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ON THE EQUIMORPHISM TYPES OF LINEAR ORDERINGS

ANTONIO MONTALBÁN

§1. Introduction.

A linear ordering (also known as *total ordering*) *embeds* into another linear ordering if it is isomorphic to a subset of it. Two linear orderings are said to be *equimorphic* if they can be embedded in each other. This is an equivalence relation, and we call the equivalence classes *equimorphism types*. We analyze the structure of equimorphism types of linear orderings, which is partially ordered by the embeddability relation. Our analysis is mainly from the viewpoints of Computability Theory and Reverse Mathematics. But we also obtain results, as the definition of equimorphism invariants for linear orderings, which provide a better understanding of the shape of this structure in general.

This study of linear orderings started by analyzing the proof-theoretic strength of a theorem due to Jullien [Jul69]. As is often the case in Reverse Mathematics, to solve this problem it was necessary to develop a deeper understanding of the objects involved. This led to a variety of results on the structure of linear orderings and the embeddability relation on them. These results can be divided into three groups.

First are the results purely about the structure of the embeddability relation on linear orderings. We start by introducing the concept of *signed tree*. Signed trees are used to represent equimorphism types of countable linear orderings. The good thing about this representation is that, in some sense, it explicitly describes the structure of the equimorphism type. So, it makes it easier, say from a computability theoretic viewpoint, to work with equimorphism types. This explicit description of the structure of the linear orderings was very important in the analysis of the proof-theoretic strength of Jullien's theorem. Signed trees also provide a different intuition about the structure of countable linear orderings.

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Then we define a way of assigning a finite object $\mathtt{Inv}(\mathcal{L})$ to each scattered linear ordering $\mathcal{L}$. (A linear ordering is *scattered* if η, the order type of the rational numbers, does not embed in it.) This assignment is an equimorphism invariant, in the sense that two linear orderings are equimorphic if and only if they are assigned the same object. This time, we work with the whole class of scattered linear orderings, and not only the countable ones. Of course, these objects are not hereditarily finite sets, as this would be impossible since there are only countably many hereditarily finite sets and there are already $\aleph_1$ many countable scattered equimorphism types. But they are finite objects with ordinal labels. More specifically, they are finite sequences of finite trees with labels in $\mathcal{ON} \times \{+,-\}$, where $\mathcal{ON}$ is the class of ordinals. These invariants are useful in studying the partial ordering of the equimorphism types. We expand on all this in Section 3.

While trying to prove Theorem 1.1 below, we had to find the minimal linear orderings of a certain Hausdorff rank. Since this result is interesting by itself, we present it in Section 4.

Second comes the analysis from the viewpoint of Computable Mathematics. This is presented in Section 5. (For background on computable sets we refer the reader to the introductory chapters of [Soa87].) From the study of the equimorphism invariants we show that if α is a computable ordinal, then the set of invariants for linear orderings of Hausdorff rank less than α is computable, and so is the embeddability relation. We then use this result to construct a computable inverse for the function $\mathtt{Inv}(\cdot)$. That is, we define a computable function $\mathrm{lin}(\cdot)$ that given an invariant for a linear ordering $\mathcal{L}$ of Hausdorff rank less than α, returns a linear ordering equimorphic to $\mathcal{L}$. So, we get that every scattered linear ordering, whose Hausdorff rank is a computable ordinal, is equimorphic to a computable linear ordering. As a corollary we get the following theorem, which is an extension of an old classical result of Spector [Spe55] that says that every hyperarithmetic well-ordering is isomorphic to a computable one.

THEOREM 1.1. *Every hyperarithmetic linear ordering is equimorphic to a computable one.*

The original proof of this result [Mon05b] used signed trees and signed forests. The proof we sketch here uses the finite invariants instead, but it is essentially the same proof.

Third are the results in Reverse mathematics. We prove that many statements about the embeddability relation of linear orderings are equivalent to each other. The most well-known of these statements being Fraïssé's conjecture, also known as Laver's theorem. A conclusion one could draw from our results is that the weakest system of second order arithmetic where one can develop a reasonable theory of equimorphism types of linear orderings is RCA_0 together with Fraïssé's conjecture. It could still be the case that this system is equivalent to ATR_0, but this is not known. One could conclude from our results that Fraïssé's conjecture is a *robust* system, in the sense that many other statement are equivalent to it. So far, only the "big five" systems, RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0, are known to be robust.

One statement we show equivalent to Fraïssé's conjecture is Jullien's classification of the extendible linear orderings. Then, there is one statement about the well-quasi-orderness of the signed trees and one about the decomposition of scattered linear orderings as finite sums of indecomposables. In unpublished work, the author has showed that other variations of these statements are also equivalent to Fraïssé's conjecture, as for example the better-quasi-orderness of the linear orderings and the fact that every indecomposable equimorphism type can be represented by a signed tree.

In the last section we study another statement about equimorphism types of linear orderings. We should mention that the results we obtain for this statement do not verify the claim we made about the robustness of Fraïssé's conjecture. We show that this statement, that we call INDEC, belongs to a class of statements that has been studied in the seventies. This is the class of *statements of hyperarithmetic analysis*, and these are the statements S such that, for every $Y \subseteq \omega$, the least ω-model of $\mathsf{RCA}_0 + \mathsf{S}$ containing the set Y is the class of sets hyperarithmetic in Y. Many systems of hyperarithmetic analysis were known before, but INDEC is the first natural example of a statement in mathematics with this strength.

Notation. We write $\mathcal{L}_1 \preccurlyeq \mathcal{L}_2$ if $\mathcal{L}_1$ embeds in $\mathcal{L}_2$, and $\mathcal{L}_1 \sim \mathcal{L}_2$ when $\mathcal{L}_1$ and $\mathcal{L}_2$ are equimorphic. Some examples of linear orderings are: $\mathbf{1}$, the linear ordering with one element; $\mathbf{m}$, the linear ordering with m many elements; ω, the order type of the natural numbers; ζ, the order type of the integers; η, the order type of the rationals; and ω_1^{CK}, the first non-computable ordinal.

We have some operations on the class of linear orderings. The *reverse* linear ordering of $\mathcal{L} = \langle L, \leqslant_L \rangle$ is $\mathcal{L}^* = \langle L, \geqslant_L \rangle$. We also let $\mathcal{L}^+ = \mathcal{L}$ and $\mathcal{L}^- = \mathcal{L}^*$. The *product*, $\mathcal{A} \cdot \mathcal{B}$, of two linear orderings $\mathcal{A}$ and $\mathcal{B}$ is obtained by substituting a copy of $\mathcal{A}$ for each element of $\mathcal{B}$. The *sum*, $\sum_{i \in \mathcal{A}} \mathcal{A}_i$, of a set of linear orderings $\{\mathcal{A}_i\}_{i \in A}$ indexed by another linear ordering $\mathcal{A}$, is constructed by substituting a copy of $\mathcal{A}_i$ for each element $i \in A$. So, for example, $\mathcal{A} \cdot \mathcal{B} = \sum_{i \in \mathcal{B}} \mathcal{A}$. When $\mathcal{A} = \mathbf{m}$, we sometimes write $\mathcal{A}_0 + \cdots + \mathcal{A}_{m-1}$ instead of $\sum_{i \in \mathbf{m}} \mathcal{P}_i$.

Basic ordinal arithmetic will be assumed. Any basic text in set theory would include background on ordinal operations. See, for instance, [Kun80].

§2. The structure of equimorphism types. Throughout this paper, except for Subsection 3.2, we will be mostly interested in countable linear orderings. We will use structural results that are proved for arbitrary cardinality, but only for the class of scattered linear orderings. In the countable case, there is only one equimorphism type which is not scattered, namely η. (Because every countable linear ordering embeds into η.)

In this section we describe previously known results about the structure of equimorphism types of scattered linear orderings. People have been interested in the class of scattered linear ordering for a long time. One of the earliest results is the following, first proved by Hausdorff [Hau08], and rediscovered by Erdös and Hajnal [EH63].

THEOREM 2.1 (Hausdorff). *Let $\mathbb{S}$ be the smallest class of linear orderings such that*

- $\mathbf{1} \in \mathbb{S}$;
- *if* $\mathcal{A}, \mathcal{B} \in \mathbb{S}$, *then* $\mathcal{A} + \mathcal{B} \in \mathbb{S}$; *and*
- *if* κ *is a regular cardinal and* $\{\mathcal{A}_\gamma : \gamma \in \kappa\} \subseteq \mathbb{S}$, *then both* $\sum_{\gamma \in \kappa} \mathcal{A}_i$ *and* $\sum_{\gamma \in \kappa^*} \mathcal{A}_i$ *belong to* $\mathbb{S}$.

Then $\mathbb{S}$ *is the class of scattered linear orderings.*

Another important contribution of Hausdorff to the study of scattered linear orderings is the definition of the Hausdorff rank (see [Ros82, Chapter 5]). He first defined an operation on linear orderings which is similar to the Cantor-Bendixon derivative on topological spaces: Given a linear ordering $\mathcal{L}$, let $\mathcal{L}'$ be the linear ordering obtained by collapsing the elements which have only finitely many elements between them. Informally, the Hausdorff rank of $\mathcal{L}$ is the least ordinal α such that the αth iterate of this operation on $\mathcal{L}$ is finite. Here is the definition we will use.

