THE BULLETIN OF SYMBOLIC LOGIC
Volume 13, Number 1, March 2007

REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Alasdair Urquhart (Managing Editor), Steve Awodey, John Baldwin, Lev Beklemishev, Mirna Džamonja, David Evans, Erich Grädel, Denis Hirschfeldt, Roger Maddux, Grigori Mints, Luke Ong, Volker Peckhaus, and Sławomir Solecki. Authors and publishers are requested to send, for review, copies of books to *ASL, Box 742, Vassar College, 124 Raymond Avenue, Poughkeepsie, NY 12604, USA.*

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

FILIP WIDEBÄCK. ***Identity of proofs.*** Acta Universitatis Stockholmiensis, Stockholm Studies in Philosophy, vol. 22. Almqvist and Wiksell International, Stockholm, 2001, 85 pp.

The identity of proofs is a profound problem on the borderline of mathematical and philosophical logic. Obviously, this problem cannot be approached without having in mind some semantics, explicit or implicit. The sources of such a semantics may belong to philosophy, be found in mathematical practice and applications, or even come from the interpretation of proofs as elements of certain concrete mathematical structures.

The book by F. Widebaeck considered in this review is a doctoral dissertation in philosophical logic published as a book. It is devoted to the conjecture that $\beta\eta$-equivalence on natural deductions captures the informal notion of identity of proofs. This conjecture was advanced in the beginning of the 70s by Martin-Löf and Prawitz.

The book is well written, contains detailed proofs and in spite of its special subject, is accessible not only to specialists. A subtle structural analysis of the conjecture and related problems is one of the most attractive aspects of the book.

One may briefly outline the main lines of the book using the author's own words.

"The conjecture can be divided into two parts, a soundness part and a completeness part. The soundness part is that $\beta\eta$-equivalent proofs are identical. The completeness part is that identical proofs are $\beta\eta$-equivalent. Soundness can not be mathematically proved. It must be taken for granted or rejected. Completeness ... must be proved or disproved, although the question is not a mathematically precise question.

1079-8986/07/1301-0006/$2.50

The main result (...) is that the completeness part of the conjecture is true for the system of minimal implicational logic, provided that soundness can be taken for granted. The result is obtained by first proving that the notion of $\beta\eta$-equivalence is Post-complete."

"The background of the conjecture is discussed in detail in Chapter 1 (...) In Chapter 2, the syntax and semantics of minimal implicational logic are reviewed. In Chapter 3, the system of natural deduction is presented. In Chapter 4, the notion of identity of proofs is axiomatized and the conjecture is presented as the question whether or not the axioms are complete and consistent. Of special interest is an axiom that separates proofs instead of identifying proofs as $\beta\eta$-equivalence does.

(...) In Chapter 5, it is proved that the axioms are consistent. (...) In Chapter 6, it is proved that the axioms are complete. (...) In Chapter 7 the main result is extended to Gentzen's sequent calculus. In Chapter 8, the axioms are discussed again. A new reduction is defined in purely structural terms without the reference to the rules of natural deduction, and it is conjectured that this new reduction coincides with $\beta\eta$-reduction."

The author's inspiration is clearly philosophical. It is interesting, how far one can advance based on very general ideas concerning the structure and properties of proofs. For example, well known Friedman's completeness result is obtained as a corollary (ch. 6). The "Crude Discharge Condition" (CDC) of Troelstra and van Dalen is shown to be incompatible with separation (or generalization) axiom (G) considered by the author (ch. 8).

In the reviewer's opinion in this "minimalistic" approach lies at the same time a strength and a weakness of the book.

In fact, there are many interesting common points with the studies of equivalence of proofs based on purely mathematical considerations as, for example, in categorical logic. See for example K. Dosen and Z. Petrić,*The maximality of typed lambda-calculus and of cartesian closed categories*, ***Publications de l'Institut Mathématique (Novy Sad)***, vol. 68 (2000), no. 82, pp. 1–19 (available at: `http://arXiv.org/math.CT/9911073`), *The maximality of cartesian categories*, ***Mathematical Logic Quaterly*** vol. 47 (2001), pp. 137–144, G. E. Mints. *Closed categories and proof theory*, ***Journal of Soviet Mathematics***, vol. 15 (1981), pp. 45–62. It is a pity that a comparison with this approach remained out of the scope of the book, while the author is well aware, for example, of the works by Statman that are closely related to the categorical models.

A very important role in the proof of main results plays the "negative axiom" (G) (or "rule for separating proofs"), which says that if two proofs of the same theorem *A generalize* in different directions then they are not equivalent. Generalizing in different directions means here that (a) two proofs are obtained by substitution of formulas for variables from proofs of two different formulas and (b) these formulas are incompatible in the sense that they are *not* substitutional instances of the same formula.

The closely related notion of *generality* was in fact suggested by J. Lambek, *Deductive systems and categories. I*, ***Mathematical Systems Theory***, vol. 2 (1968), pp. 287–318, *Deductive systems and categories. II*, ***Lecture Notes in Mathematics***, vol. 86 (1969), pp. 76–122, and it was actively studied later especially in connection with categorical equivalence of derivations in linear logic, see R. Moot and Q. Puite, *Proof nets for multimodal categorial grammars*, (G.-J. M. Kruijff and R. T. Oehrle, editors), ***Proceedings of Formal Grammar 1999***, pp. 103–114 and Sylvain Pogodalla, *Generation, Lambek calculus, Montague's semantics and semantic proof nets*, ***Proceedings of the International Conference on Computational Linguistics, Coling 2000***. The idea that an identity of proofs should be a congruence has also clear categorical overtones (it is connected with naturality and functoriality of categorical operations corresponding to the rules).

The awareness of that context suggests, in particular, taking into account the difficulties met with categorical models of the fragments of logic strictly containing the $(\rightarrow, \&)$-fragment of intuitionistic logic or the multiplicative fragment of linear logic, and indirectly casts some doubt that the author's method would successfully apply even to sufficiently large systems of propositional calculus.

