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IN THE SHADOWS OF THE LÖWENHEIM–SKOLEM THEOREM: EARLY COMBINATORIAL ANALYSES OF MATHEMATICAL PROOFS

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Abstract. The Löwenheim–Skolem theorem was published in Skolem's long paper of 1920, with the first section dedicated to the theorem. The second section of the paper contains a proof-theoretical analysis of derivations in lattice theory. The main result, otherwise believed to have been established in the late 1980s, was a polynomial-time decision algorithm for these derivations. Skolem did not develop any notation for the representation of derivations, which makes the proofs of his results hard to follow. Such a formal notation is given here by which these proofs become transparent. A third section of Skolem's paper gives an analysis for derivations in plane projective geometry. To clear a gap in Skolem's result, a new conservativity property is shown for projective geometry, to the effect that a proper use of the axiom that gives the uniqueness of connecting lines and intersection points requires a conclusion with proper cases (logically, a disjunction in a positive part) to be proved.

The forgotten parts of Skolem's first paper on the Löwenheim–Skolem theorem are the perhaps earliest combinatorial analyses of formal mathematical proofs, and at least the earliest analyses with profound results.

§1. Introduction. The famous Löwenheim–Skolem theorem was published in section 1 of Skolem's paper of 1920 that bears the long title *Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen* (Logico-combinatorial investigations into the satisfiability or provability of mathematical sentences, together with a theorem on dense sets). Despite its publication in the *Skrifter* of the ***Videnskabsakademiet i Kristiania***, the paper found its readers, among them Kurt Gödel who refers to it in his doctoral dissertation. A second paper on the Löwenheim–Skolem theorem, published in 1923, contains the famous application of the theorem to set theory, and has been widely cited. The latter paper, but only the first section of the former, were translated in Van Heijenoort's 1967 collection ***From Frege to Gödel***.

Next to the "satisfiability" part of the 1920 paper, Skolem considers the problem of derivability of an atomic formula from given atomic formulas

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by the axioms of lattice theory. His main theorem showed said derivability to be decidable. Moreover, the decision algorithm is polynomial, which was otherwise established in 1988. Skolem's third section gives a decidability result for derivability by the axioms of plane projective geometry. The fourth section deals with a "theorem on dense sets" and is, according to Skolem, "completely independent of the preceding paragraphs." It contains a well-known result on infinitary combinatorics.

What is perhaps the first serious analysis of formal derivability in mathematical axiomatic systems was soon completely forgotten. Skolem's priority on the decision problem in lattice theory by some seventy years became known in the early 1990s through Stanley Burris. Skolem's geometrical work had one follower, Oiva Ketonen in 1944, but even that was forgotten, and no one except Ketonen seems to have referred to the geometrical work before the 1990s.

We give here as a background a description of Skolem's early work that belonged to the algebraic logical tradition of Ernst Schröder. By 1919, Skolem had defined and proved the basic results of what we today call Heyting algebras, some ten years before Heyting's formalization of intuitionistic logic.

Secondly, we present Skolem's work on decidability in lattice theory. His proofs of the lemmas that lead to the theorem are very hard to follow. They give closure properties for derivability with the axioms through a specific closure construction. We introduce a notation for the representation of formal proofs with the axioms, by which Skolem's closure properties are seen to be based on the permutation of the order of forming closures and the application of the lattice axioms. All proofs with this formal notation follow strictly Skolem's arguments in his proofs. As a byproduct of the analysis of Skolem's closure construction, we find a new formalization of lattice theory, one that uses eigenvariables in two lattice rules, and in which Skolem's main result can be proved in a very concise way.

Third, we give a similar treatment of Skolem's decidability theorem on projective geometry. A gap is found here that can be closed by a conservativity theorem for projective geometry that we formulate and prove. Its contents can be expressed in logical terms as follows: The axioms contain one that has proper cases, i.e., a disjunction in a positive part of the axiom. A proof transformation shows that this axiom need not be used if the theorem to be proved does not contain proper cases.

§2. Skolem's life and work in brief. Thoralf Skolem (with both o's pronounced as in "Schoolem") was born in the southern Norwegian town of Sandsvaer on 23 May 1887. He studied mathematics and natural science at the University of Oslo, and became after graduation in 1913 an assistant to the physicist Kristian Birkeland. In 1915–1916 he visited Göttingen and worked after that year at the mathematics department in Oslo as a researcher and teacher. Between 1930 and 1938 he was a researcher at the Chr. Michelsens Institutt in Bergen, the second-biggest town of Norway. After the period in Bergen he became professor of mathematics in Oslo. He died on 23 March 1963.

The greater part of Skolem's papers was published in Norway, in the series of various scientific academies that are not always easy to find. He seems to have thought it sufficient if the papers were reviewed in standard places such as the ***Jahresbericht*** or ***Zentralblatt***, to which he himself contributed as a reviewer. His research papers are in German until the second World War, and after that mostly in English. Nearly all of his research publications in logic are available in the volume ***Selected Works in Logic***, edited by Jens Erik Fenstad and published in 1970. It includes also a review of Skolem's work by Hao Wang and a short biography by Fenstad. Besides logic, he worked in algebra from the early 1920s on. During his period in Bergen, he felt that his papers in logic remained unread and shifted his interest to number theory for a while.

As said, Skolem's early logical work is in the algebraic tradition of logic. He is without doubt the most prominent logician in this tradition, most appropriately described as belonging to lattice theory in today's terms.

Skolem's perhaps best-known result is the Löwenheim–Skolem theorem from a paper of 1920. He applied the theorem to set theory in 1923, to conclude that set-theoretic notions are "relative" and cannot offer any secure basis for the foundations of mathematics.

Skolem became aware of the axiomatic tradition of logic of Frege and Russell through his reading of Russell and Whitehead's ***Principia Mathematica*** in 1919. He had doubts about the meaningfulness of quantification over infinite domains and created instead the theory of primitive recursive arithmetic as a secure finitistic foundation of mathematics, with publication in 1923. This finitistic philosophy followed him through his whole research career.

In the late 1920s, Skolem developed further the method of Skolem functions and Skolem normal form, in connection with studies on decision problems of mathematical theories. At the same time, and before Gödel's incompleteness theorem, he expressed in print his conviction that there should be no categorical axiomatization of elementary arithmetic. This view was confirmed directly by his construction of non-standard models of arithmetic in 1933.

§3. Early work in the algebraic tradition of logic. Skolem's three earliest papers in logic (1913, 1919, 1920) belong to the algebraic tradition that he learned from Schröder's multi-volume ***Vorlesungen über die Algebra der Logik***, published between 1890 and 1905. The conception of logic is different from the Frege–Russell–Hilbert tradition that became dominant in the early decades of the 20th century. Logic is not tied to a strict formal syntax, but studies the relations between "concepts" in a direct way. Skolem, however, had by at least 1920 developed his own combinatorial notion of formal proof.

Skolem wrote in 1913 a Master's thesis in Norwegian, titled ***Undersökelser innenfor logikkens algebra*** (Investigations on the algebra of logic). A part of the results was published in 1913 in the article *Om konstitutionen av den identiske kalkuls grupper* (On the structure of groups in the identity calculus). Another paper publishing results of the thesis was ***Über die symmetrisch allgemeinen Lösungen im identischen Kalkul,*** (1931b) with a short version *Über die symmetrisch allgemeinen Lösungen im Klassenkalkul* (1931a) published in ***Fundamenta mathematicae***. Two other papers (1919, 1920) develop the algebraic methods of Schröder's calculus. Wilhelm Ackermann wrote a review (1932, in German) of Skolem's long paper of 1931, ending it with the following evaluation of the place of Schröder's approach to logic and of Skolem's results in particular:

> Let it be stated expressly that the results ... are valid not only for Schröder's calculus, but can all be carried over to the usual propositional calculus. I mention this fact, in itself obvious, because of the impression that many of the valuable results based on Schröder's ideas seem to have fallen into oblivion after Schröder's calculus was (for good reasons) given up.

Schröder's "identity calculus" is best described as Boolean algebra, and the "Klassenkalkul" as lattice theory. Skolem's contributions to Schröder's calculus were in fact prominent early works in lattice theory, even if he failed, by and large, in having this fact accepted.

An English translation of Skolem's first paper was prepared and published by Fenstad in the ***Selected Works in Logic***. In Skolem (1931b, p. 307), a publication of the whole thesis in German translation was announced, but this didn't happen. The text has been lost, according to a personal communication from Fenstad to the present author.

Skolem's 1913 paper begins with an admirably concise account of the algebraic approach to logic. It amounts to a study of logic in terms of lattice theory. The lattice ordering $a \leqslant b$ has various readings, one of which is set inclusion, another logical consequence. Skolem uses Schröder's own symbol, rather like the newly launched euro symbol, but later works use a standard order relation notation. Algebraic laws determine unique lattice operations ab and $a + b$ (product and sum) corresponding to conjunction and disjunction in the logical reading. Negation $\overline{a}$ is defined by introducing a null class 0 and a universal class 1 satisfying an algebraic law that Skolem writes as

$$(a\overline{a} \leqslant 0)(1 \leqslant a + \overline{a}).$$

Now the lattice structure becomes Schröder's "identity calculus," or a Boolean algebra in modern terms. Implication from a to b is later defined as the supremum of x such that $ax \leqslant b$.

As is seen from the algebraic law for negation, Skolem uses the basic operations (and also the order relation) on two levels: One is that of the elements $a, b, \ldots$, the other the level of assertions. Assertion of the proposition $a \leqslant b$ produces a judgment A (*Urteil*), and $A \leqslant C$, say

$$(a \leqslant b) \leqslant (c \leqslant d),$$

expresses that $c \leqslant d$ *follows from* $a \leqslant b$. The notations AB and $A + B$ are likewise used on two levels. For example, the element-level product ab is implicitly defined by a product on the judgment level,

$$(ab \leqslant a)(ab \leqslant b).$$

Toward the end of Skolem's 1913 paper, what is today called Dummett's law, $(A \supset B) \vee (B \supset A)$ in logical notation, is expressed as

$$(a \leqslant b) + (b \leqslant a)$$

It is shown to be equivalent to what is called the *disjunction property under hypotheses*,

$$(A \supset B \vee C) \supset (A \supset B) \vee (A \supset C),$$

or in Skolem's notation

$$(a \leqslant b + c) \leqslant (a \leqslant b) + (a \leqslant c).$$

A dual is also noted:

$$(ab \leqslant c) \leqslant (a \leqslant c) + (b \leqslant c).$$

The former equivalence is given in Dummett (1959), but not the latter. Dummett's law is a consequence the linearity of the order relation $a \leqslant b$.

§4. Results in lattice theory and in Heyting algebras. Skolem spent the winter of 1915–1916 in Göttingen. He was strongly influenced by Leopold Löwenheim's 1915 paper, and it is in Göttingen that he discovered, as a consequence of what is now the Löwenheim–Skolem theorem, the "Skolem paradox": If set theory is formalized, it has an interpretation in a denumerably infinite domain. Before publishing that result, he finished a long paper with an equally long title, *Untersuchungen über die Axiome des Klassenkalküls und über Produktations- und Summationsprobleme, welche gewisse Klassen von Aussagen betreffen*. It was Skolem's second paper in logic, dated 26 May 1917 and published in Norway in 1919. The work contains important early results on lattice theory, including the following:

1. A proof of the mutual independence of all the lattice axioms through graphical models, some of them infinite.

2. The same for distributive lattices. In particular, the five-element lattice that is a counterexample to distributivity is given (p. 74).
3. A method for deciding if an atomic formula is derivable from at most two atomic formulas by the axioms for a distributive lattice.

