \leq m\log(L+1). \tag{14}$$

If $k = 1$, there is nothing to show, so assume $k > 1$. By renumbering the variables if necessary, we may assume without loss of generality that $b_j > 0$ for all j $(1 \leq j \leq k)$. Now, if $I \subseteq \{1, \ldots, k\}$, define $\mathbf{v}_I$ to be the m-element column vector $(v_1, \ldots, v_m)^T$, where

$$v_i = \sum_{j \in I} \mathbf{A}_{i,j}.$$

That is, $\mathbf{v}_I$ is the sum of those columns of $\mathbf{A}$ indexed by elements of I. Since each v_i $(1 \leq i \leq m)$ is a natural number satisfying

$$v_i \leq L, \tag{15}$$

the number of vectors $\mathbf{v}_I$ (as I varies over subsets of $\{1, \ldots, k\}$) is certainly bounded by $(L+1)^m$. So suppose, for contradiction, that $k > m \log(L+1)$. Then $2^k > (L+1)^m$, whence there must exist distinct subsets I, I' of $\{1, \ldots, k\}$ such that $\mathbf{v}_I = \mathbf{v}_{I'}$. Setting $J = I \setminus I'$ and $J' = I' \setminus I$, it is evident that J and J' are distinct (and disjoint), again with $\mathbf{v}_J = \mathbf{v}_{J'}$. Since $J \neq J'$, we may assume without loss of generality that $J \neq \emptyset$. Moreover if $J' = \emptyset$, then $\mathbf{v}_J = \mathbf{v}_{J'} = \mathbf{0}$, so that all the coefficients $a_{i,j}$ for $j \in J$ must be zero, in which case it is easy to obtain a non-degenerate solution of $\mathcal{E}$ having strictly fewer non-zero entries than $\mathbf{b}$. (Just set one of the b_j to 0, for some $j \in J$.) So assume $J' \neq \emptyset$. Now define, for all j $(1 \leq j \leq L)$:

$$b'_j = \begin{cases} b_j - 1 \text{ if } j \in J, \\ b_j + 1 \text{ if } j \in J', \\ b_j \text{ otherwise,} \end{cases}$$

and write $\mathbf{b}' = (b'_1, \ldots, b'_L)^T$. Since J and J' are disjoint, the cases do not overlap; and since the b_j are all positive $(1 \leq j \leq k)$, the b'_j all lie in $\mathbb{N}$. Moreover,

$$\mathbf{A}\mathbf{b}' = \mathbf{A}\mathbf{b} - \mathbf{v}_J + \mathbf{v}_{J'} = \mathbf{A}\mathbf{b}.$$

Since J is non-empty, $\min\{b'_j | j \in J\}$, is strictly smaller than $\min\{b_j | j \in J\}$; and since J' is non-empty, at least one of the b'_j is non-zero. Generating $\mathbf{b}''$, $\mathbf{b}'''$, etc. in this way (using the same J and J') will thus eventually result in a non-zero vector—say, $\mathbf{b}^*$—with strictly fewer non-zero entries than $\mathbf{b}$, but with $\mathbf{A}\mathbf{b}^* = \mathbf{A}\mathbf{b}$—a contradiction. ⊣

By way of a digression, we strengthen Lemma 3 to obtain a bound which does not depend on L. (We ignore the non-degeneracy condition for brevity.)

PROPOSITION 1. *Let $\mathcal{E}$ be a system of m Boolean equations. If $\mathcal{E}$ has a solution over $\mathbb{N}$, then $\mathcal{E}$ has a solution over $\mathbb{N}$ with at most $\frac{5}{2} m \log m + 1$ non-zero entries.*

PROOF. The case $m = 1$ is trivial: if $\mathcal{E}$ has a solution, then it has a solution with at most one non-zero entry. So assume henceforth that $m > 1$.

In the proof of Lemma 3, the inequality (15) can evidently be strengthened to

$$v_i \leq k.$$

Proceeding exactly as for Lemma 3, we obtain, in place of (14), the inequality

$$k \leq m \log(k+1).$$

Hence, for k positive, we have

$$\frac{k}{\log(k+1)} \leq m. \tag{16}$$

Now the left-hand side of (16) is greater than or equal to unity, and since the function $x \mapsto x \log x$ is monotone increasing for $x \geq e^{-1}$, we can apply it to both sides of (16) to obtain

$$kZ(k) \leq m \log m, \tag{17}$$

where, for all $k > 0$,

$$Z(k) = \frac{\log k - \log\log(k+1)}{\log(k+1)}.$$

It is straightforward to check that Z is monotone increasing on the positive integers, and that $Z(k) \to 1$ as $k \to \infty$. (Indeed, for $x > 0$, the function $x \mapsto \log x / \log(x+1)$ is monotone increasing with limit 1 as x tends to ∞; and for $x \geq 2^e - 1$, the function $x \mapsto \log\log(x+1)/\log(x+1)$ is monotone decreasing with limit 0.)

We may now establish that $k \leq \frac{5}{2} m \log m + 1$. Calculation shows that $1/Z(7) \approx 2.4542 < \frac{5}{2}$. Therefore, since Z is monotone increasing, (17) yields, for $k \geq 7$, the inequalities $k \leq m \log m / Z(k) \leq m \log m / Z(7) < \frac{5}{2} m \log m$. Obviously, if $k \leq 6$, we have $k \leq \frac{5}{2} m \log m + 1$, since $m \geq 2$ by assumption. $\dashv$

The proof of Proposition 1 actually shows a little more than advertised: for *any* real $c > 1$, there exists a d such that, if $\mathcal{E}$ is a system of m Boolean equations with a solution over $\mathbb{N}$, then $\mathcal{E}$ has a solution over $\mathbb{N}$ with at most $cm \log m + d$ non-zero entries. (As c approaches unity, the required value of d given by the above proof quickly becomes astronomical.) It follows that none of these bounds is optimal, in the sense of being achieved infinitely often. On the other hand, the next lemma shows that, for systems of Boolean equations with variables ranging over $\mathbb{N}$, the bound of m reported in Lemma 2 is definitely not available, a fact which will prove useful in Section 5.

LEMMA 4. *Fix $m \geq 6$. Let $\boldsymbol{A}$ be the $m \times (m+1)$-matrix given by*

$$\boldsymbol{A} = \left(\begin{array}{ccccccc|ccccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & & & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \dots & & & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \dots & & & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & \dots & & & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & \dots & & & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \dots & & & 0 \\
\vdots & & & & & & & & & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \; 1 \\
\hline
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \dots & & & 0
\end{array}\right),$$

in which a pattern of three 1*s is shifted right across the first* $(m-1)$ *rows, and the last row contains the seven entries shown on the left followed by* $(m-6)$ 0*s. Let* $\boldsymbol{c}$ *be the column vector of length* m *given by*

$$\boldsymbol{c} = (3, 3, \dots, 3, 4)^T$$

consisting of $(m-1)$ $3s$ *and a single* 4. *Then the unique solution of the system of Boolean equations* $\boldsymbol{Ax} = \boldsymbol{c}$ *over* $\mathbb{N}$ *is the column vector* $(1, \ldots, 1)^T$ *consisting of* $(m+1)$ $1s$.

Proof. Evidently, $\mathbf{A}(1, \ldots, 1)^T = \mathbf{c}$. Conversely, suppose $\mathbf{b} = (b_1, \ldots, b_{m+1})^T$ is any solution of $\mathbf{Ax} = \mathbf{c}$ over $\mathbb{N}$. From the first row of $\mathbf{A}$, $b_1 + b_2 + b_3 = 3$, whence b_1, b_2, b_3 are either (i) the integers $0, 0, 3$ in some order, or (ii) the integers $0, 1, 2$ in some order or (iii) the integers $1, 1, 1$. By considering rows 2 to $m-1$ of $\mathbf{A}$, it is then easy to see that, in every case, these three values must recur, in the same order, to the end of the vector: that is, $\mathbf{b}$ must have the form

$$(b_1, b_2, b_3, b_1, b_2, b_3, b_1, \ldots)^T.$$

From the last row of $\mathbf{A}$, then, $3b_1 + b_2 = 4$. Thus, b_1, b_2, b_3 are certainly not $3, 0, 0$, in any order. Suppose, then, b_1, b_2, b_3 are $0, 1, 2$, in some order. If $b_1 = 0$, then $3b_1 + b_2$ is at most 2; if $b_1 = 1$, then $3b_1 + b_2$ equals either 3 or 5; and if $b_1 = 2$, then $3b_1 + b_2$ is at least 6. Thus, b_1, b_2, b_3 are not $0, 1, 2$, in any order, whence $\mathbf{b} = (1, \ldots, 1)^T$ as required. ⊣

Returning to the main business of this section, we have:

Theorem 1. *The fragment* $\mathcal{C}^1$ *has the finite model property. Moreover, the satisfiability* $(=$*finite satisfiability*$)$ *problem for* $\mathcal{C}^1$ *is in NP.*

Proof. If ϕ, ψ and π are $\mathcal{C}^1$-formulas, denote by $\phi[\pi/\psi]$ the result of substituting π for all occurrences of ψ (as subformulas) in ϕ.