Some other points raised by the author are debatable. For example, it can be argued that the divide between soundness and completeness is not so absolute. Indeed, soundness and completeness of certain systems of transformations like $\beta\eta$-reduction is considered w.r.t. certain semantics. If this semantics is itself to some extent "operational" (even if it is not completely formal) then each system of transformations provides a sort of semantics to the other, and the roles of soundness and completeness are reversed.

It is fair to notice that in spite of a very "strong" initial statement: "Soundness can not be mathematically proved. It must be taken for granted or rejected," F. Widebäck provides in fact an interesting discussion of soundness and completeness in his introductory chapter (ch. 1), not related to the objection sketched above by the reviewer.

As a conclusion, it is a very lively written and instructive book, "minimalistic" as to the mathematical apparatus used on a subject whose importance attracts more and more interest nowadays. In the reviewer opinion, while this "minimalistic" approach is completely justifiable as a method in the main part of the text, more complete presentation of the scientific context in the introduction and/or conclusion would only improve the quality of the book.

SERGEI SOLOVIEV
IRIT, 118, route de Narbonne, 31062, Toulouse, France. soloviev@irit.fr.

LEO CORRY. ***Modern algebra and the rise of mathematical structures.*** Second edition. Birkhäuser Verlag, Basel, 2004, xvi + 451 pp. First edition 1996.

The characteristics of modern mathematics are usually taken to include an axiomatic starting-point and a certain level of abstraction. The combination has often been seen as inevitable, especially starting from Hilbert's view on the axiomatization of geometry. The most typical representative of modern mathematics is then taken to be the series of Bourbaki texts. While there may have been something of a reaction to the insistence on abstraction in Bourbaki's pedagogy, it is against that particular attitude that even those who disagree with it are rebelling.

Leo Corry puts forward a case that this picture oversimplifies the way in which modern mathematics arose. In fact, he argues, the crucial element in mathematical 'modernity' is not axiomatization, but a certain kind of abstraction which he dubs 'structuralism'. He examines the development of abstract algebra and attributes crucial steps to figures who are not perhaps always given the credit he feels they deserve. The second part of the book then discusses various embodiments of the notion of 'structure' in a mixture of history and philosophy. His conclusions pursue some controversial lines of argument on the subject of foundations of mathematics in general.

This review is designed to serve as a guide through the sequence of topics that Corry addresses. This is the second edition of a book in history of mathematics, and the author comments that such a fate is uncommon for something that is not a textbook. It seems as though the second edition has been receiving more attention than the first, perhaps because of the earlier one's having appeared in a series called 'Science Networks'.

The book starts off with a comparison of the algebra textbook by Heinrich Weber (the ***Lehrbuch der Algebra***, whose first volume appeared in 1895) and B. L. van der Waerden's ***Moderne Algebra*** from 1930. Weber's approach to algebra in his textbook is contrasted with a more insightful treatment of abstraction in some of his articles from almost the same time. The textbook presumably reflects something of the mathematical audience for which it was written, an audience not inclined to jump directly into abstraction without the comforting discussion of familiar, concrete algebraic structures. By van der Waerden's time, algebra had moved drastically to a new level of abstraction, the culmination of changes that could also be observed in subsequent editions of Weber's text. The treatment of the textbooks is preceded by a consideration of the Jordan-Hölder theorem and the changes it underwent as it went from Jordan to Hölder and beyond. The author wants to argue that the search for algebraic invariants was crucial to the modernization of mathematics, which cannot be reduced to simple axiomatization.

The following chapters trace the rise of abstraction in algebra, especially as it arose from Galois theory. He starts with the work of Dedekind on ideals, since Dedekind is often credited with the creation of the structural approach to mathematics. Corry examines the theory of Dedekind in some detail (more than just a casual historical account), since he is interested in establishing that, even when Dedekind introduces algebraic objects that we are used to defining axiomatically, he is not treating them on a par. Instead, some of the objects (like a group) may be used as a tool for investigating other objects (like fields) which are seen as starting from the concrete examples of the real and complex numbers.

Following his making this distinction, Corry then takes on Hilbert, at least the Hilbert of the ***Grundlagen der Geometrie*** and the ***Zahlbericht***. He argues that Hilbert did not have the abstracting attitude toward algebra that was to be found in van der Waerden's text (and the work of Emmy Noether, the leading influence on van der Waerden). Instead, Hilbert's axiomatizations reflect the belief that only when a subject has reached a certain level of sophistication is it ready for an axiomatic approach. Seen in this light, Hilbert's search for axiomatizations is limited to areas which were ready for them. He notes that the formalism attributed to Hilbert does not reflect the extent to which an intuitive basis is required before axiomatic articulation. The chapter on Hilbert ends with a discussion of the postulational theorists in the United States. The influence of people like Oswald Veblen and E. V. Huntington may not have been immediate, but their introduction of issues like 'independence' for a set of axioms led to a readiness to look at cross-theoretical axiomatizations.

In Corry's pursuit of the origins of abstraction in algebra, he then follows what might be regarded as a byway through the work of Hensel on the p-adic numbers. The combination of this sort of generalization from the ordinary numbers of arithmetic with Steinitz's work on fields produced a recognition that there was something to be said for an axiomatic treatment of fields in general rather than assuming that all applications were going to be to the real and complex numbers. Those who are familiar primarily with Abraham Fraenkel for his work on set theory may be surprised to see how crucial a role Corry assigns to his earlier work on p-adic numbers and rings.

The last of the chapters in the first half of the book takes the story up to Emmy Noether and the era of van der Waerden's text. At the beginning of the period which Corry examines, groups were a tool for investigating fields, and the level of abstraction with which both were examined was limited. Rings were peripheral to the subject, and there was not much consistency in how they were defined. By Noether's time the advantages of some uniformity in axiomatization became clear, as proofs could be imported from one sort of structure to another. Groups, rings, and fields were equally the objects of algebraic investigation rather than their being allocated to different roles in a specific area like Galois theory.