Skolem's paper also introduces the notion of a *Heyting algebra*: the case of a lattice in which the inequality $ax \leqslant b$ has a maximal solution, denoted by $\frac{b}{a}$ in Skolem (p. 78). The modern notation introduces an arrow operation $a \rightarrow b$, and Skolem's inequality written with the lattice meet and arrow operations reads as $a \wedge (a \rightarrow b) \leqslant b$. In Skolem (1952, p. 195), this notation is used. In the same paper (p. 197), Skolem notes that he had introduced lattices with an implication operation already in his 1919 paper.

It is quite astonishing that Skolem had found an algebraic axiomatization of intuitionistic propositional logic well before its basic principles were definitively clarified by Heyting's axiomatization (1930). Of his motivations for introducing the algebraic axiomatization, what he called "class rings" (*Klassenringe*), Skolem writes that they are "a natural continuation and generalization of the groups of the identity calculus" (1919, p. 78).

Whatever the motivations, Skolem had invented the notion of a Heyting algebra directly from lattice theory, through a symbolic maximal solution of the defining inequality of the arrow. The "canonical" route to Heyting algebras is instead: Start from an axiomatization of intuitionistic propositional logic, as in Heyting (1930). Let $[A]$ be the class of formulas equivalent to formula A. Now define the operations of a Heyting algebra on these classes by

$$[A] \wedge [B] \equiv [A \,\&\, B],$$
$$[A] \vee [B] \equiv [A \vee B],$$
$$[A] \rightarrow [B] \equiv [A \supset B].$$

Furthermore, with $\perp$ standing for the false formula, let

$$0 \equiv [\perp] \text{ and } 1 \equiv [\perp \supset \perp].$$

The order relation $[A] \leqslant [B]$ is defined to hold if $A \supset B$ is provable, i.e., if $[A \supset B] = 1$, and we have defined a Heyting algebra with a first and last element. The open sets of a topological space have the structure of a Heyting algebra, thus, general topology contains intuitionistic logic in disguise. This connection was first presented by Tarski (1938) in his topological interpretation of intuitionistic logic, and in a somewhat different form in Stone (1937). Remarkably, at this time Tarski knew Skolem's 1919 paper.

Skolem published in 1936 a monograph titled *Utvalgte kapitler av den matematiske logikk* (Selected chapters of mathematical logic). It presents materials from lectures in Oslo, but the language and place of publication have certainly rendered it one of his less read works. Chapter 6 treats of Heyting's intuitionistic propositional logic. Skolem defines linear Heyting algebras that work as models of the logic. The order is the reverse of what one would consider today, with 0 the element corresponding to truth. Conjunction of a and b is interpreted as their maximum and disjunction as minimum, and implication $a \supset b$ has the value 0 if $b \leqslant a$, otherwise the value b (p. 31). The algebras are used for showing the unprovability of excluded middle and the independence of the intuitionistic connectives (p. 33). This publication is surely the first one that uses the connection between Heyting algebras and intuitionistic logic.

Skolem's basic results for Heyting algebras are as follows:

1. Heyting algebras obey the distributive law.
2. Finite distributive lattices always permit the definition of the arrow, i.e., they are Heyting algebras.
3. Characterizations of Heyting algebras with a linear order relation.

The last-mentioned offers an algebraic characterization of what has been called, alternatively, Dummett logic, Gödel logic, or Gödel–Dummett logic (see our 2003 for this). Some remarks by Skolem indicate that he had noticed the subtle difference between a Heyting algebra with a linear order relation and a Heyting algebra in which Dummett's law is valid (see Dyckhoff and Negri 2006 for details).

Many of Skolem's early lattice-theoretic results were rediscovered by others in the 1930s. In a paper *Über gewisse 'Verbände' oder 'Lattices,'* of 1936, he drew attention to his priority but this was not well received, as is shown by Garrett Birkhoff's dismissive review (1937). In his book ***Lattice Theory*** of 1940, Birkhoff shows a weird sense of humour by referring to Skolem (1913), a paper in Norwegian, as the latter's original contribution. It also appears from the preface to the second edition (1948) that Birkhoff considers himself the one who first gave the lattice-theoretic interpretation of intuitionistic logic in work conducted between 1937–1939, but the fact remains that he had read Skolem (1936) when working on his book.

One historian of lattice theory agreed fully with Birkhoff. He wrote, relating the contents of Skolem (1913) and (1919) to the 1936 paper claiming some priority, that "Skolem's results are elementary for lattice theory and were almost all common by 1936." Further, "Skolem had made investigations relating to Schröderian problems without having introduced a single concept of his own for the structure treated." Finally, we read that "Skolem did, nevertheless, succeed by his later paper (1936) to secure a place for his congress talk (1913) in the historical references in books on lattice theory." One wonders if this historian (Mehrtens 1979) had read a single line of the 1919 paper or even seen a definition of the notion of a Heyting algebra. The criticized 1936 Skolem paper states explicitly (on p. 418) that the basic results about implicative lattices are contained in the 1919 article. There was also, within easy reach for a historian familiar with Schröder's terminology and announced as treating decision problems in lattice theory in Van Heijenoort

(1967), Skolem's greatest achievement in the field: the 1920 polynomial-time decision algorithm for the derivability of an atom from given atoms by the axioms of lattice theory. This forgotten result was published by others as a—presumedly new—important result nine years after the said history of lattice theory appeared (Cosmadakis 1988).

Skolem's work was appreciated by Dana Scott who wrote, in a review of a late lattice-theoretic paper Skolem (1958):

> In a remarkable paper published in 1919 ... the author studied several questions about distributive lattices in a completely abstract and general form. At that time, however, he did not use the terminology we know today .... Hence, his work escaped the notice that it actually deserves. In the present article the author applies what he has known for such a long time in giving a lattice-theoretic interpretation of the Intuitionistic Propositional Calculus of Heyting and in establishing the known result that if a disjunction is provable, then at least one of the terms is provable. Nearly the same proof and exactly the same kind of interpretation is contained in, say, the two papers of J. C. C. McKinsey and A. Tarski [(1946) and (1948)]. These authors were quite aware of Skolem's early work; they refer to it in the first of the above-mentioned papers and make use of it.

Some other papers of Skolem from the 1950s continue the work in algebraic logic, notably the extensive lecture series Skolem gave in Madrid in 1952. A Spanish translation was published as Skolem (1952–1953).

It is very strange that Skolem did not mention his most prominent result in lattice theory in his 1936 survey. As we shall see, he is by this result now to be counted among the founding fathers of modern lattice theory.

§5. The Löwenheim–Skolem theorem. Skolem's original 1920 proof of the Löwenheim–Skolem theorem was followed by another, possibly more widely read proof, namely the one in his 1923 paper on Zermelo's set theory. A later paper Skolem (1929) gives yet another proof of the theorem. Some later publications discuss the significance of the result, the most extensive being Skolem (1941).

Skolem's starting point is the paper by Leopold Löwenheim (1915) that anticipated what is now known as the Skolem normal form of first-order formulas and gave a version of the Löwenheim–Skolem theorem. Löwenheim's version is given in Skolem (1941) as: "Each first-order expression that is satisfiable is satisfiable in a denumerable domain." Löwenheim also gave a positive solution to the decision problem of monadic predicate calculus, and a reduction of the general decision problem to that for formulas with only binary predicates. These results were important for Skolem's later work on decision problems.

Skolem (1920) begins with the Skolem normal form of first-order formulas, in which there is first a string of universal quantifiers, then of existential quantifiers, followed by a quantifier-free part. Satisfiability of an arbitrary formula is shown to be equivalent to the satisfiability of a corresponding formula in normal form. We have, say,

$$\forall x_1 \dots \forall x_m \, \exists y_1 \dots \exists y_n \, A(x_1, \dots, x_m, y_1, \dots, y_n) \tag{1}$$

Now the intuitive argument of Löwenheim is that if (1) is satisfiable, there is for each $(x_1, \dots, x_m)$ a non-empty set of n-tuples $(y_1, \dots, y_n)$ such that $A(x_1, \dots, x_m, y_1, \dots, y_n)$ holds. By the axiom of choice, one can choose for each $(x_1, \dots, x_m)$ one n-tuple $(y_1, \dots, y_n)$ as uniquely determined by the $(x_1, \dots, x_m)$. We have therefore functions $f_1, \dots, f_n$ such that

$$A(x_1, \dots, x_m, f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)) \tag{2}$$

holds for each $(x_1, \dots, x_m)$. Skolem (1920) filled in gaps and simplified the proof. A denumerably infinite domain is constructed, if such a verb can be used for a set-theoretic application of the axiom of choice, from the given arbitrary domain, and the interpretation of the predicates remains unchanged.

In Skolem (1923), another version of the theorem was given: If a first-order formula is satisfiable, it has an interpretation in the domain of natural numbers. This time the proof does not use the axiom of choice. The result is further strengthened into: If a denumerably infinite sequence of first-order formulas is simultaneously satisfiable, it is satisfiable in a denumerably infinite domain. A detailed presentation and discussion of the method of proof is given in Wang (1970, pp. 18–29).

Skolem (1923) comes very close to proving the completeness of the predicate calculus. In the words of Gödel (2003, p. 317), Skolem "implicitly proved: "Either A is provable or $\overline{A}$ is satisfiable" ("provable" taken in an informal sense). However, since he did not clearly formulate this result (nor, apparently, had made it clear to himself), it seems to have remained completely unknown." In Gödel's understanding, from a letter (2003, p. 398) of December 1967 to Wang, Skolem's viewpoint in logic and mathematics was that "meaning is attributed solely to propositions which speak of *concrete and finite objects*, such as combinations of symbols," and that this viewpoint prevented him from taking the necessary nonfinitary step needed for a full statement of the completeness theorem. To this view of Gödel's it should be added, however, that the necessity of nonfinitary reasoning was established only with Church's proof of the undecidability of the predicate calculus in 1936.

§6. Combinatorial analysis of mathematical deduction. The second section of Skolem (1920) is titled, in translation, *Solution of the problem of deciding whether a given proposition of the calculus of groups is provable*. Its contents have, to our knowledge, never been mentioned in print before Burris (1995), except for the review by F. Lange–Nielsen (1922) and a short passage in Van Heijenoort's ***From Frege to Gödel***. He writes in the introduction to Skolem (1920) that "the other sections deal with decision procedures for the theory of lattices and elementary geometry, as well as with dense sets." No one seems to have become curious enough by this remark to find out what exactly the decision procedures for lattices were.

Part of the reason for the complete neglect of Skolem's 1920 results lies in unfair scientific rivalry, part in the Schröderian terminology used and in the notation that is unfamiliar today. Thus, Wang in his survey (1970) did not even realize that the 1920 paper contained results on lattice theory. As to Skolem, it remains a mystery why he did not mention the decidability result in the (1936) review of his work in lattice theory. After the rediscovery by Burris, Skolem is recognized as one of the early authorities in lattice theory, together with Richard Dedekind, Garrett Birkhoff, and P. Whitman (see the book by Freese *et al.*, p. 1).

The third section of Skolem (1920) is titled, in translation, *A method for deciding whether a given sentence of descriptive elementary geometry follows from the plane axioms of incidence*. A decision method analogous to that of lattice theory, though not polynomial time, is given here for plane projective geometry. The result was the starting point of Oiva Ketonen's (1944) application of Gentzen's sequent calculus to the proof theory of elementary geometry. Ketonen's monograph, actually his doctoral dissertation, was reviewed by Paul Bernays (1945) who at this point was therefore aware of Skolem's geometric result and its extension to plane affine geometry by Ketonen.