DEFINITION 2.2. Given a linear ordering $\mathcal{L}$ and an ordinal α, we define an equivalence relation $\approx_\alpha$ on $\mathcal{L}$ by transfinite induction as follows. Let $\approx_0$ be the identity relation. For $x, y \in \mathcal{L}$, let $x \approx_\alpha y$ if and only if for some $\beta < \alpha$, there are only finitely many $\approx_\beta$-equivalence classes between x and y. Let $\mathcal{L}^{(\alpha)}$ be the linear ordering which consists of the $\approx_\alpha$-equivalence classes ordered in the obvious way. We let the *Hausdorff rank* of $\mathcal{L}$, $\mathrm{rk}_H(\mathcal{L})$, be the least ordinal α such that $\mathcal{L}^{(\alpha)}$ is finite. If no such an α exists, we let $\mathrm{rk}_H(\mathcal{L}) = \infty$.

Hausdorff proved that a linear ordering is scattered if and only if $\mathrm{rk}_H(\mathcal{L}) \neq \infty$.

The definition above is slightly different from some other definitions of Hausdorff rank found in the literature, but is essentially the same. We prefer it to other definitions because it satisfies the following three properties. Let $\mathcal{A}$ and $\mathcal{B}$ be linear orderings, then: if $\mathcal{A} \preccurlyeq \mathcal{B}$, $\mathrm{rk}_H(\mathcal{A}) \leqslant \mathrm{rk}_H(\mathcal{B})$; $\mathrm{rk}_H(\mathcal{A}+\mathcal{B}) = \max(\mathrm{rk}_H(\mathcal{A}), \mathrm{rk}_H(\mathcal{B}))$; and $\mathrm{rk}_H(\mathcal{A}\cdot\mathcal{B}) = \mathrm{rk}_H(\mathcal{A}) + \mathrm{rk}_H(\mathcal{B})$.

After Hausdorff's results, the following important structural result about the class of scattered linear orderings was conjectured by Fraïssé in [Fra48]. It was proved by Richard Laver twenty three years later.

DEFINITION 2.3. A binary relation is a *quasi-ordering* if it is reflexive and transitive. A *well-quasi-ordering* is a quasi-ordering which has no infinite descending sequences and no infinite antichains.

THEOREM 2.4. [Lav71] *The class of scattered linear orderings is well-quasi-ordered by the relation of embeddablity.*

Moreover, Laver proved that the class of scattered linear orderings is a better-quasi-ordering. Better-quasi-orderings are a particular case of well-quasi-orderings with better closure properties, introduced by Nash-Williams in [NW68]. Then, for example, using Nash-Williams' theorem on transfinite sequences [NW68], we get that the class of ideals of scattered linear orderings (i.e., downwards closed sets of linear orderings), ordered by the inclusion relation, is well-quasi-ordered too.

In Laver's proof, indecomposable linear orderings play a very important role.

DEFINITION 2.5. A linear ordering $\mathcal{L}$ is *indecomposable* if whenever $\mathcal{L} \preccurlyeq \mathcal{A} + \mathcal{B}$, either $\mathcal{L} \preccurlyeq \mathcal{A}$ or $\mathcal{L} \preccurlyeq \mathcal{B}$.

Remark 2.6. An ordinal is indecomposable if and only if it is of the form ω^δ. By the Cantor normal form, every ordinal can be written as a finite sum of indecomposable ones.

Along with the theorem above, Laver proved some structural results about the class of σ-scattered linear orderings (see Definition 8.1). When we restrict these results to the class of scattered linear orderings we obtain the following theorem.

THEOREM 2.7. [Lav71]

(1) *Every scattered linear ordering can be written as a finite sum of indecomposable linear orderings.*
(2) *Every indecomposable linear ordering is either a κ-sum or a κ^*-sum of indecomposable linear orderings of smaller Hausdorff rank, where κ is some regular cardinal.*
(3) *If $\mathcal{L}$ is a scattered linear ordering of cardinality $\kappa \geqslant \omega$, then the set of equimorphism types which are embeddable in $\mathcal{L}$ has size κ.*

Everything mentioned so far about scattered linear orderings is not really about isomorphism types of linear orderings, but actually about equimorphism types. The properties of being scattered, being indecomposable, and having a certain Hausdorff rank are preserved under equimorphisms. Also, the operation of taking finite sums, products and κ-sums are well-defined on equimorphism types.

§**3. Equimorphism invariants.** In this section we define two equimorphism invariants for linear orderings. That is, we describe a way of assigning a certain object, the invariant, to each linear ordering in a way that the equimorphism structure is respected. These invariants are in some sense simpler than the linear orderings, and therefore they provide a better understanding of the structure of equimorphism types of linear orderings.

First we define signed trees. Signed trees were introduced in [Mon06] to represent indecomposable linear orderings up to equimorphism. They have a similar flavor with the trees $T(\Psi)$ used by Laver [Lav71, page 104]. Actually, the equimorphism invariants are not the signed trees themselves, but the equimorphism types of signed trees. From a computability theoretical viewpoint, these are easier to represent than linear ordering. Also, the structure of the linear ordering is explicitly described in the signed tree representation, and the embeddability relation among signed trees is easier to visualize. Since signed trees were developed for computability theoretic applications, they have only been defined in the countable case. An extension of this notion to arbitrary cardinality is not immediate, and we do not know how to do it in a natural way, although it may be possible.

Second we define finite equimorphism invariants for the whole class of scattered linear orderings.

3.1. Signed trees.

Definition 3.1. [Mon06] A *signed tree* is pair $\langle T, s_T\rangle$, where T is a *well-founded subtree* of $\omega^{<\omega}$ (i.e., a downwards closed subset of $\omega^{<\omega}$ with no infinite paths) and s_T is a map, called a *sign function*, from T to $\{+,-\}$. We will usually write T instead of $\langle T, s_T\rangle$. A *homomorphism* from a signed tree T to another signed tree $\check{T}$ is a map $f: T \to \check{T}$ such that

- for all $\sigma \subset \tau \in T$ we have that $f(\sigma) \subset f(\tau)$ and
- for all $\sigma \in T$, $s_{\check{T}}(f(\sigma)) = s_T(\sigma)$.

(Here $\subset$ is the strict inclusion of strings.) Given signed trees T and $\check{T}$ we let $T \preccurlyeq \check{T}$ if there exists a homomorphism $f: T \to \check{T}$. We say that T and $\check{T}$ are *equimorphic*, and write $T \sim \check{T}$, if $T \preccurlyeq \check{T}$ and $\check{T} \preccurlyeq T$.

Remark 3.2. For $f: T \to \check{T}$ to be a homomorphism, we do not require that incomparable strings are mapped to incomparable strings, and also f does not need to be one-to-one.

Notation 3.3. For every $n \in \omega$ with $\langle n\rangle \in T$, we let $T_n = \{\tau : n^\frown\tau \in T\}$.

We associate to each signed tree T, a linear ordering $\mathrm{lin}(T)$.

Definition 3.4. The definition of $\mathrm{lin}(T)$ is by transfinite induction. If $T = \{\emptyset\}$, we let $\mathrm{lin}(T) = \omega$ or $\mathrm{lin}(T) = \omega^*$ depending on whether $s_T(\emptyset) = +$ or $s_T(\emptyset) = -$. Now suppose $T \supsetneq \{\emptyset\}$. If $s_T(\emptyset) = +$, we want $\mathrm{lin}(T)$ to be an ω sum of copies of $\mathrm{lin}(T_0)$, $\mathrm{lin}(T_1)$, $\dots$, where each $\mathrm{lin}(T_i)$ appears infinitely often in the sum. So, we let

$$\mathrm{lin}(T) = \mathrm{lin}(T_0) + (\mathrm{lin}(T_0) + \mathrm{lin}(T_1)) + (\mathrm{lin}(T_0) + \mathrm{lin}(T_1) + \mathrm{lin}(T_2)) + \cdots$$

If $s_T(\emptyset) = -$, we let

$$\mathrm{lin}(T) = \cdots + (\mathrm{lin}(T_2) + \mathrm{lin}(T_1) + \mathrm{lin}(T_0)) + (\mathrm{lin}(T_1) + \mathrm{lin}(T_0)) + \mathrm{lin}(T_0).$$

We say that a linear ordering, $\mathcal{L}$, is *h-indecomposable* if it is of the form $\operatorname{lin}(T)$ for some signed tree T.

Example 3.5. Here we show how the function $\operatorname{lin}(\cdot)$ behaves on small signed trees. We represent the signed trees with a picture, where the root is on top and on every node we put a $+$ or $-$ depending on the value of s_T on it.

$$\operatorname{lin}(+) = \omega; \quad \operatorname{lin}\begin{pmatrix} - \\ \mid \\ - \\ \mid \\ - \end{pmatrix} = \cdots + (\cdots + \omega^* + \omega^*) + (\cdots + \omega^* + \omega^*);$$

$$\operatorname{lin}\begin{pmatrix} + \\ \mid \\ - \end{pmatrix} = \omega^* + \omega^* + \omega^* + \cdots; \quad \operatorname{lin}\begin{pmatrix} & + & \\ \swarrow & & \searrow \\ - & & + \end{pmatrix} \sim \omega + \omega^* + \omega + \omega^* \cdots$$

It follows from the following lemma that the structure of equimorphism types of countable indecomposable linear orderings is fully represented by the signed trees. Recall that any linear ordering can be written as a finite sum of indecomposable ones, so to understand the structure of linear orderings it suffices to understand the structure of signed trees.

LEMMA 3.6. [Mon06]

(1) *Every indecomposable linear ordering is equimorphic either to* **1** *or to an h-indecomposable linear ordering.*

(2) *Given signed trees T and $\check{T}$, $T \preccurlyeq \check{T}$ if and only if* $\operatorname{lin}(T) \preccurlyeq \operatorname{lin}(\check{T})$, *and hence $T \sim \check{T}$ if and only if* $\operatorname{lin}(T) \sim \operatorname{lin}(\check{T})$.