For the second part of the book, Corry wants to put the word 'structure' under the microscope in three approaches to foundational issues. The first is the lattice theory of Oystein Ore, whose efforts to supply a uniform foundation for mathematics via lattices have largely been forgotten. He was inspired by the notion that one should be able to reconstitute a group from the structure of its lattice of certain kinds of subgroup, and he used the word 'structure' for the lattice to suggest that this was all that was needed. Corry points to a result from 1928 of Ada Rottländer that is enough to show that Ore's approach would not succeed for technical reasons.

The second approach Corry deals with is that of Bourbaki, whose foundational efforts he finds entirely unpersuasive. In particular, he argues that Bourbaki never uses the foundational machinery built up in the ***Théorie des Ensembles*** volume. Corry finds the Bourbaki approach hypocritical, and he discusses the contents of the notes from various meetings of the group to see at whose door the responsibility lies. As elsewhere in his book, Corry only takes a structural approach seriously if it affects how results are proved, not just how the foundations are laid.

The third approach with which Corry is concerned is that of category theory, and he can turn to it naturally by asking why Bourbaki never embraced an approach that might seem to fit so well with some of their attitudes. He notes that Mac Lane and Eilenberg did not start from an attempt to lay a general foundation for mathematics, but to address the issue of 'naturality' of a construction. He works from the published comments of Mac Lane in later years as well as from their original papers, although he did not have the chance to make use of Mac Lane's autobiography (which appeared in 2005). That autobiography confirms the extent to which categories were born out of a specific setting rather than a broad foundational programme.

It is clear that Corry finds the category theoretical approach to mathematics the best of the three in the second part of the book, to some extent because it arises more directly out of mathematical practice rather than foundational preconceptions. On the other hand, he also takes issue with some of the attempts following Lawvere to use category theory as the foundation for mathematics. In general, he argues, following Mac Lane, mathematics is an evolving discipline and one cannot expect a single foundational approach to succeed. This sort of perspective seems liable to the same kind of accusations that Corry himself launches at other views that emerge out of preconception rather than practice.

This summary omits many of the sidelights that the author throws on the history of algebra and abstraction. The fact that the book has been given a second edition (not heavily revised) is an expression of its substantial contributions to the history of algebra over the last century and a half. It is a little surprising that more of the typographical errors and stylistic infelicities were not cleared up in the production of the second edition. They do not, however, significantly reduce the value of Corry's book to historians, philosophers, and algebraists and logicians in search of their roots.

THOMAS DRUCKER

Department of Mathematical and Computer Sciences, University of Wisconsin-Whitewater, Whitewater, WI 53190 USA.. druckert@uww.edu.

PAUL C. GILMORE. ***Logicism renewed: logical foundations for mathematics and computer science.*** Lecture Notes in Logic, vol. 23. Association for Symbolic Logic / A K Peters, Ltd., Wellesley, Massachusetts, 2005, xvii + 230 pp.

In 1893, Frege published an inconsistent higher-order logical system. In the decades that followed, logicians such as Church (1932–33), Curry (1934), Quine (1940) and Martin-Löf (1971) inadvertently followed his example. Searching for a new, consistent higher-order logic is something like stepping into a minefield; unpleasant incidents are possible, and avoiding the missteps of previous adventurers is no guarantee that new missteps will not be made. Hence, most mathematicians and computer scientists wanting to make use of higher-order logic have recently tended to stick closely to the well-trodden paths around logics such as Church's consistent formulation of the simple theory of types (JSL V 56–68, 1940).

With the present book, Paul Gilmore blithely steps into a new area of the minefield, and emerges with several related novel, consistent, typed higher-order logics, the central one of which is named ITT (Impredicative Type Theory). As with other logics that Gilmore has described, ITT is built along nominalist lines. Nominalism, in this context, may be characterized as the view that the names of sets can sometimes be members of other sets, regardless of the type restrictions usually found in typed higher-order logic. A set is identified with a term with a type of the form $[\tau_1, \ldots, \tau_n]$; this type signifies a predicate with arguments

$\tau_1, \ldots, \tau_n$, and is the analogue of the type $\tau_1 \rightarrow \cdots \rightarrow \tau_n \rightarrow o$ of Church-style theories. Gilmore's key observation here is that if a term with a type of this form has no free variables of type other than 1 (the analogue of ι), then it is safe to consider it also to be its own name, and to consider it to have 1 as its *secondary* type.

This identification of a set with its name allows the names of sets to be smuggled into places where the set itself would not be allowed in conventional type theories. ITT thus embodies a natural impredicativity that is not dependent on explicit axioms. One consequence is that notions (such as intensional and extensional equality and ordered pairs) that must be supplied as primitive elements of other logics can be given definitions in ITT.

The purpose of the book, as announced by its title, is no less than the renewal of the grand old program of logicism. Russell characterized logicism as the thesis that mathematics should be definable with a "very small number" of "fundamental" logical concepts and principles. The question of what concepts and principles should be considered "fundamental" is of course a subject of debate. A logic is generally not considered to satisfy a logicist view if any of Peano's axioms have to be assumed *a priori*, the axiom of mathematical induction being the usual stumbling block.

The secondary type of 1 for some sets allows Gilmore to define a well-typed term that, when applied to a predicate defining a property of a set, yields the least set having that property. This term, applied to the predicate defining the property of being a zero-successor set, yields the term N, which we can treat as the set of natural numbers. Inductive properties of the natural numbers can then be proven as logical consequences of the definition of N. By page 111, Gilmore has shown that all of Peano's axioms can be proven in ITT.

This reviewer hesitates to speculate on the relationship between ITT and Quine's NF, since he is not an expert in the area. However, it should be noted that NFU requires the axiom of infinity in order to define the natural numbers, whereas ITT does not. Whether the axiom of infinity is in harmony with a logicist program is a matter of debate; however, if Gilmore's claims hold up, he may be able to argue that ITT carries out such a program more successfully than do NF and its relatives.