Skolem's paper was known in some way to Ackermann, who lists it in his book on decision problems of 1954. The next reference to Skolem's geometry we know of is 41 years later, in our paper (1995) on the axiomatization of constructive projective and affine geometry. No one in the field of automated deduction seems to have noticed Skolem's geometrical decidability result, even if elementary geometry was one of the first fields to which automated methods of proof search were applied.

§7. Skolem's axioms for lattice theory. Skolem begins his second section with the axioms of lattice theory given in a relational formulation. The axioms contain a binary order relation and two three-place relations, written in Skolem's notation as (ab), $\overline{abc}$, and $\underline{abc}$, respectively. (Skolem's lines are slightly curved.) These are called "pairs" and "triples." A modern notation would be something like $a \leqslant b$, $M(a, b, c)$, and $J(a, b, c)$, respectively, for the order relation and the two relations that we can read as "the meet of a and b is c, the join of a and b is c." The notation of Schröder's ***Algebra der Logik*** is $a ⋹ b$, $c = ab$, $c = a + b$ in place of Skolem's (ab), $\overline{abc}$, $\underline{abc}$.

TABLE 1. Skolem's axioms for lattice theory.

I.	For each x, (xx).		
II.	From (xy) in combination with (yz) follows (xz).		
$\mathrm{III}_{\times}$.	From $\overline{xyz}$ follow (zx) and (zy).	III_{+}.	From $\underline{xyz}$ follow (xz) and (yz).
$\mathrm{IV}_{\times}$.	From $\overline{xyz}$ in combination with (ux) and (uy) follows (uz).	IV_{+}.	From $\underline{xyz}$ in combination with (xu) and (yu) follows (zu).
$\mathrm{V}_{\times}$.	From $\overline{xyz}$ in combination with (xx'), $(x'x)$, (yy'), $(y'y)$, (zz'), and $(z'z)$ follows $\overline{x'y'z'}$.	V_{+}.	From $\underline{xyz}$ in combination with (xx'), $(x'x)$, (yy'), $(y'y)$, (zz'), and $(z'z)$ follows $\underline{x'y'z'}$.
$\mathrm{VI}_{\times}$.	There is for arbitrary x and y a z such that $\overline{xyz}$.	VI_{+}.	There is for arbitrary x and y a z such that $\underline{xyz}$.

We have in the axiomatization a partial order relation (axioms I–II), axioms that state in relational terms that the meet of a and b is less than a and less than b, and the join greater (III), uniqueness of meet and join (IV), and substitution of equals in the meet and join relations (V). It is seen from axioms V that equality is intended to be defined as $a \leqslant b$ and $b \leqslant a$. Axioms VI state the existence of meets and joins of any two elements.

After presenting the axioms, Skolem continues:

> The validity of a sentence in algebra based on this axiomatic foundation consists simply in the possibility of proving the following: Given these and these pairs and triples . . . , those and those pairs and triples can be derived by possibly repeated and combined applications of the axioms. In fact, *the axioms presented are principles of derivation* [Erzeugungsprinzipien] by which new pairs and triples are derived from certain initial symbols (letters) by the axioms I–V, whereas new triples are introduced by axioms VI in which there can also occur newly introduced letters. *We have here a purely combinatorial conception of deduction on which I would like to put emphasis, because it proves to be especially useful in logical investigations.* I shall show by a few examples that every proof is nothing else but a derivation of pairs and triples.

Skolem has arrived here at an inductive definition of the notion of formal derivability from a finite number of given atomic formulas. It is noteworthy that the axioms of lattice theory are not given as formulas of a logical language, but as schematic rules of derivation that conclude an atomic formula from given atomic formulas used as premisses, or in the case of axioms VI, that introduce new atomic assumptions.

Some examples show how Skolem's axiom system works. We reproduce here the first one, the derivable atomic formula that is written as $(a \wedge b) \vee (a \wedge c) \leqslant a \wedge (b \vee c)$ in the notation with lattice operations. Skolem translates it into his relational language and gets (1920, p. 117):

EXAMPLE 2. A proof in Skolem's lattice theory.

> From $\overline{abd}$ in combination with $\overline{ace}$, $\underline{bcf}$, $\underline{deg}$, and $\overline{afh}$ follows the pair (gh). Proof: From $\underline{bcf}$ follows by III_+ the pair (bf). From $\overline{abd}$ follow (da) as well as (db) by $\mathrm{III}_\times$. From (db) and (bf) follows by II (df). From (da) and (df) in combination with $\overline{afh}$ follows by $\mathrm{IV}_\times$ [original has $\mathrm{VI}_\times$] (dh). From $\underline{bcf}$ follows on the force of III_+ the pair (cf). From $\overline{ace}$ follow by $\mathrm{III}_\times$ (ea) and (ec). From (ec) and (cf) [original has (ef)] follows by II (ef). From (ea) in combination with (ef) and $\overline{afh}$ follows by $\mathrm{IV}_\times$ [original has $\mathrm{VI}_\times$] (eh). From (dh) in combination with (eh) and $\underline{deg}$ follows finally by IV_+ (gh).

Three misprints attest to how arduous it is to read such formal proofs.

The general derivability problem that Skolem considers is: Is an atomic formula derivable from $n \geqslant 0$ given atomic formulas by the rules of lattice theory? To give his decision algorithm for derivability, Skolem proves first a lemma to the effect that the existential axioms VI are conservative over rules I–V for the stated derivability problem for atomic formulas of the form $a \leqslant b$. Derivability of the other types of atomic formulas reduce to these. We can state Skolem's result as follows, thinking in term of lattice theory with the meet and join operations ($a \wedge b$ and $a \vee b$, respectively): For the derivation of $a \leqslant b$ from given atomic formulas, no new objects need to be constructed by the lattice operations. Thus, to put into effective use the existential axioms VI, one needs an existential conclusion to be proved.

Skolem's paper is written in the traditional synthetic order of presentation, and the reader has to try to find out why different things are done or assumed. For example, Schröder treats lattice theory as an equational theory with lattice operations. Skolem changes this into a relational theory, but not a word is given as motivation.

It is not easy to understand how Skolem actually proves his results. There are things he must have taken to be too easy to mention. For example, why are instances of reflexivity restricted to elements known from the assumptions of a derivability problem? One finds out by trying it that if one premiss of transitivity is a reflexivity, the conclusion is equal to the other premiss, and so on. At one stage Skolem assumes there to be just one instance of a substitution rule, as a last rule in a derivation. Sara Negri observed to the present author in the spring of 2002 that: 1. Skolem's substitution rules can be permuted down relative to the rules I–IV. 2. Several substitutions can be replaced by at most one substitution and applications of II (transitivity of order). We shall verify these properties in the next Section.

The suggested key to understanding Skolem's proofs is thus that he considered the possible permutations of the order of application of his lattice rules. That this is the case is shown in detail in the proofs we give of Skolem's two central lemmas. The first of these proofs shows that a certain construction is in fact a closure construction. In terms of the rules of proof, lattice rules applied after the closure operations permute to above the closure operations.

§8. A proof system for lattice theory. Skolem's axiomatization of lattice theory uses existence axioms for meets and joins instead of explicit meet and join operations. We give a formal system of rules that corresponds to Skolem's axioms. The notation is only slightly different from and hopefully more readable than the original: We have an infinity of parameters $a, b, c, \dots$ and variables $x, y, z, \dots$, a binary partial order relation $a \leqslant b$, and two ternary relations $M(a, b, c)$ and $J(a, b, c)$ ("c is the meet of a and b," and "c is the join of a and b"). We call atoms of these forms **O-atoms**, **M-atoms**, and **J-atoms**. Equality is defined as partial order in both directions. In the substitution rules below, we abbreviate accordingly the two premisses $a \leqslant b$ and $b \leqslant a$ by $a = b$. The first rule has zero premisses. In rules III–IV, the mnemonic letters L and R indicate that the meet and join terms appear as left and right members, respectively, of the order relation in the conclusion. The rules for lattice theory are, with the Roman numerals giving the correspondence with Skolem (1920):

TABLE 3: Rules of proof for relational lattice theory.

I–II. *Rules for partial order:*

$$\frac{}{a \leqslant a}\,Ref \qquad \frac{a \leqslant b \quad b \leqslant c}{a \leqslant c}\,Tr$$

III. *Properties of Meet and Join:*

$$\frac{M(a,b,c)}{c \leqslant a}\,LM_1 \qquad \frac{M(a,b,c)}{c \leqslant b}\,LM_2 \qquad \frac{J(a,b,c)}{a \leqslant c}\,RJ_1 \qquad \frac{J(a,b,c)}{b \leqslant c}\,RJ_2$$

IV. *Uniqueness of Meet and Join:*

$$\frac{M(a,b,c) \quad d \leqslant a \quad d \leqslant b}{d \leqslant c}\,RM \qquad \frac{J(a,b,c) \quad a \leqslant d \quad b \leqslant d}{c \leqslant d}\,LJ$$

V. *Substitution of equals in Meet and Join:*

$$\frac{M(a,b,c) \quad a = d \quad b = e \quad c = f}{M(d,e,f)}\,SM \qquad \frac{J(a,b,c) \quad a = d \quad b = e \quad c = f}{J(d,e,f)}\,SJ$$

VI. *Existential rules for Meet and Join:*

$$\dfrac{\begin{matrix}[M(a,b,x)]\\ \vdots \\ C\end{matrix}}{C}\,EM \qquad \dfrac{\begin{matrix}[J(a,b,x)]\\ \vdots \\ C\end{matrix}}{C}\,EJ$$

A derivation can begin with any O-, M-, or J-atoms as assumptions. The existential rules indicate by square brackets the **discharged** assumptions that can have been made any number of times $n \geqslant 1$. Those assumptions that have not been discharged in a derivation are the **open** assumptions of the derivation. The existential rules have the **variable restriction** that the eigenvariable x must not occur free in the conclusion C nor in any open assumption the premiss C depends on, except in the M- or J-atoms indicated as discharged. We assume that the eigenvariable of a rule appears only in the subderivation down to that rule. It follows that all the eigenvariables of existential rules in a derivation are distinct.

The existential rules are equivalent to Skolem's existence axioms for meet and join, namely $\forall x\, \forall y\, \exists z\, M(x, y, z)$ and $\forall x\, \forall y\, \exists z\, J(x, y, z)$: If the latter are assumed, the logical rules of universal and existential quantifier elimination lead to the conclusions of the existential lattice rules. In the other direction, the existence axioms are derivable by universal and existential quantifier introduction and the existential lattice rules:

$$\dfrac{\dfrac{\forall x\, \forall y\, \exists z\, M(x,y,z)}{\exists z\, M(a,b,z)}\,{\forall E,\forall E} \qquad \begin{matrix}[M(a,b,v)]\\ \vdots \\ C\end{matrix}}{C}\,\exists E \qquad \dfrac{\dfrac{\dfrac{[M(a,b,v)]}{\exists z\, M(a,b,z)}\,\exists I}{\exists z\, M(a,b,z)}\,EM}{\forall x\, \forall y\, \exists z\, M(x,y,z)}\,{\forall I,\forall I}$$

From the left derivation, we observe that an existence axiom turns into a corresponding existential rule of inference by the deletion of the existential premiss and its derivation. In Skolem (1920), rules I–V are treated formally, but existence axioms and their variable restrictions are handled somewhat intuitively. Nevertheless, one can say that Skolem treated the existential quantifier in his existential axioms in the way canonized by Gentzen's rules of natural deduction, i.e., by taking an instance with a fresh eigenvariable as in the elimination rule for existence.