It is straightforward to transform any $\mathcal{C}^1$-formula ϕ, in polynomial time, into a closed $\mathcal{C}^1$-formula ϕ' containing no occurrences of equality and no individual constants, such that ϕ is satisfiable over a given domain A if and only if ϕ' is satisfiable over A. Indeed, we may further restrict attention to such ϕ' having the form

$$\bigwedge_{1 \leq i \leq m} \exists_{\bowtie_i C_i} x \phi_i, \tag{18}$$

with the ϕ_i quantifier-free and containing no occurrences of proposition letters. For suppose ϕ' does not have this form: we process ϕ' as follows. First, non-deterministically guess the truth of any proposition letter p occurring in ϕ', replacing all its occurrences uniformly with $\top$ or $\bot$, according to the guessed truth-value. Second, choose any quantified subformula $\psi = \exists_{\bowtie C} x\pi$ with π quantifier-free, and non-deterministically replace ϕ' by either $\phi'[\top/\psi] \wedge \psi$ or $\phi'[\bot/\psi] \wedge \neg\psi$; then repeat this step until all embedded quantification has been removed. The result will be, modulo trivial logical equivalences, a formula of the form (18); and ϕ' will be satisfiable over a given domain A if and only if some formula of the form (18) obtained in this way is satisfiable over A.

We argued above that (18) is satisfiable (equivalently: finitely satisfiable) if and only if the corresponding system (13) of equations and inequalities has a non-degenerate solution over $\mathbb{N}$. By Lemma 3, this holds if and only if (13) has a non-degenerate solution over $\mathbb{N}$ in which at most $m \log(L+1) \leq m(l+1) \leq |\phi|^2$ values are non-zero. Moreover, we may obviously assume all these non-zero values to be bounded by $C = \max_i C_i$. But any such solution can be written down and checked in a time bounded by a polynomial function of the size of ϕ. $\dashv$

COROLLARY 1. *The satisfiability problem (=finite satisfiability problem) for any logic between $\mathcal{N}^1$ and $\mathcal{C}^1$ is strongly NP-complete.*

It follows that determining the validity of arguments in the numerically definite syllogistic is a co-NP-complete problem. Equipping this fragment with relative clauses, for example,

At most 3 artists who are not beekeepers are carpenters,

evidently has no effect on the complexity of determining validity, since it does not take us outside the fragment $\mathcal{C}^1$. Nor indeed has the addition of proper nouns, for the same reason. In fact, it is straightforward to show that the complexity bound for satisfiability (and finite satisfiability) given in Theorem 1 applies to extensions of $\mathcal{C}^1$ featuring a large variety of other quantifiers. The only requirement is that the truth-value of a formula $Qx(\psi_1, \ldots, \psi_n)$ be succinctly expressible as a collection of 'linear' constraints involving the cardinalities (possibly infinite) of Boolean combinations of the sets of elements satisfying the $\psi_1, \ldots, \psi_n$. In particular, we obtain the same complexity for extensions of the numerically definite syllogistic featuring such sentences as

There are more artists than beekeepers
Most artists are beekeepers
There are more than 3.7 times as many artists as beekeepers
There are finitely many carpenters.

The details are routine, and we leave them to the reader to explore. Extensions of the syllogistic with 'proportional' quantifiers are considered by Peterson [17, 18]; however, no complexity-theoretic analysis is undertaken in those papers, and certainly no analogues of Lemma 3 or Proposition 1 are provided.

We conclude this section with some remarks on a related problem. Denote by $\mathcal{S}$ the propositional language (with usual Boolean connectives) over the countable signature of proposition letters $p_1, p_2, \ldots$. A *probability assignment* for $\mathcal{S}$ is a function $P\colon \mathcal{S} \to [0,1]$ satisfying the usual axioms of probability (see, e.g., Paris [16], p. 7). The problem PSAT may now be defined as follows. Let a list of pairs $(\phi_1, q_1), \ldots, (\phi_m, q_m)$ be given, where each ϕ_i is a clause of $\mathcal{S}$, and each q_i is a rational number: decide whether there exists a probability assignment P for $\mathcal{S}$ such that

$$P(\phi_i) = q_i \qquad \text{for all } i\ (1 \leq i \leq m). \tag{19}$$

The size of any problem instance $(\phi_1, q_1), \ldots, (\phi_m, q_m)$ is measured in the obvious way, with binary coding of the q_i. By comparing (18) and (19), we see that the satisfiability problem for $\mathcal{C}^1$ is, as it were, an 'integral' version of PSAT. The problem k-PSAT is the restriction of PSAT to the case where all the clauses ϕ_i have at most k literals.

Georgakopoulos *et al.* [6] show that 2-PSAT is NP-hard, even under unary coding, and that PSAT is in NP. (Hence, k-PSAT is strongly NP-complete for all $k \geq 2$.) The proof that 2-PSAT is NP-hard is essentially the same as the proof of Lemma 1. Moreover, the proof that PSAT is in NP is similar in structure to the proof of Theorem 1. Suppose we are given an instance of PSAT in which the ϕ_i mention only the proposition letters $p_1, \ldots, p_l$: the challenge is to show that, if there exists a probability assignment P satisfying (19), then there exists one in which the number of formulas $\pm p_1 \wedge \cdots \wedge \pm p_l$ having non-zero probability is polynomially bounded as a function of l. But this is easily guaranteed by Lemma 2. By contrast, Lemma 2 does not suffice for the proof of our Theorem 1, because it does not guarantee the existence of an *integral* solution of the relevant equations—hence the need for Lemma 3 (or Proposition 1). We return to this matter in Section 5.

§4. Complexity of logics between $\mathcal{N}^2$ and $\mathcal{C}^2$. We now turn our attention to logics containing $\mathcal{N}^2$ but contained in $\mathcal{C}^2$.

LEMMA 5. *The fragment $\mathcal{N}^2$ has the finite model property.*

PROOF. Suppose $\mathfrak{A} \models \Phi$, where Φ is a finite set of $\mathcal{N}^2$-formulas. If $\phi \in \Phi$ is of the form $\exists_{\geq D} x \psi(x)$, let A_ϕ be a collection of D individuals satisfying ψ in $\mathfrak{A}$, and let

$$A_\Phi = \bigcup \{A_\phi \mid \phi \in \Phi \text{ is of the form } \exists_{\geq D} x \psi(x)\}.$$

As in Theorem 1, let the unary predicates occurring in Φ be $p_1, \ldots, p_l$, and let $\pi_1(x), \ldots, \pi_L(x)$ be all the formulas of the form $\pm p_1(x) \wedge \cdots \wedge \pm p_l(x)$, enumerated in some way. Then A is the union of the pairwise disjoint sets $A_1, \ldots, A_L$, where $A_j = \{a \in A \mid \mathfrak{A} \models \pi_j[a]\}$. Let C be the largest quantifier subscript occurring in Φ. Evidently, for all j $(1 \leq j \leq L)$,

$$|A_\Phi \cap A_j| \leq |A_\Phi| \leq C|\Phi|,$$

whence we may certainly select a set of elements A'_j such that $A_\Phi \cap A_j \subseteq A'_j \subseteq A_j$ and

$$|A'_j| = \min(|A_j|, C|\Phi| + 1).$$

Thus $A' = A'_1 \cup \cdots \cup A'_L$ is finite. We define a structure $\mathfrak{A}'$ over A' as follows. Interpret the unary predicates so that, for all j $(1 \leq j \leq L)$ and all $a' \in A'_j$, $\mathfrak{A}' \models \pi_j[a']$. Interpret each binary predicate r in such a way that, for all $a' \in A'$ and all j $(1 \leq j \leq L)$,

$$|\{b' \in A'_j : \mathfrak{A}' \models r[a', b']\}| = \min(|\{b \in A_j : \mathfrak{A} \models r[a', b]\}|, C|\Phi| + 1).$$