Chapters 1 and 2 of the book are prefatory material about more conventional logics, and will not contain any new insights for most readers of this review. Chapter 3 is the heart of the book, the description of the proof theory and semantics of ITT. Gilmore defines a model theory for the logic, adapted from Henkin's for type theory (JSL XV 81–91, 1950). He proves that models exist and are consistent (*i.e.*, that they do not assign both "true" and "false" to any formula), and defines a proof system that he shows to be sound and complete with respect to the models, thus showing consistency of the proof system. Chapter 4 demonstrates the most useful properties of the logic, including the least and greatest set operators and the characterization of N. Chapters 5 and 6 introduce choice terms (allied to Hilbert's ϵ terms) and adapt ITT to intuitionistic logic, respectively; they are less central to the main thrust of the book.

Merely capturing Peano's axioms is not the *summum bonum* of a logicist program. If a logical system is in some way not expressive enough, it may still fail to serve as a foundation for all of mathematics. Gilmore therefore presses on from this point. Chapter 7 elaborates on the use of ITT as a foundation for mathematics, taking direct aim (following Feferman) at the contention by Mac Lane that certain arguments natural in category theory cannot be formalized in higher-order logic. The points he makes here are persuasive. Finally, Chapter 8 shows applications to computer science, particularly in the areas of programming language semantics and domain theory, where the hegemony of Church has long held sway. The essentially untyped nature of digital data, and the identification of program with data in computer science, makes this a good fit. (Gilmore's more succinct quip is "Computers are consummate nominalists.")

The book is not always an easy read. The notation is sometimes idiosycratic, as when *cv* is used as a single metavariable standing for *either* a constant *or* a variable (despite the fact that c can stand for a constant, v can stand for a variable, and adjacency can represent application). Most of the proof systems and formal derivations are in the semantic tableau style, though Gilmore does supply sequent calculi in the Gentzen style for the relief of those who find them easier to work with. There are also a number of small errors that are inevitable in a work of this kind; they do not, however, detract from the overall points being made.

Gilmore is at his best when he is discussing the philosophical foundations and implications of the logic, and showing how broad a basis for mathematics and computer science his approach forms. His ideas provide a refreshingly different point of view on foundational issues, and deserve to be better known by all students of higher-order logic.

JAMES H. ANDREWS

Department of Computer Science, University of Western Ontario, London, N6A 5B7, Canada. andrews@csd.uwo.ca.

Reverse Mathematics 2001, edited by Stephen G. Simpson, Lecture Notes in Logic, vol. 21. ASL and AK Peters, Ltd., Wellesley, Massachusetts, x + 401 pp.—therein:

Reverse mathematics is a research program in mathematical logic, pioneered by Harvey Friedman, Stephen Simpson, John Steel, and others in the 1970s. The main question of this program is: *Which set existence axioms are needed to prove the theorems of "ordinary" mathematics*? But reverse mathematics asks for more: one is interested in axioms that are even equivalent to a given theorem, that is, the reverse implication, from the theorem to the axiom, should be provable as well. The first comprehensive account of reverse mathematics was given in Simpson's monograph ***Subsystems of Second Order Arithmetic*** (SOSOA), Perspectives in Mathematical Logic, Springer, 1999. The collected volume under review contains contributions of renowned logicians and mathematicians to the development of reverse mathematics after SOSOA. Many of the articles present new results within the classical framework of reverse mathematics, but there are also papers that pursue extensions to other areas like higher types or feasible mathematics.

Before describing the contents of the papers in detail, I briefly recall the subsystems of second order arithmetic that are studied most prominently in reverse mathematics. First, there are the systems RCA_0 (Δ^0_1-comprehension plus Σ^0_1-induction), ACA_0 (arithmetic comprehension plus second order induction axiom), and $\Pi^1_1{-}\mathsf{CA}_0$ (Π^1_1-comprehension plus second order induction axiom). Sandwiched between these systems are WKL_0, which is RCA_0 plus weak König's lemma (every infinite binary tree has an infinite path), and ATR_0, which is ACA_0 plus arithmetical transfinite recursion (a form of transfinite iteration of arithmetical comprehension). We have the proper inclusions $\mathsf{RCA}_0 \subset \mathsf{WKL}_0 \subset \mathsf{ACA}_0 \subset \mathsf{ATR}_0 \subset \Pi^1_1{-}\mathsf{CA}_0$. It is important to know that the first order part of ACA_0 is PA (Peano arithmetic), while the first order part of WKL_0 is just $\Sigma^0_1{-}\mathsf{PA}$ (PA with induction restricted to existential formulas). The latter result (due to Friedman) is an important contribution to Hilbert's program as it shows that the system WKL_0, which suffices to develop substantial parts of mathematics, is conservative over a fragment of Peano arithmetic. According to Simpson (SOSOA), the five systems RCA_0, WKL_0, ACA_0, ATR_0 and $\Pi^1_1{-}\mathsf{CA}_0$ correspond to the foundational programs of constructivism, finitistic reductionism, predicativism, predicative reductionism, and impredicativity, respectively.

In the following, I give an overview of the articles in the volume under review, highlighting some of the results that can be described without going too much into technical details. I arranged the articles into five subject groups: algebra and analysis, infinitary combinatorics, ordinals and comparability, recursion theory and models, feasible mathematics and higher types. Unless stated otherwise, all provability or equivalence results are to be understood relative to the base system RCA_0.

