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DESCRIPTIVE SET THEORY OF FAMILIES OF SMALL SETS

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Abstract. This is a survey paper on the descriptive set theory of hereditary families of closed sets in Polish spaces. Most of the paper is devoted to ideals and σ-ideals of closed or compact sets.

Contents

Introduction. There exist quite a lot of natural notions of smallness in analysis. To name a few, one can mention countability, smallness in the sense of measure or category, polarity, porosity, or the various types of thinness arising in harmonic analysis.

The aim of the present paper is to survey some recent and less recent results concerning the descriptive set theory of such notions of smallness. More specifically, we will be concerned with families of small *closed* or *compact* subsets of some Polish space X. We will denote by $\mathcal{F}(X)$ the family of all closed subsets of X, and by $\mathcal{K}(X)$ the family of all compact subsets of X. Then $\mathcal{K}(X)$ becomes itself a Polish space when equipped with the Vietoris topology, and $\mathcal{F}(X)$ becomes a standard Borel space when equipped with

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the Effros Borel structure; hence it makes sense to speak of the descriptive properties of a family of closed or compact sets.

If a family $\mathcal{I} \subseteq \mathcal{F}(X)$ is intended to define a notion of smallness, then it should obviously be *hereditary* for inclusion: if $F \in \mathcal{I}$, then each closed set contained in F is again in $\mathcal{I}$. Moreover, many interesting examples also have some additional structural property such as, for example, that of being stable under some set-theoretic operation. A family $\mathcal{I} \subseteq \mathcal{F}(X)$ is said to be an *ideal* of $\mathcal{F}(X)$ if it is hereditary and closed under finite unions; and $\mathcal{I}$ is said to be a σ-*ideal* if it is hereditary and stable under countable closed unions. The same terminology is used for families of compact sets. Most of the paper will be devoted to ideals and σ-ideals. We now describe the content of the paper.

In the first chapter, we collect several complexity results. Since the seminal paper [KLW] of A. S. Kechris, A. Louveau, and W. H. Woodin, it is well-known that there are strong limitations on the possible complexity of a σ-ideal of compact sets. For example, the *Dichotomy Theorem* asserts that a $\mathbf{\Pi}^1_1$ σ-ideal of compact sets is either $\mathbf{\Pi}^1_1$-complete or G_δ. We recall here some results from [KLW] and [K4] in that direction. Then we discuss the implications of the existence of a simple *basis* for a σ-ideal; again, most of the results come from [KLW] and [K4]. Next, we turn to the complexity of *ideals* of compact sets. We state a result from [MSZ] showing that for ideals, there exist some exclusion phenomena at the third level of the Borel hierarchy. Then we describe the remarkable result of F. van Engelen [vE] giving the complete list of the possible Wadge classes for ideals of compact subsets of the Cantor space 2^ω.

In the second chapter, we consider several specific examples of ideals and σ-ideals. Quite a lot of them come from harmonic analysis: sets of uniqueness and of extended uniqueness, Helson sets, and other related ideals. Actually, the study of thin sets from harmonic analysis motivated to a large extent many abstract results concerning ideals or σ-ideals. The descriptive set-theoretic point of view turned out to be very fruitful in that area, as can be seen by going through the beautiful book [KL1] or the survey paper [KL3]. However, there are many other interesting examples. Here, we describe some results concerning σ-porous sets [ZP, ZaZe1, ZeZa], smooth sets for a Borel equivalence relation [U2], Haar-null sets [S2, S5], and "sets of continuity" for Borel functions [J1, J2].

The third chapter is focused on the so-called *covering property*, which is the natural analogue of the classical perfect set property for $\mathbf{\Sigma}^1_1$ sets in the σ-ideal setting. A σ-ideal $\mathcal{I} \subseteq \mathcal{F}(X)$ has the covering property if every $\mathbf{\Sigma}^1_1$ set which cannot be covered by countably many sets from $\mathcal{I}$ contains a closed set $F \notin \mathcal{I}$. We start with a very general covering theorem of S. Solecki [S1], which allows to check the covering property on G_δ sets only. Then we describe a theorem of G. Debs and J. Saint Raymond [DSR], which relates the covering property to the existence of a "nontrivial" basis and the weaker property of *calibration*; this result led to the solution of the longstanding *category problem* for Borel sets of uniqueness, which was raised by N. K. Bary in the 1920's (see [Bary1, pp. 80 and 112]). Next, we discuss a result of Louveau [L2] about families of closed sets which are of *well-founded type*. Then we describe the work of C. Uzcátegui on the covering property [U1, U3]. Finally, we discuss some results concerning *thinness* of σ-ideals [KLW, Ze2, KS].

The fourth chapter is devoted to *polar* σ-ideals of compact sets, that is, σ-ideals whose members are the common null-sets of some family of measures. We mainly describe some of the nice results obtained by Debs in [D].

The fifth chapter contains some miscellaneous results about families of small compact sets which are *not* necessarily ideals or σ-ideals: increasing unions of thin sets from harmonic analysis [BKL], σ-ideals of continua [Ca], families of compact sets which contain a dense G_δ hereditary set [MZ], and families of compact sets defined by some "independence" property [MZ].

We conclude with several open problems.

The reader will certainly have noticed that we said nothing about ideals in the set of natural numbers ω. The main reason is that the theory of ideals of subsets of ω is by now extremely developed, and although some aspects of it are close in spirit to the topics treated in the present paper (see [F1], [FS], [S3], [S4] or [V]), a very substantial part is considerably faraway from what we intended to discuss. The reader may consult, e.g., [F2].

We do not discuss either the very interesting connections between thin sets from harmonic analysis and infinite combinatorics, for which the reader may consult [BKR]. Nor do we venture into the world of ideals or σ-ideals of *arbitrary* subsets of X; see, e.g., [Ba], [BaRo], [BRS], [Mau] or [RZ] for a sample of the kind of results one may encounter in that area.

This paper is intended to be accessible to anyone with a background in descriptive set theory; such a background and much more can be found in [K6]. Most results are stated without proofs, but in some places it seemed important to give at least the main ideas. We also included a few original results, for which we gave complete proofs.

Notations and terminology. We now fix the notations and terminology that will be used throughout the whole paper.

The symbol X will always stand for a nonempty Polish space.

We recall that the Vietoris topology on $\mathcal{K}(X)$ is the topology generated by all sets of the form $\{K;\ K \subseteq U\}$ or $\{K;\ K \cap U \neq \emptyset\}$, where U is an open subset of X. This topology is Polish, and it is compact if X is compact. The Effros Borel structure on $\mathcal{F}(X)$ is the σ-algebra generated by all sets of the form $\{F;\ F \cap U \neq \emptyset\}$, with $U \subseteq X$ open. Equipped with that structure, $\mathcal{F}(X)$ is a standard Borel space.

The *hereditary closure* of a family $\mathcal{L} \subseteq \mathcal{K}(X)$ is the smallest hereditary subset of $\mathcal{K}(X)$ containing $\mathcal{L}$, i.e., the family $\{K \in \mathcal{K}(X);\ \exists L \in \mathcal{L}\colon K \subseteq L\}$. It is easily checked that if $\mathcal{L}$ is compact (analytic, respectively), then its hereditary closure is also compact (analytic, respectively).

We will denote by $\mathcal{K}^*(X)$ the family of all *nonempty* compact subsets of X. If A is a subset of X, we denote by $\mathcal{K}(A)$ the family of all compact subsets of A, and by $\mathcal{F}_X(A)$ the family of all closed subsets of X contained in A.

The family of all meager subsets of X is denoted by $\mathrm{MGR}(X)$.

Whenever we speak of a *measure* on X, this will always mean a positive finite Borel measure. At some places we consider complex measures, and we explicitly indicate it.

We use the standard notations of descriptive set theory. In particular, we use the symbols $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$ to denote the additive and multiplicative Borel classes, and we write $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ for the analytic and coanalytic sets respectively. Occasionally, we will need the corresponding "lightface" notations Σ^1_1, Π^1_1 from effective descriptive set theory. See, e.g., [L1] or [MarKe] for a short introduction to that area.

If $\mathbf{\Gamma}$ is a class of subsets in Polish spaces then the symbol $\mathrm{D}_2(\mathbf{\Gamma})$ denotes the class of differences of sets from $\mathbf{\Gamma}$ and $\check{\mathrm{D}}_2(\mathbf{\Gamma})$ stands for the dual class.

We recall that if $\mathbf{\Gamma}$ is a class of sets in Polish spaces, then a set A in some Polish space Y is said to be $\mathbf{\Gamma}$-*hard in* Y if, for each 0-dimensional Polish space Z and each set $B \subset Z$ of class $\mathbf{\Gamma}$, one can find a continuous map $\phi\colon Z \to Y$ such that $\phi^{-1}(A) = B$. The set A is said to be $\mathbf{\Gamma}$-*complete in* Y if it is both $\mathbf{\Gamma}$-hard in Y and of class $\mathbf{\Gamma}$. We write simply $\mathbf{\Gamma}$-hard and $\mathbf{\Gamma}$-complete instead of $\mathbf{\Gamma}$-hard in Y and $\mathbf{\Gamma}$-complete in Y respectively whenever it is clear which space Y is considered.

Finally, we employ the following notations. If I is a set, we denote by $I^{<\omega}$ the set of all finite sequences of elements of I, and by I^ω the set of all infinite sequences. Given $s \in I^{<\omega}$ and $v \in I^{<\omega} \cup I^\omega$, we write $s \preceq v$ if v is an extension of s. We denote the concatenation of $s \in I^{<\omega}$ and $t \in I^{<\omega} \cup I^\omega$ by $s^\wedge t$. If $s \in I^{<\omega}$ and $i \in I$, we write $s^\wedge i$ instead of $s^\wedge(i)$. If $v = (v_0, v_1, v_2, \dots) \in I^\omega$ and $n \in \omega$, then the symbol $v_{|n}$ means the finite sequence $(v_0, v_1, \dots, v_{n-1})$. If $t \in I^{<\omega}$, then the symbol $|t|$ denotes the length of t.

§1. Complexity results.

1.1. Exclusion phenomena for σ-ideals. In this section, we state two theorems showing that there are severe limitations on the possible complexity of a σ-ideal of compact sets.

We start with a basic lemma from [KLW]. If $\mathbf{\Gamma}$ is a class of sets in Polish spaces, we denote by $\forall\mathbf{\Gamma}$ the class of all sets of the form

$$B = \forall C := \{z \in Z;\ \forall \alpha \in 2^\omega\colon (z, \alpha) \in C\},$$

where Z is Polish and $C \in \mathbf{\Gamma}(Z \times 2^\omega)$. In particular, $\forall\mathbf{\Sigma}^0_2$ is the class $\mathbf{\Pi}^1_1$.

LEMMA 1.1. *Let $\mathbf{\Gamma}$ be a class of sets in Polish spaces. If $A \subseteq X$ is $\mathbf{\Gamma}$-hard, then $\mathcal{K}(A)$ is $\forall\mathbf{\Gamma}$-hard.*

PROOF. Let B be a $\forall\mathbf{\Gamma}$ set in some 0-dimensional Polish space Z, and choose $C \in \Gamma(Z \times 2^\omega)$ with $B = \forall C$. If $A \subseteq X$ is $\mathbf{\Gamma}$-hard, one can choose a continuous map $\phi\colon Z \times 2^\omega \to X$ such that $\phi^{-1}(A) = C$. The map $\Phi\colon Z \to \mathcal{K}(X)$ defined by $\Phi(z) = \phi[\{z\} \times 2^\omega]$ is continuous, and $\Phi^{-1}(\mathcal{K}(A)) = B$. Since B is arbitrary in $\forall\mathbf{\Gamma}$, this proves the lemma. ⊣

Let $\boldsymbol{Q}$ be the set of all rational points in 2^ω, that is

$$\boldsymbol{Q} = \{\alpha \in 2^\omega;\ \exists N \in \omega\ \forall n \geq N : \alpha_n = 0\}.$$

The σ-ideal $\mathcal{K}(\boldsymbol{Q})$ is easily seen to be $\mathbf{\Pi}^1_1$. Since $\boldsymbol{Q}$ is $\mathbf{\Sigma}^0_2$-complete, we get the following classical result of W. Hurewicz [Hur].

Example 1.2. The σ-ideal $\mathcal{K}(\boldsymbol{Q})$ is $\mathbf{\Pi}^1_1$-complete.

COROLLARY 1.3. *Let $\mathcal{I}$ be a σ-ideal in $\mathcal{K}(X)$. If $\mathcal{I}$ is $\mathbf{\Sigma}^0_2$-hard, then it is $\mathbf{\Pi}^1_1$-hard.*

PROOF. Assume $\mathcal{I}$ is $\mathbf{\Sigma}^0_2$-hard, and let $\phi\colon 2^\omega \to \mathcal{K}(X)$ be a continuous map such that $\phi^{-1}(\mathcal{I}) = \boldsymbol{Q}$. The map $\Phi\colon \mathcal{K}(2^\omega) \to \mathcal{K}(X)$ defined by $\Phi(K) = \bigcup_{x\in K} \phi(x)$ is continuous, and $\Phi^{-1}(\mathcal{I}) = \mathcal{K}(\boldsymbol{Q})$ because $\mathcal{I}$ is a σ-ideal and $\boldsymbol{Q}$ is countable. By the above example, this shows that $\mathcal{I}$ is $\mathbf{\Pi}^1_1$-hard. ⊣

The following theorem gives a complete picture of the admissible classes below $\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1$ for σ-ideals of compact sets. All results are due to Kechris, Louveau and Woodin [KLW]. Part (1) is the so-called *Dichotomy Theorem* for σ-ideals of compact sets. Part (2) implies in particular that if A is a subset of X, then $\mathcal{K}(A)$ is $\mathbf{\Sigma}^1_1$ if and only if A is G_δ, a result due to J. P. R. Christensen ([Chr2], see also [SR1]).

THEOREM 1.4. *Let $\mathcal{I}$ be a σ-ideal in $\mathcal{K}(X)$.*

(1) *If $\mathcal{I}$ is $\mathbf{\Pi}^1_1$, then it is either $\mathbf{\Pi}^1_1$-complete or G_δ.*
(2) *If $\mathcal{I}$ is $\mathbf{\Sigma}^1_1$, then it is G_δ.*
(3) *Assume X is compact. If $\mathcal{I}$ is G_δ, then one of the following holds*:
    - *$\mathcal{I}$ is $\mathbf{\Pi}^0_2$-complete*;
    - *$\mathcal{I}$ is $\mathrm{D}_2(\mathbf{\Pi}^0_1)$-complete and of the form $\mathcal{K}(A)$, for some set $A \in \mathrm{D}_2(\mathbf{\Pi}^0_1)$*;
    - *$\mathcal{I}$ is $\mathbf{\Pi}^0_1$-complete and of the form $\mathcal{K}(A)$, for some $A \in \mathbf{\Pi}^0_1$*;
    - *$\mathcal{I}$ is $\mathbf{\Sigma}^0_1$-complete and of the form $\mathcal{K}(A)$, for some $A \in \mathbf{\Sigma}^0_1$*;
    - *$\mathcal{I}$ is $\mathbf{\Delta}^0_1$ and of the form $\mathcal{K}(A)$, for some $A \in \mathbf{\Delta}^0_1$.*

One can prove (1) as follows: if $\mathcal{I}$ is $\mathbf{\Pi}^1_1$ and not G_δ, then it is $\mathbf{\Sigma}^0_2$-hard by Hurewicz's Theorem (see [K6, Theorem 21.18]), and hence $\mathbf{\Pi}^1_1$-hard by Corollary 1.3. For (2) and (3), see [KLW].

Remark 1.5. It follows from the Dichotomy Theorem that if a $\mathbf{\Pi}^1_1$ σ-ideal $\mathcal{I} \subseteq \mathcal{K}(X)$ is dense and not comeager, then $\mathcal{I}$ is $\mathbf{\Pi}^1_1$-complete. For example, the σ-ideal $\mathcal{K}_\omega(X)$ of all compact countable sets is $\mathbf{\Pi}^1_1$-complete if X is perfect (Hurewicz [Hur]).

Remark 1.6. Part (2) can be put into the much wider context of *basic orders*, a notion introduced by S. Solecki and S. Todorcevic [ST]. A basic order is a separable, metrizable topological space D equipped with a partial ordering $\leq$ with the following properties, where "bounded" means "bounded from above".

- Each pair $(x, y) \in D \times D$ has a least upper bound, and the map $(x, y) \mapsto x \vee y$ is continuous.
- Each bounded sequence has a converging subsequence, and each converging sequence has a bounded subsequence.

Examples include all σ-ideal of compact sets, and all *analytic P-ideals* on ω (see [S3, S4]). Now the generalized form of (2) above reads as follows: *If D is an analytic basic order, then the topology on D is Polish.*

Above $\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1$, the situation is less well understood. However, we still have some exclusion phenomena, as shown by the next theorem, due to R. Dougherty and A. S. Kechris (see [K4]).

First, let us introduce some classes of sets, whose definition makes sense in an arbitrary fixed Polish space Z. We denote by $\mathrm{Bor}(\mathbf{\Pi}^1_1)$ the σ-algebra generated by the $\mathbf{\Pi}^1_1$ sets. Then $\mathrm{Bor}(\mathbf{\Pi}^1_1) = \bigcup_{\xi<\omega_1} \mathbf{\Sigma}^0_\xi(\mathbf{\Pi}^1_1) = \bigcup_{\xi<\omega_1} \mathbf{\Pi}^0_\xi(\mathbf{\Pi}^1_1)$, where $\mathbf{\Sigma}^0_1(\mathbf{\Pi}^1_1)$ is the class of countable unions of $\mathrm{D}_2(\mathbf{\Pi}^1_1)$ sets, $\mathbf{\Pi}^0_1(\mathbf{\Pi}^1_1)$ is the dual class, and the classes $\mathbf{\Sigma}^0_\xi(\mathbf{\Pi}^1_1)$, $\mathbf{\Pi}^0_\xi(\mathbf{\Pi}^1_1)$ are defined inductively in the usual way. Finally, if $\mathbf{\Gamma}$ is any class of sets, we denote by $\mathrm{BSU}(\mathbf{\Gamma})$ the class of all sets of the form $A = \bigcup_{n\in\omega} A_n$, where $A_n \in \mathbf{\Gamma}$ and the A_n's are Borel-separated, that is, $A_n \subseteq B_n$ for some sequence of pairwise disjoint Borel sets (B_n).

THEOREM 1.7. *Assume* $\mathrm{Bor}(\mathbf{\Pi}^1_1)$*-determinacy. If $\mathcal{I} \subseteq \mathcal{K}(X)$ is a σ-ideal and $\mathcal{I} \in$* $\mathrm{Bor}(\mathbf{\Pi}^1_1)$*, then exactly one of the following holds.*

(1) *$\mathcal{I}$ is $\mathbf{\Pi}^1_1$;*
(2) *$\mathcal{I}$ is* $\mathrm{D}_2(\mathbf{\Pi}^1_1)$*-complete;*
(3) *$\mathcal{I}$ is $\mathbf{\Pi}^0_\xi(\mathbf{\Pi}^1_1)$-complete for some ordinal ξ, $1 \leq \xi < \omega_1$;*
(4) $\mathcal{I} \in \mathrm{BSU}\left(\bigcup_{\xi<\lambda} \mathbf{\Pi}^0_\xi(\mathbf{\Pi}^1_1)\right) \setminus \bigcup_{\xi<\lambda} \mathbf{\Pi}^0_\xi(\mathbf{\Pi}^1_1)$, *for some limit ordinal $\lambda < \omega_1$.*

Remark 1.8. In [KLW] and [K4], a slightly different notion of $\mathbf{\Gamma}$-completeness is used. However, one can formulate Theorems 1.4 and 1.7 as above.

1.2. Complexity of bases. In this section, we discuss some implications of the existence of a "simple" basis for a σ-ideal $\mathcal{I}$. The following notations will be used repeatedly, in this section and afterwards.

DEFINITION 1.9. Let $\mathcal{B} \subseteq \mathcal{F}(X)$.

- We denote by $\mathcal{B}^{\text{ext}}$ the family of all sets $A \subseteq X$ which can be covered by countably many sets from $\mathcal{B}$.
- We denote by $\mathcal{B}_\sigma$ the σ-ideal of $\mathcal{F}(X)$ *generated* by $\mathcal{B}$, that is $\mathcal{B}_\sigma = \mathcal{B}^{\text{ext}} \cap \mathcal{F}(X)$.
- If $\mathcal{B}$ is hereditary, we define

  $$\mathcal{B}^{\text{loc}} := \{C \in \mathcal{F}(X);\ \exists V \subseteq X \text{ open}\colon V \cap C \neq \emptyset \text{ and } \overline{V \cap C} \in \mathcal{B}\}$$

  and we put $\mathcal{B}^{\text{perf}} := \mathcal{F}(X) \setminus \mathcal{B}^{\text{loc}}$.

It follows from the Baire Category Theorem that a closed set $C \subseteq X$ is in $\mathcal{B}^{\text{loc}}$ if and only if

$$\exists V \subseteq X \text{ open}\colon V \cap C \neq \emptyset \text{ and } V \cap C \in \mathcal{B}^{\text{ext}}. \tag{1}$$

From this, it is obvious that $\mathcal{B}^{\text{loc}} = (\mathcal{B}_\sigma)^{\text{loc}}$. It also follows from (1) that a closed set $C \subseteq X$ is not in $\mathcal{B}_\sigma$ if and only if C contains a nonempty closed set $P \in \mathcal{B}^{\text{perf}}$.

If $\mathcal{B}$ is considered as a subset of the space $\mathcal{K}(X)$ then $\mathcal{B}_\sigma$, $\mathcal{B}^{\text{loc}}$, and $\mathcal{B}^{\text{perf}}$ denote the corresponding families of *compact* sets.

DEFINITION 1.10. Let $\mathcal{I}$ be a σ-ideal in $\mathcal{K}(X)$ or in $\mathcal{F}(X)$. We say that a family $\mathcal{B} \subseteq \mathcal{I}$ is a *basis* for $\mathcal{I}$ if $\mathcal{B}$ is hereditary and $\mathcal{B}_\sigma = \mathcal{I}$.

Notice that it is part of the definition that a basis should be hereditary. If $\mathcal{B}$ is a perhaps not hereditary subfamily of $\mathcal{I}$ such that $\mathcal{B}_\sigma = \mathcal{I}$, we say that $\mathcal{B}$ is a *pre-basis* for $\mathcal{I}$.

The following proposition collects some useful facts about $\mathbf{\Pi}^1_1$ or $\mathbf{\Sigma}^1_1$ bases. Parts (1)–(4) are proved in [KLW] (see also [KL1, VI.1]), and the proof of (5) is straightforward.

PROPOSITION 1.11. *Let $\mathcal{I}$ be a σ-ideal in $\mathcal{F}(X)$ or in $\mathcal{K}(X)$.*

(1) *The following are equivalent.*
   - *$\mathcal{I}$ is $\mathbf{\Pi}^1_1$;*
   - *$\mathcal{I}$ has a $\mathbf{\Pi}^1_1$ basis;*
   - *$\mathcal{I}^{\text{loc}}$ is $\mathbf{\Pi}^1_1$.*

(2) *$\mathcal{I}$ has a $\mathbf{\Sigma}^1_1$ basis if and only if it has a $\mathbf{\Sigma}^1_1$ pre-basis. If $\mathcal{I}$ is a σ-ideal of compact sets, then $\mathcal{I}$ has a K_σ pre-basis if and only if it has a K_σ basis.*

(3) *If $\mathcal{I}$ is $\mathbf{\Pi}^1_1$ and has a $\mathbf{\Sigma}^1_1$ basis, then $\mathcal{I}$ has a Borel basis, and even a Borel basis which is an ideal.*

(4) *$\mathcal{I}$ has a Borel basis if and only if $\mathcal{I}^{\text{loc}}$ is Borel.*

(5) *Assume $\mathcal{I}$ is a σ-ideal in $\mathcal{K}(X)$. If $\mathcal{I}$ has a K_σ basis, then $\mathcal{I}^{\text{perf}}$ is G_δ.*

The next theorem gives some "reduction" results showing that in some cases, a simple basis can be modified to get an even simpler basis or pre-basis.