An implicit assumption in Skolem's proof is that the axioms and rules of logical inference do not make an atomic formula derivable from given atomic formulas if it was not already derivable with the lattice rules. Skolem seems to have taken for granted such conservativity of logic over mathematical axioms. For a proof, one would need a result such as Gentzen's normal form theorem for derivations in natural deduction. Skolem's later work gives, in fact, a result in this direction, in his 1928 anticipation of Herbrand's theorem for which see the introduction to the translation of Skolem (1928) in ***From Frege to Gödel***.

It is sufficient to consider only derivations with atoms as assumptions and conclusion, because the logical rules permute down with respect to the mathematical rules: Consider logical rules in a natural deduction formulation, with elimination rules given in the "general" form of von Plato (2001) or Negri and von Plato (2001) in which all such rules have an arbitrary conclusion. No introduction rule concludes a premiss of a lattice rule, because the latter are atomic. If an elimination rule concludes such a premiss we have, say, a part of derivation with a major premiss $A \circ B$ and an atomic minor premiss P in an E-rule followed by a two-premiss mathematical rule R. The derivation and its transformation are:

$$\dfrac{\dfrac{A \circ B \quad \overset{\vdots}{P}}{P}\,E \qquad \overset{\vdots}{Q}}{R}\,R \quad \rightsquigarrow \quad \dfrac{A \circ B \qquad \dfrac{\overset{\vdots}{P} \quad \overset{\vdots}{Q}}{R}\,R}{R}\,E$$

Repetition of such permutations brings all E-rules below mathematical rules. We can now consider parts of derivations with mathematical rules in isolation of parts of derivations with logical rules. The former have in lattice theory derivation trees with atomic assumptions and instances of rule *Ref* as leaves.

To give an example of a derivation, we consider Skolem's text-style example proof 2 in Section 7. It can be converted into a derivation tree by starting from the last sentence and the last conclusion, then adding the premisses above an inference line, etc., until assumptions are reached. We shall write atoms without commas and parentheses lest the derivation become too broad to print. The result is:

EXAMPLE 4. A formal derivation in Skolem's lattice theory.

$$\dfrac{Jdeg \qquad \dfrac{Mafh \quad \dfrac{Mabd}{d \leqslant a}\,LM \quad \dfrac{\dfrac{Mabd}{d \leqslant b}\,LM \quad \dfrac{Jbcf}{b \leqslant f}\,RJ}{d \leqslant f}\,Tr}{d \leqslant h}\,RM \qquad \dfrac{Mafh \quad \dfrac{Mace}{e \leqslant a}\,LM \quad \dfrac{\dfrac{Mace}{e \leqslant c}\,LM \quad \dfrac{Jbcf}{c \leqslant f}\,RJ}{e \leqslant f}\,Tr}{e \leqslant h}\,RM}{g \leqslant h}\,LJ$$

Skolem's promised applications of his decision method include his example 2 (p. 122) in which a purely syntactic proof of underivability of the axiom of distributivity in lattice theory is given. Distributivity is expressed in relational lattice theory as the derivability of $h \leqslant g$ from the five atomic assumptions

$$M(a, b, d), M(a, c, e), J(b, c, f), J(d, e, g), \text{ and } M(a, f, h).$$

(The reader should verify that we have here what would be expressed in terms of the lattice operations as $a \wedge (b \vee c) \leqslant (a \wedge b) \vee (a \wedge c)$.) By Skolem's main result, the conservativity of rule VI, it is sufficient to form the deductive closure of the five given atomic formulas relative to rules I–V. Rules I give eight reflexivities, rules III another ten atomic formulas. Transitivity adds two atomic formulas, and rule IV_+ one. "Now no more pairs and triples can be formed with the help of I–V from the five given ones ... by which the underivability of (hg) ... or in other words, of the distributive law, is proved." The role of distributivity had been a problem in lattice theory: Charles Peirce had announced that he can prove it from the rest of the axioms, but Schröder found counterexamples (see Thiel 1994 for a discussion of these matters).

Skolem's examples of underivability in lattice theory show how he worked out the closure of a given set of atoms. All of his examples concern the derivability of an O-atom. First all possible reflexivities are added. Next instances of rules III are added, because their premisses in derivations by rules I–IV are assumptions, then transitivities are applied in all possible ways. Last, the uniqueness axioms IV are applied. Rules V are needed only for the conclusion of M- or J-atoms, and would appear last. It seems that the clue to an analysis of the structure of formal proofs in lattice theory came from the order in which the closure could be constructed. The same order of application of lattice rules is also seen in the formal derivation tree of example 4 that we constructed.

Next to the example on underivability, a more general result is given at the end of the section on lattices (p. 124): Skolem notes that any set of atomic formulas in lattice theory is consistent.

§9. A proof of Skolem's theorem for lattice theory. The following presents a detailed proof of Skolem's theorem about the decidability of derivability by the lattice axioms. To be able to consider with ease the permutability of the order of application of the lattice axioms, we gave them as formal rules for the construction of derivation trees. Here is the statement of Skolem's theorem (1920, p. 118):

> THEOREM. *Let there be given a set S of pairs and triples built from the symbols $a_1 \dots a_n$. Let Σ be the set that contains all the pairs and triples that can be derived from the pairs and triples in S by the axioms* I–V. *Let new triples be introduced by, possibly repeated, application of* VI, *let the new symbols $b_1\ b_2\ \dots$ occur in these triples, and let Σ' be the set of all pairs and triples that can then be derived with the help of* I–V. *Then Σ' contains no other pairs formed from $a_1\ a_2 \dots a_n$ than those contained already in Σ.*

It should be emphasized that the derivability of O-atoms is concerned here. Skolem shows in the end that the derivability of "triples" reduces to the derivability of "pairs."

The theorem is proved through two lemmas. In the first, a set S of O-, M-, and J-atoms is assumed such that S is closed with respect to axioms I–V.

Existence of meets is applied once to two objects a, b from S, to get the M-atom $M(a,b,\alpha)$. Skolem's lemma 1 claims that the addition of O-atoms according to the below clauses 1–3 to S gives a set S' closed with respect to axioms I–IV. Axioms V conclude M- or J-atoms and are therefore left out at this stage.

The closure S' of S, relative to axioms I–IV when the atom $M(a,b,\alpha)$ is added, is obtained by adding also the following O-atoms (1920, p. 118):

5. Skolem's closure construction for an atom $M(a,b,\alpha)$.

(1) $\alpha \leqslant \alpha$.
(2) Each $\alpha \leqslant d$ for which d has the property that whenever $c \leqslant a$ and $c \leqslant b$ are in S, also $c \leqslant d$ is in S.
(3) Each $c \leqslant \alpha$ for which c has the property that $c \leqslant a$ and $c \leqslant b$ are in S.

Note that c in clause 2 is an arbitrary object.

The proof that S' is closed with respect to axioms I–IV is quite hard to follow. The reasons are partly the old ones: Synthetic presentations of proofs with no motivation offered for the reader. The reasons lie also partly in the absence of a formal notation for derivations. These are given as texts, as in the example proof 2 in Section 7. Further, Skolem considers closure of a set of atoms, which hides somewhat the fact that each atom in the set is the last formula in some derivation. We shall see that explicit derivation trees make Skolem's arguments understandable. Closure, in turn, is obtained by permutability arguments that correspond precisely to the steps that Skolem takes in the proof of his lemma 1. These steps assume that an O-atom with the term α has been added to S by clause 2 or 3, then move to the conditions in 2 and 3. To make all of this transparent, we formulate 1–3 as extra closure rules by which the extension to S' is achieved:

TABLE 6. Existential closure rules for an atom $M(a,b,\alpha)$.

$$\frac{}{\alpha \leqslant \alpha}\,CRef \qquad \frac{M(a,b,\alpha) \qquad \begin{array}{c}[x \leqslant a \quad x \leqslant b]\\ \vdots \\ x \leqslant d\end{array}}{\alpha \leqslant d}\,CLM \qquad \frac{M(a,b,\alpha) \quad c \leqslant a \quad c \leqslant b}{c \leqslant \alpha}\,CRM$$

The x in rule CLM is an eigenvariable, i.e., it does not occur in any open assumptions or the conclusion.

The idea with Skolem's proof is to have closure rules that always have the eigenvariable α in the conclusion. It follows from the eigenvariable condition that existential rules VI cannot be applied to derivations that end with a closure rule. Thus, it is sufficient to show that a closure is achieved by a single application of the closure rules as last rules.

Skolem's lemma 1. *Let S be a set of atoms closed with respect to the rules* I–V. *With a and b objects in S, let the atom $M(a,b,\alpha)$ be added to S. Let further S' be obtained from $M(a,b,\alpha)$ and the atoms in S by one application of the existential closure rules. Then S' is closed with respect to* I–IV.

Proof. Consider first $\alpha \leqslant \alpha$: Rule *Tr* applied after *CRef* is superfluous. Rules IV with $\alpha \leqslant \alpha$ as a premiss are excluded because α is a new term but a and b not. For the rest, we follow closely the division of Skolem's proof and show that any rule instance applied after *CLM*, *CRM* can be permuted up.

1. The conclusion of *CLM* is a left premiss in an instance of *Tr* that is permuted up:

$$\dfrac{\dfrac{M(a,b,\alpha) \qquad \begin{matrix}[x\leqslant a \quad x\leqslant b]\\ \vdots \\ x\leqslant c\end{matrix}}{\alpha\leqslant c}\,{\scriptstyle CLM} \qquad c\leqslant d}{\alpha\leqslant d}\,{\scriptstyle Tr} \quad \rightsquigarrow \quad \dfrac{M(a,b,\alpha) \qquad \dfrac{\begin{matrix}[x\leqslant a \quad x\leqslant b]\\ \vdots \\ x\leqslant c\end{matrix} \qquad c\leqslant d}{x\leqslant d}\,{\scriptstyle Tr}}{\alpha\leqslant d}\,{\scriptstyle CLM}$$

2. The conclusion of *CRM* is a right premiss in *Tr*. We have

$$\dfrac{e\leqslant c \qquad \dfrac{M(a,b,\alpha) \quad c\leqslant a \quad c\leqslant b}{c\leqslant \alpha}\,{\scriptstyle CRM}}{e\leqslant \alpha}\,{\scriptstyle Tr}$$

Rule *Tr* is permuted up:

$$\dfrac{M(a,b,\alpha) \qquad \dfrac{e\leqslant c \quad c\leqslant a}{e\leqslant a}\,{\scriptstyle Tr} \qquad \dfrac{e\leqslant c \quad c\leqslant b}{e\leqslant b}\,{\scriptstyle Tr}}{e\leqslant \alpha}\,{\scriptstyle CRM}$$

3. The term α is a middle term in rule *Tr* with the premisses derived by *CRM*, *CLM*. We have the derivation

$$\dfrac{\dfrac{M(a,b,\alpha) \quad c\leqslant a \quad c\leqslant b}{c\leqslant \alpha}\,{\scriptstyle CRM} \qquad \dfrac{M(a,b,\alpha) \qquad \begin{matrix}[x\leqslant a \quad x\leqslant b]\\ \vdots \\ x\leqslant d\end{matrix}}{\alpha\leqslant d}\,{\scriptstyle CLM}}{c\leqslant d}\,{\scriptstyle Tr}$$

We take the premisses $c \leqslant a$ and $c \leqslant b$, then substitute c for x in the derivation of $x \leqslant d$ from $x \leqslant a, x \leqslant b$, to get a derivation of $c \leqslant d$ in S:

$$\begin{matrix} c\leqslant a \quad c\leqslant b \\ \vdots \\ c\leqslant d \end{matrix}$$

4. If *CLM* is followed by rule *RM* on some atom $M(d, e, f)$ in S, we have

$$\dfrac{M(d,e,f) \qquad \dfrac{M(a,b,\alpha) \qquad \begin{matrix}[x \leqslant a \quad x \leqslant b]\\ \vdots \\ x \leqslant d\end{matrix}}{\alpha \leqslant d}\,{\scriptstyle CLM} \qquad \dfrac{M(a,b,\alpha) \qquad \begin{matrix}[x \leqslant a \quad x \leqslant b]\\ \vdots \\ x \leqslant e\end{matrix}}{\alpha \leqslant e}\,{\scriptstyle CLM}}{\alpha \leqslant f}\,{\scriptstyle RM}$$

The permutation is into

$$\dfrac{M(a,b,\alpha) \qquad \dfrac{M(d,e,f) \qquad \begin{matrix}[x \leqslant a \quad x \leqslant b]\\ \vdots \\ x \leqslant d\end{matrix} \qquad \begin{matrix}[x \leqslant a \quad x \leqslant b]\\ \vdots \\ x \leqslant e\end{matrix}}{x \leqslant f}\,{\scriptstyle RM}}{\alpha \leqslant f}\,{\scriptstyle CLM}$$

If $M(d, e, f)$ happened to be $M(a, b, \alpha)$ instead of an atom from S, the conclusion is $\alpha \leqslant \alpha$ obtained by *CRef*.