Algebra and analysis. Friedman [10] analyses the following statement about general algebras from the perspective of reverse mathematics: "Every countable algebra with an infinitely generated subalgebra has a maximal infinitely generated subalgebra" ('infinitely generated' is equivalent to 'not finitely generated'). He shows that this statement is equivalent to $\Pi^1_1{-}\mathsf{CA}_0$ for algebras with one binary function. The same holds for algebras with two unary functions, but for algebras with one unary function the statement is provable in the much weaker theory RCA_0. Downey and Solomon [7] show that Hahn's theorem, a statement about Archimedean subgroups of ordered abelian groups, is equivalent to ACA_0, and hence is not effective. Tanaka and Yamazaki [23] show that a substantial portion of field theory can be done in the weak base theory RCA_0, by proving in RCA_0 the fundamental theorem of algebra as well as quantifier elimination for the theory of real closed fields. Humphrey [15] shows that Cantor's theorem about sets of uniqueness of trigonometric series, which Cantor proved using transfinite ordinals, can in fact be proven in ACA_0, and hence without the use of (advanced) set theory. Brown [3] and Hirst [12] investigate the reverse mathematics of compactness. Brown shows, for example, that the statement "every totally bounded complete separable metric space is Heine-Borel compact" is equivalent to the system WKL_0, while Hirst answers affirmatively Friedman's question whether the equivalence between the Heine-Borel theorem and WKL_0 (proven by Friedman) still holds if the Heine-Borel theorem is restricted to the rationals. The reviewer remarks that the reverse mathematics of compactness is also being studied within the framework of Bishop's constructive mathematics, where the role of weak König's lemma is taken over by its contrapositive, the fan theorem (see, for example, Ishihara, *Constructive reverse mathematics: compactness properties*, ***From sets and types to topology and analysis***, (Laura Crosilla and Peter Schuster, editors), Oxford Logic Guides 48, Clarendon Press, Oxford, 2005, pp. 245–267).

Infinitary combinatorics. Cenzer and Remmel [4] establish equivalences of various versions of the stable marriage problem (the finite version is a classic problem in graph theory) to ACA_0, ATR_0, and WKL_0. They also prove the equivalence of WKL_0 to various combinatorial results on posets, for example Dilworth's theorem: any poset of width k can be covered by k chains. The paper of Marcone [19] gives a survey of the reverse mathematics of WQO (well quasi ordering) and BQO (better quasi ordering) theory and contains some new results about the proof theoretic strength of closure properties of well quasi orderings. WQO and BQO theory is of particular interest to reverse mathematics since it is a rich source of natural mathematical theorems that require strong set existence axioms for their proofs (for example, the graph minor theorem is not provable in $\Pi^1_1\!-\!\mathsf{CA}_0$, as shown by Friedman, Robertson and Seymour). Schmerl [20] proves theorems about (non) k-colorability of infinite graphs in the theory $\mathsf{RCA}_0\!+\!\neg\mathsf{WKL}$ (which has the recursive sets as a model). As a technical tool he uses the interesting game theoretic concept of on-line-coloring. The paper by Cholak, Giusto, Hirst, and Jokusch [5] investigates the reverse mathematics of statements related to Ramsey's theorem, namely the free set theorem and a weakening thereof, the thin set theorem. It is shown that the free set theorem has a similar strength to Ramsey's theorem. Some relationships between these theorems are proven and some questions are formulated as open problems.

Ordinals and comparability. Hirst [13] surveys the reverse mathematics of ordinal arithmetic. For example, ACA_0 is equivalent to arithmetical transfinite induction, and ATR_0 is equivalent to the comparability of well orderings. Friedman [11] strengthens some of these results and proves similar comparability results for countable metric spaces. Hirst [14] also shows that ATR_0 is equivalent to the so-called γ-lemma, stating that $\Sigma_{\alpha<\omega^\gamma} f(\alpha) = \sup\{f(\alpha)\cdot\omega^\gamma \mid \alpha<\omega^\gamma\}$ for every nondecreasing function f from ω^γ to the ordinals. If f is the identity, one obtains Sierpinski's exercise ($\Sigma_{\alpha<\omega^n}\alpha = \omega^{2n-1}$) which is already provable in RCA_0 (here, of course, α, γ range over countable well orderings and the equality sign has to be interpreted appropriately).

Recursion theory and models. Knight [16] and Arana [1] study the Turing degrees of m-diagrams of models of Peano arithmetic. Knight's main result is that to every nonstandard model of PA there exists an isomorphic model whose m-diagrams form a strict chain with respect to Turing reducibility. Arana characterizes the degrees of m-diagrams of models of true arithmetic and other models of PA. Chong, Shore, and Yang [6] translate formulas φ in the language of PA into formulas φ^* in the language of the partial order of recursively enumerable degrees such that in any model of Σ_4-induction φ holds iff φ^* holds in the recursively enumerable degrees of that model. Kossak [18] argues that arithmetic saturation relates to recursive saturation as ACA_0 relates to WKL_0. He proves the following result in reverse model theory: the statement that every consistent theory in a countable language has an arithmetically saturated model is equivalent to ACA_0^+, where ACA_0^+ extends ACA_0 by a scheme of ω-times iterated arithmetical comprehension. Schmerl [21] introduces the notion of a $\Delta_{1,n}$ set, that is, a set of natural numbers whose characteristic function is the limit of an n-trial-and-error-function. He uses $\Delta_{1,n}$ sets to give a new proof of Jokusch's result that PA has no completion in the set of boolean combinations of recursively enumerable sets. Interpreting the notion of a $\Delta_{1,n}$ set in the context of RCA_0, he also extends Simpson's classic result in reverse mathematics about the equivalence of WKL_0 to Lindenbaum's lemma stating that every countable consistent set has a completion. Schmerl shows that the equivalence still holds if Lindenbaum's lemma is formulated for completions in $\Delta_{1,n}$. Simpson's contribution [22] addresses a central foundational problem in analysis, namely the fact that important theorems of classical analysis fail in computable analysis (where only computable reals are considered). For example, the maximum principle for continuous real functions on the unit interval, which is equivalent to WKL_0, fails in computable analysis. Using the theory of Medvedev degrees and ideas from forcing, Simpson shows that there is a countable ω-model of WKL_0 such that every definable set that exists in the model is recursive. Hence recursive and classical analysis can be reconciled to a certain extent.