THEOREM 1.12. *Let $\mathcal{I}$ be a σ-ideal in $\mathcal{K}(X)$, with X compact.*

(1) *If $\mathcal{I}$ has a $\mathbf{\Sigma}_1^1$ basis, then $\mathcal{I}$ has a G_δ pre-basis.*
(2) *If $\mathcal{I}$ has a $\mathbf{\Sigma}_2^0$ basis, then $\mathcal{I}$ has a $\mathrm{D}_2(\mathbf{\Pi}_1^0)$ pre-basis.*
(3) *If $\mathcal{I}$ has a $\check{\mathrm{D}}_2(\mathbf{\Pi}_1^0)$ pre-basis, then $\mathcal{I}$ has a $\mathrm{D}_2(\mathbf{\Pi}_1^0) \cap \check{\mathrm{D}}_2(\mathbf{\Pi}_1^0)$ pre-basis.*

These results can be found in [KLW]. Notice that in (1), one cannot get a G_δ *basis*; that is, the G_δ pre-basis cannot be taken to be hereditary in general. The main point is the following simple observation: if a σ-ideal $\mathcal{I}$ has a G_δ basis, then $\mathcal{I}^{\mathrm{loc}}$ has to be $\mathbf{\Sigma}_3^0$.

Here is a simple example of a σ-ideal $\mathcal{I}$ with a Borel basis but with $\mathcal{I}^{\mathrm{loc}} \notin \mathbf{\Sigma}_3^0$. Let A be a Borel subset of 2^ω, and set

$$\mathcal{B} := \{\{z\};\ z \in 2^\omega \times 2^\omega\} \cup \bigcup_{\alpha \in A} \mathcal{K}(\{\alpha\} \times 2^\omega).$$

Then $\mathcal{B}$ is a Borel hereditary subset of $\mathcal{K}(2^\omega \times 2^\omega)$. Set $\mathcal{I} := \mathcal{B}_\sigma$. It is easily checked that a compact set of the form $K_\alpha := \{\alpha\} \times 2^\omega$ is in $\mathcal{I}^{\mathrm{loc}} = \mathcal{B}^{\mathrm{loc}}$ if and only if $\alpha \in A$. Since the map $\alpha \mapsto K_\alpha$ is continuous, it follows that the complexity of $\mathcal{I}^{\mathrm{loc}}$ is at least that of A. Starting with a Borel set $A \notin \mathbf{\Sigma}_3^0$, we thus get the desired example $\mathcal{I}$.

In the same spirit as in Theorem 1.12, the following result shows that a $\mathbf{\Sigma}_{\xi+1}^0$ basis can often be modified to get a $\mathbf{\Pi}_\xi^0$ basis.

PROPOSITION 1.13. *Let $\mathcal{I}$ be a σ-ideal in $\mathcal{K}(X)$, and let $\xi \geq 1$ be a countable ordinal. Assume that the set $\{x \in X;\ \{x\} \in \mathcal{I}\}$ is $\mathbf{\Pi}_\xi^0$, and that $\mathcal{I}$ has a basis of the form $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$, where the $\mathcal{B}_n$'s are $\mathbf{\Pi}_\xi^0$ and hereditary. Then $\mathcal{I}$ has a $\mathbf{\Pi}_\xi^0$ basis.*

PROOF. Let us fix some compatible metric on X. For each $n \in \omega$, put

$$\widetilde{\mathcal{B}}_n := \{K \in \mathcal{B}_n;\ \mathrm{diam}(K) \leq 2^{-n}\} \cup \{\emptyset\},$$

and set $\widetilde{\mathcal{B}} := \bigcup_{n\in\omega} \widetilde{\mathcal{B}}_n$. Then $\widetilde{\mathcal{B}}$ is hereditary and contained in $\mathcal{I}$. Moreover, one can split each nonempty $K \in \mathcal{B}_n$ into finitely many compact sets with diameter less than 2^{-n}, and since $\mathcal{B}_n$ is hereditary, all these sets belong again to $\mathcal{B}_n$. Thus, we have $\mathcal{B}_n \subseteq (\widetilde{\mathcal{B}}_n)_\sigma$ for each $n \in \omega$, hence $\widetilde{\mathcal{B}}$ is a basis for $\mathcal{I}$.

Put $\mathcal{F}_1 := \{\emptyset\} \cup \{\{x\};\ x \in X\}$. Notice that if $K \in \mathcal{K}(X) \setminus \mathcal{F}_1$, then K has an open neighbourhood $\mathcal{U}$ in $\mathcal{K}(X)$ which meets only finitely many $\widetilde{\mathcal{B}}_n$'s. Thus, $\widetilde{\mathcal{B}} \setminus \mathcal{F}_1$ is locally $\mathbf{\Pi}_\xi^0$ in the Polish space $\mathcal{K}(X) \setminus \mathcal{F}_1$, and this implies that $\widetilde{\mathcal{B}} \setminus \mathcal{F}_1$ is $\mathbf{\Pi}_\xi^0$ in $\mathcal{K}(X) \setminus \mathcal{F}_1$.

Assume $\xi \geq 2$. Then $\mathcal{K}(X) \setminus \mathcal{F}_1$ is $\mathbf{\Pi}_\xi^0$, hence $\widetilde{\mathcal{B}} \setminus \mathcal{F}_1$ is $\mathbf{\Pi}_\xi^0$ in $\mathcal{K}(X)$. Moreover, since $\widetilde{\mathcal{B}}$ is a basis for $\mathcal{I}$, we have $\widetilde{\mathcal{B}} \cap \mathcal{F}_1 = \mathcal{I} \cap \mathcal{F}_1$, hence $\widetilde{\mathcal{B}} \cap \mathcal{F}_1$ is $\mathbf{\Pi}_\xi^0$. Therefore, $\widetilde{\mathcal{B}} = (\widetilde{\mathcal{B}} \cap \mathcal{F}_1) \cup (\widetilde{\mathcal{B}} \setminus \mathcal{F}_1)$ is $\mathbf{\Pi}_\xi^0$ as well.

Now, assume $\xi = 1$. Then $\widetilde{\mathcal{B}} \setminus \mathcal{F}_1$ is closed in $\mathcal{K}(X) \setminus \mathcal{F}_1$, so it only remains to check that if $\{x\}$ is a singleton in the closure of $\widetilde{\mathcal{B}}$, then $\{x\} \in \widetilde{\mathcal{B}}$.

Since all sets $\widetilde{\mathcal{B}}_n$ are hereditary, one can find a sequence $(x_k) \subseteq X$ such that $\{x_k\} \in \widetilde{\mathcal{B}}_{n_k}$ for some n_k and $\{x_k\} \to \{x\}$. Since the singletons in $\mathcal{I}$ form a closed set, it follows that $\{x\} \in \mathcal{I}$, whence $\{x\} \in \widetilde{\mathcal{B}}$ because $\widetilde{\mathcal{B}}$ is a basis for $\mathcal{I}$. Thus, $\widetilde{\mathcal{B}}$ is indeed closed in $\mathcal{K}(X)$. ⊣

COROLLARY 1.14. *If $\mathcal{I} \subseteq \mathcal{K}(X)$ is a σ-ideal containing all singletons and with a K_σ pre-basis, then $\mathcal{I}$ has a closed basis.*

PROOF. This follows from Proposition 1.13 because the hereditary closure of a compact subset of $\mathcal{K}(X)$ is compact: if $\mathcal{I}$ has a K_σ pre-basis, then the hereditary closure of such a pre-basis is a basis of the required form. ⊣

Finally, we mention two results which are in some sense dual to (1) in Theorem 1.12. The first one, a weaker form of which was proved earlier by S. Zafrany [Zaf2], is due to Solecki and Todorcevic [ST]; the second one is due to Solecki [S8].

If $\mathcal{I}$ is a hereditary subset of $\mathcal{K}(X)$ or $\mathcal{F}(X)$, we say that a set $\mathcal{G} \subseteq \mathcal{I}$ is *cofinal* in $\mathcal{I}$ if each set $F \in \mathcal{I}$ is contained in some $C \in \mathcal{G}$; in other words, if $\mathcal{I}$ is the hereditary closure of $\mathcal{G}$.

THEOREM 1.15. *Each $\mathbf{\Sigma}^1_1$ ideal of subsets of ω has a cofinal G_δ subset. Similarly, each $\mathbf{\Sigma}^1_1$ ideal of compact subsets of X has a cofinal G_δ subset.*

1.3. Cantor–Bendixson ranks. Let $\mathcal{B}$ be a hereditary subset of the set $\mathcal{F}(X)$. For each closed set $C \subseteq X$, one defines the *$\mathcal{B}$-derivative $C'_{\mathcal{B}}$* of C as follows:

$$x \in C'_{\mathcal{B}} \Leftrightarrow \forall V \ni x \text{ open}\colon \overline{V \cap C} \notin \mathcal{B}.$$

Starting with $C^{(0)}_{\mathcal{B}} := C$, the transfinite derivatives $C^{(\xi)}_{\mathcal{B}}$ are defined inductively in the usual way: $C^{(\xi+1)}_{\mathcal{B}} = \left(C^{(\xi)}_{\mathcal{B}}\right)'_{\mathcal{B}}$ and $C^{(\lambda)}_{\mathcal{B}} = \bigcap_{\xi<\lambda} C^{(\xi)}_{\mathcal{B}}$ if λ is a limit ordinal. Then $C \in \mathcal{B}_\sigma$ if and only if $C^{(\xi)}_{\mathcal{B}} = \emptyset$ for some $\xi < \omega_1$. The least such ξ is called the *Cantor–Bendixson rank* of C relative to $\mathcal{B}$, and is denoted by $r_{\mathcal{B}}(C)$.

The basic example is $\mathcal{B} = \{\emptyset\} \cup \{\{x\};\ x \in X\}$. In that case, $\mathcal{B}_\sigma$ is the σ-ideal of countable sets, the $\mathcal{B}$-derivative of a closed set C is the usual derived set C', and hence $r_{\mathcal{B}}$ is the classical Cantor–Bendixson rank.

If $\mathcal{B}$ is a Borel hereditary subset of $\mathcal{K}(X)$, then the map $d_{\mathcal{B}}\colon \mathcal{K}(X) \to \mathcal{K}(X)$ defined by $d_{\mathcal{B}}(K) = K'_{\mathcal{B}}$ is easily seen to be Borel. Thus, $d_{\mathcal{B}}$ is a *Borel derivation* on $\mathcal{K}(X)$. From this, one can get the following result (see [K6, 34.D] or [KL1, VI.1]).

THEOREM 1.16. *If $\mathcal{B} \subseteq \mathcal{K}(X)$ is Borel and hereditary, then the Cantor–Bendixson rank $r_{\mathcal{B}}$ is a $\mathbf{\Pi}^1_1$ rank on $\mathcal{I} := \mathcal{B}_\sigma$.*

With the Boundedness Theorem for $\mathbf{\Pi}^1_1$-ranks in mind, it is desirable to have at hand a simple way of checking that a rank $r_{\mathcal{B}}$ is unbounded. This is the content of the next proposition (see [KL1, Theorem VI.1.6]).

PROPOSITION 1.17. *Let $\mathcal{B}$ be a hereditary subset of $\mathcal{K}(X)$, and let $\mathcal{I}$ be a σ-ideal with $\mathcal{I} \subseteq \mathcal{B}_\sigma$. Assume that in each nonempty open set $V \subseteq X$, one can find a closed set $C \in \mathcal{I}$ with $r_{\mathcal{B}}(C) > 1$. Then the rank $r_{\mathcal{B}}$ is unbounded on $\mathcal{I}$.*

If $\mathcal{B} \subseteq \mathcal{K}(X)$ is an *ideal* of compact sets, then a compact set $K \subseteq X$ is in $\mathcal{B}$ if and only if $r_{\mathcal{B}}(K) \leq 1$. Thus, we get the following corollary, which may in particular be applied to the σ-ideal of countable sets.

COROLLARY 1.18. *Let $\mathcal{B}$ be a Borel ideal in $\mathcal{K}(X)$. If each nonempty open set $V \subseteq X$ contains a compact set $K \in \mathcal{B}_\sigma \setminus \mathcal{B}$, then the σ-ideal $\mathcal{I} := \mathcal{B}_\sigma$ is true $\mathbf{\Pi}^1_1$.*

1.4. The complexity of σ-ideals with an analytic basis. Recall that a set A in some standard Borel space Z is said to be $\mathbf{\Sigma}^1_1$-*inductive* if there is some $\mathbf{\Sigma}^1_1$ relation $\Phi(n, b, z)$ on $\omega \times \mathcal{P}(\omega) \times Z$ (monotone with respect to b) and some $n_0 \in \omega$ such that

$$z \in A \Leftrightarrow \Phi^\infty(n_0, z).$$

Here, $\Phi^\infty = \bigcup_\xi \Phi^\xi$, where Φ^ξ is defined inductively as follows:

$$\Phi^0(n, z) \Leftrightarrow \Phi(n, \emptyset, z) \text{ and}$$
$$\Phi^\xi(n, z) \Leftrightarrow \Phi(n, \{k;\ \exists \eta < \xi \colon \Phi^\eta(k, z)\}, z).$$

The following theorem is due to Kechris [K4].

THEOREM 1.19. (1) *If $\mathcal{I}$ is a σ-ideal in $\mathcal{K}(X)$ with a $\mathbf{\Sigma}^1_1$ basis, then $\mathcal{I}$ is $\mathbf{\Sigma}^1_1$-inductive.*

(2) *There exists a σ-ideal in $\mathcal{K}(2^\omega)$ which is $\mathbf{\Sigma}^1_1$-inductive-complete and has a $\mathbf{\Sigma}^1_1$ basis.*

The example of a $\mathbf{\Sigma}^1_1$-inductive-complete σ-ideal given in [K4] is easy to describe. It is enough to work with $2^\omega \times 2^\omega$ instead of 2^ω since both spaces are homeomorphic. Let $G \subseteq 2^\omega \times \mathcal{K}(2^\omega)$ be a good universal set for $\mathbf{\Sigma}^1_1$ subsets of $\mathcal{K}(2^\omega)$. For each $\alpha \in 2^\omega$, denote by $G_\alpha \subseteq \mathcal{K}(2^\omega)$ the α-section of G. If $G_\alpha \neq \emptyset$, denote by $B_\alpha \subseteq \mathcal{K}(2^\omega)$ the hereditary closure of G_α; and put $B_\alpha := \{\emptyset\}$ if $G_\alpha = \emptyset$. Now, define $\mathcal{B} \subseteq \mathcal{K}(2^\omega \times 2^\omega)$ by

$$L \in \mathcal{B} \Leftrightarrow \exists (\alpha, K) \colon K \in B_\alpha \text{ and } L \subseteq K \times \{\alpha\}.$$

Then $\mathcal{B}$ is $\mathbf{\Sigma}^1_1$ and hereditary, and it can be shown that $\mathcal{I} := \mathcal{B}_\sigma$ is $\mathbf{\Sigma}^1_1$-inductive-complete.

1.5. A trichotomy for ideals. The main result of this section is a "trichotomy" theorem for ideals of compact sets, which says in essence that if an ideal is sufficiently rich and is truly not a σ-ideal, then it cannot be $\mathbf{\Delta}^0_3$.

We start with the following nice theorem due to Dougherty–Kechris and independently to Louveau. It will be generalized below, but it seems worth stating it separately.

THEOREM 1.20. *If $\mathcal{I} \subseteq \mathcal{K}(X)$ is a G_δ ideal, then $\mathcal{I}$ is a σ-ideal.*

A proof of this theorem can be found in [K5], and another one in [MSZ].

DEFINITION 1.21. We say that a family $\mathcal{A} \subseteq \mathcal{K}(X)$ is *rich in sequences at some point* $x \in X$ if there exists a dense set $\mathcal{D} \subseteq \mathcal{K}(X)$ such that $\{x\} \cup \bigcup_{n\in\omega} K_n \in \mathcal{A}$ for each sequence $(K_n) \subseteq \mathcal{D}$ converging to $\{x\}$. The family $\mathcal{A}$ is said to be *rich in sequences* if $\bigcup \mathcal{A} \neq \emptyset$ and $\mathcal{A}$ is rich in sequences at each point $x \in \bigcup \mathcal{A}$.

For a hereditary family $\mathcal{A}$, richness in sequences at x is equivalent to the existence of a dense set $D \subseteq X$ such that $\{x\} \cup \{x_n;\ n \in \omega\} \in \mathcal{A}$ for each sequence $(x_n) \subseteq D$ converging to x. Notice also that a σ-ideal is rich in sequences if and only if it is dense in $\mathcal{K}(X)$. Finally, any G_δ ideal $\mathcal{I} \subseteq \mathcal{K}(X)$ is rich in sequences when considered as an ideal in $\mathcal{K}(\bigcup \mathcal{I})$ provided $\bigcup \mathcal{I} \neq \emptyset$. This ensues at once from Theorem 1.20, or in a more elementary way from the following very useful lemma (see, e.g., [K5]).

LEMMA 1.22. *Let $\mathcal{G}$ be a G_δ subset of $\mathcal{K}(X)$. Let (K_n) be a sequence in $\mathcal{K}(X)$ converging to some compact set K. Assume that for each finite set $b \subseteq \omega$, the set $K \cup \bigcup_{n\in b} K_n$ belongs to $\mathcal{G}$. Then $K \cup \bigcup_{n\in\omega} K_n$ is the union of two elements of $\mathcal{G}$.*

At first sight, the above definition may look a bit artificial. However, the next theorem [MSZ] shows that richness in sequences is strongly connected to the classical notion of comeagerness.

THEOREM 1.23. *Let $\mathcal{I}$ be an ideal of $\mathcal{K}(X)$.*

(1) *If $\mathcal{I}$ is comeager in $\mathcal{K}(X)$, then $\mathcal{I}$ is rich in sequences at comeagerly many points of X.*
(2) *If $\mathcal{I}$ is $\mathbf{\Sigma}^0_3$ and rich in sequences at comeagerly many points of X, then $\mathcal{I}$ is comeager in $\mathcal{K}(X)$.*

Remark 1.24. The statement of this theorem is not completely symmetrical, since $\mathcal{I}$ is assumed to be $\mathbf{\Sigma}^0_3$ in (2). This hypothesis cannot be removed. For example, the ideal consisting of all compact sets $K \subseteq \mathbb{R}$ with finitely many limit points is $\mathbf{\Sigma}^0_4$ and rich in sequences, but it is meager in $\mathcal{K}(\mathbb{R})$.

We can now state the announced trichotomy [MSZ]. Recall that a set A in some Polish space Y is said to be *universally Baire* if, for every continuous map $f\colon 2^\omega \to Y$, the set $f^{-1}(A)$ has the Baire property in 2^ω.

THEOREM 1.25. *Let $\mathcal{I}$ be a universally Baire ideal of $\mathcal{K}(X)$. Assume that $\mathcal{I}$ is rich in sequences at some point $x \in X$. Then one of the following holds.*

(i) *There exists some neighborhood V of x such that $\mathcal{I} \cap \mathcal{K}(V)$ is a σ-ideal.*
(ii) *$\mathcal{I}$ is $\mathbf{\Pi}^0_3$-hard.*
(iii) *$\mathcal{I}$ is $\mathbf{\Sigma}^0_3$-hard.*

The following corollary is a sharpening of Theorem 1.20.

COROLLARY 1.26. *Let $\mathcal{I}$ be a universally Baire ideal of $\mathcal{K}(X)$ which is rich in sequences. If $\mathcal{I}$ is not a σ-ideal, then $\mathcal{I}$ is either $\mathbf{\Pi}^0_3$-hard or $\mathbf{\Sigma}^0_3$-hard.*

Remark 1.27. Let us point out that the alternatives exhibited in Theorem 1.25 and Corollary 1.26 do not exclude each other. On the other hand, for each one of the three properties used in Theorem 1.25, there exist ideals satisfying this property but neither of the other two (see [MSZ] for details).

Another consequence of Theorem 1.25 is the following criterion for a $\mathbf{\Sigma}^0_3$ ideal to be true $\mathbf{\Sigma}^0_3$. The same criterion is valid for $\mathbf{\Pi}^0_3$ ideals, but we emphasize the $\mathbf{\Sigma}^0_3$ case because it applies to many natural ideals of thin sets from harmonic analysis (see sections 2.2 and 2.3). A similar criterion was obtained in [M2].

COROLLARY 1.28. *Let $\mathcal{I} \subseteq \mathcal{K}(X)$ be a $\mathbf{\Sigma}^0_3$ ideal. Assume that $\mathcal{I} \setminus \{\emptyset\}$ is nonmeager in $\mathcal{K}(X)$, and that $\mathcal{I} \cap \mathcal{K}(U)$ is a σ-ideal for no nonempty open set $U \subseteq X$. Then $\mathcal{I}$ is $\mathbf{\Sigma}^0_3$-complete.*

PROOF. Since $\mathcal{I}$ is hereditary, it is not hard to check that it is comeager in $\mathcal{K}(O)$, for some nonempty open set $O \subseteq X$ (see [MZ, Lemma 2.11]). Thus, one can apply 1.23 together with 1.25. ⊣

We conclude this section with another result from [MSZ] where the class $\mathbf{\Sigma}^0_3$ appears naturally. A slightly weaker but essentially equivalent result was proved earlier by M. Balcerzak and U. B. Darji [BD].

PROPOSITION 1.29. *Let $\mathcal{I}$ be a hereditary subset of $\mathcal{K}(X)$ with the Baire property. If $\mathcal{I} \setminus \{\emptyset\}$ is nonmeager in $\mathcal{K}(X)$ and $\mathcal{I}^{\mathrm{perf}}$ is dense in $\mathcal{K}(X)$, then $\mathcal{I}^{\mathrm{perf}}$ is not $\mathbf{\Sigma}^0_3$.*

For example, it follows from this result that if $\mathcal{I}$ is a dense G_δ σ-ideal such that $\mathcal{I}^{\mathrm{perf}}$ is dense in $\mathcal{K}(X)$, then $\mathcal{I}^{\mathrm{perf}}$ is a true $\mathbf{\Pi}^0_3$ set. This was noticed in [BD].

1.6. Wadge classes of ideals. In this section, we briefly describe the work of F. van Engelen [vE], which gives the complete picture of the possible Wadge classes for ideals of compact sets in $\mathcal{K}(2^\omega)$. Actually, van Engelen's paper is almost exclusively about ideals of subsets of ω, but the last part explains the connection with ideals of compact sets.

We will work exclusively in the Cantor space 2^ω. Recall that a *Wadge class* of Borel sets is any family of subsets of 2^ω of the form

$$[A]_{\mathrm{w}} := \{f^{-1}(A);\ f\colon 2^\omega \to 2^\omega \text{ continuous}\}$$

for some fixed Borel set $A \subseteq 2^\omega$. If $\mathbf{\Gamma}$ is a Wadge class, we write $\check{\mathbf{\Gamma}}$ for the dual class. For Wadge classes $\mathbf{\Gamma}, \mathbf{\Gamma}'$, we write $\mathbf{\Gamma} < \mathbf{\Gamma}'$ if $\mathbf{\Gamma} \subsetneq \mathbf{\Gamma}'$.

Since $\mathcal{K}^*(2^\omega)$ $(= \mathcal{K}(2^\omega) \setminus \{\emptyset\})$ is homeomorphic to 2^ω, we may consider the Wadge class of a subset of $\mathcal{K}^*(2^\omega)$. We say that a Wadge class $\mathbf{\Gamma}$ is a *Wadge class of ideal* if there is some ideal $\mathcal{I} \subseteq \mathcal{K}(2^\omega)$ such that $\mathbf{\Gamma} = [\mathcal{I} \setminus \{\emptyset\}]_{\mathrm{w}}$.

The main obstruction for being a Wadge class of ideal is given by the following easy lemma.