This is evidently possible. Consider any formula $\theta(x)$ of either of the forms

$$\exists_{\leq D} y(q(y) \wedge r(x, y)) \qquad \exists_{\geq D} y(q(y) \wedge r(x, y)) \tag{20}$$

with $D \leq C + 1 \leq |\Phi|C + 1$. It is immediate from the construction of $\mathfrak{A}'$ that, for all $a' \in A'$,

$$\mathfrak{A} \models \theta[a'] \Rightarrow \mathfrak{A}' \models \theta[a']. \tag{21}$$

Hence, if the numerical subscript D in θ satisfies $D \leq C$, we also have:

$$\mathfrak{A} \not\models \theta[a'] \Rightarrow \mathfrak{A}' \not\models \theta[a']. \tag{22}$$

We show that $\mathfrak{A}' \models \phi$ for all $\phi \in \Phi$. If ϕ is an $\mathcal{N}^1$-formula, this result is immediate from the construction of $\mathfrak{A}'$. If ϕ has the form $\exists_{\geq D'} x(p(x) \wedge \theta(x))$, with $\theta(x)$ having one of the forms (20), the result follows from (21) and the fact that $A_\phi \subseteq A'$. If ϕ has the form $\exists_{\leq D'} x(p(x) \wedge \theta(x))$, with $\theta(x)$ having one of the forms (20), the result follows from (22). $\dashv$

Inspection of the proof of Lemma 5 shows that the size of the constructed model is bounded by an exponential function of the size of Φ. Hence, the satisfiability (= finite satisfiability) problem for $\mathcal{N}^2$ is in NEXPTIME.

It is well-known that the larger fragment $\mathcal{C}^2$ lacks the finite model property. For example, the formula

$$\exists x \forall y \neg r(x, y) \wedge \forall x \exists y r(y, x) \wedge \forall x \exists_{\leq 1} y r(x, y) \tag{23}$$

is satisfiable, but not finitely so. Thus, the satisfiability problem and the finite satisfiability problem for $\mathcal{C}^2$ do not coincide. Nevertheless, the following was shown in Pratt-Hartmann [19] (which slightly extends Pacholski, Szwast and Tendera [15]).

THEOREM 2. *The satisfiability problem and the finite satisfiability problem for $\mathcal{C}^2$ are both in* NEXPTIME.

This upper bound applies even when counting quantifiers are coded in binary—a fact which is significant here. In this section, we provide a matching lower-bound for $\mathcal{N}^2$, which is slightly surprising given the latter fragment's expressive limitations.

A *tiling system* is a triple $\langle C, H, V \rangle$, where C is a non-empty, finite set and H, V are binary relations on C. The elements of C are referred to as *colours*, and the relations H and V as the *horizontal* and *vertical* constraints, respectively. For any integer N, a *tiling* for $\langle C, H, V \rangle$ of size N is a function $t\colon \{0, \ldots, N-1\}^2 \to C$ such that, for all i, j in the range $\{0, \ldots, N-1\}$, the pair $\langle t(i, j), t(i+1, j) \rangle$ is in H and the pair $\langle t(i, j), t(i, j+1) \rangle$ is in V, with addition interpreted modulo N. A tiling of size N is to be pictured as a colouring of an $N \times N$ square grid (with toroidal wrap-around) by the colours in C; the horizontal constraints H thus specify which colours may appear 'to the right of' which other colours; the vertical constraints V likewise specify which colours may appear 'above' which other colours. By a *C-sequence*, we simply mean a sequence $\mathbf{i} = i_0, \ldots, i_{n-1}$ of elements of C

(repeats allowed). The C-sequence $\mathbf{i}$ is an *initial configuration* of a tiling t if $\mathbf{i} = t(0,0), \ldots, t(n-1, 0)$.

THEOREM 3. *The satisfiability problem for* $\mathcal{N}^2$ *is* NEXPTIME-*hard.*

PROOF. Let $\langle C, H, V\rangle$ be any tiling system and p any (non-zero) polynomial. For any C-sequence $\mathbf{i}$ of length n, we construct, in time bounded by a polynomial function of n, a set $\Theta_\mathbf{i}$ of $\mathcal{N}^2$-formulas such that $\Theta_\mathbf{i}$ is satisfiable if and only if $\langle C, H, V\rangle$ has a tiling of size $2^{p(n)}$ with initial configuration $\mathbf{i}$. Thus, we may regard $\Theta_\mathbf{i}$ as an *encoding* of the C-sequence $\mathbf{i}$ with respect to the tiling system $\langle C, H, V\rangle$ and the function $2^{p(k)}$. The existence of such an encoding suffices to show that the satisfiability problem for $\mathcal{N}^2$ is NEXPTIME-hard.

To motivate the technical details, we suppose, provisionally, that $\langle C, H, V\rangle$ does have a tiling of size $2^{p(n)}$ with initial configuration $\mathbf{i}$, and we construct the encoding $\Theta_\mathbf{i}$ in parallel with a structure $\mathfrak{A}$ in which $\Theta_\mathbf{i}$ is true. As we do so, we show that, conversely, if $\Theta_\mathbf{i}$ is satisfiable, then $\langle C, H, V\rangle$ has a tiling with the required properties. The construction of $\Theta_\mathbf{i}$ proceeds in two stages. In the first stage, we employ familiar techniques to obtain an encoding in an extension of $\mathcal{N}^2$. In the second stage, we employ some less familiar methods to obtain an encoding in $\mathcal{N}^2$.

FIRST STAGE: For convenience, we set $N = 2^{p(n)}$ and $s = 2(p(n)^2 + p(n)+1)$. Let A_1 be the set of pairs of integers (i, j) in the range $\{0, \ldots, N-1\}$, and let A_2 be the set of pairs of the forms $(i, \top)$ and $(i, \bot)$, where i is an integer in the range $\{1, \ldots, s\}$ and $\top, \bot$ are any distinct symbols. Evidently,

$$|A_1| = N^2$$
$$|A_2| = 2s.$$

Finally, let A_3 be a set disjoint from A_1 and A_2 satisfying

$$|A_3| = (M-1)N^2,$$

where $M = |C|$. We refer to A_1 as the *grid*, A_2 as the *notebook*, and A_3 as the *rubbish dump*. Our structure $\mathfrak{A}$ will have domain $A = A_1 \cup A_2 \cup A_3$.

Any natural number l in the range $\{0, \ldots, N-1\}$ can be written uniquely as $l = \sum_{i=0}^{p(n)-1} b_i 2^i$, where $b_i \in \{0, 1\}$. We say that b_i is the ith *digit of* l. Thus, digits are enumerated in order of increasing significance, starting with the zeroth. Let q, $X_0, \ldots, X_{p(n)-1}$, $\bar{X}_0, \ldots, \bar{X}_{p(n)-1}$ be new unary predicates, interpreted in the structure $\mathfrak{A}$ as follows:

$$q^\mathfrak{A} = A_1$$
$$X_i{}^\mathfrak{A} = \{(l, m) \in A_1 \mid \text{the } i\text{th digit of } l \text{ is } 1\}$$
$$\bar{X}_i^\mathfrak{A} = \{(l, m) \in A_1 \mid \text{the } i\text{th digit of } l \text{ is } 0\}.$$

We may read X_i as "has an x-coordinate whose ith digit is 1", and $\bar{X}_i$ as "has an x-coordinate whose ith digit is 0". Then $\mathfrak{A} \models \Theta_{0,X}$, where $\Theta_{0,X}$ is the set of formulas

$$\begin{array}{ll}
\exists_{\leq N^2} x q(x) & \\
\exists_{\geq N^2/2} x X_i(x) & \\
\exists_{\geq N^2/2} x \bar{X}_i(x) & \\
\forall x(X_i(x) \rightarrow q(x)) & (0 \leq i < p(n)) \\
\forall x(\bar{X}_i(x) \rightarrow q(x)) & (0 \leq i < p(n)) \\
\forall x(X_i(x) \rightarrow \neg \bar{X}_i(x)) & (0 \leq i < p(n)).
\end{array}$$

Conversely, in any model of $\Theta_{0,X}$, exactly N^2 elements satisfy q, and the extensions of X_i and $\bar{X}_i$ are complementary with respect to that collection of elements, for all i $(0 \leq i < p(n))$.