5. If *CRM* is followed by *RJ* on some atom $J(e, f, g)$, we have

$$\dfrac{J(e,f,g) \qquad \dfrac{M(a,b,\alpha) \quad e \leqslant a \quad e \leqslant b}{e \leqslant \alpha}\,{\scriptstyle CRM} \qquad \dfrac{M(a,b,\alpha) \quad f \leqslant a \quad f \leqslant b}{f \leqslant \alpha}\,{\scriptstyle CRM}}{g \leqslant \alpha}\,{\scriptstyle RJ}$$

The permutation is

$$\dfrac{M(a,b,\alpha) \qquad \dfrac{J(e,f,g) \quad e \leqslant a \quad f \leqslant a}{g \leqslant a}\,{\scriptstyle RJ} \qquad \dfrac{J(e,f,g) \quad e \leqslant b \quad f \leqslant b}{g \leqslant b}\,{\scriptstyle RJ}}{g \leqslant \alpha}\,{\scriptstyle CRM}$$

All possible cases of rules I–IV that can use conclusions of *CRef*, *CRM*, *CLM* as premisses are now covered. ⊣

The reader is invited to check that the above transformations correspond precisely to the steps found in Skolem's proof of lemma 1. How he was able to figure out closure rule 2, the one with the eigenvariable, remains to be wondered. Rule *CRM* is the same as rule *RM*, but *CLM* is different from LM_1, LM_2. If the latter were used, lattice rules would not permute up. With *CLM*, instead, a finitary definition of closure is achieved.

Skolem's second lemma shows that a system S closed with respect to axioms I–IV can be closed also with respect to V by just one application of axioms V. In terms of permutability of rules, rules V permute last, and their repeated applications contract into one:

SKOLEM'S LEMMA 2. *Let* Σ *be a set of atoms closed relative to axioms* I–IV, *and let* Σ' *be the extension of* Σ *consisting of all atoms obtained from* Σ *by at most one application of* V. *Then* Σ' *is closed relative to the axioms* I–V.

PROOF. We show that instances of rules *SM*, *SJ* permute down with respect to rules I–IV and that just one instance of *SM*, *SJ* is sufficient.

Let $M(\alpha, \beta, \gamma)$ be in Σ'. By assumption, there are some $M(a, b, c)$, $a = \alpha$, $b = \beta, c = \gamma$ derivable in Σ and an instance of a substitution SM such that

$$\dfrac{M(a,b,c) \quad a = \alpha \quad b = \beta \quad c = \gamma}{M(\alpha,\beta,\gamma)}\,SM$$

The conclusion $M(\alpha, \beta, \gamma)$ can be a premiss in LM_1, LM_2, RM, and SM. Again following Skolem's division, we have the following cases:

1. The conclusion of LM_1 is $\gamma \leqslant \alpha$. It is obtained without substitution on $M(\alpha, \beta, \gamma)$ by the derivation in Σ

$$\dfrac{\dfrac{\gamma \leqslant c \qquad \dfrac{M(a,b,c)}{c \leqslant a}\,LM_1}{\gamma \leqslant a}\,Tr \qquad a \leqslant \alpha}{\gamma \leqslant \alpha}\,Tr$$

In the case of LM_2 the derivation is analogous.

2. If $M(\alpha, \beta, \gamma)$ is a premiss in RM, we have the derivation, with all assumptions derivable in Σ,

$$\dfrac{\dfrac{M(a,b,c) \quad a = \alpha \quad b = \beta \quad c = \gamma}{M(\alpha,\beta,\gamma)}\,SM \qquad d \leqslant \alpha \quad d \leqslant \beta}{d \leqslant \gamma}\,RM$$

The conclusion is derived in Σ by

$$\dfrac{\dfrac{M(a,b,c) \qquad \dfrac{d \leqslant \alpha \quad \alpha \leqslant a}{d \leqslant a}\,Tr \qquad \dfrac{d \leqslant \beta \quad \beta \leqslant b}{d \leqslant b}\,Tr}{d \leqslant c}\,RM \qquad c \leqslant \gamma}{d \leqslant \gamma}\,Tr$$

3. If $M(\alpha, \beta, \gamma)$ is a premiss in SM, we have the derivation

$$\dfrac{\dfrac{M(a,b,c) \quad a = \alpha \quad b = \beta \quad c = \gamma}{M(\alpha,\beta,\gamma)}\,SM \qquad \alpha = \alpha' \quad \beta = \beta' \quad \gamma = \gamma'}{M(\alpha',\beta',\gamma')}\,SM$$

This is converted into transitivities and one substitution:

$$\dfrac{M(a,b,c) \qquad \dfrac{a = \alpha \quad \alpha = \alpha'}{a = \alpha'}\,Tr \qquad \dfrac{b = \beta \quad \beta = \beta'}{b = \beta'}\,Tr \qquad \dfrac{c = \gamma \quad \gamma = \gamma'}{c = \gamma'}\,Tr}{M(\alpha',\beta',\gamma')}\,SM$$

Substitutions on J-atoms are treated in the same way. In the end, if the conclusion is an O-atom, no substitutions are needed, and otherwise there is at most one substitution as a last rule. ⊣

The closure is obtained by taking all possible derivations in Σ and by continuing them by one step whenever conclusions in Σ match premisses of the closure rules. Skolem gives, in fact, proof transformations that show the eliminability of rules in a system. Eliminability is obtained through transformations that act on the assumption that derivations of the premisses of the rule to be eliminated are at hand. Thus, the concept that is used is that of the *admissibility* of a rule in a given system of rules.

To arrive at Skolem's theorem from the lemmas, we reason as follows (Skolem himself drew the conclusion of his theorem without further explanations): If an O-atom with terms known from given assumptions is derivable from these assumptions, it has by lemma 1 a derivation without existential rules. The only way left to have some new, unknown terms in a derivation would be through rule *Ref*. However, if a transitivity follows such a *Ref*, the conclusion is equal to the other premiss and the instance of *Ref* can be deleted. If a premiss of a rule that concludes an M- or J-atom were an instance of *Ref*, its term would appear in such an atom. These atoms, in turn, are assumptions because by lemma 2 no substitution rules need be used and thus no M- or J-atoms need be concluded. Therefore the terms in derivations of O-atoms by rules I–V can be restricted to those known from the assumptions.

To finish the proof of Skolem's theorem, we show that the derivability of arbitrary atoms reduces to the derivability of O-atoms. Skolem gives his proof in very brief terms (p. 121), but its content can be reproduced with explicit rules as follows:

LEMMA 3. *The problem of derivability of an M-atom* $M(a,b,c)$ *from assumptions* Γ *in Skolem's lattice theory reduces to the problem of derivability of two O-atoms from* $M(a,b,\alpha)$ *and* Γ, *with* α *a fresh variable, and similarly for a J-atom.*

PROOF. Let α be a fresh variable. We show that $M(a,b,c)$ is derivable from assumptions Γ if and only if $\alpha \leqslant c$ and $c \leqslant \alpha$ are derivable from $M(a,b,\alpha)$ and Γ.

If $M(a,b,c)$ is derivable from Γ, we have

$$\dfrac{M(a,b,\alpha) \qquad \dfrac{\begin{matrix}\Gamma\\ \vdots\\ M(a,b,c)\end{matrix}}{c \leqslant a}{\scriptstyle LM_1} \qquad \dfrac{\begin{matrix}\Gamma\\ \vdots\\ M(a,b,c)\end{matrix}}{c \leqslant b}{\scriptstyle LM_2}}{c \leqslant \alpha}{\scriptstyle RM} \qquad \dfrac{\begin{matrix}\Gamma\\ \vdots\\ M(a,b,c)\end{matrix} \qquad \dfrac{M(a,b,\alpha)}{\alpha \leqslant a}{\scriptstyle LM_1} \qquad \dfrac{M(a,b,\alpha)}{\alpha \leqslant b}{\scriptstyle LM_2}}{\alpha \leqslant c}{\scriptstyle RM}$$

In the other direction, assuming $\alpha \leqslant c$ and $c \leqslant \alpha$ derivable from $M(a,b,\alpha)$ and Γ, we have

$$\dfrac{\dfrac{[M(a,b,\alpha)] \qquad \begin{matrix}[M(a,b,\alpha)], \Gamma\\ \vdots\\ \alpha \leqslant c\end{matrix} \qquad \begin{matrix}[M(a,b,\alpha)], \Gamma\\ \vdots\\ c \leqslant \alpha\end{matrix}}{M(a,b,c)}{\scriptstyle SM}}{M(a,b,c)}{\scriptstyle EM}$$

Since α was chosen fresh, the variable restriction in rule *EM* is met. ⊣

By the lemma, rules that conclude M- or J-atoms, i.e., rules V, need not be considered and it follows that the derivability of an atom from given atoms in relational lattice theory is decidable. The derivability of arbitrary universal formulas in lattice theory now reduces to the derivability of formulas with the lattice rules and the rules of propositional logic. (Skolem argued by the use of conjunctive normal form for the reduction, i.e., he implicitly assumed classical logic.)

The decidability theorem for relational lattice theory can be proved even without the closure construction, by the use of only permutability arguments, as in Negri and von Plato (2004). However, the problem of derivability of arbitrary first-order formulas in lattice theory is not decidable, by a result announced in Tarski (1949).