LEMMA 1.30. *If $\mathbf{\Gamma}$ is a Wadge class of ideal, then $\mathbf{\Gamma}$ is stable under finite intersections.*

Proof. Assume $\mathbf{\Gamma} = [\mathcal{I} \setminus \{\emptyset\}]_{\mathrm{w}}$ for some ideal $\mathcal{I} \subseteq \mathcal{K}(2^\omega)$, and let $A, B \subseteq 2^\omega$ be in $\mathbf{\Gamma}$. Then one can find two continuous maps $f_A, f_B : 2^\omega \to \mathcal{K}^*(2^\omega)$ such that $f_A^{-1}(\mathcal{I}\setminus\{\emptyset\}) = A$ and $f_B^{-1}(\mathcal{I}\setminus\{\emptyset\}) = B$, and the map $f := f_A \cup f_B$ shows that $A \cap B$ is in $[\mathcal{I} \setminus \{\emptyset\}]_{\mathrm{w}} = \mathbf{\Gamma}$ as well. ⊣

For example, it follows from this lemma that if n is a natural number, $n \geq 3$, then the class $\mathrm{D}_n(\mathbf{\Sigma}^0_1)$ is not a Wadge class of ideal (of course, one could also use Theorems 1.4 and 1.20). On the other hand, $\mathbf{\Sigma}^0_1$ and $\mathrm{D}_2(\mathbf{\Sigma}^0_1)$ are Wadge classes of ideals, as witnessed by $\mathcal{I} = \mathcal{K}(A)$, where $[A]_{\mathrm{w}} = \mathbf{\Sigma}^0_1$ or $\mathrm{D}_2(\mathbf{\Sigma}^0_1)$.

To give a complete list of the Wadge classes of ideals, we first need to recall a description (due to Louveau [L3]) of the non–self-dual Wadge classes. To do this, we have to define some set-theoretic operations. All sets under consideration are subsets of 2^ω.

Let $\mathbf{\Gamma}$, $\mathbf{\Gamma}'$, $\mathbf{\Delta}$ be classes of sets, and let $\eta \geq 1$ be a countable ordinal.

- If $(A_\alpha)_{\alpha<\eta}$ is a nondecreasing sequence of sets, we put
$$\mathrm{D}_\eta((A_\alpha)) := \bigcup_\alpha (A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta),$$
where the union is over all ordinals $\alpha < \eta$ with parity opposite to η. The class $\mathrm{D}_\eta(\mathbf{\Gamma})$ is the class of all sets of the form $\mathrm{D}_\eta((A_\alpha))$, where $A_\alpha \in \mathbf{\Gamma}$ for all $\alpha < \eta$.
- If A_0, A_1, C are sets, we put
$$\mathrm{Sep}(C, A_0, A_1) := (A_0 \cap C) \cup (A_1 \setminus C).$$
The class $\mathrm{Sep}(\mathbf{\Delta}, \mathbf{\Gamma})$ is the class of all sets of the form $\mathrm{Sep}(C, A_0, A_1)$, where $C \in \mathbf{\Delta}$, $A_0 \in \check{\mathbf{\Gamma}}$, and $A_1 \in \mathbf{\Gamma}$.
- If A_0, A_1, C_0, C_1, B are sets and $C_0 \cap C_1 = \emptyset$, we put
$$\mathrm{Bisep}(C_0, C_1, A_0, A_1, B) := (A_0 \cap C_0) \cup (A_1 \cap C_1) \cup (B \setminus (C_0 \cup C_1)).$$
The class $\mathrm{Bisep}(\mathbf{\Delta}, \mathbf{\Gamma}, \mathbf{\Gamma}')$ is the class of all sets of the form
$$\mathrm{Bisep}(C_0, C_1, A_0, A_1, B),$$
where $C_0, C_1 \in \mathbf{\Delta}$, $A_0 \in \check{\mathbf{\Gamma}}$, $A_1 \in \mathbf{\Gamma}$ and $B \in \mathbf{\Gamma}'$.
- If (A_n) is a sequence of sets and (C_n) is a sequence of pairwise disjoint sets, we put
$$\mathrm{SU}((C_n), (A_n)) := \bigcup_{n\in\omega} (A_n \cap C_n).$$
The class $\mathrm{SU}(\mathbf{\Delta}, \mathbf{\Gamma})$ is the class of all sets of the form $\mathrm{SU}((C_n), (A_n))$, where $A_n \in \mathbf{\Gamma}$ and $C_n \in \mathbf{\Delta}$ for all n. We also define $\langle \mathbf{\Delta}, \mathbf{\Gamma} \rangle_{\mathrm{SU}}$ to be the set of all pairs $(C, A) \in \mathcal{P}(2^\omega) \times \mathcal{P}(2^\omega)$ such that $A = \mathrm{SU}((C_n), (A_n))$ and $C = \bigcup_n C_n$, for some sequences $(C_n) \subseteq \mathbf{\Delta}$ and $(A_n) \subseteq \mathbf{\Gamma}$.

- If $(A_\alpha)_{\alpha<\eta}$ and $(C_\alpha)_{\alpha<\eta}$ are nondecreasing sequences of sets with $A_\alpha \subseteq C_\alpha \subseteq A_{\alpha+1}$ for all $\alpha < \eta$, and if B is a set, we put

$$\mathrm{SD}_\eta((C_\alpha),(A_\alpha),B) := \bigcup_{\alpha<\eta}(A_\alpha \setminus \bigcup_{\beta<\alpha} C_\beta) \cup (B \setminus \bigcup_{\alpha<\eta} C_\alpha).$$

If Θ is any subset of $\mathcal{P}(2^\omega) \times \mathcal{P}(2^\omega)$, then the class $\mathrm{SD}_\eta(\Theta, \mathbf{\Gamma}')$ is the class of all sets of the form $\mathrm{SD}_\eta((C_\alpha),(A_\alpha),B)$, where $(C_\alpha, A_\alpha) \in \Theta$ for all α and $B \in \mathbf{\Gamma}'$. We will in fact only need classes of the type $\mathrm{SD}_\eta(\langle \mathbf{\Delta}, \mathbf{\Gamma}\rangle_{\mathrm{SU}}, \mathbf{\Gamma}')$, where $\langle \mathbf{\Delta}, \mathbf{\Gamma}\rangle_{\mathrm{SU}}$ was defined above.

The non–self-dual Wadge classes are described with the help of a subset D of ω_1^ω. We identify $\omega_1^\omega \times \omega_1^\omega$ and $(\omega_1^\omega)^\omega$ with pairwise disjoint subsets of ω_1^ω by means of some fixed bijections, in such a way that no $u \in \omega_1^\omega$ of the form $\xi^\wedge i^\wedge v$ with $\xi \geq 1$ and $i \in \{1,\dots,5\}$ corresponds to some pair $(u_0,u_1) \in \omega_1^\omega \times \omega_1^\omega$ or some sequence $(u_n) \in (\omega_1^\omega)^\omega$. The set D and the class $\mathbf{\Gamma}_u$ associated to some $u \in D$ are defined inductively by the following closure properties.

(i) $\underline{0} := (0,0,\dots) \in D$ and $\mathbf{\Gamma}_{\underline{0}} = \{\emptyset\}$;
(ii) if $u = \xi^\wedge 1^\wedge \eta^\wedge \underline{0}$, where $\xi \geq 1$ and $\eta \geq 1$, then $u \in D$ and $\mathbf{\Gamma}_u = \mathrm{D}_\eta(\mathbf{\Sigma}^0_\xi)$;
(iii) if $u = \xi^\wedge 2^\wedge \eta^\wedge v$, where $\xi \geq 1$, $\eta \geq 1$, $v \in D$ and $v(0) > \xi$, then $u \in D$ and $\mathbf{\Gamma}_u = \mathrm{Sep}(\mathrm{D}_\eta(\mathbf{\Sigma}^0_\xi), \mathbf{\Gamma}_v)$;
(iv) if $u = \xi^\wedge 3^\wedge \eta^\wedge (u_0,u_1)$, where $\xi \geq 1$, $\eta \geq 1$, $u_0,u_1 \in D$, $u_0(0) > \xi$, ($u_1 = 0$ or $u_1(0) \geq \xi$), and $\mathbf{\Gamma}_{u_1} < \mathbf{\Gamma}_{u_0}$, then $u \in D$ and $\mathbf{\Gamma}_u = \mathrm{Bisep}(D_\eta(\mathbf{\Sigma}^0_\xi), \mathbf{\Gamma}_{u_0}, \mathbf{\Gamma}_{u_1})$;
(v) if $u = \xi^\wedge 4^\wedge (u_n)$, where $\xi \geq 1$, $(u_n) \in D^\omega$, the sequence $(u_n(0))$ is nondecreasing with $\sup_n u_n(0) > \xi$, and $\mathbf{\Gamma}_{u_n} < \mathbf{\Gamma}_{u_{n+1}}$ for all n, then $u \in D$ and $\mathbf{\Gamma}_u = \mathrm{SU}(\mathbf{\Sigma}^0_\xi, \bigcup_n \mathbf{\Gamma}_{u_n})$;
(vi) if $u = \xi^\wedge 5^\wedge \eta^\wedge (u_0,u_1)$, where $\xi \geq 1$, $\eta \geq 2$, $u_0,u_1 \in D$, $u_0(0) = \xi$, $u_0(1) = 4$, ($u_1(0) = 0$ or $u_1(0) \geq \xi$), and $\mathbf{\Gamma}_{u_1} < \mathbf{\Gamma}_{u_0}$, then $u \in D$ and $\mathbf{\Gamma}_u = \mathrm{SD}_\eta(\langle \mathbf{\Sigma}^0_\xi, \mathbf{\Gamma}_{u_0}\rangle_{\mathrm{SU}}, \mathbf{\Gamma}_{u_1})$.

We now have the following result [L3].

THEOREM 1.31. *The non–self-dual Wadge classes of Borel sets are exactly the classes in* $\{\mathbf{\Gamma}_u;\ u \in D\} \cup \{\check{\mathbf{\Gamma}}_u;\ u \in D\}$.

Any $u \in D$ will be called a *description*. We can now proceed to define the descriptions giving rise to Wadge classes of ideals. Below, a sequence of descriptions $(u_n) \in D^\omega$ is said to be *admissible* if the sequence $(u_n(0))$ is nondecreasing and $\mathbf{\Gamma}_{u_n} < \mathbf{\Gamma}_{u_{n+1}}$ for all n. We define a set of descriptions $D^* \subseteq D$ together with a set of admissible sequences $\mathcal{D} \subseteq D^\omega$ by the following closure properties. Recall that an infinite ordinal η is said to be *indecomposable* if it cannot be written as $\eta = \eta_1 + \eta_2$ with $\eta_i < \eta$.

(1) If $u = \xi^\wedge 1^\wedge \eta^\wedge \underline{0} \in D$ and $\eta \in \{1,2\}$ or η is indecomposable, then $u \in D^*$;

(2) if $(u_n) \in D^\omega$ and $u_n = \xi^\wedge 1^\wedge \eta_n{}^\wedge \underline{0}$ for some fixed $\xi \geq 1$, where the sequence (η_n) is increasing and $\sup_n \eta_n$ is indecomposable, then $(u_n) \in \mathcal{D}$;
(3) if $u = \xi^\wedge 4^\wedge (u_n) \in D$ and $(u_n) \in \mathcal{D}$, then $u \in D^*$;
(4) if $u = \xi^\wedge 5^\wedge \eta^\wedge (u_0, \underline{0}) \in D$, where $u_0 \in D^*$ and η is indecomposable, then $u \in D^*$;
(5) if $(u_n) \in D^\omega$ is admissible and $u_n \in D^*$ for all n, then $(u_n) \in \mathcal{D}$;
(6) if $(u_n) \in D^\omega$ and $u_n = \xi^\wedge 5^\wedge \eta_n{}^\wedge (u, \underline{0}) \in D$ for some fixed $u \in D^*$, where the sequence (η_n) is increasing with $\sup_n \eta_n$ indecomposable, then $(u_n) \in \mathcal{D}$.

We can finally state van Engelen's result [vE] giving the full picture of the Wadge classes of ideals in $\mathcal{K}(2^\omega)$.

THEOREM 1.32. *A class $\mathbf{\Gamma} \neq \{\emptyset\}, \{2^\omega\}$ is a Wadge class of ideal in $\mathcal{K}(2^\omega)$ if and only if one of the following holds.*

- $\mathbf{\Gamma} = \mathbf{\Delta}^0_1$;
- $\mathbf{\Gamma} = \mathbf{\Pi}^0_\xi$, *for some $\xi \geq 1$;*
- $\mathbf{\Gamma} = \mathrm{D}_\eta(\mathbf{\Sigma}^0_1)$, *where $\eta \in \{1, 2\}$;*
- $\mathbf{\Gamma} = \mathrm{D}_\eta(\mathbf{\Sigma}^0_\xi)$ *for some $\xi \geq 2$, where $\eta \in \{1, 2\}$ or η is indecomposable;*
- $\mathbf{\Gamma} = \mathbf{\Gamma}_u$, *for some $u \in D^*$ such that $u(1) > 1$.*

Among the just described Wadge classes are the additive Borel classes $\mathbf{\Sigma}^0_\xi$. For some of them, earlier work of D. Cenzer and R. D. Mauldin [CM1, CM2] provides very natural examples of ideals with that precise complexity. For each countable ordinal ξ, set

$$\mathcal{I}^{(\xi)}_{CB} := \{K \in \mathcal{K}(X);\ K^{(\xi)} = \emptyset\},$$

where $K^{(\xi)}$ is the ξ-th Cantor–Bendixson derivative of K. Clearly, $\mathcal{I}^{(\xi)}_{CB}$ is an ideal of compact sets.

THEOREM 1.33. *Assume X is compact and uncountable.*

(1) *For each natural number $k \geq 1$, the ideal $\mathcal{I}^{(k)}_{CB}$ is $\mathbf{\Sigma}^0_{2k}$-complete.*
(2) *If λ is a limit ordinal and $k \in \omega$, then $\mathcal{I}^{(\lambda+k)}_{CB}$ is $\mathbf{\Sigma}^0_{\lambda+2k}$-complete.*

COROLLARY 1.34. *Let $d_{CB} \colon \mathcal{K}(X) \to \mathcal{K}(X)$ be the Cantor–Bendixson derivation on $\mathcal{K}(X)$. For each natural number $k \geq 1$, the k-th iterate map $d_{CB} \circ \cdots \circ d_{CB}$ has Baire class exactly $2k$.*

Finally, it should be added that earlier work of J. Calbrix [Cal1, Cal2] and Zafrany [Zaf1, Zaf2] also led to examples of ideals or filters on ω of arbitrarily high complexity in the Borel hierarchy.

§2. Examples. In this chapter, we give some specific examples of ideals and σ-ideals of closed sets: thin sets from harmonic analysis, σ-porous sets, smooth sets for a Borel equivalence relation, Haar-null sets, and "sets of continuity" for Borel functions.

2.1. Sets of uniqueness and sets of extended uniqueness. In this section, we identify the circle group $\mathbb{T}$ with the interval $[0, 2\pi]$ in the usual way.

DEFINITION 2.1. A set $A \subseteq \mathbb{T}$ is said to be a *set of uniqueness* if it has the following property: if a trigonometric series $\sum_{n\in\mathbb{Z}} c_n e^{int}$ converges to 0 at each point $t \in \mathbb{T} \setminus A$, then $c_n = 0$ for all $n \in \mathbb{Z}$. We denote by $\mathcal{U}$ the family of sets of uniqueness, and by U the family of *closed* sets of uniqueness. A set which is not of uniqueness is called a *set of multiplicity*, or an $\mathcal{M}$*-set*; a closed $\mathcal{M}$-set is called an M-set.

It is already nontrivial that countable sets or even the empty set are sets of uniqueness; these are classical results due to G. Cantor and W. H. Young. By another classical theorem due to Bary [Bary1], the union of countably many closed sets of uniqueness is again a set of uniqueness. In particular, U is a σ-ideal of compact sets. See [KL1], [Bary2] or [KahSa] for proofs of these results.

For *closed* sets, there is a very useful functional-analytic characterization of uniqueness. Let $A(\mathbb{T}) = \widehat{\ell^1(\mathbb{Z})}$ be the space of all continuous functions $f\colon \mathbb{T} \to \mathbb{C}$ with absolutely convergent Fourier series, endowed with its natural norm ($\|f\|_A = \|\hat{f}\|_1$). The dual space of $A(\mathbb{T})$ is the space of *pseudo-measures* on $\mathbb{T}$, which is denoted by $\mathrm{PM}(\mathbb{T})$; of course, $\mathrm{PM}(\mathbb{T})$ is isometric to $\ell^\infty(\mathbb{Z})$. A pseudo-measure S is said to be a *pseudo-function* if $\hat{S} \in c_0(\mathbb{Z})$; we denote by $\mathrm{PF}(\mathbb{T})$ the space of pseudo-functions.

Since $A(\mathbb{T})$ is a regular Banach algebra, there is a well defined notion of *support* for pseudo-measures: the support of $S \in \mathrm{PM}(\mathbb{T})$ is the smallest closed set $E \subseteq \mathbb{T}$ such that $\langle S, f\rangle = 0$ for all $f \in A(\mathbb{T})$ with support disjoint from E. Notice that each complex measure μ on $\mathbb{T}$ can be viewed as a pseudo-measure, and that the two notions of support at hand coincide.

We now have the following very important characterization of closed sets of uniqueness, due to I. I. Pyateckiĭ-Šapiro ([PS], see [KL1, II.4] or [KahSa, Théorème V.2.I]): *A closed set $E \subseteq \mathbb{T}$ is a set of uniqueness if and only if E does not support any nonzero pseudo-function.* One advantage of this formulation is that it makes it easy to show that U is a σ-ideal of compact sets, and also that it is $\mathbf{\Pi}^1_1$. Another advantage is that it allows to define closed sets of uniqueness in an arbitrary Hausdorff locally compact second countable nondiscrete abelian group.

DEFINITION 2.2. A *Rajchman measure* on $\mathbb{T}$ is a complex measure μ whose Fourier transform $\hat{\mu}$ belongs to $c_0(\mathbb{Z})$. A set $A \subseteq \mathbb{T}$ is said to be a set of *extended uniqueness* if $\mu(A) = 0$ for every positive Rajchman measure μ. We denote by $\mathcal{U}_0$ the family of sets of extended uniqueness, and by U_0 the family of closed sets in $\mathcal{U}_0$. A set which is not in $\mathcal{U}_0$ is called an $\mathcal{M}_0$-set; a closed $\mathcal{M}_0$-set is called an M_0-set.

It is well-known that the family of Rajchman measures is hereditary with respect to absolute continuity. It follows that a closed set E is a U_0-set if and only if it does not support any nonzero Rajchman measure, if and only if E is a "set of uniqueness for Fourier–Stieltjes series", which means that if a Fourier–Stieltjes series $\sum \hat{\mu}(n)e^{int}$ converges to 0 outside E, then all coefficients are 0. Clearly, each $\mathcal{U}$-set is a $\mathcal{U}_0$-set. It is also clear from the definition that U_0 is a σ-ideal of compact sets, and easy to check that it is $\mathbf{\Pi}^1_1$.

The following result was proved independently by R. Solovay (unpublished) and R. Kaufman [Kau2]. Theorem 2.5 below is a more precise version, but we state the Kaufman–Solovay Theorem separately because of its "historical" importance (see [KL1]). Indeed, the problem of finding a concrete characterization of sets of uniqueness had always been considered as very difficult to solve (see, e.g., [Bary1, p. 68]), and the Kaufman–Solovay Theorem formulated that feeling in a precise mathematical way.

THEOREM 2.3. *The σ-ideals U and U_0 are $\mathbf{\Pi}^1_1$-complete.*

To proceed further, we need to introduce another family of thin sets.

DEFINITION 2.4. A compact set $K \subseteq \mathbb{T}$ is said to be a *Dirichlet set* if there exists a sequence of integers $(n_k) \subseteq \mathbb{Z}$ tending to infinity such that $e^{in_k t} \to 1$ uniformly on K. We denote by D the family of Dirichlet sets.

The following facts are well-known.

- *Finite sets are Dirichlet* ("Dirichlet's Theorem").
- *D is a G_δ subset of $\mathcal{K}(\mathbb{T})$*. This is easy to check. It follows that $D \cap \mathcal{K}(E)$ is a dense G_δ subset of $\mathcal{K}(E)$ for each compact set $E \subseteq \mathbb{T}$.
- *Dirichlet sets are sets of uniqueness*. In fact, one can show that if K is a Dirichlet set, then $\|\hat{S}\|_\infty \leq \limsup_{|n|\to\infty} |\hat{S}(n)|$ for each pseudo-measure S supported on K (see [Kah, VII.8]). Notice that this is obvious if S is a measure.

We can now state the announced strengthening of the Kaufman–Solovay Theorem (see [M1]). If $\mathcal{B} \subseteq \mathcal{K}(\mathbb{T})$, we denote by $\mathcal{B}_\oplus$ the family of all compact sets $K \subseteq \mathbb{T}$ which are finite disjoint unions of sets from $\mathcal{B}$.

THEOREM 2.5. *Let E be a nonempty closed subset of $\mathbb{T}$, and let $\mathcal{I} \subseteq \mathcal{K}(E)$. Assume one of the following holds.*

(1) *E is an M-set and $D_\oplus \cap \mathcal{K}(E) \subseteq \mathcal{I} \subseteq U$.*
(2) *$E \in U_0^{\mathrm{perf}}$ and $\mathcal{G}_\oplus \subseteq \mathcal{I} \subseteq U_0$, for some dense G_δ hereditary set $\mathcal{G} \subseteq \mathcal{K}(E)$.*

Then $\mathcal{I}$ is $\mathbf{\Sigma}^0_3$-hard. If $\mathcal{I}$ is a σ-ideal, then it is $\mathbf{\Pi}^1_1$-hard.

Remark 2.6. The appearance of the class $\mathbf{\Sigma}^0_3$ is not accidental. Indeed, quite a lot of natural families of thin sets happen to be $\mathbf{\Sigma}^0_3$-complete; see below. Notice also that if $\mathcal{G} \subseteq \mathcal{K}(\mathbb{T})$ is G_δ and hereditary, then $\mathcal{G}_\oplus$ is easily seen to be $\mathbf{\Sigma}^0_3$.

Of course, the last sentence in Theorem 2.5 follows from Corollary 1.3. From this, one gets the following "local" version of the Kaufman–Solovay

Theorem, whose importance will be apparent later. The result is due to Kaufman [Kau3] for U_0, and Debs–Saint Raymond [DSR] for U. See also [KL1] for a proof of the U_0 case by a rank argument using Theorem 1.16, Corollary 1.18 and Theorem 2.15 below.

COROLLARY 2.7. *If $E \subseteq \mathbb{T}$ is an M-set, then $U \cap \mathcal{K}(E)$ is $\mathbf{\Pi}^1_1$-complete. If E is an M_0-set, then $U_0 \cap \mathcal{K}(E)$ is $\mathbf{\Pi}^1_1$-complete.*

A further consequence of Theorem 2.5 is the following striking result of Kechris [K5]. The proof in [K5] was based on a rank argument.

COROLLARY 2.8. *If E is a nonempty compact set in U_0^{perf}, then each dense G_δ σ-ideal of $\mathcal{K}(E)$ contains an M_0-set.*

Remark 2.9. V. Tardivel [T] has shown that the Kaufman–Solovay Theorem holds in every (Hausdorff) nondiscrete second countable locally compact abelian group. Theorem 2.5 holds in that setting as well.

2.2. Helson sets.

DEFINITION 2.10. A compact set $K \subseteq \mathbb{T}$ is said to be a *Helson set* if for every continuous function $f\colon K \to \mathbb{T}$ there is $\tilde{f} \in A(\mathbb{T})$ with $\tilde{f}|_A = f$. We denote by H the family of Helson sets.

By a standard duality argument involving the restriction map $R\colon A(\mathbb{T}) \to \mathcal{C}(K)$, a compact set K is Helson if and only if there exists some finite constant $c (\geq 1)$ such that

$$\|\mu\| \leq c\, \|\hat{\mu}\|_\infty \tag{2}$$

for every complex measure μ supported on K. The smallest such constant is called the *Helson constant* of K (see, e.g., [KL1, VII.3]).

An enormous amount of results on Helson sets, Dirichlet sets and other families of thin sets can be found in T. W. Körner's papers [Ko1, Ko2, Ko3, Ko4]. We will need the following important facts.

- *The family H is an ideal of $\mathcal{K}(\mathbb{T})$.* This is a deep theorem due to N. T. Varopoulos (see [LP]). On the other hand, it is very easy to show that H is closed under finite disjoint unions.
- *Finite independent sets are Helson with constant* 1. This follows from Kronecker's Theorem on diophantine approximation.
- *Helson sets are U_0-sets.* This follows because if K is Helson, then one can in fact replace $\|\hat{\mu}\|_\infty$ by $\limsup_{|n|\to\infty} |\hat{\mu}(n)|$ in (2). This is a nontrivial result (see [KL1, Theorem VII.3.4] or [KahSa, Théorème XI.4.IV]).