Further, let $X^*_0, \ldots, X^*_{p(n)}$ be new unary predicates, interpreted in the structure $\mathfrak{A}$ as follows:

$$\begin{array}{rll}
{X^*_0}^{\mathfrak{A}} &= A_1 \setminus X_0^{\mathfrak{A}} & \\
{X^*_i}^{\mathfrak{A}} &= ({X_0}^{\mathfrak{A}} \cap \cdots \cap {X_{i-1}}^{\mathfrak{A}}) \setminus {X_i}^{\mathfrak{A}} & (1 \leq i < p(n)) \\
{X^*_{p(n)}}^{\mathfrak{A}} &= {X_0}^{\mathfrak{A}} \cap \cdots \cap {X_{p(n)-1}}^{\mathfrak{A}}. &
\end{array}$$

Thus, the predicate X^*_i can be read as "has an x-coordinate in which all digits before the ith, but not the ith digit itself, are 1". Finally, for all i, j $(0 \leq i < j < p(n))$, let $X^+_{i,j}$ and $X^-_{i,j}$ be new unary predicates, interpreted in the structure $\mathfrak{A}$ as follows:

$$\begin{array}{rl}
{X^+_{i,j}}^{\mathfrak{A}} &= {X^*_i}^{\mathfrak{A}} \cap {X_j}^{\mathfrak{A}} \\
{X^-_{i,j}}^{\mathfrak{A}} &= {X^*_i}^{\mathfrak{A}} \setminus {X_j}^{\mathfrak{A}}.
\end{array}$$

Let Γ_X be the set of first-order formulas $\forall x(q(x) \rightarrow \gamma)$, where γ is any of the following clauses:

$$\begin{array}{ll}
X^*_i(x) \vee \big[X_i(x) \vee \bigvee_{0 \leq k < i} \bar{X}_k(x)\big] & (0 \leq i < p(n)) \\
X^*_{p(n)}(x) \vee \big[\bigvee_{0 \leq k < p(n)} \bar{X}_k(x)\big] & \\
X^+_{i,j} \vee \big[\bar{X}_j(x) \vee X_i(x) \vee \bigvee_{0 \leq k < i} \bar{X}_k(x)\big] & (0 \leq i < j < p(n)) \\
X^-_{i,j} \vee \big[X_j(x) \vee X_i(x) \vee \bigvee_{0 \leq k < i} \bar{X}_k(x)\big] & (0 \leq i < j < p(n)).
\end{array}$$

(The square brackets are for legibility.) It is immediate that $\mathfrak{A} \models \Gamma_X$. The formulas Γ_X, in effect, establish sufficient conditions for satisfaction of the predicates X^*_i *etc.* in terms of the predicates X_i and $\bar{X}_i$. Warning: these formulas are not in the fragment $\mathcal{N}^2$.

Similarly, let $Y_0, \ldots, Y_{p(n)-1}$, $\bar{Y}_0, \ldots, \bar{Y}_{p(n)-1}$ be new unary predicates, interpreted in the structure $\mathfrak{A}$ as follows:

$$Y_i^{\mathfrak{A}} = \{(l, m) \in A_1 \mid \text{the } i\text{th digit of } m \text{ is } 1\}$$
$$\bar{Y}_i^{\mathfrak{A}} = \{(l, m) \in A_1 \mid \text{the } i\text{th digit of } m \text{ is } 0\}.$$

We may read Y_i as "has a y-coordinate whose ith digit is 1", and $\bar{Y}_i$ as "has a y-coordinate whose ith digit is 0". Let $\Theta_{0,Y}$ be the set of formulas constructed analogously to $\Theta_{0,X}$, but with "X" replaced systematically by "Y"; and let $\Theta_0 = \Theta_{0,X} \cup \Theta_{0,Y}$. Further, let Y_i^* $(0 \le i \le p(n))$, $Y_{i,j}^+$ $(0 \le i < j < p(n))$ and $Y_{i,j}^-$ $(0 \le i < j < p(n))$ be new unary predicates, interpreted analogously to their X-counterparts; let the formulas Γ_Y be constructed analogously to Γ_X; and let $\Gamma = \Gamma_X \cup \Gamma_Y$.

We may now impose a toroidal grid structure on A_1, with the aid of a pair of binary predicates h and v. Let $\mathfrak{A}$ interpret h and v as follows:

$$h^{\mathfrak{A}} = \{\langle (l, m), (l+1, m)\rangle \mid 0 \le l < N,\ \ 0 \le m < N\}$$
$$v^{\mathfrak{A}} = \{\langle (l, m), (l, m+1)\rangle \mid 0 \le l < N,\ \ 0 \le m < N\},$$

where the addition is modulo N. It is straightforward to check that $\mathfrak{A} \models \Theta_{1,X} \cup \Theta_{1,Y}$, where $\Theta_{1,X}$ is the set of $\mathcal{N}^2$-formulas

$$\forall x(q(x) \to \exists y(q(y) \wedge h(x, y))) \tag{24}$$
$$\forall x(X_i^*(x) \to \neg\exists y(h(x, y) \wedge \bar{X}_i(y))) \qquad (0 \le i < p(n)) \tag{25}$$
$$\forall x(X_i^*(x) \to \neg\exists y(h(x, y) \wedge X_j(y))) \qquad (0 \le j < i \le p(n)) \tag{26}$$
$$\forall x(X_{i,j}^+(x) \to \neg\exists y(h(x, y) \wedge \bar{X}_j(y))) \qquad (0 \le i < j < p(n)) \tag{27}$$
$$\forall x(X_{i,j}^-(x) \to \neg\exists y(h(x, y) \wedge X_j(y))) \qquad (0 \le i < j < p(n)) \tag{28}$$
$$\forall x(Y_i(x) \to \neg\exists y(h(x, y) \wedge \bar{Y}_i(y))) \qquad (0 \le i < p(n)) \tag{29}$$
$$\forall x(\bar{Y}_i(x) \to \neg\exists y(h(x, y) \wedge Y_i(y))) \qquad (0 \le i < p(n)), \tag{30}$$

and $\Theta_{1,Y}$ is defined analogously, but with "X" and "Y" interchanged, and "h" replaced by "v". Let $\Theta_1 = \Theta_{1,X} \cup \Theta_{1,Y}$. Formula (24) ensures that every element a satisfying q is related via h to some other such element b. In the presence of Θ_0 and Γ, (25)–(28) then ensure that the 'x-coordinate' of b is one greater (modulo N) than the 'x-coordinate' of a; and (29)–(30) likewise ensure that a and b have the same 'y-coordinate'. Similar remarks apply, *mutatis mutandis*, to the formulas $\Theta_{1,Y}$. Since Θ_0 ensures that at most N^2 elements satisfy q, it follows that, in any model of $\Theta_0 \cup \Gamma \cup \Theta_1$, the extension of q contains exactly one element with any given pair of (x, y)-coordinates in the range $\{0, \ldots, N-1\}$, and moreover that this collection of elements is organized by the interpretations of h and v into an $N \times N$ toroidal grid in the expected way.

Having set up our grid, we proceed to colour it. Recall that the 'rubbish dump', A_3, is a set containing $(M-1)N^2$ elements, where $M = |C|$. Let $C = \{c_1, \ldots, c_M\}$. Assuming, provisionally, that (C, H, V) has a tiling of size N with initial segment $\mathbf{i} = i_0, \ldots, i_{n-1}$, choose some such tiling t. For all k $(1 \leq k \leq M)$, let $n_k \leq N^2$ be the number of grid-squares to which t assigns colour c_k, and let B_k be a subset of A_3 with cardinality $N^2 - n_k$. From the cardinality of A_3, and the fact that $\sum n_k = N^2$, we may choose the B_k to be pairwise disjoint; and, in that case, the B_k will together exactly cover A_3. Now treat the elements of C as new unary predicates, and set

$$c_k^{\mathfrak{A}} = \{a \in A_1 \mid t \text{ assigns the colour } c_k \text{ to } a\} \cup B_k,$$

for all k $(1 \leq k \leq M)$. Let o be a new unary predicate and set

$$o^{\mathfrak{A}} = A_1 \cup A_3.$$

It is simple to check that $\mathfrak{A} \models \Theta_2$, where Θ_2 is the set of formulas

$$\begin{array}{ll}
\forall x(q(x) \rightarrow o(x)) & \\
\exists_{\leq MN^2} x o(x) & \\
\exists_{\geq N^2} x c_k(x) & (1 \leq k \leq M) \\
\forall x(c_k(x) \rightarrow o(x)) & (1 \leq k \leq M) \\
\forall x(c_k(x) \rightarrow \neg c_{k'}(x)) & (1 \leq k < k' \leq M).
\end{array}$$

Conversely, in any model of Θ_2, the interpretations of the predicates c_k form a pairwise disjoint cover of the interpretation of o, and, therefore, of the interpretation of q.