Feasible mathematics and higher types. The remaining five papers of this collection represent developments in reverse mathematics beyond the classical framework of subsystems of second order arithmetic between RCA_0 and $\Pi^1_1{-}\mathsf{CA}_0$. The papers by Fernandes, Ferreira, and Yamazaki are set within Ferreira's theory BTFA of basic feasible analysis whose provable recursive functions are all polynomial time computable. Fernando and Ferreira [9] show that the Heine-Borel theorem is equivalent to $\Pi^b_1{-}\mathsf{WKL}$, a version of weak König's lemma for trees defined by Π^b_1-formulas (formulas with one binary length bounded universal quantifier). They also show that the statement "every real function on the unit interval is uniformly continuous" lies between WKL and $\Pi^b_1{-}\mathsf{WKL}$. Fernando [8] uses a forcing technique to show that the Baire category theorem is conservative (over BTFA) for sentences of the form $\forall X\ \exists! Y\varphi(X, Y)$ with arithmetical φ. Yamazaki [24] proves that, for real functions on the unit interval, the intermediate value theorem is provable (in BTFA) and a version of the maximum principle is equivalent to Σ^b_1-comprehension. The articles by Avigad and Kohlenbach have in common that they both extend the universe of discourse from natural numbers and sets of natural numbers to functionals of finite types over the natural numbers (sets of natural numbers are represented by their characteristic functions). Higher type systems are attractive because they are at the same time expressive and proof theoretically weak (often conservative over their first order parts). Avigad [2] uses a forcing technique to obtain nonstandard extensions of weak systems of arithmetic in higher types, including polynomial-time computable arithmetic. He proves a number of conservation results and describes how to formalize weak systems of analysis in such systems. Kohlenbach [17] stresses the need for higher types to express e.g., statements about noncontinuous functions. He sketches a development of reverse mathematics in higher types where set existence principles are replaced by existence principles for higher type functionals. One of his results in this direction is, for example, the pairwise equivalence (over a suitable higher type version of RCA_0) of the uniform weak König's lemma, the uniform intermediate value theorem, and the existence of the type 2 functional $\exists^2$ representing quantification over numbers.

I found this book very inspiring and enjoyable. The articles give a broad account of the state of the art in reverse mathematics and highlight interesting new developments in current research on Hilbert's program. The book is therefore essential reading for logicians and philosophers working in the area of the foundations of mathematics. All articles are clearly written and well motivated. Most of them are also fairly self contained and provide short introductions into the formal frameworks they use. Therefore, this book is accessible and of interest not only to specialists, but also to "ordinary" mathematicians with a basic logic background.

ULRICH BERGER

Department of Computer Science, University of Wales Swansea, SA2 8PP, UK.
u.berger@swansea.ac.uk.

V. F. Hendricks. ***Mainstream and formal epistemology.*** Cambridge University Press, 2006, xii + 188 pp.

Vincent Hendricks' book is an interesting and original attempt to bring together different traditions in epistemology. The term "mainstream epistemology" is used by the author to refer to the philosophical tradition of conceptual analysis of knowledge, which the author characterizes as the seeking of "necessary and sufficient conditions for the possession of knowledge using largely common-sense considerations and folksy examples and counter-examples" (p. 4). Under the head of "formal epistemology", Hendricks subsumes first and foremost work by epistemic and modal logicians on knowledge representation, information processing and the mathematical modelling of uncertainty (including, although more briefly in the book, work in Bayesian epistemology).

Main argument. The book rests on two observations. The first is that analytic epistemologists and epistemic logicians seem largely to pursue different goals: while analytic epistemologists are interested mostly in the precise *definition* of the concept of knowledge, with a concern to distinguish it from near approximations such as true belief or justified true belief (E. Gettier, *Is justified true belief knowledge?*, ***Analysis***, vol. 23 (1963), no. 6), epistemic logicians focus rather on how belief and knowledge are *processed*, and usually set aside foundational issues concerning the precise definition of the concept of knowledge. Hintikka thus argues that "epistemic logic and its applications (…) are completely independent of the issues concerning the definition (precise criteria) of knowledge" (*Knowledge acknowledged: knowledge of propositions vs knowledge of objects*, ***Philosophy and phenomenological research***, vol. 61, 1996, p. 254—a quote not from Hendricks' book), by pointing out that the concepts of information and belief are more fundamental to epistemic logic than the concept of knowledge itself.

Despite this—and this is Hendricks' second observation—a common perspective may still be taken on both enterprises if one considers the way both analytic epistemologists and epistemic logicians assign truth conditions to knowledge. Since Hintikka's seminal work—discussed in Chapter 6 of the book, and undeniably a main bridge between the two traditions—knowledge and belief are characterized in semantic terms as *elimination of uncertainty*: a subject S knows that P if and only if P is true in all scenarios or possible worlds compatible with the subject's information. Equivalent definitions are at the heart of much of the philosophical literature dealing with the problem of skepticism, but with a different emphasis. For instance, D. Lewis, in his influential paper *Elusive knowledge* (***Australasian Journal of Philosophy*** vol. 74 (1996), no. 4)—discussed in Chapter 5 of the book—takes as his starting point the idea that S knows that P if "S's evidence eliminates every possibility in which not-P". The skeptical challenge to philosophers, however, concerns the grounds that allow a given subject to be sure that she has taken all possibilities of error into consideration, namely what epistemic alternatives should count as relevant or irrelevant for knowledge and attributions thereof.

As Hintikka emphasized, the problem of skepticism is usually not of interest to epistemic logicians, because "most of the questions concerning the logic of knowledge are independent of how the dichotomy between admitted and excluded alternatives concerning the world is set up. Only the dichotomy itself matters" (*ibid.*). Belief therefore is the primitive category, and knowledge is defined as true belief, leaving issues of external justification implicit. The account of epistemic logic given in Chapter 6 adequately underlines this internalist perspective prevailing on knowledge in epistemic logic. More precisely, Hendricks stresses the link of Hintikka's work to G. E. Moore and N. Malcolm's tradition of "autoepistemology", and puts the right emphasis on the fact that most of the philosophical issues connected to epistemic logic have to do with self-knowledge and rationality. Such is the case in static epistemic logic, with issues concerning introspection or closure of belief under logical consequence, but also in dynamic epistemic logic, where mechanisms of belief revision and informational updates do not depend on whether the information received is true, as much as on whether it is consistent with previous information.