It is not hard to check that for each $c \geq 1$, the family H_c of Helson sets with constant not greater than c is a G_δ subset of $\mathcal{K}(\mathbb{T})$. It follows that $H = \bigcup_{p=1}^{\infty} H_p$ is $\mathbf{\Sigma}^0_3$. Moreover, H_1 is dense in $\mathcal{K}(P)$, for each perfect set $P \subseteq \mathbb{T}$, because finite independent sets are in H_1. Applying (2) in

Theorem 2.5 with $\mathcal{G} = H_1$, we now get the following result. Alternatively, one can use Corollary 1.28, since it is well-known that $H \cap \mathcal{K}(V)$ is a σ-ideal for no nonempty open set $V \subseteq \mathbb{T}$ (see, e.g., [Kah, III.5]), and likewise inside any M_0-set (see [K5]).

PROPOSITION 2.11. *The family H is a true $\mathbf{\Sigma}^0_3$ subset of $\mathcal{K}(\mathbb{T})$. In fact, H is true $\mathbf{\Sigma}^0_3$ within any M_0-set.*

Notice that it is *not* true that H is true $\mathbf{\Sigma}^0_3$ within any *M*-set. This follows from the following fundamental theorem.

THEOREM 2.12. *There exist Helson sets which are not sets of uniqueness. In fact, any M-set contains a Helson M-set.*

The first part of this theorem is due to T. W. Körner [Ko2, Ko3]. Almost simultaneously, R. Kaufman obtained the second part using completely different arguments [Kau1].

Helson sets of multiplicity are typical examples of sets which fail harmonic synthesis. A compact set $K \subseteq \mathbb{T}$ is said to be a *set of harmonic synthesis* if each pseudo-measure S supported on K can be w^*-approximated by measures supported on K. The existence of sets which are *not* sets of synthesis was established in 1959 by P. Malliavin. Actually, Malliavin proved that each *M*-set contains a set of non-synthesis ([Mal], see also [Kah, V.8]).

We now use Theorems 2.5 and 2.12 to outline a descriptive-set-theoretic proof of the following result of T. W. Körner [Ko5]; see [M1].

COROLLARY 2.13. *There exists a compact set $K \subseteq \mathbb{T}$ which is both Dirichlet and Helson, but not of harmonic synthesis.*

PROOF. Let us denote by SYN the family of all sets of synthesis. The basic observation is the following: if $E \subseteq \mathbb{T}$ is a Helson set with constant c, then a compact set $K \subseteq E$ is in SYN if and only if each pseudo-measure S supported on K is in fact a measure, with $\|S\|_{\mathrm{M}(\mathbb{T})} \leq c\,\|\hat{S}\|_\infty$. Using this, one can verify that $SYN \cap \mathcal{K}(E)$ is a G_δ subset of $\mathcal{K}(E)$ (cf. [KL1, X.1]). Moreover, SYN is easily seen to be closed under finite disjoint unions. Starting with a Helson *M*-set $E \subseteq \mathbb{T}$ and applying Theorem 2.5, we get that $SYN \cap \mathcal{K}(E)$ does not contain $D \cap \mathcal{K}(E)$, which gives the desired result. ⊣

2.3. U', U'_0 and U'_1 sets.

DEFINITION 2.14. A compact set $K \subseteq \mathbb{T}$ is said to be a *U'-set* if there exists some finite constant c such that $\|\hat{S}\|_\infty \leq c\, \limsup_{|n|\to\infty} |\hat{S}(n)|$ for each pseudo-measure S with support in K. If K has the same property with "pseudo-measure" replaced by "positive measure", then K is said to be a *U'_0-set*.

We have already pointed out that Dirichlet sets are in U' and Helson sets are in U'_0. To each compact set in U' is associated in a natural way a "U'-constant", and likewise for U'_0-sets. It is not too hard to check that in both cases, the family of compact sets with constant not greater than some fixed number c is G_δ in $\mathcal{K}(\mathbb{T})$. It follows that U' and U'_0 are countable unions of hereditary G_δ sets, and hence $\mathbf{\Sigma}^0_3$ sets. Finally, it can be shown (see [KL1]) that U' and U'_0 are ideals of compact sets. Using Theorem 2.5 or Corollary 1.28 in the same way as for Helson sets, it follows that U' and U'_0 are true $\mathbf{\Sigma}^0_3$ subsets of $\mathcal{K}(\mathbb{T})$. See [M2] for details.

The following theorem is due to Kechris and Louveau [KL2].

THEOREM 2.15. *The family U'_0 is a basis for U_0.*

Remark 2.16. Debs and Saint Raymond [DSR] independently proved that U_0 has a Borel basis. Later on, Lyons showed that the Borel basis they used coincides in fact with U'_0 (see [Ly3]).

Since U'_0 is a countable union of G_δ hereditary sets, an application of Proposition 1.13 yields the following corollary.

COROLLARY 2.17. *The σ-ideal U_0 has a G_δ basis.*

Remark 2.18. It follows from this (or simply from the fact that U'_0 is a countable union of G_δ hereditary sets), that U_0^{perf} is a $\mathbf{\Pi}^0_3$ set. Applying Proposition 1.29, we get that U_0^{perf} is true $\mathbf{\Pi}^0_3$.

We finally mention one more interesting family of thin sets. If K is a compact subset of $\mathbb{T}$, let us denote by $\mathrm{M}(K)$ the family of all complex measures supported on K (viewed as a subspace of $\mathrm{PM}(\mathbb{T})$), and by $\mathrm{N}(K)$ the w^*-closure of $\mathrm{M}(K)$ in $\mathrm{PM}(\mathbb{T})$.

DEFINITION 2.19. A compact set $K \subseteq \mathbb{T}$ is said to be a U'_1-set if there exists some finite constant c such that $\|\hat{S}\|_\infty \leq c\ \limsup_{|n|\to\infty} |\hat{S}(n)|$ for all $S \in \mathrm{N}(K)$.

By a duality argument (see [KL1, VI.2]), a compact set $K \subseteq \mathbb{T}$ is a U'_1-set if and only if the ideal

$$I(K) := \{f \in A(\mathbb{T});\ f \equiv 0 \text{ on } K\}$$

is w^*-sequentially dense in $A(\mathbb{T})$ (note that $A(\mathbb{T}) \simeq \ell^1(\mathbb{Z})$ is the dual space of $\mathrm{PF}(\mathbb{T}) \simeq c_0(\mathbb{Z})$). Using this, it is not difficult to check that U'_1 is a $\mathbf{\Sigma}^1_1$ ideal of $\mathcal{K}(\mathbb{T})$.

The exact complexity of U'_1 is unknown. A related class of thin sets is the family of U_1-*sets*, defined as follows: a compact set K is in U_1 if and only if $I(K)$ is w^*-dense in $A(\mathbb{T})$. Equivalently (by a duality argument) K is in U_1 if and only if $\mathrm{N}(K)$ contains no nonzero pseudo-function. It was proved by Pyateckiĭ-Šapiro ([PS], see [KL1]) that each U_1-set is a countable union of U'_1-sets. Therefore, U_1 and U'_1 generate the same σ-ideal of compact sets, which is denoted by U_1^*. Using the Kaufman–Körner Theorem, one can show that U_1 is not a σ-ideal, in other words that $U_1 \neq U_1^*$ (see [KL1, Proposition VI.3.6]). Having a $\mathbf{\Sigma}_1^1$ basis, the σ-ideal U_1^* is $\mathbf{\Sigma}_1^1$-inductive by Theorem 1.19. It seems likely that U_1^* is not $\mathbf{\Pi}_1^1$, and hence that U_1' is not Borel, but no proof has been found yet. This problem is extensively studied in [KLT].

Remark 2.20. In [Ly3], R. Lyons defines and studies two other classes of thin sets, denoted by U_2 and U_2'. The class U_2 is strictly intermediate between U_1 and U_0, and U_2' is strictly intermediate between U_1' and U_0'. In [KLT], a nondecreasing transfinite family of intermediate classes $(U_1^\xi)_{\xi<\omega_1}$ is defined in a very natural way, with $U_1^0 = U_1$, $U_1^1 = U_2$ and $U_1^\xi \subseteq U_0$ for all ξ. It is shown that U_1 is $\mathbf{\Pi}_1^1$ if and only if this nondecreasing family is stationary.

2.4. Porous and σ-porous sets. In this section, X is a metric Polish space, that is, a Polish space for which we have fixed some compatible Polish metric d. The open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$.

DEFINITION 2.21. A set $A \subseteq X$ is said to be *porous* at some point $x \in X$ if the following relation between A and x holds: there exists a fixed positive number c such that, for every $r > 0$, one can find $z \in B(x, r)$ and $r' > 0$ with $B(z, r') \cap A = \emptyset$ and $r' > c \cdot d(x, z)$. The set A is said to be *porous* if it is porous at each point $x \in A$, and *σ-porous* if it can be covered by countably many porous sets.

Remark 2.22. If the definition of porosity is satisfied with the same positive constant c at each point $x \in A$, then the set A is said to be *c-porous*.

Clearly, each porous set is nowhere dense, so that each σ-porous set is meager in X. In $\mathbb{R}^n$, porous sets also have Lebesgue measure 0, by the Lebesgue Density Theorem. Notice also that if X is perfect, then each finite set is porous. We refer to [Za1, Za2] and [Za4] for very nice surveys on porous and σ-porous sets.

Let us denote by $\mathcal{I}_{\sigma\mathbf{p}}$ the σ-ideal in $\mathcal{K}(X)$ consisting of all compact σ-porous sets. It can be shown (see [ZP]) that if X is compact, then $\mathcal{I}_{\sigma\mathbf{p}}$ is $\mathbf{\Pi}_1^1$. Moreover, we have the following result [ZP], which shows in particular that the σ-ideal $\mathcal{I}_{\sigma\mathbf{p}}$ is true $\mathbf{\Pi}_1^1$ (hence $\mathbf{\Pi}_1^1$-complete) if X is perfect.

THEOREM 2.23. *Assume the metric space X is compact, and let E be a subset of X. Assume one of the following holds.*

(1) *E is dense in X.*

(2) *E is compact and not σ-porous.*

Then there is no $\mathbf{\Sigma}_1^1$ set $\mathcal{A} \subseteq \mathcal{K}(X)$ such that $\mathcal{K}_\omega(E) \subseteq \mathcal{A} \subseteq \mathcal{I}_{\sigma\mathbf{p}}$.

Remark 2.24. When $X = [0, 1]$, it is not difficult to show that $\mathcal{I}_{\sigma\mathbf{p}}$ is $\mathbf{\Pi}_1^1$-hard. For each sequence $\varepsilon = (\varepsilon_n)_{n\in\omega}$ with $0 < \varepsilon_n < 1$, let K_ε be the Cantor set constructed in the same way as the classical Cantor ternary set, starting from $K_0 = [0, 1]$ and using at step n the dissection ratio ε_n. By a result of

P. D. Humke and B. S. Thomson [HT], K_ε is not σ-porous if and only if $\varepsilon_n \to 0$. Since the map $\varepsilon \mapsto K_\varepsilon$ is continuous, it follows that the σ-ideal $\mathcal{I}_{\sigma\mathbf{p}}$ is $\mathbf{\Pi}^0_3$-hard, and hence $\mathbf{\Pi}^1_1$-hard.

Let us point out at least one "concrete" consequence of Theorem 2.23 [ZP]. Other results of the same type can be found in [ZP].

COROLLARY 2.25. *There exists a compact set $K \subseteq \mathbb{T}$ which is of uniqueness, but not σ-porous.*

PROOF. It follows from a classical result of L. H. Loomis [Loo] that each countable compact set is in U' (see also [KL1, V.5]). Since U' is Borel, one can apply (2) in Theorem 2.23 to get that some U'-sets are not σ-porous. $\dashv$

Remark 2.26. The compact set K in Corollary 2.25 cannot be a Dirichlet set, since Dirichlet sets are easily seen to be porous. More generally, it cannot be of type $H^{(n)}$, for any $n \geq 1$ (see [KL1, III.1] for the definition). Indeed, it was shown by P. Šleich [Sl] that $H^{(n)}$-sets are σ-porous. See [Za3] for a proof of Šleich's result.

Remark 2.27. It is also true that there exist closed porous sets in $\mathbb{T}$ which are not sets of uniqueness. More precisely, it follows from the classical Salem–Zygmund Theorem that there exist symmetric Cantor sets of constant dissection ratio which are M_0-sets (see [KL1, Theorem III.4.1] or [KahSa]), and such sets are easily seen to be porous. Using Theorem 2.5, one can also prove that each M_0-set contains a porous M_0-set, and that each M-set contains a porous M-set: consider the family $\mathcal{G}$ of all compact $\frac{1}{4}$-porous sets, which is easily seen to be G_δ and stable under finite disjoint unions.

It follows from Theorem 2.23 that if G is a nonmeager G_δ subset of X (with X compact), then G contains a closed, non–σ-porous set. One can in fact prove a stronger result [ZP], which we quote for future reference.

THEOREM 2.28. *Let Y be a complete metric space (possibly nonseparable). If $A \subseteq Y$ is Souslin and not σ-porous, then A contains a closed set which is not σ-porous.*

Apart from the ordinary porosity defined above, there are many other natural notions of "porosity", which can be studied within a general "abstract" framework. The basic concept is the following.

DEFINITION 2.29. Let $\mathbf{P}$ be a point-set relation on X, that is, a relation between points of X and subsets of X. We say that $\mathbf{P}$ is a *porosity-like relation* if it satisfies the following three axioms.

(P1) If $\mathbf{P}(x, B)$ and $A \subseteq B$, then $\mathbf{P}(x, A)$.
(P2) $\mathbf{P}(x, A)$ if and only if there exists $r > 0$ such that $\mathbf{P}(x, B(x, r) \cap A)$.
(P3) $\mathbf{P}(x, A)$ if and only if $\mathbf{P}(x, \overline{A})$.

Given a porosity-like relation $\mathbf{P}$, a set $A \subseteq X$ is said to be $\mathbf{P}$-*porous* if $\mathbf{P}(x, A)$ for all $x \in A$, and σ-$\mathbf{P}$-porous if it can be covered by countably many **P**-porous sets. In that setting, one can prove abstract analogues of 2.23 and 2.28 for porosity-like relations satisfying additional properties; see, e.g., [ZeZa] or [ZaZe1].

A different definition of abstract porosity can be found in [FZ]. J. Zapletal found a game characterization of σ-porosity (see [FZ]) and even a characterization of his abstract porosity [Zap]. Connections between the Banach–Mazur game and σ-porosity are investigated in [Ze1].

Finally, it should be added that porosity makes sense in an arbitrary metric space, not necessarily separable. And in fact, it can be useful in nonseparable Banach space theory as well; see [Za4].

2.5. Haar-null sets. In this section, G is a Polish abelian group.

Definition 2.30. A universally measurable set $A \subseteq G$ is said to be *Haar-null* if one can find some probability measure μ on G such that $\mu(A+x) = 0$ for all $x \in G$. An arbitrary set is Haar-null if it is contained in a universally measurable Haar-null set.

Haar-null sets were introduced by Christensen (see [Chr2] and [Chr1]) and have received much attention in the last few years. Some authors call them *shy* sets rather than Haar-null sets, and call a set *prevalent* if its complement is Haar-null (e.g., [HSY]). See [OY] for a recent survey, and [BL], [Mva] for very interesting connections with Banach space geometry.

It is easy to check (using Fubini's Theorem) that if the group G is locally compact, then the Haar-null sets in G are exactly the null-sets for the Haar measure of G. On the other hand, it is proved in [Chr2] that if G is not locally compact, then all compact subsets of G are Haar-null.

Clearly, Haar-null sets have empty interior. Moreover, it is shown in [Chr2] that the family of Haar-null sets is a translation-invariant σ-ideal of subsets of G. Thus, we see that Haar-negligibility provides a reasonable notion of "almost-everywhere". Among other things, Christensen used it to prove automatic continuity results for group homomorphisms and an infinite dimensional version of the classical Rademacher Theorem on the differentiability of Lipschitz functions. Since then, many other notions of negligibility were introduced to get differentiability results for Lipschitz functions; see [BL] and [LPr].

Let us denote by HN the family of all closed Haar-null sets in G. If G is locally compact, then HN is Borel in $\mathcal{F}(G)$ since Haar-nullness can be tested on the Haar measure only. If G is not locally compact, then the exact complexity of HN is unknown. By its very definition, HN is $\mathbf{\Sigma}^1_2$ in $\mathcal{F}(G)$. Solecki [S5] has shown that HN is not Borel, so that Borelness of HN characterizes local compactness of G. Solecki's result gives in fact a much sharper lower bound for the complexity of HN. To state it precisely, we need the following definition.

Definition 2.31. A set $\Gamma \subseteq \omega^\omega$ is said to be *dominating* if it has the following property: for each $\alpha = (\alpha_n) \in \omega^\omega$, one can find $\gamma = (\gamma_n) \in \Gamma$

such that α is eventually dominated by γ, i.e., there exists $N \in \omega$ such that for every $n \geq N$ we have $\alpha_n \leq \gamma_n$. We denote by ND the family of all closed, *non*dominating subsets of ω^ω.

The following result is due to Saint Raymond [SR4].

THEOREM 2.32. ND *is a complete* $\mathbf{\Sigma}^1_1$*-inductive set in* $\mathcal{F}(\omega^\omega)$.

Now, we can state Solecki's result [S5].

THEOREM 2.33. *Assume G is not locally compact. Then one can find a closed, non–Haar-null set* $E \subseteq G$, *and a continuous, open surjection* $\phi\colon E \to \omega^\omega$ *such that for each closed set* $C \subseteq \omega^\omega$:

$$C \in \mathrm{ND} \Leftrightarrow \phi^{-1}(C) \in \mathrm{HN}.$$

COROLLARY 2.34. *If G is not locally compact, then the complexity of* HN *is at least* $\mathbf{\Sigma}^1_1$*-inductive. In particular,* HN *is not* $\mathbf{\Pi}^1_1$ *in* $\mathcal{F}(G)$.

We conclude this section with another result of Solecki [S2]. With the terminology of section 3.6, it says that if the underlying group is not locally compact, then the σ-ideal of Haar-null sets is not *thin*. Special cases of this result (e.g., when G is a Banach space) were obtained earlier by Dougherty [Do2].

THEOREM 2.35. *If G is not locally compact, then one can find in G an uncountable family of pairwise disjoint, closed non–Haar-null sets.*

Remark 2.36. The notion of Haar-negligibility makes sense in non-abelian Polish groups as well, but then three reasonable definitions come to mind: one can consider left Haar-null sets, right Haar-null sets, or "two-sided" Haar-null sets, where one requires $\mu(xAy) = 0$ for all $x, y \in G$. Then 2.33 and 2.35 hold for two-sided Haar-null sets provided the group G admits a translation-invariant metric. The three definitions are not always equivalent, and some properties of HN (in particular, the validity of *Pettis' Lemma*) are closely related to the structure of the group G. See [S6] and [S7] for very interesting results along these lines.

2.6. Smooth sets for a Borel equivalence relation.

DEFINITION 2.37. A Borel equivalence relation E on X is said to be *smooth* (or *Borel-separated*) if there exists a countable family of Borel sets $(A_i)_{i\in I}$ such that xEy if and only if for all $i \in I$ we have $x \in A_i \Leftrightarrow y \in A_i$.

For example, the equality relation $=_X$ is smooth, as witnessed by a countable basis (V_i) for the topology of X.

On the other hand, it is not difficult to show that if there exists some nonzero measure μ on X such that $\mu(A) = 0$ or $\mu(X \setminus A) = 0$ for each E-invariant, μ-measurable set $A \subseteq X$ and each E-equivalence class is μ-null, then the equivalence relation E cannot be smooth. Such a measure μ is said to be *E-ergodic*. A typical example is the Vitali equivalence relation E_0 on 2^ω, defined as follows: if $x = (x_n)$, $y = (y_n) \in 2^\omega$, then

$$xE_0y \Leftrightarrow \exists N \in \omega\ \forall n \geq N : x_n = y_n.$$

By Kolmogorov's 0-1 law, the Lebesgue measure on 2^ω is E_0-ergodic, so that E_0 is not smooth.

By a fundamental result due to L. Harrington, A. S. Kechris, and A. Louveau [HKL], a Borel equivalence relation E is non-smooth if and only if E_0 can be continuously embedded in E, which means that there exists a continuous embedding $\mathbf{i}\colon 2^\omega \to X$ such that

$$xE_0y \Leftrightarrow \mathbf{i}(x)\, E\, \mathbf{i}(y).$$

It follows that a Borel equivalence relation E is non-smooth *if and only if* it admits a non-zero ergodic measure.

Assume that the Polish space X is compact, and let E be a non-smooth equivalence relation on X. A compact set $K \subseteq X$ is said to be *E-smooth* if $\mu(K) = 0$ for each E-ergodic measure μ on X. Equivalently, K is E-smooth if and only if the equivalence relation $E_{|K}$ is smooth, if and only if E_0 does not embed in $E_{|K}$. We denote by $\mathcal{I}_E$ the family of all compact E-smooth sets $K \subseteq X$. By definition, $\mathcal{I}_E$ is a σ-ideal in $\mathcal{K}(X)$. This σ-ideal has been extensively studied by Uzcátegui [U2]. Among other things, Uzcátegui obtained the following theorem.

THEOREM 2.38. *Let E be a Borel non-smooth relation on a compact space X. The σ-ideal $\mathcal{I}_E$ is $\mathbf{\Pi}^1_1$-complete in any compact set $K \notin \mathcal{I}_E$. More precisely, if $\mathcal{I} \subseteq \mathcal{K}(X)$ is a σ-ideal such that $\mathcal{I} \subseteq \mathcal{I}_E$ and $\mathcal{I}$ contains all singletons of some non-smooth compact set, then $\mathcal{I}$ is not $\mathbf{\Sigma}^1_1$.*

The proof that $\mathcal{I}_E$ is $\mathbf{\Pi}^1_1$ given in [U2] seems to require effective descriptive set theory. We sketch a proof of $\mathbf{\Pi}^1_1$-hardness of $\mathcal{I}_{E_0}$. For each $\alpha \in 2^\omega$, set

$$K_\alpha := \{x \in 2^\omega;\ x \leq \alpha\},$$

where $\leq$ is the product ordering on 2^ω. The map $\alpha \mapsto K_\alpha$ is continuous from 2^ω into $\mathcal{K}(2^\omega)$. Obviously, K_α is a finite set if $\alpha \in \mathbf{Q}$. On the other hand, if $\alpha \in 2^\omega \setminus \mathbf{Q}$, then the canonical homeomorphism $\mathbf{i}_\alpha\colon 2^\omega \to K_\alpha$ is an embedding of E_0 into $(E_0)_{|K_\alpha}$ and hence K_α is not E_0-smooth. Thus, $\mathbf{Q}$ can be continuously reduced to $\mathcal{I}_{E_0}$, whence the σ-ideal $\mathcal{I}_{E_0}$ is $\mathbf{\Pi}^1_1$-hard by Corollary 1.3.

2.7. Sets of continuity for Borel functions. In this section, X and Y are Polish spaces. For an arbitrary function $f\colon X \to Y$, one may define

$$\mathcal{I}_{\mathbf{cont}}(f) := \{K \in \mathcal{K}(X);\ f_{|K} \text{ is continuous}\}.$$

Clearly, $\mathcal{I}_{\mathbf{cont}}(f)$ is an ideal of compact sets. In [J1] and [J2], F. Jordan has studied the connections between the complexity of $\mathcal{I}_{\mathbf{cont}}(f)$ and that of the function f itself. We outline here some of his results.

PROPOSITION 2.39. *The ideal $\mathcal{I}_{\mathbf{cont}}(f)$ is G_δ if and only if the function f is continuous.*

PROOF. If f is not continuous, one can find a sequence $(x_n) \subseteq X$ converging to some $x \in X$ and such that $f(x)$ is not a cluster point of the sequence $(f(x_n))$. Then $E := \{x_n;\ n \in \omega\} \cup \{x\}$ is compact and $\mathcal{I}_{\mathbf{cont}}(f) \cap \mathcal{K}(E)$ is just the family of all finite subsets of E. It follows at once that $\mathcal{I}_{\mathbf{cont}}(f)$ is not G_δ.