Turning to the input $\mathbf{i} = i_0, \ldots, i_{n-1}$, let $o_0, \ldots, o_{n-1}$ be new unary predicates. We interpret these so as to pick out the squares $(0,0), \ldots, (n-1,0)$ of the grid, respectively. Formally:

$$o_i^{\mathfrak{A}} = \{(i,0)\}$$

for all i $(0 \leq i < n)$. It is a simple matter to write formulas specifying the coordinates of these predicates. For example, define $\Theta_{3,0}$ to be the set of formulas

$$\begin{array}{ll}
\exists x(o_0(x) \wedge q(x)) & \\
\forall x(o_0(x) \rightarrow \bar{X}_i(x)) & (0 \leq i < p(n)) \\
\forall x(o_0(x) \rightarrow \bar{Y}_i(x)) & (0 \leq i < p(n)) \\
\forall x(o_0(x) \rightarrow i_0(x)). &
\end{array}$$

(Remember that i_0, being an element of C, is also a predicate interpreted by $\mathfrak{A}$.) It is easy to see that $\mathfrak{A} \models \Theta_{3,0}$. Conversely, in any model of $\Gamma \cup \Theta_0 \cup \Theta_1 \cup \Theta_2 \cup \Theta_{3,0}$, o_0 must be interpreted as the (unique) element in the extension of q with 'coordinates' $(0,0)$; moreover, that element must be assigned the 'colour' i_0. Let the sets of formulas $\Theta_{3,1}, \ldots, \Theta_{3,n-1}$ be constructed analogously, fixing the interpretations of $o_1, \ldots, o_{n-1}$, respectively, with colours assigned as specified in $\mathbf{i}$; and let Θ_3 be $\Theta_{3,0} \cup \cdots \cup \Theta_{3,n-1}$.

Finally, let Θ_4 be the set of formulas

$$\forall x(c_j(x) \to \neg\exists y(c_k(y) \wedge h(x,y))) \quad (1 \le j \le M,\ 1 \le k \le M,\ (j,k) \notin H)$$
$$\forall x(c_j(x) \to \neg\exists y(c_k(y) \wedge v(x,y))) \quad (1 \le j \le M,\ 1 \le k \le M,\ (j,k) \notin V).$$

Since the interpretations of the c_k were taken from a tiling t, we certainly have $\mathfrak{A} \models \Theta_4$. Conversely, any model of $\Gamma \cup \Theta_0 \cup \cdots \cup \Theta_4$ defines a tiling for (C, H, V) of size N with initial segment $\mathbf{i}$, in the obvious way.

The set of formulas $\Gamma \cup \Theta_0 \cup \cdots \cup \Theta_4$ is *almost* the required encoding $\Theta_{\mathbf{i}}$: the only problem is that the formulas Γ are not in the fragment $\mathcal{N}^2$. Massaging them into the appropriate form is the task of the second stage of the proof.

SECOND STAGE: Recall that, in the first stage, we gave each of the predicates X_i and Y_i $(0 \le i < p(n))$, a "barred" counterpart $\bar{X}_i$, and $\bar{Y}_i$, with a complementary interpretation in $\mathfrak{A}$ with respect to $q^{\mathfrak{A}} = A_1$. Moreover, we provided a set of formulas Θ_0, guaranteeing that such pairs have complementary interpretations with respect to the extension of q. Let us do the same for the predicates X^*_i, $X^+_{i,j}$, $X^-_{i,j}$, Y^*_i, $Y^+_{i,j}$, $Y^-_{i,j}$ (with indices in the ranges specified above), letting Θ_5 be the requisite set of formulas. The construction of Θ_5 is completely routine.

Now enumerate the various predicates X_i, X^*_i, $X^+_{i,j}$, $X^-_{i,j}$, Y_i, Y^*_i, $Y^+_{i,j}$, $Y^-_{i,j}$, in some order, as

$$q_1, \ldots, q_s. \qquad (31)$$

(There are indeed $s = 2(p(n)^2 + p(n) + 1)$ of these, if you tot them all up.) And enumerate their barred counterparts, in the corresponding order, as

$$\bar{q}_1, \ldots, \bar{q}_s. \qquad (32)$$

Recall that the 'notebook', A_2, consists of the elements $(1, \top), \ldots, (s, \top)$ and $(1, \bot), \ldots, (s, \bot)$. Referring to the enumerations (31) and (32), think of the element $(h, \top)$ as standing for the atom $q_h(x)$, and of the element $(h, \bot)$ as standing for the atom $\bar{q}_h(x)$, for all h $(1 \le h \le s)$. Let $l, l_1, \ldots, l_s$ and $\bar{l}_1, \ldots, \bar{l}_s$ be new unary predicates, interpreted in $\mathfrak{A}$ as follows:

$$l^{\mathfrak{A}} = A_2$$
$$l_h^{\mathfrak{A}} = \{(h, \top)\} \qquad (1 \le h \le s)$$
$$\bar{l}_h^{\mathfrak{A}} = \{(h, \bot)\} \qquad (1 \le h \le s).$$

It is simple to check that $\mathfrak{A} \models \Theta_6$, where Θ_6 is the set of formulas

$$\begin{array}{lll}
\exists_{\leq 2s} x l(x) & & \\
\exists x l_h(x) & \exists x \bar{l}_h(x) & (1 \leq h \leq s) \\
\forall x(l_h(x) \rightarrow l(x)) & \forall x(\bar{l}_h(x) \rightarrow l(x)) & (1 \leq h \leq s) \\
\forall x(l_h(x) \rightarrow \neg l_{h'}(x)) & \forall x(\bar{l}_h(x) \rightarrow \neg \bar{l}_{h'}(x)) & (1 \leq h < h' \leq s) \\
\forall x(l_h(x) \rightarrow \neg \bar{l}_{h'}(x)) & & (1 \leq h \leq s, 1 \leq h' \leq s).
\end{array}$$

Conversely, in any model of Θ_6, the predicates $l_1, \ldots, l_s, \bar{l}_1, \ldots, \bar{l}_s$ are uniquely instantiated, and pick out the $2s$ elements satisfying l.

Fix any formula $\forall x(q(x) \rightarrow \gamma) \in \Gamma$. Note that the clause γ is actually a disjunction of atoms featuring only the predicates in (31) and (32). Let r_γ be a new binary predicate. Since $\mathfrak{A} \models \forall x(q(x) \rightarrow \gamma)$, define, for each $a \in A_1$, the element $a_\gamma \in A_2$ as follows. Choose a literal (atom) L of γ satisfied by a: if L is $q_h(x)$ for some h $(1 \leq h \leq s)$, set $a_\gamma = \langle h, \top \rangle$; if, on the other hand, L is $\bar{q}_h(x)$ for some h $(1 \leq h \leq s)$, set $a_\gamma = \langle h, \perp \rangle$. Think of the object a_γ as representing some literal of γ satisfied by a. Having defined a_γ for all $a \in A_1$, set

$$r_\gamma^{\mathfrak{A}} = \{\langle a, a_\gamma \rangle | a \in A_1\}.$$

It is then easy to check that $\mathfrak{A} \models \Theta_\gamma$, where Θ_γ consists of the formula

$$\forall x(q(x) \rightarrow \exists y(l(y) \wedge r_\gamma(x, y))), \tag{33}$$

together with the following formulas, for all h $(1 \leq h \leq s)$:

$$\forall x(q(x) \rightarrow \neg \exists y(l_h(y) \wedge r_\gamma(x, y))) \qquad (\text{if } q_h(x) \text{ not a literal of } \gamma) \tag{34}$$

$$\forall x(q(x) \rightarrow \neg \exists y(\bar{l}_h(y) \wedge r_\gamma(x, y))) \qquad (\text{if } \bar{q}_h(x) \text{ not a literal of } \gamma) \tag{35}$$

$$\forall x(q_h(x) \rightarrow \neg \exists y(\bar{l}_h(y) \wedge r_\gamma(x, y))) \tag{36}$$

$$\forall x(\bar{q}_h(x) \rightarrow \neg \exists y(l_h(y) \wedge r_\gamma(x, y))). \tag{37}$$

Conversely, in any model of $\Theta_6 \cup \Theta_\gamma$, (33) guarantees that every object a in the extension of q is related via r_γ to some object a_γ in the extension of l (representing a literal); (34) and (35) then state that the literal represented by a_γ is a literal of the clause γ; and (36) and (37) state that a satisfies this literal. Together, Θ_0, Θ_1, Θ_5, Θ_6 and Θ_γ thus guarantee that any object satisfying q also satisfies the clause γ; in other words:

$$\Theta_0 \cup \Theta_1 \cup \Theta_5 \cup \Theta_6 \cup \Theta_\gamma \models \forall x(q(x) \rightarrow \gamma). \tag{38}$$

Let $\Theta_7 = \bigcup\{\Theta_\gamma \mid \forall x(q(x) \rightarrow \gamma) \in \Gamma\}$.