The ranks of T and of $\operatorname{lin}(T)$ are very closely related too. We define $\operatorname{rk}(T) = \sup\{\operatorname{rk}(T_i) + 1 \colon \langle i \rangle \in T\}$.

LEMMA 3.7. [Monb] *Let T be a signed tree. If T has finite rank, then* $\operatorname{rk}(T) + 1 = \operatorname{rk}_H(\operatorname{lin}(T))$. *If T has infinite rank, then* $\operatorname{rk}(T) = \operatorname{rk}_H(\operatorname{lin}(T))$.

Another advantage of representing linear orderings by signed trees is that proofs and constructions which are inductive on the Hausdroff rank of the linear ordering became simpler. See, for instance, the proof in ATR_* that every indecomposable linear ordering is extendible [Mon06, Theorem 6.1].

3.2. Finite invariants. Now we go back to the uncountable case. We use Laver's work and assign to each scattered linear ordering $\mathcal{L}$ a finite sequence $\mathtt{Inv}(\mathcal{L})$ of finite trees labeled by ordinals and signs in $\{+, -\}$. This assignment is an equimorphism invariant, that is, given scattered linear orderings $\mathcal{A}$ and $\mathcal{B}$, we have that

$$\mathcal{A} \sim \mathcal{B} \iff \mathtt{Inv}(\mathcal{A}) = \mathtt{Inv}(\mathcal{B}).$$

Let $\mathbb{S}$ denote the class of equimorphism types of scattered linear orderings and $\mathbb{H}$ the class of equimorphism types of scattered indecomposable linear orderings. From now on, indecomposable means scattered and indecomposable linear ordering, unless otherwise stated. Let

$$\mathbb{H}_\alpha = \{\mathcal{L} \in \mathbb{H} \colon \operatorname{rk}_H(\mathcal{L}) < \alpha\}.$$

Jullien [Jul69, Theorem IV.6.2] proved the following. Let $\mathcal{L}$ be a scattered linear ordering and let $\langle \mathcal{A}_0, \dots, \mathcal{A}_{n-1} \rangle$ be a sequence of indecomposables such that $\mathcal{L} = \mathcal{A}_0 + \cdots + \mathcal{A}_{n-1}$ and n is minimum possible. Then the tuple $\langle \mathcal{A}_0, \dots, \mathcal{A}_{n-1} \rangle$ is unique up to equimorphism (see also [Mon06, Subsection 3.2]). The tuple $\langle \mathcal{A}_0, \dots, \mathcal{A}_{n-1} \rangle$ is called a *minimal decomposition* of $\mathcal{L}$. So, to define $\mathtt{Inv}(\mathcal{L})$, it is enough to define invariants for the class of indecomposable linear orderings. We will assign a finite tree $\mathtt{T}(\mathcal{A}_i)$ to each indecomposable linear ordering and then take

$$\mathtt{Inv}(\mathcal{L}) = \langle \mathtt{T}(\mathcal{A}_0), \dots, \mathtt{T}(\mathcal{A}_{n-1}) \rangle.$$

A linear ordering is *indecomposable to the left* (*right*) if, whenever $\mathcal{A}$ and $\mathcal{B}$ are linear orderings such that $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we have that $\mathcal{L}$ is equimorphic to $\mathcal{A}$ (to $\mathcal{B}$). Another result of Jullien [Jul69, Theorem IV.3.3] is that every indecomposable linear ordering is either indecomposable to the right or to the left. (See also [Fra00, 6.3.4(3)] and [Ros82, Lemma 10.3], and see Section 7 below for a reverse mathematics analysis of this statement.)

Definition 3.8. Let $\epsilon_{\mathcal{L}}$ be $+$ if $\mathcal{L}$ is indecomposable to the right, and let $\epsilon_{\mathcal{L}}$ be $-$ if it is indecomposable to the left. Given $\mathcal{L} \in \mathbb{H}$, let $\mathbb{I}_{\mathcal{L}} = \{\mathcal{A} \in \mathbb{H} \colon \mathbf{1} + \mathcal{A} + \mathbf{1} \prec \mathcal{L}\}$.

Note that $\mathbb{I}_{\mathcal{L}} \subseteq \mathbb{H}_{\mathrm{rk}_H(\mathcal{L})}$ and that $\mathbb{I}_{\mathcal{L}}$ is closed downwards. Subsets of $\mathbb{H}$ which are closed downwards are called *ideals of* $\mathbb{H}$.

Lemma 3.9. [Monb] *If $\mathcal{A}, \mathcal{B} \in \mathbb{H}$ are such that $\epsilon_{\mathcal{A}} = \epsilon_{\mathcal{B}}$ and $\mathbb{I}_{\mathcal{A}} = \mathbb{I}_{\mathcal{B}}$, then $\mathcal{A} = \mathcal{B}$.*

The proof of this lemma is not too complicated in the countable case, using signed trees. It requires a little more work in the general case. We use it to define $\mathtt{T}(\mathcal{L})$, the invariant of $\mathcal{L}$.

Definition 3.10. We assign a finite tree, $\mathtt{T}(\mathcal{L})$, with labels in $\mathcal{ON} \times \{+, -\}$, to each $\mathcal{L} \in \mathbb{H}$. Let $\{\mathcal{L}_1, \dots, \mathcal{L}_k\}$ be the set of minimal elements of $\mathbb{H}_{\mathrm{rk}_H(\mathcal{L})} \smallsetminus \mathbb{I}_{\mathcal{L}}$. Define

$$\mathtt{T}(\mathcal{L}) \quad = \quad \begin{array}{ccccc} & & \langle \mathrm{rk}_H(\mathcal{L}), \epsilon_{\mathcal{L}} \rangle & & \\ \mathtt{T}(\mathcal{L}_1) & \cdots & \cdots & \cdots & \mathtt{T}(\mathcal{L}_k) \end{array}$$

That is, $\mathcal{T}(\mathcal{L})$ is a tree with a root labeled $\langle \mathrm{rk}_H(\mathcal{L}), \epsilon_{\mathcal{L}} \rangle$ and with k branches $\mathtt{T}(\mathcal{L}_1), \dots, \mathtt{T}(\mathcal{L}_k)$.

The set of minimal elements of $\mathbb{H}_{\mathrm{rk}_H(\mathcal{L})} \smallsetminus \mathbb{I}_{\mathcal{L}}$ is finite because there are no infinite antichains in $\mathbb{H}$, since $\mathbb{H}$ is well-quasi-ordered. Moreover, it determines $\mathbb{I}_{\mathcal{L}}$ because, since $\mathbb{H}$ is well-founded, for $\mathcal{A} \in \mathbb{H}_{\mathrm{rk}_H(\mathcal{L})}$, $\mathcal{A} \in \mathbb{I}_{\mathcal{L}}$ if and only if for no $i \leqslant k$, $\mathcal{L}_i \preccurlyeq \mathcal{A}$. This representation of ideals of $\mathbb{H}_\alpha$ by finite antichains in $\mathbb{H}_\alpha$ is one of the key properties that is being exploited.

The rest of [Monb] is dedicated to proving that these invariants are somewhat constructive. We do it by showing that the definition of the embeddablity relation on the invariants is relatively simple, and that we can easily characterize the finite trees that correspond to invariants. We also compute the invariants of every linear ordering which is a product of linear orderings of the form ω^α or $(\omega^\alpha)^*$.

3.2.1. *Ordering of invariants.* Let $\mathcal{Tr}$ be the class $\{\mathtt{T}(\mathcal{L})\colon \mathcal{L} \in \mathbb{H}\}$ and let $\mathcal{In} = \{\mathtt{Inv}(\mathcal{L})\colon \mathcal{L} \in \mathbb{S}\}$.

Now, we define a relation $\preccurlyeq$ on $\mathcal{Tr}$ that such that $\mathtt{T}$ is an isomorphism

$$T\colon \langle \mathcal{Tr}, \preccurlyeq\rangle \to \langle \mathbb{H}, \preccurlyeq\rangle.$$

We then define a relation $\preccurlyeq$ on $\mathcal{In}$ such that $\mathtt{Inv}\colon \langle \mathbb{S}, \preccurlyeq\rangle \to \langle \mathcal{In}, \preccurlyeq\rangle$ is an isomorphism. We define $\preccurlyeq$ in a way such that, given $S, T \in \mathcal{In}$, we can tell whether $S \preccurlyeq T$ via a finite manipulation of symbols, assuming we can compare the ordinals that appear in the labels of S and T and their cofinalities. To give the reader a flavor of how this definitons work, we include the definition of $\preccurlyeq$ in $\mathcal{Tr}$. See [Monb] for more information.

Let $\mathcal{A}, \mathcal{B} \in \mathbb{H}$, and let $\mathtt{T}(\mathcal{A}) = S = [\langle \alpha, \epsilon_S\rangle; S_0, \dots, S_{l-1}]$ and $\mathtt{T}(\mathcal{B}) = T = [\langle \beta, \epsilon_T\rangle; T_0, \dots, T_{k-1}]$. We are using $[\langle \alpha, \epsilon_S\rangle; S_0, \dots, S_{l-1}]$ to denote the tree with a root labeled $\langle \alpha, \epsilon_S\rangle$ and l branches $S_0, \dots, S_{l-1}$ coming out of the root.