If f is continuous then $\mathcal{I}_{\mathbf{cont}}(f) = \mathcal{K}(X)$, hence $\mathcal{K}(X)$ is G_δ. ⊣

THEOREM 2.40. *The ideal $\mathcal{I}_{\mathbf{cont}}(f)$ is Borel if and only if f has a G_δ graph.*

One half of this result is easy, once it is observed that $\mathcal{I}_{\mathbf{cont}}(f) = \pi_X[\mathcal{K}(G_f)]$ and that π_X is 1-1 when restricted to G_f. The more difficult converse implication uses Hurewicz's Theorem and a variant of Blumberg's Theorem; see [J1].

The next results can be found in [J1] for f Borel. Using a result of V. Vlasák [Vl] saying that if $\mathcal{I}_{\mathbf{cont}}(f)$ is $\mathbf{\Sigma}^1_1$, then the map f is Borel, we can state them as follows.

THEOREM 2.41. *Let $f\colon X \to Y$ be an arbitrary map.*

(i) *If $\mathcal{I}_{\mathbf{cont}}(f)$ is $\mathbf{\Pi}^0_3$, then f is Baire 1; if f is Baire 1, then $\mathcal{I}_{\mathbf{cont}}(f)$ is $\mathbf{\Pi}^0_4$. Moreover, these implications cannot be reversed.*

(ii) *The ideal $\mathcal{I}_{\mathbf{cont}}(f)$ is $\mathbf{\Delta}^0_3$ if and only if it is $\mathbf{\Sigma}^0_2$, and this happens exactly when f has the following property: for each point $x \in X$, one can find $U \subseteq X$ and $V \subseteq Y$ open such that $x \in U \cap f^{-1}(V)$ and the restriction of f to $\overline{U \cap f^{-1}(V)}$ is continuous.*

Further results can be found in [J1] and [J2].

§3. The covering property. A σ-ideal $\mathcal{I} \subseteq \mathcal{F}(X)$ is said to have the *covering property* if the following statement holds for all $\mathbf{\Sigma}^1_1$ sets $A \subseteq X$: either A can be covered by countably many sets from $\mathcal{I}$, or A contains a closed set $F \notin \mathcal{I}$.

Recall the definition of $\mathcal{I}^{\text{ext}}$: a set $A \subseteq X$ is in $\mathcal{I}^{\text{ext}}$ if it can be covered by countably many sets from $\mathcal{I}$. Now, define the family $\mathcal{I}^{\text{int}}$ as follows: a set $A \subseteq X$ is in $\mathcal{I}^{\text{int}}$ if all closed subsets of X contained in A are in $\mathcal{I}$. With these notations, $\mathcal{I}$ has the covering property if and only if $\mathcal{I}^{\text{int}} \cap \mathbf{\Sigma}^1_1 \subseteq \mathcal{I}^{\text{ext}}$.

A typical example of a σ-ideal with the covering property is the σ-ideal of countable sets: this is the content of the classical Perfect Set Theorem. Notice that it is consistent with ZFC that this theorem can fail for $\mathbf{\Pi}^1_1$ sets, and this is the main reason for restricting the covering property to $\mathbf{\Sigma}^1_1$ sets.

On the other hand, the σ-ideal of closed meager sets in $\mathbb{R}$ does not have the covering property, since there exist dense G_δ sets $G \subseteq \mathbb{R}$ with empty interior. Similarly, the σ-ideal of closed Lebesgue-null sets does not have the covering property, because one can find dense G_δ sets with measure 0.

This chapter will be focused on the covering property and its interactions with other properties of σ-ideals. Here we will use repeatedly the notation

$\mathcal{B}^{\text{perf}}$, for a family $\mathcal{B} \subseteq \mathcal{F}(X)$: an arbitrary set $A \subseteq X$ is in $\mathcal{B}^{\text{perf}}$ if and only if

$$\forall V \subseteq X \text{ open}: V \cap A \neq \emptyset \Rightarrow V \cap A \notin \mathcal{B}^{\text{ext}}.$$

By the Baire Category Theorem, this definition is compatible with the one given in Definition 1.9 for closed sets (and a hereditary family $\mathcal{B}$).

3.1. Solecki's covering theorem. The following very general result was proved by Solecki. Due to its importance, we discuss it in some details.

THEOREM 3.1 (Solecki [S1]). *Let $\mathcal{I}$ be a family of closed subsets of X. If $A \subseteq X$ is $\mathbf{\Sigma}^1_1$ and $A \notin \mathcal{I}^{\text{ext}}$, then A contains a G_δ set which is not in $\mathcal{I}^{\text{ext}}$.*

The proof of Theorem 3.1 relies on the next two lemmas. The first one is essentially due to G. Petruska [Ptr1, Ptr2]. If $(B_s)_{s\in\omega^{<\omega}}$ is a Souslin scheme of subsets of X, we denote by $\mathcal{A}((B_s))$ the set $\bigcup_{\alpha\in\omega^\omega}\bigcap_{n\in\omega} B_{\alpha|n}$. The Souslin scheme is called *regular* if $B_t \subseteq B_s$ whenever $s \preceq t$.

LEMMA 3.2. *Let $\mathcal{I}$ be a family of closed subsets of X, and let $A \subseteq X$ be $\mathbf{\Sigma}^1_1$. Assume $A \notin \mathcal{I}^{\text{ext}}$. Then one can find a regular Souslin scheme $(C_s)_{s\in\omega^{<\omega}}$ consisting of closed sets such that*

(i) $C_\emptyset \neq \emptyset$;
(ii) $\mathcal{A}((C_s)) \subseteq A$;
(iii) *if $C_s \neq \emptyset$, then $A \cap C_s \in \mathcal{I}^{\text{perf}}$ and $A \cap C_s$ is dense in C_s;*
(iv) $\bigcup_{k\in\omega} C_{s^\wedge k}$ *is dense in C_s.*

The second lemma comes from [S1]. The basic idea used in the proof is the following simple observation: if $F \subseteq X$ is a (nonempty) closed nowhere dense set, then one can find a (nonempty) discrete set $D \subseteq X$ such that $D \cap F = \emptyset$ and $\overline{D} = D \cup F$. This idea appears in several papers [S1, KS, U2], where variants of Lemma 3.3 can be found.

LEMMA 3.3. *Let $(G_s)_{s\in\omega^{<\omega}}$ be a regular Souslin scheme with $G_s \in \mathbf{\Pi}^0_2(X)$ for all $s \in \omega^{<\omega}$ and $G_\emptyset \neq \emptyset$. Set $\widetilde{A} := \mathcal{A}((G_s))$. Let also $\mathcal{F}$ and $\mathcal{B}$ be two families of closed subsets of X, with $\mathcal{B}$ hereditary. Assume that the following properties hold true for each $s \in \omega^{<\omega}$:*

(i) $\bigcup_{n\in\omega} G_{s^\wedge n}$ *is dense in G_s;*
(ii) *for each open set $V \subseteq X$ such that $V \cap G_s \neq \emptyset$, one can find a closed set $F \subseteq V \cap G_s$ with F meager in G_s and $F \in \mathcal{F} \setminus \mathcal{B}$.*

Then there exists a nonempty G_δ set $H \subseteq \widetilde{A}$ such that $H \in \mathcal{B}^{\text{perf}}$ and $\overline{H} \setminus H \in \mathcal{F}^{\text{ext}}$.

PROOF OF THEOREM 3.1. Let A be a $\mathbf{\Sigma}^1_1$ set not in $\mathcal{I}^{\text{ext}}$, and let (C_s) be a Souslin scheme given by Lemma 3.2. Then two cases can occur: either there exists $s_0 \in \omega^{<\omega}$ and an open set V such that $V \cap C_{s_0} \neq \emptyset$ and $\text{MGR}(V \cap C_{s_0}) \subseteq \mathcal{I}^{\text{ext}}$, or not. In the first case, $A' := V \cap C_{s_0} \cap A$ is nonempty and in $\mathcal{I}^{\text{perf}}$ by condition (iii) in Lemma 3.2; since A' is $\mathbf{\Sigma}^1_1$, it has the Baire property in $V \cap C_{s_0}$, so A' contains a G_δ set G such that $A' \setminus G$ is meager in $V \cap C_{s_0}$; then $A' \setminus G \in \mathcal{I}^{\text{ext}}$ by assumption, hence $G \notin \mathcal{I}^{\text{ext}}$. In the second case, one can apply Lemma 3.3 with $G_s = C_s$, $\mathcal{F} := \mathcal{F}(X)$ and $\mathcal{B} := \mathcal{I}^{\text{ext}} \cap \mathcal{F}(X)$ to get a nonempty G_δ set $H \subseteq \widetilde{A} \subset A$ with $H \in \mathcal{I}^{\text{perf}}$. ⊣

Remark 3.4. It is also proved in [S1, Corollary 2] that under the extra set-theoretic assumption

$$\forall x \in \omega^\omega : \omega_1^{L[x]} < \omega_1,$$

Theorem 3.1 holds for all $\mathbf{\Sigma}^1_2$ sets A provided family of closed sets under consideration is $\mathbf{\Sigma}^1_2$ with respect to the Effros Borel structure.

The role of separability of the underlying space X is discussed in [HZZ].

As far as the covering property is concerned, the usefulness of Solecki's Theorem is apparent, since it allows to check the property on G_δ sets only: to prove that a σ-ideal $\mathcal{I}$ has the covering property, one just has to show that if G is a nonempty G_δ set in $\mathcal{I}^{\text{perf}}$, then $G \notin \mathcal{I}^{\text{int}}$; or equivalently, that if G is a G_δ set which is dense in some nonempty closed set $P \in \mathcal{I}^{\text{perf}}$, then $G \notin \mathcal{I}^{\text{int}}$. Here is a simple illustration (see [K4]). Recall that we write $\mathcal{K}^*(P)$ for $\mathcal{K}(P) \setminus \{\emptyset\}$.

Corollary 3.5. *Let $\mathcal{I}$ be a σ-ideal in $\mathcal{K}(X)$. Assume that $\mathcal{I} \cap \mathcal{K}^*(P)$ is meager in $\mathcal{K}^*(P)$ for each compact set $P \in \mathcal{I}^{\text{perf}}$. Then $\mathcal{I}$ has the covering property. This holds in particular if $\mathcal{I}$ has a K_σ basis.*

Proof. Let P be a nonempty closed set in $\mathcal{I}^{\text{perf}}$, and let G be a dense G_δ subset of P. We have to show that $G \notin \mathcal{I}^{\text{int}}$, and this follows from the Baire Category Theorem: $\mathcal{K}(P) \setminus \mathcal{I}$ is comeager in $\mathcal{K}^*(P)$ and $\mathcal{K}(G)$ is a dense G_δ subset of $\mathcal{K}(P)$, whence $\mathcal{K}(G) \cap (\mathcal{K}(P) \setminus \mathcal{I}) \neq \emptyset$.

The last assertion follows from Proposition 1.11(5). ⊣

3.2. Calibration, bases and the covering property.

Definition 3.6. [KLW] Let $\mathcal{I}$ be a σ-ideal in $\mathcal{F}(X)$. Then $\mathcal{I}$ is said to be *calibrated* if the following property holds true: whenever H is a G_δ set in $\mathcal{I}^{\text{int}}$ and (F_n) is a sequence in $\mathcal{I}$ such that $F := H \cup \bigcup_n F_n$ is closed, it follows that $F \in \mathcal{I}$. Equivalently, $\mathcal{I}$ is calibrated if and only if for each G_δ set $H \in \mathcal{I}^{\text{int}}$ and each F_σ set $M \in \mathcal{I}^{\text{int}}$, the set $H \cup M$ is again in $\mathcal{I}^{\text{int}}$.

One can also define calibration for a σ-ideal of *compact* sets (in a not necessarily compact X) with the obvious modifications: the sets F and F_n should be compact, and $H \in \mathcal{I}^{\text{int}}$ is replaced by $\mathcal{K}(H) \subseteq \mathcal{I}$.

The following lemma gives an intuitive characterization of calibration for σ-ideals of compact sets. (see [KLW, Section 3.2, Proposition 1], [KL1, Proposition VI.1.10]).

Lemma 3.7. *Let $\mathcal{I}$ be a σ-ideal in $\mathcal{K}(X)$. Then $\mathcal{I}$ is calibrated if and only if $\mathcal{I}^{\text{int}} \cap \mathbf{\Pi}^0_2(X)$ is a σ-ideal of $\mathbf{\Pi}^0_2$ sets. If it is so, then in fact $\mathcal{I}^{\text{int}} \cap \mathbf{\Sigma}^0_3(X)$ is a σ-ideal of $\mathbf{\Sigma}^0_3$ sets.*

Here are some examples of calibrated and non-calibrated σ-ideals.

- If A is any subset of X, then $\mathcal{F}_X(A)$ and $\mathcal{K}(A)$ are calibrated.
- If $\mathcal{I}$ is the σ-ideal of common null-sets of some family of measures, then $\mathcal{I}$ is calibrated. In particular, U_0 is calibrated.
- If γ is a subadditive capacity, then the closed null-sets for γ form a calibrated σ-ideal.
- *The σ-ideal U is calibrated.* (Debs–Saint Raymond [DSR], Kechris–Louveau).
- *The σ-ideal of closed σ-porous sets is calibrated.* This follows at once from Theorem 2.28.
- If X is perfect, then the σ-ideal of meager sets is *not* calibrated, since one can find in X dense G_δ sets with empty interior.

Before stating the main result of this section, we need a definition: we shall say that a basis $\mathcal{B}$ for a σ-ideal $\mathcal{I} \subseteq \mathcal{F}(X)$ is *nontrivial* in some set $E \subseteq X$ if one can find a closed set $F \subseteq E$ with $F \in \mathcal{I} \setminus \mathcal{B}$. Typically, if $\mathcal{I}$ is a σ-ideal in $\mathcal{F}(X)$ which has a Borel basis $\mathcal{B}$, and if E is a closed subset of X such that $\mathcal{I} \cap \mathcal{F}(E)$ is true $\mathbf{\Pi}_1^1$, then the basis $\mathcal{B}$ is nontrivial for $\mathcal{I}$ in E.

THEOREM 3.8. *Let $\mathcal{I}$ and $\mathcal{J}$ be two σ-ideals in $\mathcal{F}(X)$. Assume that $\mathcal{I}$ is calibrated and has a basis $\mathcal{B}$ which is nontrivial for $\mathcal{I}$ in each closed set $E \notin \mathcal{J}$. Then every G_δ set in $\mathcal{I}^{\text{int}}$ is in $\mathcal{J}^{\text{ext}}$.*

PROOF. It is enough to show that if $G \subseteq X$ is a nonempty G_δ set in $\mathcal{J}^{\text{perf}}$, then $G \notin \mathcal{I}^{\text{int}}$. Replacing X by $\overline{G}$, we may assume that G is dense in X. Then $X \in \mathcal{J}^{\text{perf}}$. Write $G = \bigcap_{n\in\omega} O_n$, where the O_n's are open and $O_{n+1} \subseteq O_n$ for all n. It follows that all sets in $\mathcal{B}$ are meager in X, hence all sets in $\mathcal{I}$ are meager as well. Moreover, the basis $\mathcal{B}$ is nontrivial in any open set. Thus, one can apply Lemma 3.3 with $G_s := O_{|s|}$, $\mathcal{F} := \mathcal{I}$, and $\mathcal{B} := \mathcal{B}$. This gives a nonempty G_δ set $H \subseteq G$ such that $H \in \mathcal{B}^{\text{perf}} = \mathcal{I}^{\text{perf}}$ and $\overline{H} \setminus H \in \mathcal{I}^{\text{ext}}$. Since $\mathcal{I}$ is calibrated, this shows that $G \notin \mathcal{I}^{\text{int}}$. ⊣

Applying Theorem 3.8 with $\mathcal{J} = \mathrm{MGR}(X) \cap \mathcal{F}(X)$, we get the following result (see [KL3]).

COROLLARY 3.9. *Let $\mathcal{I}$ be a σ-ideal in $\mathcal{F}(X)$. Assume that $\mathcal{I}$ is calibrated and has a basis which is nontrivial in each nonempty open set. Then every set $A \in \mathcal{I}^{\text{int}}$ with the Baire property is meager in X.*

As another consequence of Theorem 3.8, we can now state the following criterion for the covering property. It applies in particular if the σ-ideal $\mathcal{I}$ has a Borel basis and if $\mathcal{I} \cap \mathcal{F}(E)$ is true $\mathbf{\Pi}_1^1$ for each closed set $E \notin \mathcal{I}$.

COROLLARY 3.10 (Debs–Saint Raymond [DSR]). *Let $\mathcal{I}$ be a σ-ideal in $\mathcal{F}(X)$. Assume that $\mathcal{I}$ is calibrated and has a basis which is nontrivial in each closed set $E \notin \mathcal{I}$. Then $\mathcal{I}$ has the covering property.*

PROOF. By Solecki's Theorem 3.1, it is enough to check the covering property on G_δ sets. So one can apply Theorem 3.8 with $\mathcal{J} = \mathcal{I}$. ⊣

The Debs–Saint Raymond criterion can be used in two ways: either to show that a given σ-ideal has the covering property, or to show that it has no nontrivial basis. Let us give some examples.

Example 3.11. *The σ-ideal U_0 has the covering property.*

This follows immediately from Corollary 3.10: U_0 is calibrated and true $\mathbf{\Pi}^1_1$ in any M_0-set, and it has a Borel basis, namely U_0'. This is the original proof of Debs and Saint Raymond [DSR]. A bit later, a very different proof was found by Kechris and Louveau [KL2] using functional analytic arguments. Notice that the result also follows from 2.8 and Solecki's Theorem.

The covering property for U_0 implies in particular that any Borel set of extended uniqueness is meager in $\mathbb{T}$. This is the solution of the longstanding *category problem* raised by Bary. See [KL1, VIII.3] for a detailed discussion and interesting applications.

Example 3.12. *The σ-ideal U has no Borel basis.*

This is also due to Debs and Saint Raymond [DSR]. It is proved in [DSR] that if $E \subseteq \mathbb{T}$ is a Helson set, then E has a dense G_δ subset in U^{int}. Applying this to some (nonempty) Helson set $E \in U^{\mathrm{perf}}$ provided by Körner–Kaufman's Theorem 2.12, it follows that U does not have the covering property. Since U is calibrated and true $\mathbf{\Pi}^1_1$ in any M-set, this gives the result.

Example 3.13. *If (X, d) is a perfect compact metric space, then the σ-ideal $\mathcal{I}_{\sigma\mathbf{p}} \subseteq \mathcal{K}(X)$ of all compact σ-porous sets has no Borel basis.*

This can be found in [ZP]. The proof follows the same scheme as for U.

Example 3.14. *Assume X is compact, and let $A \subseteq X$. Then the σ-ideal $\mathcal{K}(A)$ has a Borel basis if and only if A is the difference of two $\mathbf{\Pi}^0_2$ sets.*

This is due to Kechris, Louveau, and Woodin [KLW, Section 2.3, Theorem 10]. It is not hard to check that if $A \in \mathrm{D}_2(\mathbf{\Pi}^0_2)$, then $\mathcal{K}(A)$ has a Borel basis. For the converse, it is enough to show (by a Hurewicz-type result due to Saint Raymond [SR3]) that if

$$A = \{(\alpha, \beta) \in 2^\omega \times 2^\omega;\ \alpha \in \boldsymbol{Q} \text{ or } \beta \notin \boldsymbol{Q}\},$$

then $\mathcal{K}(A)$ has no Borel basis, as a σ-ideal in $2^\omega \times 2^\omega$. Now, $\mathcal{I} := \mathcal{K}(A)$ is true $\mathbf{\Pi}^1_1$ in any nonempty open set, since A is Borel and not G_δ in any nonempty open set; and $G := 2^\omega \times (2^\omega \setminus \boldsymbol{Q})$ is a dense G_δ subset of $2^\omega \times 2^\omega$ in $\mathcal{I}^{\mathrm{int}}$. By Corollary 3.9, it follows that $\mathcal{K}(A)$ has no Borel basis.

3.3. Well-founded type and porosity-like relations. The following notion of smallness was introduced by Louveau [L2]. If R is a tree on some set Λ, we denote by $[R]$ the set of infinite branches of R, and if C is a closed subset of Λ^ω, we denote by T_C the tree on Λ canonically associated to C (see [K6, 2.B]).

DEFINITION 3.15. Assume X is a G_δ subset of Λ^ω, for some countable set Λ. A family $\mathcal{B}$ of closed subsets of X is said to be *of well-founded type* if there exists some countable family of trees $(R_i)_{i\in I}$ on Λ such that the following properties hold.

(a) $[R_i] \cap X = \emptyset$ for each $i \in I$.
(b) A closed set $F \subseteq X$ is in $\mathcal{B}$ if and only if the tree $T_{\overline{F}}$ does not contain any tree R_i (where $\overline{F}$ is the closure of F in Λ^ω); in other words $F \in \mathcal{B}$ if and only if
$$\forall i \in I \; \exists s \in R_i \colon V_s \cap F = \emptyset,$$
where $V_s = \{x \in \Lambda^\omega;\ s \preceq x\}$.

If $X = \Lambda^\omega$, then (a) means that each tree R_i is well-founded, which explains the terminology.

Notice that if R is a tree on Λ such that $[R] \cap X = \emptyset$, then the family $\mathcal{V}_R := \{V_s \cap X;\ s \in R\}$ is locally finite in X. This suggests to define the notion of well-founded type in an arbitrary Polish space X, as follows: a family $\mathcal{B} \subseteq \mathcal{F}(X)$ will be said to be of well-founded type if there exists a family $(\mathcal{V}_i)_{i\in I}$ of locally finite families of nonempty open sets such that
$$F \in \mathcal{B} \Leftrightarrow \forall i \in I \; \exists V \in \mathcal{V}_i \colon F \cap V = \emptyset.$$
Notice that the family $(\mathcal{V}_i)_{i\in I}$ is not required to be countable.

Here are some examples of families of well-founded type.

Example 3.16. *The family $\mathcal{F}_1 := \{F \in \mathcal{F}(X);\ \sharp F \le 1\}$ is of well-founded type.* To see this, choose a basis $(O_j)_{j\in J}$ for the topology of X (with $O_j \neq \emptyset$ for all j), and let $I := \{(j,j') \in J \times J;\ O_j \cap O_{j'} = \emptyset\}$. For each $i = (j,j') \in I$, put $\mathcal{V}_i = \{O_j, O_{j'}\}$. Then the family $(\mathcal{V}_i)_{i\in I}$ witnesses that $\mathcal{F}_1$ is of well-founded type.

Example 3.17. *If X is not compact, then the family of all compact subsets of X is of well-founded type.* To prove this, let $\hat{X}$ be some metrizable compactification of X. For each $z \in Z := \hat{X} \setminus X$, choose a decreasing sequence $(\hat{V}_k(z))_{k\in\omega}$ of open neighbourhoods of z in $\hat{X}$ such that $\overline{\hat{V}_{k+1}(z)} \subseteq \hat{V}_k(z)$ and $\bigcap_k \hat{V}_k(z) = \{z\}$, and put $\mathcal{V}_z := (\hat{V}_k(z) \cap X)_{k\in\omega}$. Since a closed set $F \subseteq X$ is compact if and only if it is closed in $\hat{X}$, if and only if $z \notin \overline{F}^{\hat{X}}$ for all $z \in Z$, the family $(\mathcal{V}_z)_{z\in Z}$ does the job.

Example 3.18. In [L2], a closed set $F \subseteq \omega^\omega$ is called *perforated* (*troué*) if
$$\forall s \in \omega^{<\omega} \; \exists n \in \omega \colon V_{s^\frown n} \cap F = \emptyset.$$
Clearly, the family of all closed perforated subsets of ω^ω is of well-founded type.