It is possible to give a formulation of lattice theory with constructions in place of existential axioms for meet and join. The effect of this change on the proof of decidability is surprising: Most of the difficulties of the above proof were covered by lemmas that simply disappear, and the whole proof is just one short permutability argument (Negri and von Plato 2002). The argument can be simplified further:

A concise proof of Skolem's theorem: Skolem's closure rule CLM turns out to be equivalent to the two standard lattice rules LM_1, LM_2 of table 3: With $d \equiv a$ resp. $d \equiv b$ in CLM, the old rules come out as special cases. In the other direction, $M(a, b, \alpha)$ gives $\alpha \leqslant a, \alpha \leqslant b$ by the old rules. Because x in CLM is arbitrary, we can substitute α for x in the derivation of $x \leqslant d$ from $x \leqslant a, x \leqslant b$, to obtain the conclusion $\alpha \leqslant d$ of CLM by LM_1, LM_2. If a rule *CRJ* is formulated in the dual way, a handy rule system of lattice theory is obtained by which Skolem's theorem on the decidability of derivability with the lattice axioms follows. Things are simplified further if lattice theory is formulated with the lattice operations: We have the two rules *Ref* and *Tr* of the order relation and the following four rules for the meet and join operations, with x an eigenvariable in rules LM, RJ:

TABLE 7. A handy system of rules for lattice theory.

$$
\frac{c \leqslant a \quad c \leqslant b}{c \leqslant a \wedge b}\,RM
\qquad
\frac{\begin{matrix}[x \leqslant a \quad x \leqslant b]\\ \vdots \\ x \leqslant d\end{matrix}}{a \wedge b \leqslant d}\,LM
\qquad
\frac{\begin{matrix}[a \leqslant x \quad b \leqslant x]\\ \vdots \\ c \leqslant x\end{matrix}}{c \leqslant a \vee b}\,RJ
\qquad
\frac{a \leqslant d \quad b \leqslant d}{a \vee b \leqslant d}\,LJ
$$

THEOREM 4. *Terms in derivations in the handy system for lattice theory can be restricted to terms known from assumptions or the conclusion and a bounded number of eigenvariables.*

PROOF. Consider the last occurrence of a new term t in a derivation. It is removed by rule Tr and the premisses are $c \leqslant t$ and $t \leqslant d$ for some c, d. As long as at least one premiss is derived by another Tr, the step removing t is permuted up, and the same if the left premiss is derived by LM or LJ, or the right premiss by RM or RJ. There remain the cases that the left premiss is derived by RM and the right by LM, and similarly for join. We have $t \equiv a \wedge b$ for some a, b and the part of derivation

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ c \leqslant a\end{matrix} \quad \begin{matrix}\vdots \\ c \leqslant b\end{matrix}}{c \leqslant a \wedge b}\,RM \qquad \dfrac{\begin{matrix}[x \leqslant a \quad x \leqslant b] \\ \vdots \\ x \leqslant d\end{matrix}}{a \wedge b \leqslant d}\,LM}{c \leqslant d}\,Tr$$

We take the premisses $c \leqslant a$ and $c \leqslant b$, then substitute c for x in the derivation of $x \leqslant d$ from the assumptions $x \leqslant a, x \leqslant b$, to get a derivation of $c \leqslant d$ without the new term $a \wedge b$:

$$\begin{matrix}\vdots & & \vdots \\ c \leqslant a & & c \leqslant b \\ & \vdots & \\ & c \leqslant d & \end{matrix}$$

Thus, in the end the instance of Tr that removes the new term disappears, or else at least one of its premisses is an assumption and t not a new term.

Each time a rule with eigenvariables is applied, a lattice operation appears in the conclusion. Therefore the number of lattice operations in the atom to be derived gives an upper bound to the number of eigenvariables in a possible derivation. ⊣

Connection to logic programming: We can turn Skolem's axioms I–IV into logic programming clauses in which the assumptions of an axiom form an antecedent list of formulas $P, \dots, Q$ in a clause of the form $P, \dots, Q \to a \leqslant b$, with an O-atom $a \leqslant b$ as a conclusion. Derivations by the clauses consist of the composition of instances of the axioms by the rule

$$\dfrac{\Gamma \to a \leqslant b \quad a \leqslant b, \Delta \to c \leqslant d}{\Gamma, \Delta \to c \leqslant d}\,Comp$$

The question whether an atom $a \leqslant b$ is derivable from given atoms $P, \dots, Q$ is decidable by Skolem's result: In a derivation of $P, \dots, Q \to a \leqslant b$, the only clause instances that need be used are those with known terms. Further, because all clauses have just one formula in the succedent part, proof search is polynomial by known results of logic programming.

A somewhat peculiar coincidence can be recorded here. Namely, Gentzen's first paper dealt with precisely the kind of clauses and their composition as in lattice theory. In the words of Schroeder–Heister (2002, p. 246), its main results contain a normal form for derivations that amounts, in modern terms, "to the completeness of propositional SLD-resolution in logic programming." Skolem published a detailed review (1932) of Gentzen's paper, but gave no indication that Gentzen's calculus would apply to his own earlier work.

§10. Skolem's axioms for projective geometry. We come now to the third, geometrical section of Skolem (1920). The axiomatization is again a relational one, with a two-sorted language with points that are denoted by $A, B, C, \dots$ and lines that are denoted by $a, b, c, \dots$. The basic relations are equality of points (AB), equality of lines (ab), and incidence of a point on a line (Aa), the last also written as (aA). Skolem refers to the atoms as "pairs." The axioms read:

TABLE 8. Skolem's axioms for plane projective geometry.

I_p.	For each A, (AA).	I_l.	For each a, (aa).
II_p.	From (AB) and (BC) follows (AC).	II_l.	From (ab) and (bc) follows (ac).
III_p.	From (AB) and (Ac) follows (Bc).	III_l.	From (ab) and (Ca) follows (Cb).
IV.	From the pairs $(Aa)(Ba)(Ab)(Bb)$ follows either (AB) or (ab).		
V_p.	For each A and B, there is a c such that (Ac) as well as (Bc) obtain.	V_l.	For each a and b, there is a C such that (aC) as well as (bC) obtain.

By axioms I–II, equality is an equivalence relation, symmetry being taken care of by considering (AB) and (ab) to be notational variants of (BA) and (ba), respectively. Axioms III permit the substitution of equals in the incidence relation. Axiom IV gives the uniqueness of connecting lines and intersection points, the existence of which is guaranteed by the existential axioms V. In words, the elegantly formulated uniqueness axiom states that if two points are incident with two lines, either the points or the lines are equal. The disjunction here is a strictly classical one, in the sense that it need not be decidable which of (AB), (ab) is the case. (Such decidability would imply the decidability of equality of any two real numbers: Consider for this a model of projective geometry that uses pairs of real numbers as points.) Skolem explains in careful terms the necessity to always consider the two cases, and to draw a single conclusion only if it follows from both. We see here perhaps for the first time the generalization of the derivability relation in mathematics, previously from given assumptions to a conclusion, and now from given assumptions to a number of *possible cases*. The same generalization is the most essential feature of Gentzen's classical sequent calculus.

The proof of the decidability of derivability by the rules of plane projective geometry is similar to that for lattice theory. The existential rules are shown conservative relative to the other rules, and the closure of derivability of atomic cases by rules I–IV from given atomic assumptions is easily seen to be bounded. A formulation in terms of constructions in place of existential axioms can also be given, as in our (2005).

Skolem gives as an application of his decision method a proof of the independence of what is known as the property of Desargues in plane projective geometry. He notes that it is not easy to find examples that would not grow unduly complicated.

Ketonen's thesis of 1944 reformulates Skolem's geometry as a system in sequent calculus. The axioms appear through their instances in the left, assumption part of sequents, and Gentzen's cut elimination theorem for sequent calculus applies. Ketonen had studied with Gentzen in Göttingen in 1938–1939, but it is unknown how he found Skolem's 1920 paper. He extends the treatment to plane affine geometry and gives as an application a proof of the underivability of Euclid's parallel postulate from the rest of the affine axioms (see also our 2004 article for Ketonen's work).

Skolem's and Ketonen's axiom systems are purely universal; They lack the standard axiom of *noncollinearity* by which the existence of at least three noncollinear points is postulated. In Negri (2003), a proof-theoretical way of dealing with such axioms within sequent calculus is given. Skolem's geometry with the axiom of noncollinearity added is given as one example.

We wrote Skolem's rules for lattice theory as clauses of logic programming. By Skolem's main result, if a clause is derivable, its derivation uses only a bounded number of instances of the clauses. In lattice theory, each clause has a single formula at right, and proof search is therefore polynomial in time. With projective geometry, one clause is

$$(Aa), (Ba), (Ab), (Bb) \rightarrow AB, ab$$

By the results of logic programming, derivations by such clauses are not in general polynomial.

§11. A proof of Skolem's theorem for projective geometry. We shall give a proof of Skolem's theorem on the decidability of derivability by the axioms of plane projective geometry in which the steps of proof are presented as explicit permutations. Skolem uses the traditional writing of upper case letters for points, lower case letters for lines, but his notation has some variation. We use here lower case letters $a, b, c \dots$ for points where Skolem uses $A, B, C, \dots$ and $P, Q, R, \dots$, and $l, m, n, \dots$ where Skolem uses $a, b, c, \dots$ and $r, s, t, \dots$. We write $a = b$ and $l = m$ for equality and $a \in l$ for incidence in place of Skolem's idiosyncratic notation of equality as (AB), (ab), and for incidence

(Aa). He considers the "pairs" (AB) and (ab) the same as (BA) and (ba), so that there is no explicit axiom of symmetry.

Skolem's use of existential axioms is again somewhat intuitive: The proofs are texts in which it can be indicated when a new line l is introduced through axiom V_l, and what the points a, b were on which l depends, and the same for new points. We shall use two relations $L(a, b, l)$ and $P(l, m, a)$ to this purpose, with additional incidence rules that conclude $a \in l$ and $b \in l$ from $L(a, b, l)$ and $a \in l$ and $a \in m$ from $P(l, m, a)$. Thus, the existential axioms are $\forall x \, \forall y \, \exists z \, L(x, y, z)$ and $\forall x \, \forall y \, \exists z \, P(x, y, z)$, instead of $\forall x \, \forall y \, \exists z \, (x \in z \,\&\, y \in z)$ and $\forall x \, \forall y \, \exists z \, (z \in x \,\&\, z \in y)$. Finally, in contrast to lattice theory, the rule system has one rule with a multiple conclusion.

TABLE 9: Rules of proof for plane projective geometry.

I–II. *Rules for equality:*

$$\frac{}{a = a}\,Ref \qquad \frac{a = b}{b = a}\,Sym \qquad \frac{a = b \quad b = c}{a = c}\,Tr$$

$$\frac{}{l = l}\,Ref \qquad \frac{l = m}{m = l}\,Sym \qquad \frac{l = m \quad m = n}{l = n}\,Tr$$

III. *Substitution rules:*

$$\frac{a \in l \quad a = b}{b \in l}\,SP \qquad \frac{a \in l \quad l = m}{a \in m}\,SL$$

IV. *Uniqueness rule:*

$$\frac{a \in l \quad a \in m \quad b \in l \quad b \in m}{a = b \quad l = m}\,Uni$$

V. *Incidence rules and Existential rules for lines and points:*

$$\frac{L(a,b,l)}{a \in l}\,IL \qquad \frac{L(a,b,l)}{b \in l}\,IL \qquad \frac{\begin{matrix}[L(a,b,x)]\\ \vdots \\ C\end{matrix}}{C}\,EL \qquad \frac{P(l,m,a)}{a \in l}\,IP \qquad \frac{P(l,m,a)}{a \in m}\,IP \qquad \frac{\begin{matrix}[P(l,m,x)]\\ \vdots \\ C\end{matrix}}{C}\,EP$$

The application of rules V is now clearly indicated. In the existential rules *EL*, *EP*, x is an eigenvariable and any number $n \geqslant 1$ of assumptions with the eigenvariable can be discharged. As with lattice theory, we assume that the eigenvariable of a rule appears only in the upward subderivation that is defined by the rule, so that all eigenvariables are distinct.