The key observation is that $\mathcal{A} \preccurlyeq \mathcal{B}$ if and only if

- either $\tau(\mathcal{A}) \preccurlyeq \tau(\mathcal{B})$ and $\mathbb{I}_\mathcal{A} \subseteq \mathbb{I}_\mathcal{B}$,
- or $\tau(\mathcal{A}) \not\preccurlyeq \tau(\mathcal{B})$ and $\mathcal{A} \in \mathbb{I}_\mathcal{B}$.

where $\tau(\mathcal{A}) = \mathrm{cf}(\alpha)^{\epsilon_S}$ and $\tau(\mathcal{B}) = \mathrm{cf}(\beta)^{\epsilon_T}$. (This is proved in [Monb].) Then, we need the following observation. Let $\{\mathcal{A}_0, \dots, \mathcal{A}_{l-1}\}$ be the set of minimal elements of $\mathbb{H}_\alpha \smallsetminus \mathbb{I}_\mathcal{A}$, and let $\{\mathcal{B}_0, \dots, \mathcal{B}_{k-1}\}$ be the set of minimal elements of $\mathbb{H}_\beta \smallsetminus \mathbb{I}_\mathcal{B}$. Then, $\mathbb{I}_\mathcal{A} \subseteq \mathbb{I}_\mathcal{B}$ if and only if $\alpha \leqslant \beta$ and for each $i < k$, either $\mathcal{B}_i \notin \mathbb{H}_\alpha$ or there exists $j < l$ such that $\mathcal{A}_j \preccurlyeq \mathcal{B}_i$. Also, $\mathcal{A} \in \mathbb{I}_\mathcal{B}$ if and only if $\alpha < \beta$ and for each $i < k$, $\mathcal{B}_i \not\preccurlyeq \mathcal{A}$. So, we get the following definition.

Definition 3.11. Given

$$S = [\langle \alpha, \epsilon_S\rangle; S_0, \dots, S_{l-1}]$$

and

$$T = [\langle \beta, \epsilon_T\rangle; T_0, \dots, T_{k-1}] \in \mathcal{Tr}$$

we let $S \preccurlyeq T$ if,

- either $\alpha \leqslant \beta$, $\mathrm{cf}(\alpha) \leqslant \mathrm{cf}(\beta)$, $\epsilon_S = \epsilon_T$ and $\forall i < k$ $(\mathrm{rk}(T_i) \geqslant \alpha \vee \exists j < l(S_j \preccurlyeq T_i))$,
- or $\alpha < \beta$, $(\mathrm{cf}(\alpha) > \mathrm{cf}(\beta) \vee \epsilon_S \neq \epsilon_T)$ and $\forall i < k$ $(T_i \not\preccurlyeq S)$.

(Given a $T \in \mathcal{Tr}$, we use $\mathrm{rk}(\mathcal{T})$ to denote the ordinal that is labeling the root of T. So, if T is as above, $\mathrm{rk}(T) = \beta$.)

3.2.2. *The class of invariants.* Now we are interested in characterizing the finite sequences of finite trees with labels in $\mathcal{ON} \times \{+,-\}$ which belong to $\mathcal{In}$. This characterization is based on Proposition 3.12 where we characterize the finite trees with labels in $\mathcal{ON} \times \{+,-\}$ which belong to $\mathcal{Tr}$. All the conditions in these characterizations but one can be checked using a finite algorithm, namely 3.12.(4), which requires the computation of the cofinality of an ideal. This condition always holds when we are dealing with countable linear orderings. So, we do have a characterization of the elements of $\mathcal{In}_{\omega_1} = \{\mathtt{Inv}(\mathcal{L})\colon \mathcal{L} \in \mathbb{S} \text{ and } \mathrm{rk}_H(\mathcal{L}) < \omega_1\}$ via a finite algorithm. This will be very useful in the next section.

Given an ordinal α and $T_0, \dots, T_{k-1} \in \mathcal{Tr}$, let $\mathcal{I}^{\alpha}_{T_0,\dots,T_{k-1}} = \{S \in \mathcal{Tr}\colon \mathrm{rk}(S) < \alpha \text{ and } \forall i < k(T_i \not\preccurlyeq S)\}$. In other words, $\mathcal{I}^{\alpha}_{T_0,\dots,T_{k-1}}$ is the ideal of $\mathcal{Tr}_\alpha = \{T \in \mathcal{Tr}\colon \mathrm{rk}(T) < \alpha\}$ which has $T_0, \dots, T_k$ as the set of minimal elements of its complement. Given and ideal $\mathcal{I} \subset \mathcal{Tr}$, let $\mathrm{rk}(\mathcal{I}) = \sup\{\mathrm{rk}(T) + 1\colon T \in \mathcal{I}\}$ and we let $\mathrm{cf}(\mathcal{I})$ be the least cardinal such that there is a cofinal subset of $\mathcal{I}$ of that cardinality.

PROPOSITION 3.12. [Monb] *A tree $T = [\langle \alpha, \epsilon\rangle; T_0, \dots, T_{k-1}]$ with labels in $\mathcal{ON} \times \{+,-\}$ belongs to $\mathcal{Tr}$ if and only if*

(1) *for each i, $T_i \in \mathcal{Tr}$ and* $\mathrm{rk}(T_i) < \alpha$;
(2) $T_0, \dots, T_{k-1}$ *are mutually $\preccurlyeq$-incomparable*;
(3) $\mathrm{rk}(\mathcal{I}^{\alpha}_{T_0,\dots,T_{k-1}}) = \alpha$;
(4) $\mathrm{cf}(\mathcal{I}^{\alpha}_{T_0,\dots,T_{k-1}}) \vee \omega = \mathrm{cf}(\alpha) \vee \omega$;
(5) *for no i, $\tau(T_i) \prec \tau(T)$.*

Notation 3.13. If $T = [\langle \alpha, \epsilon\rangle; T_0, \dots, T_{k-1}] \in \mathcal{Tr}$, we let $\mathcal{I}_T = \mathcal{I}^{\alpha}_{T_0,\dots,T_{k-1}}$.

PROPOSITION 3.14. [Monb] *Let $J = \langle T_0, \dots, T_k\rangle \in \mathcal{Tr}^{<\omega}$. Then, $J \in \mathcal{In}$ if and only if for no $i < k$ we have that*

(1) *either $\epsilon_i = -$ and $T_{i+1} \in \mathcal{I}_{T_i}$,*
(2) *or $\epsilon_{i+1} = +$ and $T_i \in \mathcal{I}_{T_{i+1}}$.*

§4. Minimal linear orderings.

In this section we show an extension of the following result of Hausdroff. The proofs are in [Monb]. In the countable case this follows from some lemmas in [Mon05b].

THEOREM 4.1 (Hausdorff, see [Ros82]). *Let κ be a regular cardinal and $\mathcal{L}$ a scattered linear ordering. Then, $\kappa \leqslant |\mathcal{L}|$ if and only if either $\kappa \preccurlyeq \mathcal{L}$ or $\kappa^* \preccurlyeq \mathcal{L}$.*

Since a scattered linear ordering has rank $\geqslant \kappa$ if and only if it has size $\geqslant \kappa$, it follows that $\{\kappa, \kappa^*\} \subset \mathbb{S}$, is the set of minimal equimorphism types of rank κ. For each ordinal α, since $\mathbb{S}$ is well-quasi-ordered, there exists a finite set $\mathbb{F}_\alpha$ of minimal equimorphism types of rank α. We explicitly define the elements of $\mathbb{F}_\alpha$ for each α.

DEFINITION 4.2. [Monb] Given an indecomposable ordinal $\alpha > 1$, and two signs $\epsilon_0, \epsilon_1 \in \{+, -\}$, we define an equimorphism type $\mathrm{lin}(\alpha, \epsilon_0, \epsilon_1)$ as follows. Let $\{\alpha_\gamma : \gamma < \mathrm{cf}(\alpha)\}$ be an increasing sequence cofinal in α. Define

$$\mathrm{lin}(\alpha, \epsilon_0, \epsilon_1) = \sum_{\gamma \in \mathrm{cf}(\alpha)^{\epsilon_1}} (\omega^{\alpha_\gamma})^{\epsilon_0}.$$

Observe that, up to equimorphism, this definition is independent of the cofinal sequence chosen. For example, $\mathrm{lin}(\omega^\omega, +, -) = \cdots + \omega^{\omega^n} + \cdots + \omega^{\omega^2} + \omega^\omega$ and $\mathrm{lin}(\alpha, +, +) = \omega^\alpha$.

For α indecomposable, let

$$\bar{\mathbb{F}}_\alpha = \{\mathrm{lin}(\alpha, +, +), \mathrm{lin}(\alpha, +, -), \mathrm{lin}(\alpha, -, +), \mathrm{lin}(\alpha, -, -)\}.$$

We also let $\bar{\mathbb{F}}_1 = \{\omega, \omega^*\}$.

Now, consider δ, an ordinal with Cantor normal form $\delta = \omega^{\alpha_0} + \cdots + \omega^{\alpha_{k-1}}$ where $\alpha_0 \geqslant \alpha_1 \geqslant \cdots \geqslant \alpha_{k-1}$. Then, let

$$\bar{\mathbb{F}}_\delta = \{\mathcal{L}_0 \cdot \mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_{k-1} : \forall i < k (\mathcal{L}_i \in \bar{\mathbb{F}}_{\omega^{\alpha_i}})\}.$$

PROPOSITION 4.3. [Monb] *Let $\mathcal{L} \in \mathbb{S}$ and δ be an ordinal. Then,*

$$\delta \leqslant \mathrm{rk}(\mathcal{L}) \iff (\exists \mathcal{A} \in \bar{\mathbb{F}}_\delta)\, \mathcal{A} \preccurlyeq \mathcal{L}.$$

This only shows that the set of minimal scatterd equimorphism types of rank δ is included in $\bar{\mathbb{F}}_\delta$. Note, for example, that when κ is a regular cardinal $\kappa = \mathrm{lin}(\kappa, +, +) \prec \mathrm{lin}(\kappa, -, +)$. We define $\mathbb{F}_\delta$ by picking out the minimal elements of $\bar{\mathbb{F}}_\delta$.