It was shown by Louveau [L2] that if X is a G_δ subset of ω^ω and if $\mathcal{B} \subseteq \mathcal{F}(X)$ is of well-founded type, then the σ-ideal $\mathcal{B}_\sigma$ has the covering property. Louveau's proof relies on the fact that one can associate to each set $A \subseteq \omega^\omega$ an infinite game $G^{\mathcal{B}}_A$ which is determined if and only if $\mathcal{B}_\sigma$ has the covering property for A, and is indeed determined when A is $\mathbf{\Sigma}^1_1$. Using the same strategy, Kechris [K3] had proved a bit earlier a regularity result of the same type for another class of families of small sets in ω^ω. Both results cover the case of *σ-bounded sets* in ω^ω (subsets of ω^ω which can be covered by a K_σ set), and hence give the following theorem, due independently to Kechris [K3] and Saint Raymond [SR2]: *If A is a $\mathbf{\Sigma}^1_1$ subset of ω^ω, then either A is contained in a K_σ set, or A contains a closed set homeomorphic to ω^ω.*

We now give a different proof of Louveau's result, based on Solecki's Theorem 3.1 and a Cantor-like argument. Such a proof was suggested in [S1].

Theorem 3.19. *If $\mathcal{B} \subseteq \mathcal{F}(X)$ is of well-founded type, then the σ-ideal $\mathcal{B}_\sigma$ has the covering property.*

Proof. We fix a complete compatible metric on X. Let $\mathcal{B} \subseteq \mathcal{F}(X)$ be of well-founded type with witness $(\mathcal{V}_i)_{i\in I}$. Then each family $\mathcal{V}_i$ is countable, possibly finite. Adding the empty set infinitely many times if necessary, each family $\mathcal{V}_i$ can be extended to a countably infinite, locally finite family $\widetilde{\mathcal{V}}_i$. By Solecki's Theorem, it is enough to check that if $G \subseteq X$ is a nonempty G_δ set and $G \in \mathcal{B}^{\mathrm{perf}}$, then G contains a nonempty closed set $F \in \mathcal{B}^{\mathrm{perf}}$. Let G be such a G_δ set, and write $G = \bigcap_{n\in\omega} O_n$, where the O_n's are open with $O_{n+1} \subseteq O_n$ for all n. If W is any open set such that $W \cap G \neq \emptyset$, one can find $i \in I$ such that

$$\forall V \in \mathcal{V}_i\colon\ V \cap W \cap G \neq \emptyset.$$

It follows that one can construct inductively for $t \in \omega^{<\omega}$, an open set $W_t \subseteq X$ and some $i_t \in I$ such that

(1) $W_\emptyset \cap G \neq \emptyset$;
(2) $\overline{W}_t \subseteq O_{|t|}$;
(3) $\operatorname{diam}(W_t) \leq 2^{-|t|}$, if $W_t \neq \emptyset$;
(4) $\overline{W}_{t^\wedge k} \subseteq W_t$ for all $k \in \omega$;
(5) $\overline{W}_{t^\wedge k} \subseteq V_k$ for all k, where $\widetilde{\mathcal{V}}_{i_t} = (V_k)_{k\in\omega}$;
(6) $W_{t^\wedge k} \cap G \neq \emptyset$ if $W_t \neq \emptyset$ and $V_k \neq \emptyset$.

Now, put $F = \mathcal{A}((\overline{W}_t)) = \bigcup_{\alpha\in\omega^\omega} \bigcap_{n\in\omega} \overline{W}_{\alpha|n}$. By (1), (3), (4), (6), the set F is nonempty, and $F \subseteq G$ by (2). Since for each $t \in \omega^{<\omega}$ the family $(\overline{W}_{t^\wedge k})_{k\in\omega}$ is point-finite, it follows from König's Lemma that we also have $F = \bigcap_{n\in\omega} \bigcup_{t\in\omega^n} \overline{W}_t$. Moreover, using (4), (5) and local finiteness of the families $\mathcal{V}_i$, one checks by induction that for each $n \in \omega$, the family $(W_t)_{t\in\omega^n}$ is locally finite and hence the set $\bigcup_{t\in\omega^n} \overline{W}_t$ is closed in X. Therefore, F is a closed subset of X. Finally, if W is an open set such that $W \cap F \neq \emptyset$, then one can find $t \in \omega^{<\omega}$ such that $\emptyset \neq W_t \subseteq W$, and it follows from (3), (4), (5), (6) that $V \cap W \cap F \neq \emptyset$ for each $V \in \mathcal{V}_{i_t}$. This shows that $F \in \mathcal{B}^{\mathrm{perf}}$. $\dashv$

It turns out that Theorem 3.19 can be put into the general framework of porosity-like relations. Let $\mathcal{B} \subseteq \mathcal{F}(X)$ be of well-founded type with witness $(\mathcal{V}_i)_{i\in I}$. Then the point-set relation defined by

$$\mathbf{P}(x, A) \Leftrightarrow \exists r > 0 \colon \overline{A \cap B(x,r)} \in \mathcal{B}$$
$$(\Leftrightarrow \exists r > 0\, \forall i \in I\; \exists V \in \mathcal{V}_i \colon A \cap B(x,r) \cap V = \emptyset\ )$$

is easily seen to be a porosity-like relation.

Obviously, each closed set $F \in \mathcal{B}$ is $\mathbf{P}$-porous, and each $\mathbf{P}$-porous set is in $\mathcal{B}^{\text{ext}}$. Thus, the σ-$\mathbf{P}$-porous sets are exactly the sets in $\mathcal{B}^{\text{ext}}$, and each σ-$\mathbf{P}$-porous set can be covered by countably many *closed* $\mathbf{P}$-porous sets. Finally, using the fact that each family $\mathcal{V}_i$ is locally finite, it is not hard to check that $\mathbf{P}$ has the following additional property, where S' denotes the set of all accumulation points of a set $S \subseteq X$:

$(*)$ If $A \subseteq X$ is not $\mathbf{P}$-porous at some point $x \in A'$, then one can find a set $D \subseteq A$ such that $D' = \{x\}$ and D is not $\mathbf{P}$-porous at x.

Therefore, Theorem 3.19 appears to be a special case of the following result [ZaZe2].

THEOREM 3.20. *Let* $\mathbf{P}$ *be a porosity-like relation on* X. *Assume that each* σ-$\mathbf{P}$-*porous set can be covered by countably many* closed $\mathbf{P}$-*porous sets, and that* $\mathbf{P}$ *satisfies property* $(*)$. *Then each non–*σ-$\mathbf{P}$-*porous* $\boldsymbol{\Sigma}^1_1$ *set* $A \subseteq X$ *contains a closed non–*σ-$\mathbf{P}$-*porous set*.

The proof of this theorem is not very difficult, and it is in some sense similar to that of Theorem 3.19. It should be added that for many interesting and natural porosity-like relations, it is *not* true that σ-$\boldsymbol{P}$-porous sets can be covered by countably many closed $\mathbf{P}$-porous sets. It *is* nevertheless true that the conclusion of Theorem 3.20 holds for a large class of porosity-like relations (see [ZeZa]). But the proof is much more complicated than that of Theorem 3.20, even in the case of ordinary porosity.

3.4. The covering property and small Π^1_1 sets. In this section, the Polish space X is assumed to be *compact*. Moreover, each time we use the "lightface" notations of effective descriptive set theory, the spaces under consideration are assumed to be recursively presented.

It is well-known that if A is a countable Σ^1_1 subset of 2^ω, then one can find a Δ^1_1 sequence of Δ^1_1 binary trees (T_n) such that $[T_n]$ is countable for each n and $A \subseteq \bigcup_n [T_n]$. It is also well-known that there is a largest Π^1_1 subset of 2^ω without nonempty perfect subsets, which is usually denoted by C_1. It turns out that both results can be extended to any Π^1_1 σ-ideal of compact sets with the covering property. More precisely, we have the following results, due to Uzcátegui [U1].

THEOREM 3.21. *Let* $\mathcal{I}$ *be a* Π^1_1 σ-*ideal in* $\mathcal{K}(X)$ *with the covering property.*

(1) *Assume* $X = 2^\omega$. *If* $A \subseteq 2^\omega$ *is a* Σ^1_1 *set in* $\mathcal{I}^{\text{int}}$, *then there exists a* Δ^1_1 *sequence of* Δ^1_1 *trees* (T_n) *such that* $[T_n] \in \mathcal{I}$ *for all* n *and* $A \subseteq \bigcup_n [T_n]$.

(2) *If $\mathcal{I} \subseteq \mathrm{MGR}(X)$, then there is a largest Π^1_1 set in $\mathcal{I}^{\mathrm{int}}$, denoted by $C_1(\mathcal{I})$. If $X = 2^\omega$, then $C_1(\mathcal{I})$ can be described as follows*:

$$x \in C_1(\mathcal{I}) \Leftrightarrow \exists T \text{ binary tree}: x \in [T] \in \mathcal{I} \text{ and } T \in L_{\omega_1^x}.$$

The proof of (1) relies on a lemma due to R. Barua and V. V. Srivatsa [BS]: *If $\mathcal{I}$ is a Π^1_1 σ-ideal in $\mathcal{K}(2^\omega)$ and if T is a Σ^1_1 binary tree such that $[T] \in \mathcal{I}$, then one can find a Δ^1_1 binary tree S such that $T \subseteq S$ and $[S] \in \mathcal{I}$.* Using this and the Baire Category Theorem for the Gandy–Harrington topology, one shows that if A is a Σ^1_1 set in $\mathcal{I}^{\mathrm{int}}$, then $A \subseteq \bigcup\{[T];\ T \text{ is a } \Delta^1_1 \text{ binary tree and } [T] \in \mathcal{I}\}$, from which (1) follows easily.

The main tool for showing the existence of a largest "small" Π^1_1 set is the following theorem of Kechris [K2, Theorem 1.A-2]. It was used by Kechris [K3] and Louveau [L2] to show that for many interesting σ-ideals $\mathcal{I} \subseteq \mathcal{F}(\omega^\omega)$, there exists a largest Π^1_1 set in $\mathcal{I}^{\mathrm{int}}$. For example, this holds if $\mathcal{I}$ is the σ-ideal generated by a family $\mathcal{B}$ of well-founded type and parametrized by some Π^0_1 relation [L2].

A hereditary family $\mathcal{M}$ of subsets of some Polish space Z is said to be $\mathbf{\Pi}^1_1$-*additive* if the following property holds: if $(A_\xi)_{\xi<\alpha}$ is any transfinite sequence of sets in $\mathcal{M}$ such that the associate pre-well-ordering is $\mathbf{\Pi}^1_1$ (as a subset of $Z \times Z$), then $\bigcup_{\xi<\alpha} A_\xi \in \mathcal{M}$. For example, if τ is a Polish topology on Z finer than the original one, then the family of τ-meager sets is $\mathbf{\Pi}^1_1$-additive. Similarly, the family of negligible sets for a measure μ is $\mathbf{\Pi}^1_1$-additive. This can be proved by a simple application of Kuratowski–Ulam's or Fubini's Theorem; see Proposition 1.5.1 in [K1].

Now, Kechris' theorem reads as follows.

THEOREM 3.22. *Let $\mathcal{M}$ be a hereditary family of subsets of 2^ω or ω^ω. Assume $\mathcal{M}$ has the following properties*:

(a) *$\mathcal{M} \cap \mathbf{\Sigma}^1_1$ is Π^1_1 in the codes of $\mathbf{\Sigma}^1_1$ sets*;

(b) *$\mathcal{M}$ is $\mathbf{\Pi}^1_1$-additive.*

Then there exists a largest Π^1_1 set in $\mathcal{M}$.

It is proved in [U1] that if $\mathcal{I}$ has the covering property and $\mathcal{I} \subseteq \mathrm{MGR}(X)$, then $\mathcal{M} = \mathcal{I}^{\mathrm{int}}$ satisfies properties (a), (b) above. This gives the existence of a largest Π^1_1 set in $\mathcal{I}^{\mathrm{int}}$.

The proof of (a) uses the notion of *strong calibration*, which was introduced in [KLW].

DEFINITION 3.23. A σ-ideal $\mathcal{I} \subseteq \mathcal{K}(X)$ is said to be *strongly calibrated* if, for any compact metric space E and any $\mathbf{\Sigma}^1_1$ set $A \subseteq X \times E$ such that $\pi_X[A] \notin \mathcal{I}^{\mathrm{int}}$, there exists a compact set $K \subseteq A$ such that $\pi_X[K] \notin \mathcal{I}$.

For example, if $\mathcal{I}$ is the σ-ideal of common null-sets of some family of measures, or the σ-ideal of null-sets for some subadditive capacity, then $\mathcal{I}$ is strongly calibrated: this follows from a capacitability argument. It is also true, though not immediately apparent, that strong calibration implies calibration. Indeed, if H and M are arbitrary $\mathbf{\Sigma}^1_1$ sets in $\mathcal{I}^{\rm int}$, then $A = (H \times \{0\}) \cup (M \times \{1\}) \subseteq X \times \{0, 1\}$ is a $\mathbf{\Sigma}^1_1$ set such that $\pi_X[K] \in \mathcal{I}$ for each compact set $K = (K \cap (X \times \{0\})) \cup (K \times (X \times \{1\})) \subseteq A$, so $H \cup M = \pi_X[A] \in \mathcal{I}^{\rm int}$ by strong calibration. The following results from [U1] are interesting for their own sake.

Proposition 3.24. *Let $\mathcal{I}$ be a σ-ideal in $\mathcal{K}(X)$.*

(1) *If $\mathcal{I}$ has the covering property, then it is strongly calibrated.*
(2) *If $\mathcal{I}$ is Π^1_1 and strongly calibrated, then $\mathbf{\Sigma}^1_1 \cap \mathcal{I}^{\rm int}$ is Π^1_1 in the codes of $\mathbf{\Sigma}^1_1$ sets.*
(3) *If $\mathcal{I}$ is strongly calibrated, then $\mathcal{I}^{\rm int} \cap \mathbf{\Sigma}^1_1$ is a σ-ideal of $\mathbf{\Sigma}^1_1$ sets.*

Strong calibration is strictly weaker than the covering property. Yet, one can still get results similar to those in 3.21 for strongly calibrated σ-ideals [U1].

Proposition 3.25. *Let $\mathcal{I} \subseteq \mathcal{K}(X)$ be a Π^1_1 strongly calibrated σ-ideal.*

(1) *Any Σ^1_1 set $A \in \mathcal{I}^{\rm int}$ is contained in some Δ^1_1 set $B \in \mathcal{I}^{\rm int}$. Any $\mathbf{\Sigma}^1_1$ set in $\mathcal{I}^{\rm int}$ is contained in in some Borel set in $\mathcal{I}^{\rm int}$.*
(2) *The set $H(\mathcal{I}) := \bigcup\{B;\ B \in \Delta^1_1 \cap \mathcal{I}^{\rm int}\}$ is a Π^1_1 set in $\mathcal{I}^{\rm int}$ which contains all Σ^1_1 sets in $\mathcal{I}^{\rm int}$.*

Finally, we say a word about the covering property for beyond $\mathbf{\Sigma}^1_1$ sets. Part (1) of the following theorem is contained in [S1] (see Remark 3.4), and part (2) is again taken from [U1].

Theorem 3.26. *Let $\mathcal{I} \subseteq \mathcal{K}(X)$ be a σ-ideal with the covering property.*

(1) *If $\omega_1^{L(x)} < \omega_1$ for all $x \in \omega^\omega$ and $\mathcal{I}$ is $\mathbf{\Sigma}^1_2$, then $\mathcal{I}$ has the covering property for $\mathbf{\Sigma}^1_2$ sets.*
(2) *Assume $\mathcal{I} \subset \mathrm{MGR}(2^\omega)$. If $\omega_1^L < \omega_1$ and $\mathcal{I}$ is Π^1_1, then there is a largest Σ^1_2 set in $\mathcal{I}^{\rm int}$, namely*

$$C_2(\mathcal{I}) := \{x \in 2^\omega;\ \exists T \in L\colon T \textit{ is a binary tree and } x \in [T] \in \mathcal{I}\}.$$

Since $C_2(\mathcal{I}) \in \mathcal{I}^{\rm ext}$, it follows that $\mathcal{I}$ has the covering property for Σ^1_2 sets.

Remark 3.27. Let μ be the Lebesgue measure on 2^ω, and let $\mathcal{I}_\mu$ be the σ-ideal of compact μ-null sets. Dougherty and Kechris have shown that for any $x \in 2^\omega$, the set $\{y \in 2^\omega;\ x \text{ is recursive in } y\}$ is not in $\mathcal{I}_\mu^{\rm ext}$. From this, one can prove that in L, the set C_1 is not in $\mathcal{I}_\mu^{\rm ext}$. In particular, if $\mathcal{I}$ is any σ-ideal containing all singletons and contained in $\mathcal{I}_\mu$, then (in L) $\mathcal{I}$ does not have the covering property for Π^1_1 sets. This result can be transferred to $[0, 1]$ or to the circle group $\mathbb{T}$, hence one gets that in L, the σ-ideal U_0 does not have the covering property for Π^1_1 sets. See [U1] for details. The precise strength of the statement "U_0 has the covering property for Π^1_1 sets" seems to be unknown.

3.5. Smooth sets for a Borel equivalence relation. In this section, we assume that the Polish space X is compact, and E is a non-smooth Borel equivalence relation on X (see section 2.6). We denote by $\mathcal{I}_E$ the σ-ideal of E-smooth compact subsets of X.

Assume that X is recursively presented, and that E is Δ^1_1. Then $\mathcal{I}_E$ is Π^1_1. Moreover, since $\mathcal{I}_E$ is the σ-ideal of common null-sets of a family of measures, it is strongly calibrated (whence $\mathbf{\Sigma}^1_1 \cap \mathcal{I}_E^{\text{int}}$ is Π^1_1 in the codes of $\mathbf{\Sigma}^1_1$ sets by 3.24), and $\mathcal{I}^{\text{int}}$ is $\mathbf{\Pi}^1_1$-additive. By 3.22, it follows that there exists a largest Π^1_1 set in $\mathcal{I}^{\text{int}}$. On the other hand, we have the following result [U2].

THEOREM 3.28. *The σ-ideal $\mathcal{I}_E$ does not have the covering property, and hence it has no Borel basis.*

The second assertion follows from the Debs–Saint Raymond Theorem, since $\mathcal{I}_E$ is calibrated, and true $\mathbf{\Pi}^1_1$ in any non-smooth compact set by 2.38. The proof that $\mathcal{I}_E$ does not have the covering property uses a variant of Lemma 3.3. Finally, let us point out that the σ-ideal $\mathcal{I}_{E_0}$ does satisfy a weak form of the covering property: each set $A \in \mathcal{I}_{E_0}^{\text{int}}$ with the Baire property is meager in 2^ω. On the other hand, there exist non-smooth equivalence relations E for which one can find dense G_δ sets in $\mathcal{I}_E^{\text{int}}$. See [U2] for details.

3.6. Thinness of σ-ideals.

DEFINITION 3.29. Let $\mathcal{I}$ be a σ-ideal in $\mathcal{K}(X)$. A set $A \subseteq X$ is said to be *$\mathcal{I}$-thin* if A does not contain any uncountable family of pairwise disjoint compact sets not in $\mathcal{I}$. The σ-ideal $\mathcal{I}$ is said to be *thin* if the whole space X is $\mathcal{I}$-thin.

The notion of thinness makes sense also for a σ-ideal of *closed* sets, in the obvious way: just replace "compact" by "closed" in the above definition.

Let us denote by $\mathcal{I}^{\text{thin}}$ the family of all $\mathcal{I}$-thin subsets of X. It is easy to check that $\mathcal{I}^{\text{thin}} \cap \mathcal{K}(X)$ is a σ-ideal, whose study turns out to be very interesting (see [Del] and [KLW]). We shall not give any detail here. Our purpose is rather to point out some results which are close in spirit to those already presented in this chapter. However, we mention the following basic fact from [KLW].

THEOREM 3.30. *Let $\mathcal{I}$ be a $\mathbf{\Pi}^1_1$ σ-ideal in $\mathcal{K}(X)$. If $\mathcal{I}$ is not thin, then there exists a continuous map $\Phi\colon 2^\omega \to \mathcal{K}(X)$ such that the sets $\Phi(\alpha)$ are pairwise disjoint and $\Phi(\alpha) \notin \mathcal{I}$ for each $\alpha \in 2^\omega$.*

COROLLARY 3.31. *If $\mathcal{I}$ is a $\mathbf{\Pi}^1_1$ σ-ideal in $\mathcal{K}(X)$, then any G_δ set $G \notin \mathcal{I}^{\text{thin}}$ contains a compact set $K \notin \mathcal{I}^{\text{thin}}$.*

PROOF. Apply Theorem 3.30 to the σ-ideal $\mathcal{I} \cap \mathcal{K}(G) \subseteq \mathcal{K}(G)$ and put $K = \Phi[2^\omega]$. $\dashv$

Remark 3.32. If $\mathcal{I}$ is strongly calibrated, then 3.31 holds for any $\mathbf{\Sigma}^1_1$ set $A \notin \mathcal{I}^{\text{thin}}$; see [KLW].

The following very simple lemma is useful.

LEMMA 3.33. *Let $\mathcal{I}$ be a thin σ-ideal in $\mathcal{K}(X)$ (resp. in $\mathcal{F}(X)$). Then, for any set $H \subseteq X$, one can find a K_σ set (resp. an F_σ set) M such that $M \subseteq H$ and $H \setminus M \in \mathcal{I}^{\text{int}}$.*

PROOF. Let $(K_i)_{i \in I}$ be a maximal family of pairwise disjoint compact (resp. closed) sets not in $\mathcal{I}$ and contained in H. Then (K_i) is countable by thinness of $\mathcal{I}$, so $M := \bigcup_i K_i$ is a K_σ set (resp. an F_σ set), which has the required property. ⊣

Remark 3.34. It is proved in [KLW] that this lemma is actually a characterization of thinness for calibrated G_δ σ-ideals.

The following theorem [Ze2] "explains" why many natural σ-ideals happen to be non-thin. The simple proof below is due to Solecki [S9].

THEOREM 3.35. *Calibrated thin $\mathbf{\Pi}^1_1$ σ-ideals of compact sets are G_δ.*

PROOF. Let $\mathcal{I}$ be a calibrated, thin, $\mathbf{\Pi}^1_1$ σ-ideal in $\mathcal{K}(X)$, and assume that $\mathcal{I}$ is not G_δ. Since $\mathcal{I}$ is $\mathbf{\Pi}^1_1$, Hurewicz's Theorem provides a continuous map $\Phi\colon 2^\omega \to \mathcal{K}(X)$ such that $\Phi^{-1}(\mathcal{I}) = \mathbf{Q}$. Put $H = X \setminus \bigcup_{\alpha \in \mathbf{Q}} \Phi(\alpha)$, and let M be a K_σ set such that $M \subseteq H$ and $H \setminus M \in \mathcal{I}^{\text{int}}$. Now, set $G := X \setminus M$. If K is any compact subset of G, then $K \cap H$ is in $\mathcal{I}^{\text{int}}$ by the choice of M, and $K \setminus H \in \mathcal{I}^{\text{ext}}$ by definition of Φ. Since $\mathcal{I}$ is calibrated, this shows $K \in \mathcal{I}$. Consequently, $G \in \mathcal{I}^{\text{int}}$. It follows that if $\alpha \in 2^\omega \setminus \mathbf{Q}$, then $\Phi(\alpha)$ cannot be contained in G. Since we obviously have $\Phi(\alpha) \subseteq G$ if $\alpha \in \mathbf{Q}$, we conclude that $\mathbf{Q} = \Phi^{-1}(\mathcal{K}(G))$, which is impossible since $\mathbf{Q}$ is not G_δ in 2^ω. ⊣

This theorem can be used to recover two results of R. Kaufman (see [KL1, Theorem VII.1.7]) concerning sets of uniqueness and of extended uniqueness. Recall that the σ-ideals U and U_0 are calibrated, that U is true $\mathbf{\Pi}^1_1$ in any M-set, and that U_0 is true $\mathbf{\Pi}^1_1$ in any M_0-set. Applying Theorem 3.35, we get:

COROLLARY 3.36. *Any M-set $E \subseteq \mathbb{T}$ contains uncountably many pairwise disjoint M-sets. Any M_0-set contains uncountably many pairwise disjoint M_0-sets.*

Similarly, one can use Theorem 3.35 to show that if E is a non-smooth Borel equivalence relation on some compact metric space X, then the σ-ideal $\mathcal{I}_E$ is not thin. This result (with a different proof) is due to Uzcátegui [U2].