One of the main originalities of the book, in this context, is the attempt made to relate the internalist perspective of epistemic logicians to the externalist concern of philosophers addressing the problem of skepticism. More precisely, Hendricks proposes to subsume under a common concept, which he calls *epistemological forcing*, the definitions and characterizations of knowledge proposed by different authors within the two traditions. The second main originality is the room given to the paradigm of *computational epistemology*, a branch of formal epistemology which Hendricks proposes to regiment in a modal logic framework.

Organization of the book. The book consists of nine chapters which can be grouped into four main parts. The first part (Chapters 1 and 2) delimits the two paradigms of mainstream and formal epistemology and introduces the notion of epistemological forcing. The idea is that knowledge can always be forced to hold over some space of relevant possibilities, and therefore be *consistently* attributed to an agent, even if the agent "commits many errors, even grave ones, in other but irrelevant possibilities" (Preface, p. ix). The notion is introduced by way of an analogy with Cohen's set-theoretical notion of forcing (forcing an axiom to hold in a generic extension of a model of a given theory, in order to show that the axiom is consistent with the theory), even though Hendricks denies any deeper substantial import from the analogy in a footnote, and presents it as a metaphor in Chapter 9. Indeed, from a model-theoretic point of view, Hendricks' idea of epistemological forcing generally corresponds to evaluating knowledge over a *restricted* subset of the set of all possibilities (as in the anti-skeptical strategies discussed in Chapters 3–5), rather than considering something like an extension. The analogy becomes more substantial in Chapters 7 and 8, however, where a notion of limiting convergence to knowledge is formally defined.

The second part of the book (Chapters 3–5) gives a survey and critical examination of the epistemological theories of three "mainstream" epistemologists, namely A. Goldman (Chap. 3), R. Nozick (Chap. 4) and D. Lewis (Chap. 5), and of their respective attempts to defeat skepticism. Chapter 3 discusses the successive versions of Goldman's epistemic reliabilism, in particular his idea that a belief-forming method is reliable if it produces a high ratio of true beliefs in "normal" worlds. Chapter 4 examines Nozick's characterization of knowledge in terms of true belief that is counterfactually sensitive to truth. Chapter 5 focusses on Lewis's version of contextualism in epistemology, in which knowledge depends on which possibilities or error can be properly ignored depending on the context.

The third part of the book, on formal epistemology (Chapters 6–8), can itself be divided into two blocks. Chapter 6 ("Logical Epistemology") is a self-contained introduction to modal epistemic logic and some of its recent developments and applications, including in game theory. The emphasis of the chapter is on canonical systems of propositional epistemic logic of the S4 and S5 type, and on discussions about the plausibility of the respective axioms. Hendricks also stresses the contrast between static and dynamic epistemic logics, namely combined logics of time and knowledge.

The temporal perspective on epistemic logic is the basis of Chapters 7 and 8, which together make a separate block and are generally more technical than the rest of the book. Chapter 7 introduces the paradigm of computational epistemology developed in particular by K. Kelly (***The logic of reliable inquiry***, Oxford UP, 1996), E. Martin and D. Osherson (***Elements of scientific inquiry***, MIT Press, 1998), and Hendricks himself in several papers and in an earlier book (***The convergence of scientific knowledge, a view from the limit***, Trends in logic, Studia Logica Library Series, Kluwer, 2001). With this chapter, Hendricks turns to yet another epistemological framework, in which the notions of inductive knowledge and knowledge acquisition become central. Chapter 7 gives some useful background on computational epistemology and its origins, in particular in the work of E. M. Gold on formal language learning and the identification of formal grammars on the basis of inductive evidence. The cornerstone is Gold's notion of *identification in the limit*, namely the existence of a finite time after which all guesses made on the basis of partial evidence about the identity of a language are the same and correct.

In Chapter 8 ("Modal Operator Epistemology"), Hendricks presents a possible worlds model of time and evidence developed with S. A. Pedersen, which he uses to define a semantics of propositional epistemic logic with temporal and alethic modalities, based on the same idea. The semantics is original in that it incorporates notions of evidence and heuristics, traditionally wanting in epistemic logic, via the concepts of *discovery method* (function from finite evidence sequences to propositions) and *assessment method* (fonctions from finite evidence sequences and propositions into truth values). By definition, a hypothesis h is *knowable in the limit* relative to a discovery method δ, given some sequence of evidence and time, if there is some time such that at all later times and for all further allowable inputs, δ entails h and h is true. The formula $K_\delta h$ is interpreted as "δ knows h in the limit". The rest of the chapter examines the validity of the axioms of S5, in particular axioms 4 and 5, when K_δ is thus interpreted, depending on the structural constraints imposed on δ. Further results are mentioned about *knowledge transmissibility*, namely the interaction of epistemic operators indexed by different methods.

Chapter 9, entitled " 'Plethoric' Epistemology", concludes on methodological considerations about mainstream and formal epistemology. To some extent, Hendricks converges with the criticisms found in the writings of naturalized epistemologists against the excessive use of thought experiments (a recent illustration, not from Hendricks' book, is M. Bishop and J. D. Trout's polemical *The pathologies of standard analytic epistemology*, ***Noûs***, vol. 39 (2005), no. 4, whose delineation of "Standard Analytic Epistemology" happens to overlap Hendricks' characterization of "Mainstream Epistemology"). Hendricks explicitly distances himself from naturalized epistemologists, however, by presenting formal methods as a way of controlling and giving structure to normative intuitions, better than simply dispensing with them (in which I side entirely with him). More generally, Hendricks puts forward a collaborative program of "plethoric" epistemology, in which conceptual analysis and formal methods are encouraged to regulate each other. The notion of "plethora" is quite sensibly opposed to the ecumenical fiction of a unity or complete merge between the two traditions.