DEFINITION 4.4. Let δ be an ordinal with Cantor normal form $\delta = \omega^{\alpha_0} + \cdots + \omega^{\alpha_k}$ where $\alpha_0 \geqslant \alpha_1 \geqslant \cdots \geqslant \alpha_k$. Let $\mathbb{F}_\delta$ be the set consisting of the equimorphism types of the form $\mathcal{L}_0 \cdot \mathcal{L}_1 \cdot \cdots \cdot \mathcal{L}_{k-1}$, where for each i, $\mathcal{L}_i = \mathrm{lin}(\omega^{\alpha_i}, \epsilon_{i,0}, \epsilon_{i,1})$ and

(1) if ω^{α_i} is a regular cardinal, then $\epsilon_{i,0} = \epsilon_{i,1}$,
(2) if $\omega^{\alpha_{i+1}} > \mathrm{cf}(\omega^{\alpha_i})$, then $\epsilon_{i,1} = \epsilon_{i+1,0}$.

THEOREM 4.5. [Monb] *For each ordinal δ, $\mathbb{F}_\delta$ is the set of minimal equimorphism types of rank δ.*

§5. Up to equimorphism, hyperarithmetic is computable. Computable Mathematics deals with the computable aspects of mathematical theorems and objects. The question "given a mathematical structure, which is the simplest way to represent it?" is of great importance in this area. Part of our work in Computable Mathematics is related to this question.

The way we *present* linear ordering, and in particular well-orderings, is by a pair $\langle L, \leqslant_L\rangle$, where $L \subseteq \omega$ and $\leqslant_L \subseteq \omega \times \omega$. So, it makes sense to talk about computable or hyperarithmetic presentations of a linear ordering. We only deal with countable linear orderings.

Clifford Spector proved the following well known classical theorem in Computable Mathematics.

THEOREM 5.1. [Spe55] *Every hyperarithmetic well-ordering is isomorphic to a computable one.*

In less technical terms this says that, if an ordinal has a representation of a certain complexity (hyperarithmetic, which is quite high), it has a very simple (computable) representation.

We prove a generalization of Spector's result to all countable linear orderings. Before explaining our result we give the basic definitions on hyperarithmetic theory. Standard references for Hyperarithmetic Theory are [AK00] and [Sac90].

5.1. Hyperarithmetic sets. We give different equivalent definitions of ω_1^{CK} and the class of hyperarithmetic sets to emphasize that these concept are natural, but it is enough for the reader to understand one of these definitions.

We say that an ordinal is *computable* if it has a computable presentation. It is not hard to observe that computable ordinals form an initial segment of the class of ordinals. We use ω_1^{CK} to denote the least non-computable ordinal, where CK stands for Church-Kleene. It follows from Spector's theorem above that ω_1^{CK} is also the least non-hyperarithmetic ordinal. It is also known that ω_1^{CK} is the least admisible ordinal, that is, the least ordinal μ such that $L(\mu)$ is a model of Kripke-Platek set theory, where $L(\mu)$ is set of Gödel constructible sets up to level μ, and Kripke-Platek set theory consists of the axioms of ZFC, but with comprehension and replacement restricted to only Δ_0 formulas (see [Sac90, Chapter VII]).

THEOREM 5.2. [Kle55, Ash86] *Given $X \subseteq \omega$, the following are equivalent*

(1) *X is computable in $0^{(\alpha)}$ for some computable ordinal α, where $0^{(\alpha)}$ is the αth iteration of the Turing jump of 0.*
(2) *$X \in \Delta_1^1$, that is, there exists Σ_1^1 formulas ψ and φ of second order arithmetic, such that $(\forall n)\, n \in X \iff \psi(n) \iff \neg\varphi(n)$.*
(3) *There is a computable infinitary formula φ such that $X = \{n : \varphi(n)\}$.*
(4) *$X \in L(\omega_1^{CK})$.*

(A *computable infinitary formula* is a formula where infinite disjunctions and infinite conjunctions are allowed, so long as they are taken over computably enumerable sets of computable infinitary formulas. See [AK00, Chapter 7].)

DEFINITION 5.3. When $X \subseteq \omega$ satisfies any of the conditions in the theorem above we say that *X is hyperarithmetic*.

5.2. An extension of Spector's result. The direct generalization of Theorem 5.1 to the class of linear orderings does not hold. It is not the case that every linear ordering with a hyperarithmetic presentation is isomorphic to a computable one. Feiner constructed in [Fei67] and [Fei70] (see also

[Dow98, Theorem 2.5]) a Π^0_1 subset of $\mathcal{Q}$ that, as a linear ordering, is not isomorphic to a computable one. Other examples were given later. It follows from the work of Lerman [Ler81] that for every Turing degree **a** such that $\mathbf{a}'' >_T 0''$ there is a linear ordering of degree **a** without a computable copy. This result was later extended, first to any non-computable computably enumerable degree **a** by Jockusch and Soare [JS91], then to any non-computable Δ^0_2 degree **a** by Downey [Dow98] and Seetapun (unpublished), and finally to any non-computable degree **a** by Knight [AK00]. Many other results have been proved about presentations of linear orderings; we refer the reader to [Dow98] for a survey on the computable mathematics of linear orderings.

But there are other ways in which we can generalize Theorem 5.1. Observe that if a linear ordering $\mathcal{L}$ is equimorphic to an ordinal α, then $\mathcal{L}$ and α are actually isomorphic. (It is clear that two equimorphic well orderings are isomorphic. Note that $\mathcal{L}$ has to be a well ordering because it is isomorphic to a subset of α.) So, actually, we can state Theorem 5.1 as "every hyperarithmetic well ordering is equimorphic to a computable linear ordering." The main theorem of [Mon05b] is the following generalization of Theorem 5.1.

THEOREM 1.1. *Every hyperarithmetic linear ordering is equimorphic to a computable one.*

We will sketch a proof of this result below, but first we mention some related results. Greenberg and Montalbán showed in [GM] that the theorem above is also true for other classes of structures, like superatomic Boolean algebras, countable compact metric spaces and Abelian p-groups. The proofs for superatomic Boolean algebras and countable compact metric spaces are just observations of previously known results: Every superatomic Boolean algebra is the linear algebra of some ordinal and every countable compact metric space comes from a natural metrization of some ordinal. The class of equimorphism types of Abelian p-groups is not as simple. So a bit more work is required in this case, although, not as much as for the case of linear orderings.

We note that the equimorphism between the hyperarithmetic and the computable linear ordering in Theorem 1.1 does not need to be hyperarithmetic. However, this only happens in the case when the linear ordering is not scattered, and hence equimorphic to $\mathbb{Q}$. If $\mathcal{L}$ is computable in $0^{(\alpha)}$ and has Hausdorff rank β, with $\alpha, \beta < \omega_1^{CK}$, then we can get the equimorphism to be computable in something like $0^{(\alpha+\beta+\omega)}$ (unpublished).

5.3. Idea of the proof. We now sketch the proof of Theorem 1.1. The whole proof can be find in [Mon05b]. Here we use terminology that was not used in [Mon05b], since it was later developed in [Monb].

The first step to prove Theorem 1.1 is to prove the following lemma.

LEMMA 5.4. *If $\mathcal{L}$ is a hyperarithmetic scattered linear ordering, then* $\mathrm{rk}_H(\mathcal{L}) < \omega_1^{CK}$.

The proof of this lemma is a very standard overspill argument. The second step is the following theorem.

THEOREM 5.5. *A scattered linear ordering has Hausdorff rank less than ω_1^{CK} if and only if it is equimorphic to a computable linear ordering.*

The direction from right to left follows easily from the lemma above. The other implication is the hard one, and together with the lemma above, implies Theorem 1.1: Let $\mathcal{L}$ be a hyperarithmetic linear ordering. If $\mathcal{L}$ is not scattered, it is equimorphic to $\mathbb{Q}$ which has a computable presentation. Otherwise, by Lemma 5.4, $\mathcal{L}$ has Hausdorff rank less than ω_1^{CK} and then by Theorem 5.5 it is equimorphic to a computable linear ordering. Moreover, using the relativized version of Lemma 5.4 we get that if $\mathcal{L}$ is computable in $X \subseteq \omega$ and $\omega_1^X = \omega_1^{CK}$, or in particular if X is hyperarithmetically-low, then $\mathcal{L}$ is equimorphic to a computable linear ordering.[1]

The key point of the proof of Theorem 5.5 is to show the following result.

PROPOSITION 5.6. *Let α be a computable ordinal. Represent the ordinals which appear as labels in the elements of $\mathcal{T}r_\alpha$ as elements of α. Then, $\mathcal{T}r_\alpha$ is a computable set. (Recall that $\mathcal{T}r_\alpha = \{T \in \mathcal{T}r\colon \operatorname{rk}(T) < \alpha\} = \{\mathtt{T}(\mathcal{L})\colon \mathcal{L} \in \mathbb{H}_\alpha\}$.)*

SKETCH OF THE PROOF. We use computable transfinite induction on α to show that $\mathcal{T}r_\alpha$ is uniformly computable. Suppose that we know that $\mathcal{T}r_\alpha$ is computable; we want to show that $\mathcal{T}r_{\alpha+1}$ is too. Given a tree $T = [\langle\alpha, \epsilon\rangle; T_0, \dots, T_{k-1}]$ with labels in $\mathcal{ON} \times \{+, -\}$, we have to check that the conditions in Proposition 3.12 are satisfied. Condition (4) is trivially satisfied since we are in the countable case. The only condition that is not easy to check computably is (3), and here is where most of the work in this proof goes. What we do is to define, uniformly computably in α, the finite set of minimal ideals of $\mathcal{T}r_\alpha$ of rank α, and then to check whether an ideal has rank α all we have to do is to compare it with one of these minimal ideals.