Here is another consequence of Theorem 3.35, which gives in particular a descriptive-set-theoretic proof of the following well-known fact: if μ is any measure on $\mathbb{T}$, then one can find an M_0-set K such that $\mu(K) = 0$. More general results can be proved using the covering property of U_0, see [KL1, Theorem VIII.3.3].

COROLLARY 3.37. *Let $\mathcal{I}$ be a calibrated, true $\mathbf{\Pi}^1_1$ σ-ideal in $\mathcal{K}(X)$, and let μ be a measure on X. Then one can find a compact set $K \subseteq X$ such that $K \notin \mathcal{I}$ and $\mu(K) = 0$.*

PROOF. Since μ is finite, the σ-ideal $\mathcal{I}_\mu := \{K \in \mathcal{K}(X);\ \mu(K) = 0\}$ is thin. If $\mathcal{I}_\mu$ were contained in $\mathcal{I}$, then $\mathcal{I}$ would be thin as well, hence G_δ by Theorem 3.35. ⊣

The following simple result shows that thinness and the covering property are essentially incompatible.

PROPOSITION 3.38. *Let $\mathcal{I}$ be a σ-ideal in $\mathcal{K}(X)$ or in $\mathcal{F}(X)$.*

(i) *If X is perfect and $\mathcal{I}$ is thin, then there exists a dense G_δ set $G \subseteq X$ with $G \in \mathcal{I}^{\text{int}}$.*

(ii) *Assume $\mathcal{I}$ is a σ-ideal in $\mathcal{F}(X)$. Then $\mathcal{I}$ is thin and has the covering property if and only if $\mathcal{I}$ has the form $\mathcal{F}_X(A)$, where $A \subseteq X$ is $\mathbf{\Sigma}^0_2$ and $X \setminus A$ is countable.*

PROOF. (i) Let H be a dense G_δ subset of X with empty interior, and let M be a $\mathbf{\Sigma}^0_2$ set such that $M \subseteq H$ and $H \setminus M \in \mathcal{I}^{\text{int}}$. Then M is meager in X, so $G := H \setminus M$ is a dense G_δ set in $\mathcal{I}^{\text{int}}$. This proves (i).

(ii) We just have to check that if $\mathcal{I}$ is thin with the covering property, then $\mathcal{I}$ has the required form (the converse implication is clear). Put $A := \{x \in X\colon \{x\} \in \mathcal{I}\}$. Then $X \setminus A$ must be countable because $\mathcal{I}$ is thin. If $\mathcal{I} \neq \mathcal{F}_X(A)$, then one can find a nonempty closed $\mathcal{I}$-perfect set P inside A. Then P is perfect since $\{x\} \in \mathcal{I}$ for all $x \in P$. By (i), one can find a dense G_δ set $G \subseteq P$ in $\mathcal{I}^{\text{int}}$. Since $\mathcal{I}$ has the covering property, we have $G \in \mathcal{I}^{\text{ext}}$, hence G is meager in P, a contradiction. Thus, $\mathcal{I} = \mathcal{F}_X(A)$. Since $\mathcal{I}$ has the covering property, it follows that A must be $\mathbf{\Sigma}^0_2$. ⊣

It follows from (ii) that if $\mathcal{I}$ has the covering property and contains all singletons, then $\mathcal{I}$ cannot be thin unless $X \in \mathcal{I}$. This can be applied to $\mathcal{I} \cap \mathcal{F}(E)$, for any closed set $E \subseteq X$, so we get the following result from [U1], which gives in particular the second half of Corollary 3.36 since U_0 has the covering property.

COROLLARY 3.39. *Let $\mathcal{I}$ be a σ-ideal in $\mathcal{F}(X)$. If $\mathcal{I}$ has the covering property and contains all singletons, then any closed set $E \notin \mathcal{I}$ contains uncountably many pairwise disjoint closed sets not in $\mathcal{I}$.*

To conclude this chapter, we now say a word about the so-called *countable chain condition*. Recall that a σ-ideal $\mathcal{J}$ of subsets of X is said to have the *ccc* if each family of pairwise disjoint Borel sets not in $\mathcal{J}$ is countable. Obviously, if $\mathcal{I}$ is a σ-ideal in $\mathcal{F}(X)$ and $\mathcal{I}^{\text{ext}}$ has the ccc, then $\mathcal{I}$ is thin, but the converse needs not be true.

If $\mathcal{C}$ is a family of closed subsets of X, we set

$$\text{MGR}(\mathcal{C}) := \{A \subseteq X;\ A \cap C \text{ is meager in } C \text{ for each } C \in \mathcal{C}\}.$$

Clearly, σ-ideals of the form $\text{MGR}(\mathcal{C})$ for some countable family $\mathcal{C}$ have the ccc. The following theorem of Kechris and Solecki [KS] is a strong converse for σ-ideals generated by closed sets. Let us say that a σ-ideal $\mathcal{J} \subseteq 2^X$ is *proper* if $\mathcal{J} \neq 2^X$ and $\mathcal{J}$ contains all singletons.

Theorem 3.40. *Let $\mathcal{J} \subseteq 2^X$ be a proper σ-ideal generated by a family of closed sets. Then precisely one of the following holds.*

(1) *$\mathcal{J} = \mathrm{MGR}(\mathcal{C})$ for some countable family $\mathcal{C}$ of closed subsets of X, which can be assumed to be well-ordered by reverse inclusion.*
(2) *There exists a homeomorphic embedding $\mathbf{i}\colon 2^\omega \times \omega^\omega \to X$ such that $\mathbf{i}[\{\varepsilon\} \times \omega^\omega] \notin \mathcal{J}$ for any $\varepsilon \in 2^\omega$.*

This theorem provides several characterization of the ccc for σ-ideals generated by closed sets [KS].

Corollary 3.41. *Let $\mathcal{J} \subseteq 2^X$ be a proper σ-ideal generated by a family of closed sets. The following are equivalent.*

(i) *$\mathcal{J}$ has the ccc.*
(ii) *$\mathcal{J} = \mathrm{MGR}(\mathcal{C})$, for some countable family $\mathcal{C} \subseteq \mathcal{F}(X)$.*
(iii) *$\mathcal{J} \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Delta}^1_1$ in the codes of Borel sets.*
(iv) *$\mathcal{J} \cap \mathbf{\Delta}^1_1$ is $\mathbf{\Sigma}^1_1$ in the codes of Borel sets.*

We conclude with the following striking consequence of Theorem 3.40 [KS].

Corollary 3.42. *Let G be a Polish group, and let $\mathcal{J} \subseteq 2^G$ be a proper σ-ideal. Assume that $\mathcal{J}$ is generated by a family of closed sets, has the ccc, and is translation-invariant. Then $\mathcal{J}$ is the σ-ideal of meager sets.*

Proof. By translation invariance, $\mathcal{J}$ contains no nonempty open set; and since $\mathcal{J}$ is generated by a family of closed sets, it follows that $\mathcal{J} \subseteq \mathrm{MGR}(X)$. Let $\mathcal{C} = (C_\xi)_{\xi\leq\alpha}$ be a countable decreasing family of closed sets such that $\mathcal{J} = \mathrm{MGR}(\mathcal{C})$, with $\alpha \geq 1$ and $C_\alpha = \emptyset$. Then $C_0 = X$, otherwise $\mathcal{J}$ would contain the nonempty open set $X \setminus C_0$. It follows that $\mathrm{MGR}(X \setminus C_1) \subseteq \mathcal{J}$, whence $\mathrm{MGR}(X) \subseteq \mathcal{J}$ by translation-invariance. ⊣

§4. Polar σ-ideals.

Some important σ-ideals of compact sets are closely related to families of measures. For example, this is true for U_0 by its very definition; and this is also true for the σ-ideal of countable compact sets $\mathcal{K}_\omega(X)$, since a compact set $K \subseteq X$ is countable if and only if $\mu(K) = 0$ for all continuous measures μ. These observations are the starting point of [D], where Debs studies the relationships between σ-ideals of compact sets and families of measures. In this section, we describe some of the results of [D].

Throughout this section, the Polish space X is assumed to be *compact*. The set of all (positive, finite, Borel) measures on X is denoted by $\mathrm{M}^+(X)$, or simply by M^+. On $\mathrm{M}^+(X)$, two natural topologies are available: the norm-topology, and the w^*-topology induced by the duality with $\mathcal{C}(X)$. Unless otherwise stated, all topological notions refer to the w^*-topology. We recall that the space $\mathrm{M}^+(X)$ is Polish and locally compact.

Definition 4.1. A family of measures $M \subseteq \mathrm{M}^+(X)$ is said to be a *band* if it has the following properties:

- M is a norm-closed convex cone in M^+;
- M is hereditary with respect to the order of M^+: if $\mu' \leq \mu \in M$, then $\mu' \in M$.

A band M is said to be a *strong band* if it is strongly convex, which means that any probability measure concentrated on M has its barycenter in M; or equivalently, that the closed convex hull of every compact subset of M is contained in M (see [DelMe] or [KL1]).

Example 4.2. The family of all continuous measures on X and the family of all positive Rajchman measures on $\mathbb{T}$ are strong bands. If μ is any measure on X, then the family $L_1^+(\mu)$ of all measures absolutely continuous with respect to μ is a strong band. The family of all discrete measures is a band, which is not a strong band in general. If M is an arbitrary subset of M^+, then the family

$$M' = \{\mu' \in \mathrm{M}^+(X);\ \mu' \perp \mu \text{ for all } \mu \in M\}$$

is a band, which is called the *orthogonal band of* M. We note that if M is a band, then $M = (M')'$ and $\mathrm{M}^+(X) = M \oplus M'$.

Now, we define two polarity operations as follows: if $P \subseteq \mathrm{M}^+(X)$ and $\mathcal{J} \subseteq \mathcal{K}(X)$, we set

$$P^\circ = \{K \in \mathcal{K}(X);\ \mu(K) = 0 \text{ for all } \mu \in P\};$$

$$\mathcal{J}_\circ = \{\mu \in \mathrm{M}^+(X);\ \mu(K) = 0 \text{ for all } K \in \mathcal{J}\}.$$

Definition 4.3. A σ-ideal $\mathcal{I} \subseteq \mathcal{K}(X)$ is said to be *polar* if $\mathcal{I} = P^\circ$ for some family $P \subseteq \mathrm{M}^+(X)$. A band $M \subseteq \mathrm{M}^+(X)$ is said to be *polar* if $M = \mathcal{J}_\circ$ for some $\mathcal{J} \subseteq \mathcal{K}(X)$.

Example 4.4. The σ-ideals U_0 and $\mathcal{K}_\omega(X)$ are polar. On the other hand, U and $\mathcal{I}_{\sigma\mathbf{p}}$ are not polar (see [D] and [HP]). The σ-ideal of Lebesgue-null sets in [0, 1] is of course polar, but the σ-ideal of meager sets in [0, 1] is not because its polar band is $\{0\}$.

It is easily seen that each polar band is a strong band, and that a band M is polar iff $M = (M^\circ)_\circ$ (observe that the inclusion $M \subseteq (M^\circ)_\circ$ always holds). The next theorem gives a characterization of $\mathbf{\Sigma}_1^1$ polar bands (see [KL1, Theorem IX.1.2]). Unfortunately, there is no similar characterization of polar σ-ideals.

Theorem 4.5. *If* $M \subseteq \mathrm{M}^+(X)$ *is a* $\mathbf{\Sigma}_1^1$ *strong band, then* $M = (M^\circ)_\circ$.

Remark 4.6. As observed by Louveau, Theorem 4.5 is a simple consequence of a result of G. Mokobodzki. The full statement of Mokobodzki's theorem reads as follows: *If* P *is a* $\mathbf{\Sigma}_1^1$ *subset of* M^+, *then the strong band* $M(P)$ *generated by* P *is equal to* $(P^\circ)_\circ$. *Moreover,* $M(P)$ *is also* $\mathbf{\Sigma}_1^1$. For a proof, see [KL1, Theorem IX.1.3], [DelMe, IX.3.34] or [D]. An elementary proof of the last statement can be found in [D].

Remark 4.7. From Theorem 4.5, one gets in particular that a measure $\mu \in \mathrm{M}^+(\mathbb{T})$ is a Rajchman measure if and only if $\mu(K) = 0$ for all U_0-sets $K \subseteq \mathbb{T}$. This characterization of Rajchman measures is due to R. Lyons [Ly1] (Lyon's result is in fact much more precise; see [Ly1] for details). The fact that Mokobodzki's theorem can be used to recover Lyon's result was noticed by Louveau.

DEFINITION 4.8. Let $\mathcal{I}$ be a σ-ideal of $\mathcal{K}(X)$. A subset $\mathcal{B}$ of $\mathcal{I}$ is said to be a *polarity basis* for $\mathcal{I}$ (in short, a *p-basis*) if $\mathcal{B}_\circ = \mathcal{I}_\circ$; that is, any measure which is null on all sets from $\mathcal{B}$ has to be null on all sets from $\mathcal{I}$.

Clearly, any basis is also a p-basis, but the two concepts are quite different. For example, if A is a Borel subset of X, then the σ-ideal $\mathcal{K}(A)$ has a Borel p-basis: let $\phi\colon \omega^\omega \to X$ be a continuous, 1-1 map with $\phi[\omega^\omega] = A$, and set $\mathcal{B} := \{\phi[L];\ L \in \mathcal{K}(\omega^\omega)\}$. On the other hand, we have seen that $\mathcal{K}(A)$ has a Borel basis only if A is the difference of two $\mathbf{\Pi}^0_2$ sets. One can even find $\mathbf{\Pi}^1_1$ σ-ideals with a G_δ hereditary p-basis, but with no Borel basis; see [D].

Example 4.9. (i) Lyons has shown in [Ly2] that the family of Helson sets and the family of *H-sets* are not p-bases for U_0; see the next chapter for the definition of H-sets. This last result was "quantified" by Kechris and Lyons [KLy], who showed that the polar band $(H\text{-sets})_\circ$ is in fact non-Borel in $\mathrm{M}^+(\mathbb{T})$, hence very different from the family of Rajchman measures, which is $\mathbf{\Pi}^0_3$.

(ii) By a remarkable result of R. Kaufman [Kau4], U is not a p-basis either for U_0. The complexity of the polar band $(U)_\circ$ is unknown.

The following consequence of Mokobodzki's Theorem will be needed below.

PROPOSITION 4.10. *Let M be a $\mathbf{\Sigma}^1_1$ strong band. If $\mathcal{J}$ is a p-basis for M° which is an ideal, then, for any measure $\mu \in M'$, one can find compact sets $K \in \mathcal{J}$ with $\mu(X \setminus K)$ arbitrarily small.*

PROOF. If $\mathcal{J}$ is *any* hereditary p-basis for M°, then each measure $\mu \in M'$ is carried by some K_σ set which is a countable union of sets from $\mathcal{J}$: consider a maximal family (K_i) of pairwise disjoint sets from $\mathcal{J}$ with positive μ-measure, and apply Theorem 4.5 to show that $\mu(X \setminus \bigcup_i K_i) = 0$. ⊣

The next results are similar to Proposition 1.11 (3) and Theorem 1.12 (1).

PROPOSITION 4.11. *Let $\mathcal{I} \subseteq \mathcal{K}(X)$ be a σ-ideal.*

(1) *If $\mathcal{I}$ has a $\mathbf{\Sigma}^1_1$ p-basis, then $\mathcal{I}$ has a G_δ p-basis.*
(2) *If $\mathcal{I}$ is $\mathbf{\Pi}^1_1$ and has a Borel p-basis, then $\mathcal{I}$ has a Borel p-basis which is an ideal.*

We now state four results which highlight the mutual relationships between σ-ideals and bands.

THEOREM 4.12. *If $M \subseteq \mathrm{M}^+$ is a Borel strong band, then the polar σ-ideal M° has a Borel p-basis.*

An interesting feature of the proof of this theorem given in [D] is that it makes an extensive use of effective descriptive set theory. For G_δ bands (not necessarily strong bands), one can get a stronger result.

PROPOSITION 4.13. *If M is a G_δ band, then the polar σ-ideal M° has a Borel basis.*

PROOF. It is not hard to check that a compact set $K \subseteq X$ is in $(M^\circ)^{\mathrm{perf}}$ if and only if $M \cap \mathrm{M}^+(K)$ is dense in $\mathrm{M}^+(K)$. If moreover $M = \bigcap_n M_n$, where the M_n's are open in M^+, it follows from this observation and the Baire Category Theorem that $K \in \mathcal{K}(X)$ is in $(M^\circ)^{\mathrm{perf}}$ if and only if $M_n \cap \mathrm{M}^+(K)$ is dense in $\mathrm{M}^+(K)$ for each $n \in \omega$. From this, it is easy to check that $(M^\circ)^{\mathrm{perf}}$ is $\mathbf{\Pi}^0_3$ in $\mathcal{K}(X)$, which implies that M° has a Borel basis by Proposition 1.11 (4). ⊣

The next theorem characterizes the $\mathbf{\Sigma}^1_1$ strong bands whose polar σ-ideal is G_δ.

THEOREM 4.14. *Let M be a $\mathbf{\Sigma}^1_1$ strong band. Then the following are equivalent.*

(a) *The polar σ-ideal M° is G_δ.*
(b) *The polar σ-ideal M° has a p-basis which is a G_δ ideal.*
(c) *The orthogonal band M' is G_δ.*

If M is a Borel strong band whose polar σ-ideal is G_δ, then one may expect to be able to bound the Borel complexity of M. The following theorem shows that this is indeed the case under a stronger assumption. The formulation and the proof below are slightly different from those in [D].

THEOREM 4.15. *Let M be a $\mathbf{\Sigma}^1_1$ strong band and suppose that the polar σ-ideal M° has a p-basis $\mathcal{J}$ with the following property: for each sequence $(K_n) \subseteq \mathcal{J}$, one can find a G_δ set $G \subseteq X$ such that $\bigcup_n K_n \subseteq G$ and $\mathcal{K}(G) \subseteq \mathcal{J}$. Then M is $K_{\sigma\delta}$.*

PROOF. Since M is norm-closed, it is enough to show that for each $\varepsilon > 0$, one can find a K_σ set $\Lambda \subseteq \mathrm{M}^+$ such that $M \subseteq \Lambda \subseteq \{\lambda \in \mathrm{M}^+; \mathrm{dist}(\lambda, M) \leq \varepsilon\}$, where the distance is relative to the norm of M^+. Assume this fails for some ε. Then, by Solecki's covering Theorem, M contains a nonempty G_δ set H such that the set $\{\lambda \in \overline{\mathrm{H}};\ \mathrm{dist}(\lambda, M) > \varepsilon\}$ is dense in $\overline{\mathrm{H}}$. Since $\mathcal{J}$ is obviously an ideal, this means by Proposition 4.10 that the set $\{\lambda \in \overline{\mathrm{H}};\ \exists K \in \mathcal{J}: \lambda(K) > \varepsilon\}$ is dense in $\overline{\mathrm{H}}$. Let (O_n) be a countable basis of nonempty open sets for $\overline{\mathrm{H}}$. For each $n \in \omega$, one can pick a compact set $K_n \in \mathcal{J}$ such that $\mathrm{O}_n \cap \{\lambda;\ \lambda(K_n) > \varepsilon\} \neq \emptyset$. By assumption on $\mathcal{J}$, it follows that one can find a G_δ set $G \subseteq X$ such that $\mathcal{K}(G) \subseteq \mathcal{J}$ and the set $\{\lambda \in \overline{\mathrm{H}};\ \lambda(G) > \varepsilon\}$ is dense in $\overline{\mathrm{H}}$. Since the set $\{\lambda \in \mathrm{M}^+;\ \lambda(V) > \varepsilon\}$ is open in M^+ for each open set $V \subseteq X$, the set $\{\lambda \in \overline{\mathrm{H}};\ \lambda(G) \geq \varepsilon\}$ is then comeager in $\overline{\mathrm{H}}$, and disjoint from H because $\mathcal{K}(G) \subseteq \mathrm{H}^\circ$. Since H is G_δ, this is a contradiction. ⊣

Remark 4.16. The property of $\mathcal{J}$ described above is introduced by Solecki in [S9], where it is called *property* $(*)$. It is obvious that if a family $\mathcal{J} \subseteq \mathcal{K}(X)$ has property $(*)$, then $\mathcal{J}$ is a σ-ideal. It is shown in [S9] that if $\mathcal{J}$ is $\mathbf{\Pi}^1_1$ and has property $(*)$, then $\mathcal{J}$ is G_δ. The proof is the same as that of Theorem 3.35; and it is in fact also proved in [S9] that any calibrated thin σ-ideal of compact sets has property $(*)$.

Remark 4.17. It is not hard to check that a family $\mathcal{J} \subseteq \mathcal{K}(X)$ has property $(*)$ provided it has the form $\mathcal{J} = \bigcap_{n\in\omega} \mathcal{J}_n$, where each $\mathcal{J}_n$ satisfies the following properties:

- $\mathcal{J}_n$ is a hereditary open set in $\mathcal{K}(X)$;
- If $K \in \mathcal{J}$ and $L \in \mathcal{J}_n$, then $K \cup L \in \mathcal{J}_n$.

In [D], Theorem 4.15 is proved for bands whose polar σ-ideal has a p-basis of that form.

Remark 4.18. In [D], it is proved by a direct argument that if Λ is a compact convex subset of M^+, then the band M generated by Λ is a $K_{\sigma\delta}$ strong band. For such a band M, it is very easy to check that $M^\circ = \Lambda^\circ$ is G_δ, and not too hard to show that M° has the form $\bigcap_n \mathcal{J}_n$, where the $\mathcal{J}_n$'s are as above; the details can be found in [D]. It seems that all known examples of G_δ polar σ-ideals have the form Λ° with Λ compact convex.

To conclude this chapter, we explain how Borel strong bands can be described in a rather canonical way using limits along nice filters on ω.

Let us say that a filter $\mathcal{F}$ on ω *satisfies the Bounded Convergence Theorem* if the following property holds true: for any probability space $(\Omega, \Sigma, \mathbb{P})$ and any uniformly bounded sequence (φ_n) of measurable functions $\varphi_n : \Omega \to \mathbb{R}$ such that $\varphi(t) = \lim_{\mathcal{F}} \varphi_n(t)$ $\mathbb{P}$-a.e., we have

$$\int \varphi(t)\, d\mathbb{P}(t) = \lim_{\mathcal{F}} \int \varphi_n(t)\, d\mathbb{P}(t).$$

It is easy to check that if $\mathcal{F}$ is a Borel filter on ω satisfying the Bounded Convergence Theorem and if (f_n) is a sequence of Borel functions, $f_n : X \to [0, 1]$, then the family

$$M_{\mathcal{F}} := \Big\{\mu \in \mathrm{M}^+(X);\ \lim_{\mathcal{F}} \int f_n\, d\mu = 0\Big\}$$

is a Borel strong band in M^+. Conversely, we have the following result.

THEOREM 4.19. *If $M \subseteq \mathrm{M}^+(X)$ is a Borel strong band, then there exists a sequence (f_n) of continuous functions on X, with $0 \le f_n \le 1$, and a Borel* *filter $\mathcal{F}$ on ω satisfying the Bounded Convergence Theorem, such that*

$$M = \Big\{\mu \in \mathrm{M}^+;\ \lim_{\mathcal{F}} \int f_n\, d\mu = 0\Big\} \quad \textit{and}$$
$$M' = \Big\{\mu \in \mathrm{M}^+;\ \limsup_{\mathcal{F}} \int f_n\, d\mu = \|\mu\|\Big\}.$$

In fact, it is shown in [D] that one can take for $\mathcal{F}$ some transfinite iterate of the Fréchet filter (see [L4] for a description of iterated Fréchet filters). Thus, any Borel strong band has a "c_0-like" representation using some iterated Fréchet filter. It follows from 4.14 that if M and M' can be represented as above via the Fréchet filter itself, then the polar σ-ideal M° is G_δ. For example, this cannot be done for Rajchman measures since U_0 is not G_δ. However, the family of Rajchman measures has an obvious c_0-like representation involving the Fréchet filter on $\mathbb{Z}$ and the complex-valued functions e^{int}, $n \in \mathbb{Z}$. We will return to this in the next chapter.

§**5. Other results.** In the previous chapters we dealt mainly with ideals and σ-ideals. This final chapter is devoted to families of compact sets which are not of this type.