Derivations by rules I–V are no more trees, but have downward binary branchings when rule *Uni* is applied. We shall accordingly talk about **threads** in derivations, in place of branches. We shall call the lowest formulas of a derivation its **cases**. The assumptions and cases of a derivation form finite multisets that we denote by $\Gamma, \Delta, \ldots$. (A detailed study of this system of derivation is given in von Plato 2005.)

Skolem gives rules for forming the closure of a given set of atoms S relative to axioms I-IV, whenever an existential axiom has been applied once. The method is exactly the same as in lattice theory. We formulate the closure operation through a set of rules. The assumption is that axiom V_l has been applied once to form the connecting line λ of points a, b as expressed by $L(a,b,\lambda)$:

TABLE 10. Existential closure rules for an atom $L(a,b,\lambda)$.

$$\frac{L(a,b,\lambda)}{\lambda=\lambda}\,CRef \qquad \frac{L(a,b,\lambda) \quad a=c}{c\in\lambda}\,CSP \qquad \frac{L(a,b,\lambda) \quad b=d}{d\in\lambda}\,CSP$$

$$\frac{L(a,b,\lambda) \quad e\in l \quad a=c \quad b=d \quad c\in l \quad d\in l}{e\in\lambda}\,CSL$$

$$\frac{L(a,b,\lambda) \quad a=c \quad b=d \quad c\in l \quad d\in l}{\lambda=l}\,CUni \qquad \frac{L(a,b,\lambda) \quad c=a \quad d=b \quad c\in l \quad d\in l}{l=\lambda}\,CUni$$

In Skolem's rules, symmetry of equality is built into the notation, but here we need two closure rules for *Uni* to handle the case that its conclusion is followed by rule *Sym*.

The idea of Skolem's proof is to show that all new atoms that are derivable by the addition of a line λ through points a, b contain the new eigenvariable λ. Therefore no new atoms with only the previously known points and lines become derivable so that extension by the existential axioms is conservative.

THEOREM 1. *Let S be a set of pairs closed with respect to* I–IV. *Let further a and b be points that occur in atoms of S, but so that $a=b$ is not in S. We form a more comprehensive set S' by applying once closure rules with $L(a,b,\lambda)$ as a premiss to derivations of atoms in S to obtain a set of atoms S'. Then S' is closed with respect to* I–IV.

PROOF. We show that a single application of a closure rule with the premiss $L(a,b,\lambda)$ followed by rules I–IV of projective geometry can be permuted down until it is the last rule, or removed. The cases and steps of proof follow closely Skolem's presentation. To keep the derivations printable, we may leave the premiss $L(a,b,\lambda)$ unwritten.

1. Rule *Sym* is applied to the conclusion $\lambda=l$ (or $l=\lambda$) of rule *CUni*. We apply rule *SP* to the premisses $a=c$ and $b=d$ and use the other form of rule *CUni*.

In what follows, we shall mostly not consider *Sym* explicitly, but reverse equalities as needed.

2. Rule *Tr* is applied to the conclusion $\lambda = l$ of rule *CUni*. The derivation is

$$\dfrac{\dfrac{L(a,b,\lambda)\quad a=c\quad b=d\quad c\in l\quad d\in l}{\lambda=l}\,CUni\qquad l=m}{\lambda=m}\,Tr$$

The right premiss of rule *Tr* and the premisses of rule *CUni* except the L-atom are derivable in S. The permutation is into

$$\dfrac{L(a,b,\lambda)\quad a=c\quad b=d\quad \dfrac{c\in l\quad l=m}{c\in m}\,SP\quad \dfrac{d\in l\quad l=m}{d\in m}\,SP}{\lambda=m}\,CUni$$

The transformation is similar if a right premiss of *Tr* is derived by the second *CUni* rule.

3. Let now λ be a middle term in *Tr*, so that both of its premisses are derived by *CUni*:

$$\dfrac{\dfrac{e=a\quad f=b\quad e\in l\quad f\in l}{l=\lambda}\,CUni\qquad \dfrac{a=c\quad b=d\quad c\in m\quad d\in m}{\lambda=m}\,CUni}{l=m}\,Tr$$

The transformed derivation is

$$\dfrac{\dfrac{e\in l\quad \dfrac{e=a\quad a=c}{e=c}\,Tr}{c\in l}\,SP\qquad c\in m\qquad \dfrac{f\in l\quad \dfrac{f=b\quad b=d}{f=d}\,Tr}{d\in l}\,SP\qquad d\in m}{c=d\quad l=m}\,Uni$$

Skolem now argues that the case $c = d$ follows only if $a = b$ follows, but the latter was excluded. Therefore only the conclusion $l = m$ remains. We shall return to this argument in the next Section.

4. The conclusion of *CSP* is a premiss in *SP*. The derivation and its transformation are

$$\dfrac{\dfrac{L(a,b,\lambda)\quad a=c}{c\in\lambda}\,CSP\qquad c=d}{d\in\lambda}\,SP \quad\rightsquigarrow\quad \dfrac{L(a,b,\lambda)\qquad \dfrac{a=c\quad c=d}{a=d}\,Tr}{d\in\lambda}\,CSP$$

The transformation is similar if the second premiss of *CSP* is $b = c$.

5. The conclusion of *CSL* is a premiss in *SP*. The derivation is

$$\dfrac{\dfrac{L(a,b,\lambda)\quad e\in l\quad a=c\quad b=d\quad c\in l\quad d\in l}{e\in\lambda}\,CSL\qquad e=f}{f\in\lambda}\,SP$$

This is transformed into

$$\dfrac{L(a,b,\lambda)\quad \dfrac{e\in l\quad e=f}{f\in l}\,SP\quad a=c\quad b=d\quad c\in l\quad d\in l}{f\in\lambda}\,CSL$$

6. Rule *SL* is applied. Now λ can appear in one or two premisses:

$$\frac{e \in l \quad l = \lambda}{e \in \lambda}\,SL \qquad \frac{e \in \lambda \quad \lambda = m}{e \in m}\,SL$$

In the first derivation, the premiss $l = \lambda$ has been derived by *CUni* with premisses $L(a, b, \lambda)$, $c = a$, $d = b$, $c \in l$, and $d \in l$. The second and third are inverted by *Sym*, and the conclusion follows by rule *CSL*:

$$\frac{L(a, b, \lambda) \quad e \in l \quad a = c \quad b = d \quad c \in l \quad d \in l}{e \in \lambda}\,CSL$$

If λ appears in both premisses of rule *SL*, the first premiss is derived by *CSP* or *CSL*. In the latter case we have the derivation

$$\dfrac{\dfrac{e \in l \quad a = c \quad b = d \quad c \in l \quad d \in l}{e \in \lambda}\,CSL \quad \dfrac{a = f \quad b = g \quad f \in m \quad g \in m}{\lambda = m}\,CUni}{e \in m}\,SL$$

The equalities $c = f$ and $d = g$ follow, and therefore also the incidences $c \in m$ and $d \in m$. The transformed derivation is

$$\dfrac{e \in l \quad \dfrac{c \in l \quad c \in m \quad d \in l \quad d \in m}{l = m}\,Uni}{e \in m}\,SL$$

Here rule *Uni* concludes *a priori* two possible cases $c = d$ and $l = m$ of which $c = d$ is excluded by the assumption that $a = b$ be not derivable.

There is a final possibility under case 6, not treated by Skolem, namely that the first premiss has been derived by *CSP* as in

$$\dfrac{\dfrac{L(a, b, \lambda) \quad a = e}{e \in \lambda}\,CSP \quad \dfrac{a = f \quad b = g \quad f \in m \quad g \in m}{\lambda = m}\,CUni}{e \in m}\,SL$$

This is transformed into

$$\dfrac{\dfrac{f \in m \quad a = f}{a \in m}\,SP \quad a = e}{e \in m}\,SP$$

7. The last case is that a closure rule is followed by rule *Uni*. The step is, say,

$$\frac{c \in l \quad c \in \lambda \quad d \in l \quad d \in \lambda}{c = d \quad l = \lambda}\,Uni$$

If $c \in \lambda$ and $d \in \lambda$ are derived by *CSP* we have, say, $a = c$ and $b = d$. We then have the derivation

$$\dfrac{c = a \quad b = d \quad \dfrac{c \in l \quad c = a}{a \in l}\,SP \quad \dfrac{d \in l \quad d = b}{b \in l}\,SP}{l = \lambda}\,CUni$$

Let next $c \in \lambda$ be derived by *CSL* and $d \in \lambda$ by *CSP*:

$$\dfrac{c \in l \quad \dfrac{c \in m \quad a=e \quad b=f \quad e \in m \quad f \in m}{c \in \lambda}\,CSL \quad d \in l \quad \dfrac{L(a,b,\lambda) \quad a=d}{d \in \lambda}\,CSP}{c=d \quad l=\lambda}\,Uni$$

The transformed derivation is

$$\dfrac{\dfrac{c \in l \quad c \in m \quad d \in l \quad \dfrac{e \in m \quad \dfrac{e=a \quad a=d}{e=d}\,Tr}{d \in m}\,SP}{c=d \quad l=m}\,Uni \quad \dfrac{a=e \quad b=f \quad e \in m \quad f \in m}{m=\lambda}\,CUni}{l=\lambda}\,Tr$$

By case 2, the instance of rule *Tr* after rule *CUni* permutes up.

Finally, let both $c \in \lambda$ and $d \in \lambda$ be derived by *CSL*, with the derivations

$$\dfrac{c \in m \quad a=e \quad b=f \quad e \in m \quad f \in m}{c \in \lambda}\,CSL \qquad \dfrac{d \in n \quad a=g \quad b=h \quad g \in n \quad h \in n}{d \in \lambda}\,CSL$$

The transformed derivation is

$$\dfrac{\dfrac{c \in l \quad c \in m \quad d \in l \quad \dfrac{d \in n \quad \dfrac{\dfrac{a=g \quad b=h \quad g \in n \quad h \in n}{n=\lambda}\,CUni \quad \dfrac{a=e \quad b=f \quad e \in m \quad g \in m}{\lambda=m}\,CUni}{n=m}\,Tr}{d \in m}\,SL}{c=d \quad l=m}\,Uni \quad m=\lambda}{l=\lambda}\,SL$$

The second premiss of the last rule is derived with *CUni* similarly to the right *CUni* on top of the derivation. All rules that follow the instances of *CUni* permute to above by previous cases. ⊣

We found that Skolem missed one case in 6 with his hard-to-follow verbal proof.

Skolem's next concern is to consider the case that the equality $a = b$ is derivable in S. Then the connecting line λ of points a and b is degenerate, and closure rules *CRef* and *CSP* of table 10 suffice:

TABLE 11. Existential closure rules for the degenerate case $a = b$.

$$\dfrac{L(a,b,\lambda)}{\lambda=\lambda}\,CRef \qquad \dfrac{L(a,b,\lambda) \quad a=c}{c \in \lambda}\,CSP$$

The second form of rule *CSP* follows by $a = b$: Given that $b = c$, conclude $a = c$ by *Tr*, then apply rule *CSP* to conclude $c \in \lambda$.