Let $\Theta_{\mathbf{i}} = \Theta_0 \cup \cdots \cup \Theta_7$. The construction of $\Theta_{\mathbf{i}}$ evidently proceeds in time bounded by a polynomial function of the length n of $\mathbf{i}$. Every formula in $\Theta_{\mathbf{i}}$ is an $\mathcal{N}^2$-formula, modulo trivial logical manipulations. And (C, H, V) has a tiling of size $N = 2^{p(n)}$ with initial segment $\mathbf{i}$ if and only if $\Theta_{\mathbf{i}}$ is satisfiable. $\dashv$

We remark that the proof of Theorem 3 makes essential use of binary coding of quantifier subscripts. For example, the subscript MN^2 has size $\lfloor 2p(n) + \log M \rfloor + 1$, and hence is bounded by a polynomial function of n.

COROLLARY 2. *The satisfiability problem and finite satisfiability problem for any logic between $\mathcal{N}^2$ and $\mathcal{C}^2$ are both* NEXPTIME-*complete.*

It follows that determining the validity of arguments in the numerically definite relational syllogistic is a co-NEXPTIME-complete problem. Equipping this fragment with relative clauses, for example,

At most 3 artists whom at least 4 beekeepers admire despise at least 5 dentists who envy at most 6 electricians,

evidently has no effect on the complexity of determining validity, since it does not take us outside the fragment $\mathcal{C}^2$. Nor do proper nouns or negated verb-phrases, for example

At most 3 artists do not despise (= fail to despise) at least one beekeeper
At least 3 artists despise Fred.

In fact, we may add a certain amount of anaphora to the fragment while still remaining within $\mathcal{C}^2$, thus:

At most 3 artists who despise themselves admire at least 4 beekeepers who envy them,

$$\exists_{\leq 3} x(\text{artist}(x) \wedge \text{despise}(x, x) \wedge \\ \exists_{\geq 4} y(\text{beekeeper}(y) \wedge \text{envy}(y, x) \wedge \text{admire}(x, y)))$$

though care evidently has to be taken in specifying the precise interpretation of pronouns. However, the complexity-theoretic consequences of extending the repertoire of quantifiers in $\mathcal{C}^2$—for example, to include such constructions as "for most x, ϕ"—are unknown.

Although adding relative clauses to $\mathcal{N}^2$ does not increase the complexity of satisfiability, it nevertheless has other repercussions of a logical nature. For one thing, we lose the finite model property: the sentences

At least 1 p rs at most 0 ps
At most 0 ps are ps which at most 0 ps r
At most 0 ps r at least 2 ps,

which, in essence, reproduce the content of formula (23), are satisfiable, but not finitely so. Hence the satisfiability and finite satisfiability problems, though both NEXPTIME-complete, are distinct. Interestingly, the addition of relative clauses also affects the question of *strong* NEXPTIME-completeness. Inspection of the proof of Theorem 3 shows that binary coding of quantifier subscripts is required only to overcome the lack of Boolean connectives in $\mathcal{N}^2$ (specifically, in simulating the effect of the formulas Γ with the formulas of Θ_5–Θ_7, or stating that the colours must exhaust the grid).

Adding relative clauses obviates the need for these contortions; and it is in fact easily checked that the satisfiability problem and the finite satisfiability problem for this fragment are strongly NEXPTIME-complete. This difference is noteworthy, because some other fragments with counting quantifiers discussed in the literature have satisfiability and finite satisfiability problems whose complexity is insensitive to whether quantifier subscripts are coded in unary or binary, for example, graded modal logic (Tobies [22]), the guarded two-variable fragment with counting (Kazakov [11], Pratt-Hartmann [20]) or indeed $\mathcal{C}^2$ itself.

The following related facts regarding fragments of English not featuring numerical determiners are shown in Pratt-Hartmann and Third [21]: if sentences involving transitive verbs are added to the ordinary syllogistic (without numerical quantifiers), the satisfiability problem for the resulting fragment remains in PTIME; if sentences involving both transitive verbs and relative clauses are added to the ordinary syllogistic, the satisfiability problem for the resulting fragment is EXPTIME-complete.

§5. **Numerically definite syllogisms.** After the initial investigations by de Morgan, Boole and Jevons, interest in the numerically definite syllogistic waned. In recent decades, however, various proof-systems have been proposed, based on numerical generalizations of the traditional syllogisms. Good examples are the deduction systems of Murphree [13], for the language $\mathcal{N}^{1+}$, and of Hacker and Parry [8] for the language $\mathcal{N}^1$. In this section, we use the results of the foregoing analysis to explore the possibility of developing a system of numerically definite syllogisms which is complete, in the sense that all valid sequents become derivable.

We start by adapting some familiar Aristotelian syllogisms in the obvious way. In the sequel, L, L_1, L_2 and L_3 range over non-ground literals of $\mathcal{C}^1$—i.e., formulas of the forms $p(x)$ or $\neg p(x)$. Thus, the formulas of $\mathcal{N}^{1+}$ simply have the forms

$$\exists_{\geq C} x(L_1 \wedge L_2) \qquad\qquad \exists_{\leq C} x(L_1 \wedge L_2).$$

For convenience, we regard $\exists_{\geq C} x(L_1 \wedge L_2)$ and $\exists_{\geq C} x(L_2 \wedge L_1)$ as identical, and similarly for their $\exists_{\leq C}$-quantified counterparts. In addition, we allow negative numbers to appear in quantifier subscripts, again with the obvious semantics: for $C < 0$, $\exists_{\leq C} x(L_1 \wedge L_2)$ is trivially false, and $\exists_{\geq C} x(L_1 \wedge L_2)$ trivially true. If L is a literal, let $\bar{L}$ denote its opposite—that is, the literal formed by removing any double negation from $\neg L$. Under these conventions, define $\mathcal{M}$ to be the deduction system with axiom schemas

$$\frac{}{\exists_{\geq 0} x(L_1 \wedge L_2)}\ (\text{Ax1}) \qquad\qquad \frac{}{\exists_{\leq C} x(L \wedge \bar{L})}\ (\text{Ax2}) \qquad (\text{if } C \geq 0)$$

and rules of inference

$$\frac{\exists_{\leq C}x(L_1 \wedge L_2) \quad \exists_{\leq D}x(\bar{L}_2 \wedge L_3)}{\exists_{\leq (C+D)}x(L_1 \wedge L_3)} \text{ (B)} \qquad \frac{\exists_{\leq C}x(L_1 \wedge L_2) \quad \exists_{\geq D}x(L_3 \wedge L_1)}{\exists_{\geq (D-C)}x(L_3 \wedge \bar{L}_2)} \text{ (D)}$$

$$\frac{\exists_{\leq C}x(L_1 \wedge L_1) \quad \exists_{\geq D}x(L_1 \wedge L_2)}{\exists_{\leq (C-D)}x(L_1 \wedge \bar{L}_2)} \text{ (N)}$$

$$\frac{\exists_{\leq C}x(L_1 \wedge L_2) \quad \exists_{\geq D}x(L_1 \wedge L_2)}{\psi} \text{ (X)} \qquad \text{(if } C < D\text{)}$$

Very roughly, Rule (B) may be thought of as a numerical form of *Barbara*, and Rule (D), a numerical form of *Darii*. Rule (N) has no classical counterpart. Rule (X), which applies only if $C < D$, allows the derivation of any formula ψ whatsoever from contradictory premises: i.e., it is a numerical version of *ex falso quodlibet*. We write $\Phi \vdash_{\mathcal{M}} \phi$ if there is a deduction from premises Φ to conclusion ϕ in $\mathcal{M}$.