5.1. Monotone unions of compact sets. Increasing unions of compact sets have been studied in the context of descriptive set theory by S. Kahane [KahS2, KahS3, KahS1] and by H. Becker, S. Kahane, and A. Louveau [BKL]. In this section, we mainly describe some results from [BKL] showing that the operation of increasing union leads to natural $\mathbf{\Sigma}^1_2$-complete sets.

We start with the following notation.

NOTATION 5.1. Let $\mathcal{C} \subseteq \mathcal{K}(X)$ be hereditary. Then we define $\mathcal{C}^\uparrow$ as the family of all sets of the form $\bigcup_{n\in\omega} K_n$, where (K_n) is a nondecreasing sequence of elements of $\mathcal{C}$, and we set $\mathcal{C}^{\uparrow\mathcal{K}} = \mathcal{C}^\uparrow \cap \mathcal{K}(X)$.

It is easy to check that if $\mathcal{C}$ is Borel (or even $\mathbf{\Sigma}^1_2$), then the family $\mathcal{C}^{\uparrow\mathcal{K}}$ is $\mathbf{\Sigma}^1_2$. The following result from [BKL] shows that this cannot be improved, and hence that the operation $\uparrow \mathcal{K}$ has a much more complicated behaviour than the operation of countable union.

THEOREM 5.2. *There exists a hereditary open set $\mathcal{C} \subseteq \mathcal{K}(2^\omega)$ such that $\mathcal{C}^{\uparrow\mathcal{K}}$ is $\mathbf{\Sigma}^1_2$-complete.*

Several other examples of complete $\mathbf{\Sigma}^1_2$ sets are given in [BKL]. All proofs ultimately rely on the following lemma.

LEMMA 5.3. *Let $A \subset 2^\omega$ be any $\mathbf{\Sigma}^1_2$ set. Then there exists a sequence of continuous functions $g_n\colon 2^\omega \times 2^\omega \to \{0,1\}$ such that for each $\alpha \in 2^\omega$, the following equivalence holds true: $\alpha \in A$ if and only if (g_n) has a subsequence (g_{n_k}) such that $g_{n_k}(\alpha, x) \to 0$ for all $x \in 2^\omega$.*

SKETCH OF PROOF OF 5.2. Let us denote by $\pi_n : 2^\omega \to \{0,1\}$ the coordinate functions on 2^ω. Using Lemma 5.3, one can show that the set

$$\mathcal{Z} := \{K \in \mathcal{K}(2^\omega);\ \text{some subsequence of } (\pi_n) \text{ converges to 0 pointwise on } K\}$$

is a $\mathbf{\Sigma}^1_2$-complete subset of $\mathcal{K}(2^\omega)$. Now, set

$$\mathcal{C} := \{K \in \mathcal{K}(2^\omega);\ \exists n \in \omega\colon \pi_n \equiv 0 \text{ on } K\}.$$

Then $\mathcal{C}$ is an open hereditary subset of $\mathcal{K}(2^\omega)$, and $\mathcal{C}^{\uparrow\mathcal{K}} = \mathcal{C} \cup \mathcal{Z}$. Moreover $\mathcal{Z}$ is continuously reduced to $\mathcal{C} \cup \mathcal{Z}$ via the map $K \mapsto K \cup K_0$, where K_0 is any compact set such that $\pi_n \to 0$ on K_0 and $K_0 \notin \mathcal{C}$. Thus, we get that $\mathcal{C}^{\uparrow\mathcal{K}}$ is $\mathbf{\Sigma}^1_2$-complete. ⊣

The motivation for studying the operation $\uparrow \mathcal{K}$ comes from harmonic analysis. Let us introduce the following three important families of thin sets.

DEFINITION 5.4. A set $A \subseteq \mathbb{T}$ is said to be

- an N_0-set if there exists an increasing sequence of natural numbers (n_i) such that $\sum \sin n_i t$ converges pointwise absolutely on A;
- an A-*set* if for some increasing sequence (n_i) of natural numbers, $n_i t \to 0$ pointwise on A;
- an H-*set* if there exist an increasing sequence of natural numbers (n_i) and a nonempty open interval $I \subseteq \mathbb{T}$ such that $n_i A \cap I = \emptyset$ for all $i \in \omega$.

The families of N_0-sets, A-sets, and H-sets are denoted by $\mathcal{N}_0$, $\mathcal{A}$ and $\mathcal{H}$-sets respectively, and the corresponding families of *compact* sets are denoted by N_0, A, and H-sets.

It is not difficult to see that

$$D^{\uparrow\mathcal{K}} \subseteq N_0 \subseteq A \subseteq (H\text{-sets})^{\uparrow\mathcal{K}},$$

where D is the family of all compact Dirichlet sets. It is also easy to see that N_0 and A are $\mathbf{\Sigma}^1_2$. Moreover, we know that D is G_δ, and H-sets is easily seen to be $\mathbf{\Sigma}^0_3$ (it is actually $\mathbf{\Sigma}^0_3$-complete by a result of T. Linton [Lin]). Since $\mathcal{C}^{\uparrow\mathcal{K}}$ is $\mathbf{\Sigma}^1_2$ whenever $\mathcal{C}$ is, it follows that $\mathcal{D}^{\uparrow\mathcal{K}}$ and $(H\text{-sets})^{\uparrow\mathcal{K}}$ are $\mathbf{\Sigma}^1_2$. The next result (also taken from [BKL]) says that these simple estimates are sharp.

THEOREM 5.5. *The families $D^{\uparrow\mathcal{K}}$, N_0, A, and H-sets$^{\uparrow\mathcal{K}}$ are $\mathbf{\Sigma}^1_2$-complete.*

Remark 5.6. A set $A \subseteq \mathbb{T}$ is said to be an N-*set* if for some sequence of positive numbers (a_n) with $\sum a_n = \infty$, the series $\sum a_n \sin nt$ converges pointwise absolutely on A. It is shown in [KahS1] that unlike N_0, the family N of all compact N-sets is descriptively very simple, namely G_δ.

Families of the form $\mathcal{C}^\uparrow$ (whose members are not necessarily compact sets) are studied in detail in [KahS2] and [KahS3]. Moreover, everything there is carried out in the much more general context of *Hausdorff operations*.

Among many interesting results, it is shown that in general, one has to iterate ω_1 times a Hausdorff operation to get a class which is stable with respect to this operation, and that the families $D^{\uparrow}$, $\mathcal{N}_0$, and $\mathcal{N}$ have no stability property whatsoever with respect to Hausdorff operations. We will not give any detail here, since it would take us too far away. However, we will answer below a question posed by Kahane in [KahS2].

NOTATION 5.7. Let $\mathcal{C} \subseteq \mathcal{K}(X)$ be hereditary. We define inductively the classes $\mathcal{C}^{\alpha\uparrow}$ and $\mathcal{C}^{\alpha\uparrow\mathcal{K}}$ for each ordinal $\alpha < \omega_1$ as follows.

$$\mathcal{C}^{0\uparrow} = \mathcal{C}, \qquad \mathcal{C}^{\alpha\uparrow} = \Big(\bigcup_{\beta<\alpha} \mathcal{C}^{\beta\uparrow}\Big)^{\uparrow};$$

$$\mathcal{C}^{0\uparrow\mathcal{K}} = \mathcal{C}, \qquad \mathcal{C}^{\alpha\uparrow\mathcal{K}} = \Big(\bigcup_{\beta<\alpha} \mathcal{C}^{\beta\uparrow\mathcal{K}}\Big)^{\uparrow\mathcal{K}}.$$

We can now state

PROPOSITION 5.8. *Assume X is compact, and let $\mathcal{C} \subseteq \mathcal{K}(X)$ be hereditary. Then $\mathcal{C}^{\alpha\uparrow\mathcal{K}} = \mathcal{C}^{\alpha\uparrow} \cap \mathcal{K}(X)$ for all $\alpha < \omega_1$.*

PROOF. We will proceed by induction on α. For $\alpha = 0$ there is nothing to prove. Now assume that $\alpha > 0$ and $\mathcal{C}^{\beta\uparrow\mathcal{K}} = \mathcal{C}^{\beta\uparrow} \cap \mathcal{K}(X)$ for every $\beta < \alpha$. The inclusion $\mathcal{C}^{\alpha\uparrow\mathcal{K}} \subseteq \mathcal{C}^{\alpha\uparrow} \cap \mathcal{K}(X)$ is obvious. To prove the other inclusion, let us fix $K \in \mathcal{C}^{\alpha\uparrow} \cap \mathcal{K}(X)$. Thus $K = \bigcup_{n\in\omega} F_n$, where $F_n \in \mathcal{C}^{\beta_n\uparrow}$ for some $\beta_n < \alpha$ and the sequence (F_n) is nondecreasing.

CLAIM. *If $H \in \mathcal{K}(K)$, then there exists a nondecreasing sequence of compact sets (H_n) such that $H = \bigcup_{n\in\omega} H_n$ and for every $n \in \omega$, there exists $k_n \in \omega$ with $H_n \subseteq F_{k_n}$.*

Granting this Claim we can finish the proof as follows. The set K can be written as $K = \bigcup K_n$, where (K_n) is a nondecreasing sequence of compact sets such that $K_n \subseteq F_{d_n}$ for some $d_n \in \omega$. Then we have $K_n \in \mathcal{C}^{\beta_{d_n}\uparrow} \cap \mathcal{K}(X) = \mathcal{C}^{\beta_{d_n}\uparrow\mathcal{K}}$ by the induction hypothesis. Consequently, $K \in \mathcal{C}^{\alpha\uparrow\mathcal{K}}$. ⊣

PROOF OF CLAIM. For $H \in \mathcal{K}(K)$ we define the following derivative.

$$H' = \{x \in H;\ \forall V \ni x \text{ open } \forall n \in \omega\colon \overline{H \cap V} \not\subseteq F_n\}.$$

As usual, we set $H^{(0)} = H$, $H^{(\xi+1)} = (H^{(\xi)})'$, $H^{(\xi)} = \bigcap_{\beta<\xi} H^{(\beta)}$ for ξ limit; and we define the rank of H by $\mathrm{rk}(H) = \min\{\xi;\ H^{(\xi)} = H^{(\xi+1)}\}$.

First observe that $H' \subsetneq H$ whenever $H \in \mathcal{K}^*(K)$. Indeed, the F_n's are obviously F_σ, hence according to the Baire Category Theorem there exist $n \in \omega$ and $V \subseteq E$ open such that $\emptyset \neq \overline{H \cap V} \subseteq F_n$. It follows that $\mathrm{rk}(H) < \omega_1$ and $H^{(\mathrm{rk}(H))} = \emptyset$ for every $H \in \mathcal{K}(K)$.

We will proceed by induction on $\mathrm{rk}(H)$. If $\mathrm{rk}(H) = 0$, then $H = \emptyset$ and the assertion obviously holds. Now we assume that the claim holds for all $H \in \mathcal{K}(K)$ with $\mathrm{rk}(H) < \xi$, where $\xi > 0$. Suppose that $H \in \mathcal{K}(K)$ satisfies

$\mathrm{rk}(H) = \xi$. Then $\mathrm{rk}(H') < \xi$, so we can write $H' = \bigcup \widetilde{H}_n$, where $(\widetilde{H}_n)$ is an increasing sequence of compact sets such that $\widetilde{H}_n \subseteq F_{l_n}$ for some $l_n \in \omega$.

Let (V_j) be a sequence of open sets such that $H \setminus H' \subseteq \bigcup_{j\in\omega} V_j$ and $\overline{H \cap V_j} \subseteq F_{p_j}$ for some $p_j \in \omega$. Set $H_n := \left(\bigcup_{j=0}^{n} \overline{H \cap V_j}\right) \cup \widetilde{H}_n$. Then (H_n) is a nondecreasing sequence of compact sets such that $H = \bigcup H_n$ and

$$H_n \subseteq \bigcup_{j=0}^{n} F_{p_j} \cup F_{l_n} \subseteq F_{\max\{p_0,\dots,p_n,l_n\}}.$$

This finishes the proof of the claim. $\dashv$

5.2. σ-ideals of continua. In the last few years, quite a lot of work has been done concerning the descriptive-set-theoretic aspects of continuum theory. A nice account can be found in A. Marcone's survey paper [Marc]. Here we briefly mention some results of R. Camerlo [Ca] on σ-ideals of continua.

Recall that a *continuum* in X is a nonempty, compact, connected subset of X. We denote by $\mathbf{C}(X)$ the space of all continua in X. A family $\mathcal{I} \subset \mathbf{C}(X)$ is said to be a *σ-ideal of continua* if, with respect to $\mathbf{C}(X)$, it is hereditary and closed under countable unions; in other words, if it has the following properties:

- if $L \in \mathbf{C}(X)$ and $L \subseteq K \in \mathcal{I}$, then $L \in \mathcal{I}$;
- if $(C_n)_{n\in\omega}$ is a sequence in $\mathcal{I}$ and if $\bigcup_{n\in\omega} C_n \in \mathbf{C}(X)$, then we have $\bigcup_{n\in\omega} C_n \in \mathcal{I}$.

The following theorem from [Ca] shows that parts (1) and (2) of Theorem 1.4 are essentially true in the continua setting. However, an additional assumption on the σ-ideal of continua is needed.

THEOREM 5.9. *Let $\mathcal{I}$ be a σ-ideal of continua in $\mathbb{R}^n$ or in $[0,1]^n$, $2 \leq n \leq \omega$. Assume $\mathcal{I}$ is invariant under homeomorphisms.*

(1) *If $\mathcal{I}$ is $\mathbf{\Pi}^1_1$, then it is either $\mathbf{\Pi}^1_1$-complete or G_δ.*
(2) *If $\mathcal{I}$ is $\mathbf{\Sigma}^1_1$, then it is G_δ.*

Remark 5.10. It is not true that any σ-ideal of continua is either $\mathbf{\Pi}^1_1$-complete or G_δ. For example, if A is any subset of X, then the family $\mathcal{I}_A := \{\{x\};\ x \in A\}$ is a σ-ideal of continua, and it has the same complexity as A. This example is a bit extreme because it consists entirely of degenerate continua, but one can easily produce examples of $\mathbf{\Sigma}^0_2$-complete σ-ideals of continua which contain many nondegenerate continua. For example, one can consider the family $\mathcal{I} \subseteq \mathbf{C}(\mathbb{R}^2)$ consisting of all vertical segments $L \subseteq \mathbb{R} \times \mathbb{R}$ projecting to a rational point.

5.3. Nonmeager hereditary families of compact sets. Some results from [M2] inspired the following question: if $\mathcal{I} \subseteq \mathcal{K}(X)$ is a hereditary comeager set, is it possible to find a dense G_δ set $\mathcal{G} \subseteq \mathcal{I}$ which is also hereditary? The following result is proved in [MZ].

THEOREM 5.11. *Every comeager hereditary $\mathbf{\Pi}^1_1$ subset of $\mathcal{K}(X)$ contains a dense G_δ hereditary set.*

The proof of this result goes as follows. If $\mathcal{I} \subseteq \mathcal{K}(X)$ is $\mathbf{\Pi}^1_1$ and does not contain any dense G_δ hereditary set, then $\mathcal{I}$ is contained in an F_σ set $\mathcal{M}$ with the same property, by Solecki's covering Theorem applied to $\mathcal{K}(X) \setminus \mathcal{I}$. Using the Banach–Mazur game, it is then possible to show that the co-hereditary closure of $\mathcal{K}(X) \setminus \mathcal{M}$ is nonmeager in $\mathcal{K}(X)$. Therefore, $\mathcal{I}$ cannot be comeager and hereditary.

Remark 5.12. In the above problem, no definability assumption is made on $\mathcal{I}$, but it is easily seen that a positive answer for $\mathbf{\Sigma}^1_1$ sets would give a positive answer for all comeager hereditary sets $\mathcal{I}$. Indeed, if $\mathcal{I} \subseteq \mathcal{K}(X)$ is comeager and hereditary, then it contains a comeager hereditary $\mathbf{\Sigma}^1_1$ set, namely the hereditary closure of any dense G_δ set contained in $\mathcal{I}$. Since Solecki's Theorem holds for all $\mathbf{\Sigma}^1_2$ sets and $\mathbf{\Sigma}^1_2$ families of closed sets if one assumes that $\omega_1^{L[x]} < \omega_1$ for all $x \in \omega^\omega$, it follows from the proof we sketched above that one can remove the complexity assumption in Theorem 5.11 under this extra set-theoretic axiom.

5.4. Rudin-like sets. By a well-known result of W. Rudin [Ru], there exist M_0-sets which are linearly independent over $\mathbb{Q}$ when viewed as subsets of $[0, 2\pi)$. This result is formally similar to the Debs–Saint Raymond Theorem on the covering property for U_0, which says in essence that each nonmeager G_δ subset of $\mathbb{T}$ contains an M_0-set, and also to the following classical result of J. Mycielski [My]: *If R is a meager binary relation on X, then there exist an uncountable compact set $K \subseteq X$ such that $\neg R(x, y)$ for all $x, y \in K$ with $x \neq y$.* Indeed, each of these three results asserts that one can find a "big" compact set which satisfies some "smallness" property of a particular type (in each case, a set is small if and only if all its finite subsets are). In this section, we state an abstract theorem from [MZ] which encompasses these three results.

Let us denote by $\mathrm{P}(X)$ the space of all Borel probability measures on X. Then $\mathrm{P}(X)$ becomes a Polish space when endowed with the Prokhorov topology τ_P (the weak topology induced by the bounded continuous functions on X). If $M \subseteq \mathrm{P}(X)$ then, as in Section 5, we denote by M° the polar σ-ideal $\{K \in \mathcal{K}(X);\ \mu(K) = 0 \text{ for all } \mu \in M\}$.

We will say that a family of measures $M \subseteq \mathrm{P}(X)$ is *nicely* $\mathbf{\Pi}^0_3$ if one can find a map $\phi\colon \mathrm{P}(X) \to B_{\ell^\infty}$ which is both (τ_P, w^*)-continuous and $(\|\,.\,\|, \|\,.\,\|)$-Lipschitz such that $\phi^{-1}(c_0) = M$. This disputable terminology comes from [MZ].

For example, the family of continuous measures on X and the family of Rajchman measures on $\mathbb{T}$ are nicely $\mathbf{\Pi}^0_3$. Notice also that a nicely $\mathbf{\Pi}^0_3$ family of measures has to be both $\|\,.\,\|$-closed and τ_P-$\mathbf{\Pi}^0_3$, but the converse is not true. This should be compared with the fact that c_0 is $\mathbf{\Pi}^0_3$-complete in (B_{ℓ^∞}, w^*). We can now state:

THEOREM 5.13. *Let $M \subseteq \mathrm{P}(X)$. Assume that M is nicely $\mathbf{\Pi}^0_3$ and hereditary for absolute continuity, and that X is the support of some continuous measure in M. If $\mathcal{I}$ is a $\mathbf{\Pi}^1_1$ hereditary subset of $\mathcal{K}(X)$ such that $\mathcal{I} \setminus \{\emptyset\}$ is nonmeager, then one can find a compact set $K \subseteq X$ such that $K \notin M^\circ$ and all finite subsets of K are in $\mathcal{I}$.*

The proof of this theorem uses Theorem 5.11 together with some ad hoc constructions inspired by harmonic analysis. Besides the three applications mentioned above, one may use Theorem 5.13 to prove the existence of M_0-sets satisfying various smallness conditions; for example, it allows to find M_0-sets in $\mathbb{R}^d$ which meet each hyperplane in at most d points.

Remark 5.14. It follows from Theorem 5.13 that if m is the Lebesgue measure on $\mathbb{R}$, then the family $L^1(m) \cap \mathrm{P}(\mathbb{R})$ is not nicely $\mathbf{\Pi}^0_3$, although it is both $\|\,.\,\|$-closed and τ_P-$\mathbf{\Pi}^0_3$. This observation is sharpened in [MZ], where it is proved that if $M \subseteq \mathrm{P}(X)$ is a hereditary, nicely $\mathbf{\Pi}^0_3$ family containing at least one continuous measure, then the polar σ-ideal M° is not thin. By essentially reproducing some arguments from [M1], one can in fact prove the following result: *Let $M \subseteq \mathrm{P}(X)$ be convex, hereditary and nicely $\mathbf{\Pi}^0_3$. Then the polar σ-ideal M° is G_δ if and only if it has the form $\mathcal{K}(A)$, for some $\mathbf{\Delta}^0_2$ set $A \subseteq X$.* By Theorem 3.35 and since polar σ-ideals are calibrated, this is a more general result if one forgets the additional convexity assumption.

Remark 5.15. Several results similar to Theorem 5.13 were obtained recently by T. W. Körner [Ko6].

§6. Open problems.

Let us conclude our paper with several open problems.

Problem 6.1. [K5] Let $\mathcal{I} \subseteq \mathcal{K}(X)$ be a G_δ σ-ideal containing all singletons. Does there exist a dense G_δ set $G \subseteq E$ with $\mathcal{K}(G) \subseteq \mathcal{I}$?

This is arguably the most intriguing problem in the area. Clearly, any nontrivial G_δ σ-ideal with the covering property would be a counterexample. On the other hand, the answer would be obviously positive if one could show that any G_δ σ-ideal has Solecki's property $(*)$ introduced in 4.15. This would hold if one could prove that any G_δ σ-ideal has the form described in Remark 4.17. For example, this holds for polar σ-ideals of the form Λ°, for some K_σ family of measures Λ; and likewise for σ-ideals of the form $\mathrm{MGR}(\mathcal{C})$, for some K_σ family $\mathcal{C} \subseteq \mathcal{K}(X)$.

Problem 6.2. [KLW] Let $\mathcal{I}$ be a σ-ideal in $\mathcal{K}(X)$. A G_δ pre-basis $\mathcal{B}$ of $\mathcal{I}$ is said to be *homogeneous* if for every $K \in \mathcal{K}(X)$, the set $\{K \cap L;\ L \in \mathcal{B}\}$ is G_δ. Which σ-ideals admit a homogeneous G_δ pre-basis?

Problem 6.3. Let G be a Polish abelian group which is not locally compact. What is the exact complexity of the family of closed Haar-null sets in G?

Problem 6.4. [D] Does U have a Borel p-basis?

Problem 6.5. [KL1] What is the complexity of U_1 and U_1'?

Problem 6.6. [KL1] Let SYN be the family of all compact sets of harmonic synthesis in $\mathbb{T}$. In [KL1], it is proved by a rank argument that SYN is a true $\mathbf{\Pi}_1^1$ set. Can one prove in ZFC that SYN is $\mathbf{\Pi}_1^1$-complete?

Problem 6.7. [KL1] A compact set $K \subseteq \mathbb{T}$ is said to be a *set of spectral resolution* if all compact subsets of K are sets of harmonic synthesis. It is proved in [KL1] that the family RE of all sets of spectral resolution is $\mathbf{\Pi}_1^1$-complete in $\mathcal{K}(\mathbb{T})$. What about the family of *perfect* sets in RE?

Problem 6.8. What is the complexity of the polar band $(U)_\circ$? Likewise, what is the complexity of $(H)_\circ$, where H is the family of Helson sets in $\mathbb{T}$?

Problem 6.9. [MZ] Let $\mathcal{I}$ be a hereditary comeager subset of $\mathcal{K}(X)$. Is it possible to prove in ZFC that there exists a hereditary G_δ set $\mathcal{G} \subseteq \mathcal{I}$ which is dense in $\mathcal{K}(X)$?

Problem 6.10. Is there any natural example of a calibrated σ-ideal, which is not strongly calibrated? In particular, is U strongly calibrated?

Problem 6.11. For each countable ordinal α, let $U_0^{(\alpha)}$ be the family of all U_0-sets whose Cantor–Bendixson rank relative to the basis U_0' is not greater than α. What is the exact complexity of $U_0^{(\alpha)}$?

Problem 6.12. Let $\mathcal{I}$ be a σ-ideal in $\mathcal{K}(X)$. Is true that if $\mathcal{I}^{\text{perf}}$ is $\mathbf{\Pi}_3^0$ and $\bigcup \mathcal{I}$ is G_δ, then $\mathcal{I}$ has a G_δ basis?

Problem 6.13. Is every $\mathbf{\Sigma}_3^0$ ideal of compact sets expressible as a countable union of G_δ hereditary sets? By a result of Dougherty [Do1], there exist $\mathbf{\Sigma}_3^0$ hereditary families which are not expressible in that way.

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