More precisely, uniformly in α, we define a finite set $X_\alpha^1, \dots, X_\alpha^{k_\alpha}$ of finite antichains of $\mathcal{T}r_\alpha$ such that $\mathcal{I}^\alpha_{X_\alpha^1}, \dots, \mathcal{I}^\alpha_{X_\alpha^{k_\alpha}}$ are the minimal ideals of $\mathcal{T}r_\alpha$ of rank α. (Recall that $\mathcal{I}^\alpha_X = \{S \in \mathcal{T}r_\alpha\colon \forall T \in X\ (T \not\preccurlyeq S)\}$.) Then, given an ideal $\mathcal{I} \subseteq \mathcal{T}r_\alpha$, we have that

$$\operatorname{rk}(\mathcal{I}) = \alpha \iff \exists i \leqslant k_\alpha\ (\mathcal{I}^\alpha_{X_\alpha^i} \subseteq \mathcal{I}).$$

Notice that checking whether $\mathcal{I}^\alpha_{X_\alpha^i} \subseteq \mathcal{I}^\alpha_{T_0,\dots,T_{k-1}}$ can be done computably:

$$\mathcal{I}^\alpha_{X_\alpha^i} \subseteq \mathcal{I}^\alpha_{T_0,\dots,T_{k-1}} \iff \forall j < k\ \exists S \in X_\alpha^i\ (S \preccurlyeq T_j).$$

The definition of $\{X_\alpha^1, \dots, X_\alpha^{k_\alpha}\}$ is done in both [Mon05b] and [Monb], so we do not include it here. The idea is similar to the one for the definition of minimal equimorphims types of a given Hausdroff rank in Section 4. ⊣

[1] Liang Yu [Yu] has recently proved that this result is also true for any Σ_1^1 linear ordering $\mathcal{L}$.

Now, to prove Theorem 5.5 we define a computable map $\mathrm{lin}(\cdot)$ that assigns a linear ordering to each element of $\mathcal{T}r_\alpha$ such that $\mathrm{inv}(\mathrm{lin}(T)) = T$. The definition of $\mathrm{lin}(\cdot)$ is by computable transfinite recursion on the rank of T. Given $T = [\langle\beta, \epsilon\rangle; T_0, \dots, T_{k-1}] \in \mathcal{T}r_\alpha$, let $\{S_0, S_1, \dots\}$ be an computable enumeration of $\mathcal{I}^\beta_{T_0,\dots,T_{k-1}}$. This computable enumeration exists because $(\mathcal{T}r, \preccurlyeq)$ is computable. Then, if $\epsilon = +$, let

$$\mathrm{lin}(T) = \mathrm{lin}(S_0) + (\mathrm{lin}(S_0) + \mathrm{lin}(S_1)) + (\mathrm{lin}(S_0) + \mathrm{lin}(S_1) + \mathrm{lin}(S_2)) + \cdots$$

and if $\epsilon = -$, let

$$\mathrm{lin}(T) = \cdots + (\mathrm{lin}(S_2) + \mathrm{lin}(S_1) + \mathrm{lin}(S_0)) + (\mathrm{lin}(S_1) + \mathrm{lin}(S_0)) + \mathrm{lin}(S_0).$$

This shows that every indecomposable equimorphism type of Hausdorff rank less than α has a computable presentation, and this is for every $\alpha < \omega_1^{CK}$. Since every scattered linear ordering can be written as a finite sum of indecomposable, this proves Theorem 5.5.

As a corollary we also get that if α is a computable ordinal, then $\langle \mathbb{S}_\alpha, \preccurlyeq\rangle$ is computably presentable [Mon05b, Corollary 4.3].

§6. Reverse mathematics of Fraïssé's conjecture.

The main result of [Mon06], is the following one.

THEOREM 6.1. *The following are equivalent over* RCA_0:

(1) *Fraïssé's conjecture*;
(2) *The signed trees are well-quasi-ordered under* $\preccurlyeq$;
(3) *Every scattered linear ordering is equimorphic to a finite sum of h-indecomposables.*

The following statement is also equivalent to the previous ones but over RCA_*

(4) *Jullien's classification of extendible linear orderings.*

This result shows that Fraïssé's conjecture is sufficient and necessary to prove basic results about equimorphism types of linear orderings. This makes it an interesting system of second order arithmetic.

We start by describing the program of Reverse Mathematics. Then, we describe the statements mentioned in the theorem above.

6.1. Reverse Mathematics. The questions of what axioms are necessary to do mathematics is of great importance in Foundations of Mathematics and is the main question behind Friedman and Simpson's program of Reverse Mathematics. Old known examples along this line of investigation are Euclid's question of whether the fifth postulate was necessary to do geometry and the question of the necessity of the Axiom of Choice to do mathematics. To analyze this question formally it is necessary to fix a logic system. Reverse Mathematics deals with subsystems of Z_2, the system of second-order arithmetic. Second-order Arithmetic, even though it is much weaker than set theory, is rich enough to be able to express an important fragment of classical mathematics. This fragment includes number theory, calculus, countable algebra, real and complex analysis, differential equations and combinatorics among others. Almost all of mathematics that can be modeled with, or coded by, countable objects can be done in Z_2. The basic reference for this subject is [Sim99].

The idea of Reverse Mathematics is as follows. We start by fixing a basic system of axioms. The most commonly used system is RCA_0 which is closely related to Computable Mathematics. In RCA_0, the only sets we can assume exist are the ones that we can describe via an effective algorithm. Now, given a theorem of "ordinary" mathematics, the question is what axioms do we need to add to the basic system to prove this theorem. It is often the case in Reverse Mathematics that we can prove that a certain set of axioms is needed to prove a theorem by proving the axioms from the theorem using the basic system. Many different system of axioms have been defined and studied, but a very interesting fact is that most of the theorems that have been analyzed are equivalent over RCA_0 to one of five systems. These five systems are RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0, listed in increasing order of strength.

The language of second order arithmetic is the usual language of first order arithmetic (which contains non-logical symbols 0, 1, $+$, $\times$ and $\leqslant$) augmented with set variables and a membership relation $\in$. (We use the letters $x, y, z, n, m, \dots$ for number variables and capital letters $X, Y, Z, A, \dots$ for set variables.) The axioms of Z_2 are divided in three groups. First we have the *Basic axioms* which say that the natural numbers form an ordered semiring. Then we have the *Induction axioms*. Given a formula $\varphi(x)$ of second-order arithmetic we have the axiom:

$$\varphi(0) \text{ and } \forall x(\varphi(x) \Longrightarrow \varphi(x+1)) \Longrightarrow \forall x \varphi(x). \qquad (\mathsf{IND}(\varphi))$$

Last, we have the *Comprehension axioms*. These axioms are *set existence axioms* in the sense that they say that sets with certain properties exist. Again, we have one for each formula $\varphi(x)$:

$$\exists X \forall x \; (x \in X \iff \varphi(x)). \qquad (\mathsf{CA}(\varphi))$$

If φ has free variables other than x, then we take $\mathsf{IND}(\varphi)$ and $\mathsf{CA}(\varphi)$ to be the universal closure of the formulas shown above. Subsystems of Z_2 are obtained by restricting the induction and comprehension axioms to certain classes of formulas. The basic system RCA_0 consist of the basic axioms, and the schemes of Σ^0_1-induction and Δ^0_1-comprehension. Σ^0_1-induction is the scheme of axioms that contains a sentence $\mathsf{IND}(\varphi)$ for each Σ^0_1 formula $\varphi(x)$. (A formula ψ is Σ^0_0 if it contains no set quantifiers and all the first order quantifiers are bounded, that is, of the form either $(\forall y < t)$ or $(\exists y < t)$. A formula φ is Σ^0_1 if it is of the form $\exists z \psi(z)$, where ψ is a Σ^0_0 formula.) The *Recursive Comprehension Axiom scheme* or Δ^0_1-comprehension consist of the axioms of the form

$$\forall x(\varphi(x) \iff \neg\psi(x)) \Longrightarrow \exists X \forall x\ (x \in X \iff \varphi(x)).$$

where φ and ψ are Σ^0_1 formulas. Another important system is ACA_0. Its axioms are the ones of RCA_0 plus the *Arithmetic Comprehension Axiom scheme*, which consist of the sentences $\mathsf{CA}(\varphi)$ for arithmetic formulas $\varphi(x)$. (A formula is *arithmetic* if it contains no second order quantifiers.) The scheme of arithmetic comprehension is equivalent to the sentence that says that for every set X, there exists a set X' which is the Turing jump of X. For other classes, Γ, of formulas, like Π^1_1 for example, the system Γ-CA_0 is defined analogously. A system that will be important in this paper is ATR_0. It consist of RCA_0 and the axiom scheme of *Arithmetic Transfinite Recursion*. The scheme of Arithmetic Transfinite Recursion is a little technical so we omit the details. What it says is that arithmetic comprehension can be iterated along any ordinal, which is equivalent to saying that the Turing jump can be iterated along any ordinal. ATR_0 is the natural subsystem of second order arithmetic in which one can develop a decent theory of ordinals [Sim99]. For example, ATR_0 is equivalent to the fact that any two ordinals are comparable.