The system $\mathcal{M}$ is at least as powerful as that of Murphree, once notational differences are taken into account. And Murphree's system is in turn at least as powerful as that of Hacker and Parry. Nevertheless, $\mathcal{M}$ is not complete for the language $\mathcal{N}^{1+}$. For example, the valid sequent

$$\exists_{\geq C}x(p(x) \wedge q(x)), \exists_{\geq D}x(p(x) \wedge \neg q(x)) \models \exists_{\geq (C+D)}x(p(x) \wedge p(x)) \quad (39)$$

is not derivable in $\mathcal{M}$ if $C + D > 0$. To see this, let $\Phi = \{\exists_{\geq C}x(p(x) \wedge q(x)), \exists_{\geq D}x(p(x) \wedge \neg q(x))\}$. Since Φ is consistent, Rule (X) can never be used in a deduction from Φ, and so may, for present purposes, be ignored. By a simple induction on the lengths of proofs, $\Phi \vdash_{\mathcal{M}} \exists_{\leq E}(L_1 \wedge L_2)$ implies $L_1 = \bar{L}_2$. Hence, in a derivation from Φ, Rule (N) can never be used, and any application of Rule (D) in must take the form

$$\frac{\exists_{\leq C}x(L_2 \wedge \bar{L}_2) \quad \exists_{\geq D}x(L_1 \wedge L_2)}{\exists_{\geq (D-C)}x(L_1 \wedge L_2)} \text{ (D)}.$$

It then follows that, for $E > 0$, $\Phi \vdash_{\mathcal{M}} \exists_{\geq E}x(L_1 \wedge L_2)$ implies that $\{L_1, L_2\}$ is equal either to $\{p(x), q(x)\}$ or to $\{p(x), \neg q(x)\}$.

On the other hand, it might be argued that the systems of numerical syllogisms found in the literature were never in fact designed to handle examples such as this. To make the analysis of these systems fairer, therefore, we consider the special case of the validity problem in which it is known how many objects satisfy each predicate in question, and how many objects fail to do so. Formally, we consider only inference problems from *numerically explicit* premise sets, in the following sense.

DEFINITION 2. Let Φ be a set of $\mathcal{N}^{1+}$-formulas, and let $p_1, \ldots, p_n$ be the predicates appearing in Φ. We say that Φ is *numerically explicit* if there exist natural numbers $C, C_1, \ldots, C_n$ with $C > 0$ such that, for all i $(1 \leq i \leq n)$, (i) $C_i \leq C$, and (ii) Φ contains the formulas

$$\begin{array}{ll} \exists_{\leq C_i} x(p_i(x) \wedge p_i(x)) & \exists_{\leq (C-C_i)} x(\neg p_i(x) \wedge \neg p_i(x)) \\ \exists_{\geq C_i} x(p_i(x) \wedge p_i(x)) & \exists_{\geq (C-C_i)} x(\neg p_i(x) \wedge \neg p_i(x)). \end{array}$$

The list of numerically definite syllogisms given in de Morgan [4], p. 161, all of which can be generated by combinations of our rules (B), (D) and (N), suggests that de Morgan primarily had in mind situations in which information about the numbers of objects satisfying various relevant properties is available.

Might the system $\mathcal{M}$ be complete for numerically explicit sets of premises? Certainly, the sequent (39) is derivable under this restriction. Specifically, if Φ is *any* numerically explicit premise set containing $\exists_{\geq C} x(p(x) \wedge q(x))$ and $\exists_{\geq D} x(p(x) \wedge \neg q(x))$, then $\Phi \vdash_{\mathcal{M}} \exists_{\geq (C+D)} x(p(x) \wedge p(x))$. For suppose $\exists_{\leq E} x(p(x) \wedge p(x)), \exists_{\geq E} x(p(x) \wedge p(x)) \in \Phi$. If $E < C + D$, we have the derivation

$$\cfrac{\cfrac{\exists_{\leq E} x(p(x) \wedge p(x)) \quad \exists_{\geq C} x(p(x) \wedge q(x))}{\exists_{\leq (E-C)} x(p(x) \wedge \neg q(x))}\,(\mathrm{N}) \qquad \exists_{\geq D} x(p(x) \wedge \neg q(x))}{\exists_{\geq (C+D)} x(p(x) \wedge p(x))}\,(\mathrm{X}).$$

If, on the other hand, $E \geq C + D$, we have the derivation

$$\cfrac{\cfrac{}{\exists_{\leq (E-C-D)} x(p(x) \wedge \neg p(x))}\,(\mathrm{Ax2}) \qquad \exists_{\geq E} x(p(x) \wedge p(x))}{\exists_{\geq (C+D)} x(p(x) \wedge p(x))}\,(\mathrm{D}).$$

Unfortunately, the prospects for a complete system of numerical syllogisms, even for the special case of numerically explicit premise sets, are not bright. For we shall exhibit a numerically explicit set of $\mathcal{N}^{1+}$-formulas Φ and an $\mathcal{N}^{1+}$-formula ϕ such that $\Phi \models \phi$, but $\Phi \nvdash_{\mathcal{M}} \phi$. Moreover, this incompleteness result will be seen to be robust under a wide range of conceivable extensions of $\mathcal{M}$.

For readability, we shall henceforth contract $\mathcal{N}^{1+}$-formulas with repeated literals: thus, $\exists_{\leq C} x(p(x) \wedge p(x))$ becomes $\exists_{\leq C} x p(x)$, etc. We use the quantifier $\exists_{=C}$ to abbreviate the obvious pair of formulas involving $\exists_{\leq C}$ and $\exists_{\geq C}$. And we write $\mathcal{N}^1$-formulas of the form $\exists_{\leq 0} x(p(x) \wedge \pm q(x))$ in their more familiar guise: $\forall x(p(x) \rightarrow \mp q(x))$. Fix $m \geq 6$. Let $\mathbf{A}$ and $\mathbf{c}$ be as in Lemma 4, and let Φ_1 be the set of $\mathcal{N}^1$-formulas consisting of

$$\exists_{\leq 3(m+1)} x t(x) \tag{40}$$

$$\exists_{\geq 3} x t_j(x) \qquad (1 \leq j \leq m+1) \tag{41}$$

$$\forall x(t_j(x) \rightarrow t(x)) \qquad (1 \leq j \leq m+1) \tag{42}$$

$$\forall x(t_j(x) \to \neg t_{j'}(x)) \qquad (1 \le j < j' \le m+1) \tag{43}$$

$$\forall x(s_i(x) \to t(x)) \qquad (1 \le i \le m) \tag{44}$$

$$\forall x(t_j(x) \to s_i(x)) \qquad (1 \le i \le m, 1 \le j \le m+1, \mathbf{A}_{i,j} = 1) \tag{45}$$

$$\forall x(t_j(x) \to \neg s_i(x)) \qquad (1 \le i \le m, 1 \le j \le m+1, \mathbf{A}_{i,j} = 0) \tag{46}$$

$$\exists_{=3} x(s_i(x) \wedge r(x)) \qquad (1 \le i \le m-1) \tag{47}$$

$$\exists_{=4} x(s_m(x) \wedge r(x)). \tag{48}$$

Note that the list of quantifier subscripts in the m formulas of (47) and (48) matches the vector $\mathbf{c}$.

Claim 1. *For all j $(1 \le j \le m+1)$, $\Phi_1 \models \exists_{\ge 1} x(t_j(x) \wedge r(x))$.*

Proof. Suppose $\mathfrak{A} \models \Phi_1$. From (40)–(43), $t^{\mathfrak{A}}$ is partitioned into the pairwise disjoint sets $t_1^{\mathfrak{A}}, \ldots, t_{m+1}^{\mathfrak{A}}$. And so, from (44)–(46), we have, for all i $(1 \le i \le m)$,

$$s_i^{\mathfrak{A}} = \bigcup \{t_j^{\mathfrak{A}} \mid \mathbf{A}_{i,j} = 1\},$$

and hence

$$|s_i^{\mathfrak{A}} \cap r^{\mathfrak{A}}| = \sum \{|t_j^{\mathfrak{A}} \cap r^{\mathfrak{A}}| \colon \mathbf{A}_{i,j} = 1\}.$$

Therefore, (47) implies, for all i $(1 \le i \le m-1)$,

$$\sum \{|t_j^{\mathfrak{A}} \cap r^{\mathfrak{A}}| \colon \mathbf{A}_{i,j} = 1\} = 3,$$

while (48) implies

$$\sum \{|t_j^{\mathfrak{A}} \cap r^{\mathfrak{A}}| \colon \mathbf{A}_{m,j} = 1\} = 4.$$