THEOREM 2. *If S is closed with respect to* I–IV, *if a and b are points such that $a = b$ is in S, and if S' is the set of atoms obtained from S through one application of the degenerate closure rules with $L(a, b, \lambda)$ as one premiss, a set S' closed with respect to* I–IV *is obtained.*

PROOF. We show that geometrical rules applied after the closure rules permute up. The only line equality obtained by the closure rules is a reflexivity and any rule that uses it as a premiss has one premiss equal to the conclusion. The other cases are:

1. Rule *SP* follows rule *CSP*. The derivation and its transformation are

$$\cfrac{\cfrac{L(a,b,\lambda) \quad a=c}{c\in\lambda}\,CSP \qquad c=d}{d\in\lambda}\,SP \quad\rightsquigarrow\quad \cfrac{L(a,b,\lambda) \qquad \cfrac{a=c \quad c=d}{a=d}\,Tr}{d\in\lambda}\,CSP$$

2. With line substitution, the second premiss has to be $\lambda=\lambda$, which makes the rule superfluous.

3. Rule *Uni* is applied. We can have two premisses with eigenvariables, as in

$$\cfrac{c\in l \qquad \cfrac{L(a,b,\lambda) \quad a=c}{c\in\lambda}\,CSP \qquad d\in l \qquad \cfrac{L(a,b,\lambda) \quad a=d}{d\in\lambda}\,CSP}{c=d \quad l=\lambda}\,Uni$$

We get $c=d$ in S from the premisses $a=c, a=d$. If all premisses of *Uni* have the eigenvariable, the same conclusion follows. ⊣

Skolem does not mention in case 3 the second case of the conclusion of rule *Uni*, namely $l=\lambda$. We shall comment on this matter in the next Section.

By theorems 1 and 2, Skolem concludes that all new atoms that become derivable contain an eigenvariable, by which his main result follows (1920, p. 127–128):

> THEOREM 3. *Given a set S of pairs closed with respect to axioms* I–IV, *but not with respect to* V, *if* V *is applied once or repeatedly, a more comprehensive set S' is obtained through the addition of all pairs that can then be derived by using* I–IV, *such that S' is again closed with respect to* I–IV *and contains no other pairs formed from the original symbols of the pairs occurring in S than those that already appear in S.*

The decision algorithm is of a different character as compared to the one for lattice theory, because rule *Uni* has two possible cases as conclusions (1920, p. 128):

> Therefore the process must be executed in all possible ways, by choosing at each application of IV between two alternatives. For this reason, one obtains in differing ways closed sets of pairs through the application of axioms I–IV to a given set S of pairs; Let $S_1, S_2, \dots$ be these sets. In order for a pair (ab) to necessarily follow from the pairs in S, (ab) must appear in each of the sets $S_1, S_2, \dots$. If the task is to find out whether at least one of the pairs $(ab), (cd)$, etc. follows from the pairs of a set S, one has to check whether in each of $S_1, S_2, \dots$ there is at least one of $(ab), (cd)$,

> etc. It does not, though, have to be the *same* one of these latter pairs in each of the sets $S_1, S_2, \ldots$. The disjunction
>
> $$(ab) + (cd) + \ldots$$
>
> follows from the conjunction of all the pairs in S when one of the pairs $(ab), (cd), \ldots$ occurs within S_1, one within S_2, but not necessarily the same, etc.

The passage is a remarkable anticipation of the generalization of the derivability relation from one conclusion to a finite number of cases. Said generalization was the most essential feature of Gentzen's classical sequent calculus, as compared to the calculus of Hertz and the system of intuitionistic natural deduction that Gentzen formulated before inventing sequent calculus. (All of this is detailed out in von Plato 2007.)

Skolem concludes his presentation by noting that the derivability of universal formulas is decidable and then gives as an example the application to Desargues' theorem. Skolem's decidability result for plane projective geometry concerns the universal theory; The axiomatization does not contain any axiom of noncollinearity, an axiom by which there exist two distinct points and a third point outside the connecting line of the first two.

There is a reason intrinsic to Skolem's approach that prevents the extension of his results to the full theory of plane projective geometry. Namely, Skolem's method is to analyze formal derivations in a "logic-free" manner: The geometrical axioms would traditionally be used through a system of logic, be it axiomatic or natural deduction. For example, an instance of the axiom of transitivity is used to conclude $a = c$ from the assumptions $a = b, b = c$ as in the following derivation written in natural deduction:

$$\dfrac{a = b \,\&\, b = c \supset a = c \qquad \dfrac{a = b \quad b = c}{a = b \,\&\, b = c}\,{\scriptstyle \&I}}{a = c}\,{\scriptstyle \supset E}$$

Skolem formulates transitivity directly as a rule that acts on the atomic assumptions:

$$\dfrac{a = b \quad b = c}{a = c}\,Tr$$

Skolem gave his "logic-free" approach in an intuitive way, without even noticing explicitly that the logical structure in axioms vanished. A similar analysis of derivations within a sequent calculus for the intuitionistic theories of apartness and order was developed (without knowledge of Skolem's work) by Negri (1999). It was generalized by Negri and the present author to any universal theories, in Negri and von Plato (1998). In her (2003), Negri extended the approach to typical existential axioms such as that of noncollinearity. It turned out that the relational axiomatizations of lattice theory and projective geometry that Skolem used were incompatible with the addition of further existential axioms such as noncollinearity: A relational axiomatization requires existence axioms in place of constructions, as in Skolem's existence axioms for meets, joins, connecting lines, and intersection points. In these axioms, atoms or conjunctions of atoms are quantified existentially. In the axiom of noncollinearity, instead, negations of atoms or conjunctions of negations of atoms are quantified existentially, and no way of converting these two types of existential axioms into rules within one system was found to give an adequate proof system. If elementary geometry is formulated with constructions, it becomes instead possible to add the axiom of noncollinearity in a "logic-free" calculus of mathematical rules and to extend Skolem's decidability result to universal formulas in the full theory of projective geometry, as is done in von Plato (2005). As with lattice theory, the first-order theory of projective geometry is undecidable (Tarski 1949).

§**12. The problematic points of Skolem's proofs.** As we saw, the proof of theorem 1 in Section 11 excluded the case $a = b$ (end of case 3) because it was by assumption not derivable. In the proof of theorem 2, the case $l = \lambda$ was ignored (again at the end of case 3) because the first case $c = d$ was already otherwise derivable. These are somewhat problematic steps in Skolem's proofs, and behind them is the question of what a closure of a set of atoms is in projective geometry. By the above quotation, there is no unique closure of axioms I–IV, but we have instead a (finite) number of possible extensions $S_1, \dots, S_k$ of a given set of atoms S. Case 3 of the proof of theorem 1 shows that even the *case* of $a = b$ in a derivation is excluded. This means that Skolem in fact assumes a and b to be distinct, formally, $\neg a = b$. Such an assumption is needed to close the case $a = b$ in a derivation. In theorem 2, $a = b$ is assumed derivable, so that *it has to appear in each extension* S_i. We show now something Skolem apparently had not thought of:

THEOREM 4. *If an atom is derivable by* I–IV *from the atomic assumptions S in projective geometry, it is derivable without rule Uni.*

PROOF. Assume there to be an atom not derivable without rule *Uni*. We can assume each instance of *Uni* to be essential in the derivation, in the sense that the derivable atom appears, as a downmost formula, only after downmost instances of *Uni*. (If not, delete the superfluous *Uni*'s.) We consider such a downmost instance of *Uni* and show that it can be eliminated:

$$\frac{a \in l \quad a \in m \quad b \in l \quad b \in m}{a = b \quad l = m}\,Uni$$

After downmost instances of *Uni*, only rules for equality and substitution are applied. These are permuted to above *Uni*, and we have the following cases:

1. Rule *Sym* is applied to $a = b$. We change the order of the premisses so that the conclusion of *Uni* becomes $b = a$. If *Sym* is applied to $l = m$, we do similarly to conclude by *Uni* $m = l$.

2. Rule *Tr* is applied to $a = b$, as in

$$\frac{a = b \quad b = c}{a = c}\,Tr$$

The transformation is into

$$\frac{a \in l \quad a \in m \quad \dfrac{b \in l \quad b = c}{c \in l}\,SP \quad \dfrac{b \in m \quad b = c}{c \in m}\,SP}{a = c \quad l = m}\,Uni$$

If *Tr* is applied to $l = m$, the transformation is similar.

3. Rule *SP* is applied to $a = b$. The conclusion is an incidence $b \in n$. It can be a premiss in *SP* or *SL*. With *SP*, we have the transformation

$$\frac{\dfrac{a \in n \quad a = b}{b \in n}\,SP \qquad b = d}{d \in n}\,SP \quad \rightsquigarrow \quad \frac{a \in n \qquad \dfrac{a = b \quad b = d}{a = d}\,Tr}{d \in n}\,SP$$

The result is covered by case 2. With *SL*, we have the transformation

$$\frac{\dfrac{a \in n \quad a = b}{b \in n}\,SP \qquad n = k}{b \in k}\,SL \quad \rightsquigarrow \quad \frac{\dfrac{a \in n \quad n = k}{a \in k}\,SL \qquad a = b}{b \in k}\,SP$$

Now transformation 3 with *SP* is applied, until there is just one instance of *SP* after *Uni* as a last step in the derivation.

4. Rule *SL* is applied to $l = m$. The conclusion is an incidence $c \in m$. It can be a premiss in *SP* or *SL* and we have transformations similar to those in case 3 until there is just one instance of *SL* after *Uni*.

After the above transformations, we have just one instance of *SP* and of *SL* after *Uni*, which forces the conclusions, say $b \in n$ and $c \in m$, to be the same. But then the conclusion is $b \in m$, the fourth premiss of *Uni*, and the number of instances of *Uni* in the derivation was reduced. ⊣

This conservativity result shows that an essential use of rules with cases requires a theorem with cases. Such a theorem would usually be expressed in a form that has a disjunction in a positive part. As to Skolem's main theorem, his proofs of the preliminary theorems that lead to it are based on the dichotomy: The case $a = b$ is either forbidden or derivable. What if neither is the case? The proof of decidability goes through without the said dichotomy, because an analogous proof with geometric constructions does (von Plato 2005). What the correct existential closure rules for such a proof for Skolem's geometry are, is still open.

§13. Conclusion. Why didn't Skolem's results become generally known, despite their adequate review in Lange–Nielsen (1922)? It seems impossible to think that the first publication of the celebrated Löwenheim–Skolem theorem would not have found readers sufficiently curious to see what the other parts of the paper contained. If they took a look, they found that Skolem's presentation was not exactly what you call "user friendly:" He is clear about having developed "a purely combinatorial conception of mathematical deduction," but there are no reflections about the method used in the conservative extension theorems, nor even about the reasons for a relational axiomatization. Thus, the reading and understanding of his proofs becomes a tedious undertaking. Finally, Skolem did not as much as mention his lattice-theoretic decidability result in his (1936) survey. Had he himself perhaps forgotten in the intervening close to twenty years how his results had been proved?

Skolem's polynomial-time decision algorithm for derivability with the axioms of lattice theory was forgotten until 1992 when Burris located it in the ***Selected Works in Logic***. Meanwhile, a polynomial algorithm had been derived in Cosmadakis (1988). Burris showed Skolem's paper to Freese and Jezek who, in the words of Burris in an e-mail to the present author, 2 April 2002,

> ... immediately recognized that Skolem had proved his result by showing that a finite structure closed under the "production rules" could be embedded in a lattice, essentially into the free completion of a partially ordered set. The free completion was a familiar tool to them, but it was thought to be of relatively recent origins.

In the book ***Free Lattices*** by Freese, Jezek, and Nation, Skolem's work is further described from this point of view (pp. 241–242).

A reading of Skolem's proofs against the ones given here in Sections 9 and 11 shows that he proved the conservativity of extension by existential axioms through combinatorial arguments on the set of all possible derivations: If the conclusion of an existential closure rule is a premiss in another lattice rule or geometric rule, their order can be permuted. He lacked a formal notation for writing down the permutations, which made his arguments hard to follow.

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