Critical comments. Hendricks' book appears in a context of numerous and active publications in different areas of epistemology, and the first question the book raises concerns the adequacy of the opposition drawn between "mainstream" and "formal" epistemology. The distinction is not meant to be exclusive, as Hendricks points out (p. 159), and in some cases it is a matter of stylistic or discursive polarity. Indeed, prominent philosophers like D. Lewis, R. Stalnaker or T. Williamson all contributed both formal logical work *and* analytic argumentative pieces in epistemology. Nevertheless, the distinction made by Hendricks makes very good sense in my opinion, notably from a sociological point of view: the community of formal epistemologists generally include logicians, computer scientists and economists who are not professional philosophers and who ignore the concerns of philosophers; conversely, epistemologists in the analytic tradition are often unaware of advances in epistemic logic.

The book covers a lot of useful material in a concise, efficient and very clearly structured manner; it proposes an original synthesis between the two traditions, and as such it should be brought to the attention of both communities. The chapters are written with a palpable intention to reduce technicalities and make the formal results accessible to non-technically minded readers. The book could easily be used as an introduction to epistemology, especially Chapters 1 to 7. Chapter 8, the most technical, contains several diagrams which help to understand the formal definitions. All technical results are explained, but formal proofs are deferred to relevant papers and to Hendrick's earlier book: a technical appendix would have been useful to make the chapter entirely self-contained, however, especially for the section concerning the status of axioms 4 and 5 (pp. 144–145), which is quite dense.

One aspect the book does not cover, in particular in Chapter 6 on epistemic logic, concerns first-order modal logics of belief and issues of intensionality, which played a significant part in Hintikka's original treatment. While the focus on propositional logics is legitimate, a brief mention would have been welcome (for instance of *de re* vs *de dicto* beliefs, a distinction relevant to one of Gettier's original puzzles). The selection of "mainstream" epistemologists given in Chapters 3–5, although representative of three main trends in contemporary analytic epistemology, is not supposed to be exhaustive either, a point Hendricks should have emphasized and motivated more clearly at the start. Thus some readers will probably regret the lack of discussion of more recent theories and ongoing debates in "mainstream" epistemology, in particular on the topic of contextualism.

Chapter 3 on Goldman's epistemology is entitled "Mainstream Epistemology" and would better be named "Reliabilist Epistemology" to avoid any specific identification of "mainstream" with a specific theory. Two slight historical inaccuracies should be corrected in the next edition of the book: on p. 20, Russell is said to have anticipated Gettier on the difference between knowledge and true belief in the late 1940s; it is right that Russell anticipated Gettier, but he actually did so as early as 1912 in ***The problems of philosophy***, Chap. 13; on p. 3 and in Chapter 4, F. Dretske's paper *Epistemic operators* is referred to as the anticipation of Nozick's counterfactual conditions on knowledge, but Dretske's relevant paper is actually *Conclusive reasons*, published 1971. Conversely, more emphasis might have been placed on Dretske's *Epistemic operators* directly within Chapter 5 on contextualist epistemology, especially for the discussion of Lewis's views about epistemic closure.

The part of the book on computational epistemology is the most personal and stimulating in my opinion, in particular because this is where the concept of epistemological forcing gets its bite, both formally and conceptually. As is the tendency with metaphilosophical concepts, Hendricks' notion of epistemological forcing strikes me as too unifying rather than too narrow in the book (a criticism Hendricks lucidly entertains at the end, p. 165: "Perhaps the forcing metaphor is too narrow and too unifying at the same time"), although the idea of finding a common strategy behind all replies to skepticism remains original and forceful. I do not think, in particular, that Hendricks' logic will be the right logic to analyze and solve the original Gettier examples, which do not involve learning, as much as justification (even though the concept of limiting knowledge may be *immune* to structurally analogous puzzles, as Hendricks suggests, somewhat elliptically, on p. 140).

Despite this, the concept of *knowledge in the limit* introduced in Chapters 7 and 8 appears as the clearest illustration of Hendricks' forcing idea, and justifies his remark that computational epistemology "balances delicately between pessimistic skepticism and optimistic epistemology" (p. 128): on the one hand, discovery methods are always fallible (below the limit); on the other hand, categorical knowledge can always be ascribed to learning agents in principle (namely in the limit), even if the agents themselves are not necessarily in a position to know that they know (due to the threat of negative evidence). This distinction between first-person and third-person perspective, brought up in Chapter 2, is another useful clarification made in the book, notably for the discussion of logical omniscience given in Chapter 6.

One can foresee several interesting perspectives to the notion of knowability in the limit and the way it is regimented in Hendricks' system, in particular with regard to two other current areas of research that are at the cross-roads of formal and mainstream epistemology. The first concerns the issue of knowability in principle, as raised by the Fitch Paradox ("if all truths are knowable, then all truths are known"). The paradox is not discussed in the book, but it has been a topic of growing interest to both formal and mainstream epistemologists recently. It raises the problem of the combination of epistemic modalities with temporal and alethic modalities, and as such it should belong in an extension of what Hendricks calls "modal operator epistemology" (Hendricks' logic contains special modalities of *empirical* possibility and necessity, but no *metaphysical* modalities). Another area with which one suspects there might be interesting connections is the emerging paradigm of evidence-based logics of knowledge, namely epistemic logics with explicit justifications, as developed by S. Artemov and his school. In Hendricks' logic, epistemic agents are identified with discovery methods, which relate received evidence to propositions. Evidence segments come close to epistemic justifications therefore, even though they cannot be referred to in the object-language, and remain implicit in that sense. The logic may be enriched, however, to allow explicit reference to evidence.

I mention these further links not to detract from the main contribution of the book, but on the contrary, and in line with what Hendricks calls "mergers" between traditions, to emphasize the wealth of perspectives brought up by Vincent Hendricks' stimulating and valuable book.

Paul Égré

Institut Jean-Nicod, UMR 8129 du CNRS - 29, rue d'Ulm, 75005, Paris, France.
paulegre@magic.fr.