In other words, $(|t_1^{\mathfrak{A}} \cap r^{\mathfrak{A}}|, \ldots, |t_{m+1}^{\mathfrak{A}} \cap r^{\mathfrak{A}}|)^T$ is a solution of $\mathbf{A}\mathbf{x} = \mathbf{c}$. Applying Lemma 4, $|t_j^{\mathfrak{A}} \cap r^{\mathfrak{A}}| = 1$ for all j $(1 \le j \le m+1)$, which proves the claim. ⊣

Now let Φ_2 be the set of $\mathcal{N}^{1+}$-formulas

$$\begin{array}{lll}
\exists_{=3(m+1)} x\, t(x) & \exists_{=3(m+1)} x \neg t(x) & \\
\exists_{=3} x\, t_j(x) & \exists_{=6m+3} x \neg t_j(x) & (1 \le j \le m+1) \\
\exists_{=9} x\, s_i(x) & \exists_{=6m-3} x \neg s_i(x) & (1 \le i \le m-1) \\
\exists_{=12} x\, s_m(x) & \exists_{=6m-6} x \neg s_m(x) & \\
\exists_{=3(m+1)} x\, r(x) & \exists_{=3(m+1)} x \neg r(x), &
\end{array}$$

and let $\Phi = \Phi_1 \cup \Phi_2$. Thus, Φ is numerically explicit. Moreover, Φ is satisfiable: Fig. 1(a) depicts a (in fact, *the*) model $\mathfrak{A}$ of Φ. The domain A has cardinality $6(m+1)$, equally split between $t^{\mathfrak{A}}$ and its complement; the sets $t_j^{\mathfrak{A}}$ $(1 \le j \le m+1)$ partition $t^{\mathfrak{A}}$ into 3-element sets; the set $r^{\mathfrak{A}}$ has cardinality $3(m+1)$; and the sets $t_j^{\mathfrak{A}} \cap r^{\mathfrak{A}}$ are all singletons. The extensions of the s_i (not indicated in Fig. 1(a), for clarity) are all unions of various $t_j^{\mathfrak{A}}$, as specified by the matrix $\mathbf{A}$. Nevertheless, the validities reported in Claim 1 cannot all be reproduced by the proof-system $\mathcal{M}$.

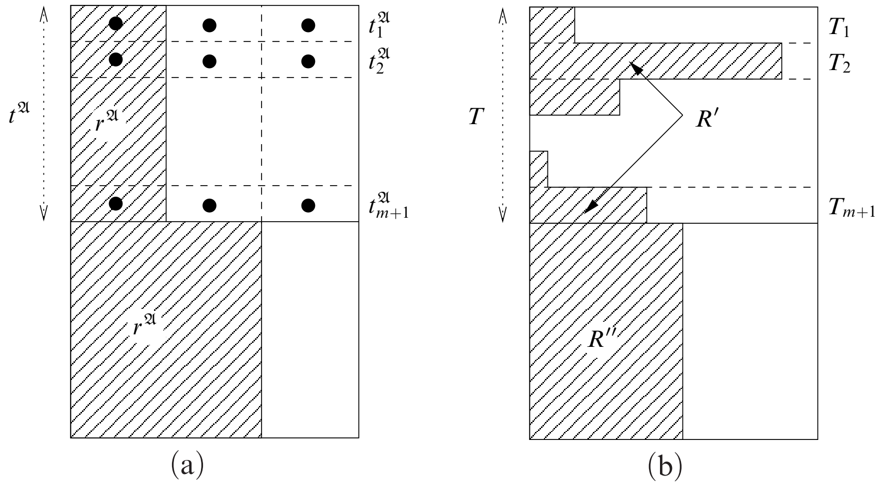

FIGURE 1. Models of Φ: (a) standard semantics; (b) probabilistic semantics. In (b), one of the sets $R_j = R' \cap T_j$ is empty.

CLAIM 2. *There exists a* j $(1 \leq j \leq m+1)$ *such that* $\Phi \nvdash_{\mathcal{M}} \exists_{\geq 1} x(t_j(x) \wedge r(x))$.

PROOF. Fix $N = 6(m+1)$. We give a completely new semantics for the language $\mathcal{N}^{1+}$ as follows. Let Σ be the (assumed countable) set of all unary predicates available to $\mathcal{N}^{1+}$. Let us re-badge the elements $p_1, p_2, \ldots$ of Σ as proposition letters; and, as before, we let $\mathcal{S}$ denote the propositional language over this signature. If P is a probability assignment for $\mathcal{S}$, we interpret $\mathcal{N}^{1+}$ with respect to P by writing

$$P \approx\!\!\!| \; \exists_{\geq C} x(p_1(x) \wedge p_2(x)) \text{ if and only if } P(p_1 \wedge p_2) \geq C/N,$$

and similarly for all the other forms of $\mathcal{N}^{1+}$. (Remember that N is a constant here.) It is readily verified that the proof-system $\mathcal{M}$ is sound for the $\approx$-semantics: all instances of the axiom schemas are true, and all instances of the inference rules are truth-preserving. We proceed to construct a probability assignment P such that: (i) $P \approx \Phi$, and (ii) for some j $(1 \leq j \leq m+1)$, $P \not\approx \exists_{\geq 1} x(t_j(x) \wedge r(x))$. It follows that, for this j, $\Phi \nvdash_{\mathcal{M}} \exists_{\geq 1} x(t_j(x) \wedge r(x))$.

By Lemma 2, the equations $\mathbf{Ax} = \mathbf{c}$ have a solution $u_1, \ldots, u_{m+1}$ over $\mathbb{Q}^+$ with at least one zero value. On the other hand, it is obvious from examination of $\mathbf{A}$ and $\mathbf{c}$ that u_j must be less than or equal to 3 for all j $(1 \leq j \leq m+1)$. Let u be the least common multiple of all the (non-zero) denominators in the u_j; let W be a set of $uN = 6u(m+1)$ objects (henceforth: "worlds"); and let T be a subset of W of cardinality $3u(m+1)$. Now let T be partitioned into cells $T_1, \ldots, T_{m+1}$, each of which contains

$3u$ worlds. For each j $(1 \leq j \leq m+1)$, let R_j be a subset of T_j of cardinality uu_j (which must be a natural number no greater than $3u$), and let $R' = \bigcup\{R_j \mid 1 \leq j \leq m+1\}$. Since $R' \subseteq T$, $|T| = 3u(m+1)$, and $|W| = 6u(m+1)$, we may choose a set $R'' \subseteq W \setminus T$ such that the set $R = R' \cup R''$ has cardinality 3u(m+1). Finally, for each i $(1 \leq i \leq m)$, let $S_i = \bigcup\{T_j \mid \mathbf{A}_{i,j} = 1\}$. Thus, S_i has cardinality $9u$ for all i $(1 \leq i \leq m-1)$, and S_m has cardinality $12u$. This arrangement is depicted, schematically, in Fig. 1(b), except that the S_i are not indicated, for clarity. Note, however, that, because $u_1, \ldots, u_{m+1}$ is a solution of $\mathbf{Ax} = \mathbf{c}$, $|S_i \cap R| = 3u$, for all i $(1 \leq i \leq m-1)$, and $|S_m \cap R| = 4u$.

We associate with each world $w \in W$ a truth-value assignment θ_w for the propositional language $\mathcal{S}$ by setting $\theta_w(s_i) = \top$ if and only if $w \in S_i$, $\theta_w(t_j) = \top$ if and only if $w \in T_j$, $\theta_w(t) = \top$ if and only if $w \in T$, $\theta_w(r) = \top$ if and only if $w \in R$, and $\theta_w(p) = \bot$ for all other proposition-letters p. Then we define the probability assignment P by taking a flat distribution on W. That is, for every $\phi \in \mathcal{S}$, set

$$P(\phi) = |\{w \in W : \theta_w \models \phi\}|/(uN).$$

It is simple to check that $P \models_{\approx} \Phi$. On the other hand, at least one of the u_j is zero; and for this value of j, $P(t_j \wedge r) = 0$, whence $P \not\models_{\approx} \exists_{\geq 1} x(t_j(x) \wedge r(x))$. $\dashv$

Hence we have:

THEOREM 4. *The proof-system $\mathcal{M}$ is not complete, even for numerically explicit sets of premises.*

Theorem 4 is robust with respect to *any* strengthening of $\mathcal{M}$ that is sound under the probabilistic interpretation in the proof of Claim 2. We mention that, in another paper, Murphree presents a language similar to our $\mathcal{N}^2$ (Murphree [14]); however, no systematic proof theory is developed. In fact, we are not aware of any published system of numerically definite syllogisms which has been shown to be complete.

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