All the systems we have described have restricted induction. The subindex 0 in the notation of a system means that the induction scheme the system contains is Σ^0_1-induction. If we drop the subindex 0, and for example get RCA or ATR, we are adding the Full induction scheme to the system. The *Full induction scheme* consists of the sentences $\mathsf{IND}(\varphi)$, for all formulas $\varphi(x)$. A subindex $*$, as in ATR_*, indicates that the system has the scheme of Σ^1_1-induction. (Σ^1_1-induction is defined analogously to Σ^0_1-induction. A formula φ is Σ^1_1 if it is of the form $\exists X\psi(X)$, where ψ is an arithmetic formula.) When this program started, RCA, which is slightly stronger than RCA_0, was often used as the basic system.

It happens often that the analysis of theorems from the viewpoint of reverse math gives a deeper understanding of the theorems and sometimes leads to new proofs. This is the case here.

6.2. Fraïsé's conjecture. A quasi-ordering $(P, \leqslant_P)$ is a *well-quasi-ordering* if, for every sequence $\{x_n\}_{n\in\mathcal{N}}$ of elements of P, there exists $i < j$ such that $x_i \leqslant_P x_j$. An equivalent definition of well-quasi-ordering, that might be easier to visualize, is that $\leqslant_P$ contains no infinite strictly descending chains and no infinite antichains. The proof of the equivalence between the two definitions follows from Ramsey's theorem.

Statement 6.2. Fraïsé's conjecture, which we denote by FRA, is the statement of second order arithmetic that says that the countable linear orderings form a well-quasi-ordering under the relation of embeddablity.

Roland Fraïssé conjectured in [Fra48] that there are no sequences of countable linear orderings which are strictly descending under embeddablity. Although this statement is slightly different from FRA, FRA became known as

Fraïssé's conjecture. Moreover, FRA is still known as Fraïssé's conjecture even though it is not a conjecture anymore. Richard Laver proved FRA in [Lav71] using Nash-Williams complicated notion of better-quasi-ordering [NW68].

The theory of well-quasi-orderings has been of interest to people studying Reverse Mathematics because it seems to require very strong systems compared with results from other areas of mathematics. Many of the proofs use Π^1_2-CA_0. However, none of these theorems have been proved to be equivalent to Π^1_2-CA_0 and for most of them the exact proof-theoretic strength is unknown. A very interesting example is Kruskal's theorem [Kru60] which says that the class of finite trees is well-quasi-ordered under embeddablity. (The embeddability of trees used in Kruskal's theorem is not the relation $\preccurlyeq$ we defined for signed trees; this embedding has to be one-to-one and has to preserve gratest lower bounds.) Harvey Friedman proved that Kruskal's theorem can not be proved in ATR_0. (See [Sim85] for a proof of Friedman's result and Rathjen and Weiermann [RW93] for an analysis of the exact proof-theoretic strength of Kruskal's theorem.) The reader can find a survey on the theory of well-quasi-orderings from the viewpoint of reverse mathematics in [Mar05].

The exact proof-theoretic strength of FRA is also unknown. It is known that Laver's proof of FRA can be carried out in Π^1_2-CA_0, and that since FRA is a true Π^1_2 statement, it cannot imply Π^1_1-CA_0. Shore [Sho93] proved that the assumption that the class of well orderings is well-quasi-ordered under embeddablity implies ATR_0, getting as a corollary that FRA implies ATR_0. But we still do not know whether FRA could be proved using just ATR_0 (or even Π^1_1-CA_0), as has been conjectured by Peter Clote [Clo90], Stephen Simpson [Sim99, Remark X.3.31] and Alberto Marcone [Mar05]. The known results are:

$$
\begin{tikzcd}
\Pi^1_2\text{-}\mathsf{CA}_0 \arrow[d] \arrow[ddr] & \\
\Pi^1_1\text{-}\mathsf{CA}_0 \arrow[dd] & \\
& \mathsf{FRA} \arrow[ul, "\times"] \arrow[dl] \\
\mathsf{ATR}_0 &
\end{tikzcd}
$$

6.3. Jullien's Theorem on the extendibility of linear orderings. A *linearization* of a partial ordering $\mathcal{P} = \langle P, \leqslant_P \rangle$ is a linear ordering $\langle P, \leqslant_L \rangle$ such that $\forall x, y \in P(x \leqslant_P y \Longrightarrow x \leqslant_L y)$. A linear ordering $\mathcal{L}$ is *extendible*[2] if every

[2]This property is sometimes called *weakly extendability* and extendibility refers to the same property but considering all partial orderings $\mathcal{P}$, and not only the countable ones. A characterization of these linear orderings has been given by Bonnet [BP82]. Since we are only interested in countable objects, we omit the word "weakly". Other names given to this property in the literature are *enforceable* and *Szpilrajn*.

countable partial ordering $\mathcal{P}$ which does not embed $\mathcal{L}$ has a linearization which does not embed $\mathcal{L}$. For example, the extendibility of ω^* is a well known result and it can be translated as every well-founded partial ordering has a well-ordered linearization. But for instance, **2**, the linear ordering with two elements, is not extendible. Other linear orderings which are not extendible are the ones of the form $\langle \rightarrow, \leftarrow \rangle$. We say that $\mathcal{L}$ is of the form $\langle \rightarrow, \leftarrow \rangle$ if $\mathcal{L}$ can be written as a sum of two linear orderings, $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ is indecomposable to the right and $\mathcal{B}$ is indecomposable to the left; for example $\mathcal{L} = \omega + \omega^*$. The extendibility of η was proved by Bonnet and Pouzet in [BP69] (see also [BP82, p. 140]).

With respect to the extendibility of linear orderings, people have been interested not only in its reverse mathematical strength, but also in the effective content of certain theorems. For example, Szpilrajn proved in [Szp30] that every partial ordering has a linearization. This can be done in an effective way; that is, for every partial ordering we can effectively construct a linearization of it (see [Dow98, Observation 6.1]). The effectiveness of the extendibility of ω^* has also been studied: Rosenstein and Kierstead proved that every computable well-founded partial ordering has a computable well-ordered linearization; and Rosenstein and Statman proved that there is a computable partial ordering without computable descending sequences which has no computable linearization without computable descending sequences. (For proofs of these results and other related ones see [Ros84], and see [Ros82] for more background.) The proof theoretic strength of the fact that ω^* is extendible was studied by Downey, Hirschfeldt, Lempp and Solomon in [DHLS03]. They showed that the extendibility of ω^* can be proved in ACA_0, that it implies WKL_0, and that it is not implied by WKL_0. It is not known whether it is equivalent to ACA_0, or it is strictly in between WKL_0 and ACA_0. In that same paper they studied the extendibility of ζ and η. They prove that the extendibility of ζ is equivalent to ATR_0 over RCA_0. For η, they adapted Bonnet and Pouzet's proof of its extendibility to work in Π^1_2-CA_0 and then they give a modification of their proof, due to Howard Becker, that uses only Π^1_1-CA_0. Joseph Miller (unpublished) proved that the extendibility of η implies WKL_0 and that over Σ^1_1-AC_0, it implies ATR_0. We prove in [Mon06] that the extendibility of η is provable in ATR_*, which is strictly weaker than Π^1_1-CA_0, using a completely new proof. Our proof is based on a general analysis of the extendibility of h-indecomposable linear orderings and on the fact that if a partial ordering does not emebed η, there is some h-indecomposable linear ordering that it does not embed either.

A characterization of exactly which linear orderings are extendible has been given by Jullien in his Ph.D. thesis [Jul69]. Rod Downey and R. B. Remmel asked about the effective content of the Bonnet-Jullien result (that here we call Jullien's theorem) in [DR00, Question 4.1] and also in [Dow98, Question 6.1]. In [DR00] they observe that Jullien's proof requires Π^1_2-CA_0, and they mention that it would be remarkable if Jullien's theorem was equivalent to Π^1_2-CA_0. It follows from our results that this is not the case because it is implied by $\mathsf{RCA}_* + \mathsf{FRA}$ which does not imply Π^1_2-CA_0.

Jullien [Jul69] proved that, up to equimorphism, every scattered linear ordering has a unique minimal decomposition, and then characterized the extendible linear orderings by putting conditions on their minimal decompositions:

STATEMENT 6.3. JUL(min-dec) is the statement that says that if $\langle \mathcal{F}_1, \ldots, \mathcal{F}_n \rangle$ is a minimal decomposition of $\mathcal{L}$, then $\mathcal{L}$ is extendible if and only if there is no i such that either $\mathcal{F}_i = \mathcal{F}_{i+1} = \mathbf{1}$ or $\mathcal{F}_i$ is indecomposable to the right and $\mathcal{F}_{i+1}$ is indecomposable to the left.

The problem with this statement is that, without knowing that minimal decompositions always exists, JUL(min-dec) is not enough to classify all the extendible linear orderings as Jullien did. So, from the viewpoint of Reverse Mathematics, this is not a satisfactory formulation of Jullien's classification of the extendible linear orderings. We could say that Jullien's theorem, as stated in [Jul69], is the conjunction of JUL(min-dec) and the sentence that says that every scattered linear ordering has a minimal decomposition. However the fact that every scattered linear ordering has a minimal decomposition is already too strong; it is equivalent to FRA. We proved in [Mon06] that JUL(min-dec) is equivalent to ATR_* over RCA_*. This proof is divided in two parts. In one we proved that every h-indecomposable linear ordering is extendible. Moreover, we prove that for all h-indecomposable linear orderings $\mathcal{L}$, any partial ordering $\mathcal{P}$ which does not embed $\mathcal{L}$ has a linearization hyperarithmetic in $\mathcal{P} \oplus \mathcal{L}$ which does not embed $\mathcal{L}$. Next we used this result to prove that every linear ordering which is a finite sum of h-indecomposable ones satisfying the right properties is extendible. We also get that the linearizations can be taken to be hyperarithmetic in $\mathcal{P} \oplus \mathcal{L}$. The fact that we are getting hyperarithmetic linearizations not only is interesting in itself from the viewpoint of computable mathematics, but also it is useful to reduce the complexity of some formulas we need to prove by induction. We use the fact that existential quantification over the hyperarithmetic sets is, in certain cases, equivalent to universal second order quantification. This allow us to transform some complicated formulas into Π^1_1 equivalents and then prove them by Σ^1_1-induction.

Because of the problem we mentioned earlier, we study an alternative formulation, JUL, of Jullien's theorem. We find this formulation more natural than the original one, and it has the advantage that it does not mention minimal decompositions.

DEFINITION 6.4. A segment $\mathcal{B}$ of a linear ordering $\mathcal{L} = \mathcal{A} + \mathcal{B} + \mathcal{C}$ is *essential* if whenever we have $\mathcal{L} \preccurlyeq \mathcal{A} + \mathcal{B}' + \mathcal{C}$ for some linear ordering $\mathcal{B}'$, it has to be the case that $\mathcal{B} \preccurlyeq \mathcal{B}'$.

STATEMENT 6.5. JUL is the statement: A scattered linear ordering $\mathcal{L}$ is extendible if and only if it does not have an essential segment $\mathcal{B}$ which is either $\mathbf{2}$ or of the form $\langle \rightarrow, \leftarrow \rangle$.

The left-to-right direction of JUL can be proved in RCA_0; it is the other direction that is proof theoretically strong. It is also not to hard to show that JUL follows from JUL(min-dec) and the existence of minimal decompositions for every scattered linear ordering. Using this, we show that JUL follows from FRA and Σ^1_1-induction. We also prove that JUL implies FRA over RCA_0, getting that JUL and FRA are equivalent over RCA_*(recall that RCA_* is RCA_0 together with Σ^1_1-induction). RCA_* is still a very weak system and, as RCA_0 and RCA, is closely related to Computable mathematics. From our work, one can still get that the amount of set existence axioms needed to prove JUL and FRA is the same.

6.4. H-indecompsables and finite decompositions. The other two statements of Theorem 6.1 that we prove are equivalent to FRA are very useful when working with linear orderings. The fact that every scattered linear ordering is equimorphic to a finite sum of h-indecomposables is very useful to prove results about linear orderings, as for instance, to prove Jullien's theorem. So, showing that the finite decomposability of scattered linear orderings follows from Fraïssé's conjecture was key to proving Jullien's theorem from FRA.

On the other hand, the statement that says that the signed trees are well-quasi-ordered is simpler to deal with than FRA. We used it to show that Jullien's theorem implies FRA. This statement might be useful when studying the strength of Fraïssé's conjecture.

§7. Statements of hyperarithmetic analysis. RCA_0 resembles Computable Mathematics in the sense that, when working in RCA_0, all the sets we can assume exist are the ones that are computable from the ones we already know exist. It can be proved that the ω-models of RCA_0 are exactly the ones whose second order part is closed under Turing reduction and disjoint union, where the *disjoint union of two sets* $X, Y \subseteq \omega$ is the set $X \oplus Y = \{2n \colon n \in X\} \cup \{2n+1 \colon n \in Y\}$. The models of second order arithmetic whose first order part is the standard one $(\omega, 0, 1, +, \times)$, are called ω-models. We will identify these models with their second order parts. The system of Arithmetic Comprehension, ACA_0, has a similar behavior, but with respect to arithmetic reducibility. The ω-models of ACA_0 are exactly the ones whose second order part is closed under arithmetic reduction and disjoint union. As are the classes of computable sets and of arithmetic sets, the class of hyperarithmetic sets is a very natural one and enjoys many closure properties. This is the class that will concern us in this section. A set X is *hyperarithmetically reducible to* (or *hyperarithmetic in*) a set Y if it is $\Delta^1_1(Y)$. We could also relativize the other conditions in Proposition 5.2 to get alternative definitions. We say that an ω-model is *hyperarithmetically closed* if it is closed under disjoint union and for every $X, Y \subseteq \omega$, if X is hyperarithmetically reducible to Y and Y is in the model, then X is in the model too.

DEFINITION 7.1. A system of axioms of second order arithmetic T is a *Theory of hyperarithmetic analysis* if

- it holds in $HYP(\mathrm{Y})$ for every $Y \subseteq \omega$, where $HYP(\mathrm{Y})$ is the ω-model consisting of the sets hyperarithmetic in Y ; and
- all its ω-models are hyperarithmetically closed.

Note that this is equivalent to say that every for every set $Y \subseteq \omega$, $HYP(\mathrm{Y})$ is the least ω-model of T which contains Y, and that every ω-model of T is closed under disjoint unions.

In [Ste78, Section 5], Steel defines "theories of hyperarithmetic analysis" as the ones which have $HYP = HYP(\emptyset)$ as their minimum ω-model. People were interested in these theories because they characterize the class HYP. Our definition is a relativized version of the previous one, and it characterizes not only HYP, but also the relation of hyperarithmetic reduction: When T is a theory of hyperarithmetic analysis, a set X is hyperarithmetically reducible to a set Y if and only if every ω-model of T which contains Y also contains X.

The bad news is that there is no theory whose ω-models are exactly the ones that are hyperarithmetically closed. This follows from a more general result of Van Wesep [Van77, 2.2.2]: For every theory T whose ω-models are all hyperarithmetically closed, there is another theory T' whose models are all also hyperarithmetically closed and which has more ω-models than T does. Examples of known theories of hyperarithmetic analysis are the following schemes: Σ^1_1-dependent choice (Σ^1_1-DC_0), Σ^1_1-choice (Σ^1_1-AC_0), Δ^1_1-comprehension (Δ^1_1-CA_0), and weak-Σ^1_1-choice (weak-Σ^1_1-AC_0). The unrelativezed versions of these results were proved by Harrison [Har68], Kreisel [Kre62], [Kle59] and [Sim99, Theorem VIII.4.16]. As listed, these statements go from strongest to weakest, they all imply ACA_0, and, except for Σ^1_1-DC_0, they are implied by ATR_0 (see [Sim99, VIII.3 and VIII.4]). Moreover, the implications Σ^1_1-$\mathsf{DC}_0 \Longrightarrow \Sigma^1_1$-$\mathsf{AC}_0$, Σ^1_1-$\mathsf{AC}_0 \Longrightarrow \Delta^1_1$-$\mathsf{CA}_0$, and Δ^1_1-$\mathsf{CA}_0 \Longrightarrow$ weak-Σ^1_1-AC_0 can not be reversed as proved by Friedman [Fri67], Steel [Ste78] and van Wesep [Van77], respectively.

We say that a sentence S is a *sentence of hyperarithmetic analysis* if $\mathsf{RCA}_0+\mathsf{S}$ is a theory of hyperarithmetic analysis. In [Fri75, Section II], Friedman mentions two sentences related to hyperarithmetic analysis. These sentences, ABW (arithmetic Bolzano-Weierstrass) and SL (sequential limit systems), use the concept of arithmetic set of reals, which is not used outside logic. Another previously known sentence of hyperarithmetic analysis is Game-AC studied by Van Wesep [Van77]. He studied it in a more general context than second order arithmetic. But if we restrict it to second order arithmetic, it essentially says that if we have a sequence of open games such that player II has a winning strategy in each of them, then there exists a sequence of strategies for all of them. He proved that, when restricted to second order arithmetic, Game-AC is equivalent to Σ^1_1-AC_0.

However, to the author's knowledge, no previously published mathematical theorem, which does not mention concepts from logic, has been proved a statement of hyperarithmetic analysis. In [Monc] we present an example of such a theorem. This theorem, that we call INDEC, was first proved by Jullien in [Jul69, Theorem IV.3.3]. INDEC is published in English in, for example, [Fra00, 6.3.4(3)] and [Ros82, Lemma 10.3].

STATEMENT 7.2. Let INDEC be the statement: Every scattered indecomposable linear ordering is either indecomposable to the right or indecomposable to the left.

In [Monc], not only did we prove that INDEC is a statement of hyperarithmetic analysis, but also that, over RCA_0, INDEC is implied by Δ^1_1-CA_0 and implies ACA_0. Note that since HYP is the minimum ω-model of INDEC, neither ACA_0nor $\mathsf{ACA}_0{}^+$ can imply it.

Another interesting fact about INDEC is that it is incomparable over ACA_0 to other natural statements of mathematics. This is probably the first example of incomparable previously published purely mathematical statements which are between ACA_0 and ATR_0. The statements we have in mind are the following: The existence of elementary equivalence invariants for Boolean Algebras, and Ramsey Theorem. The former statement was studied by Shore [Sho06]. He first analyzed how to work with the statement in second order arithmetic and then proved that it is equivalent to $\mathsf{ACA}_0{}^+$ over RCA_0. ($\mathsf{ACA}_0{}^+$ is equivalent to ACA_0 plus the sentence $\forall X(X^{(\omega)}$ exists), where $X^{(\omega)}$ is the ωth Turing jump of X.) The latter statement, Ramsey's Theorem, has been extensively studied in the context of reverse mathematics (see [Sim99, III.7], [CJS01], or [Mil04, Chapter 7]). It is known that it is slightly stronger than ACA_0. (Essentially, the reason why these statements are incomparable with INDEC is that Σ^1_1-AC_0 is conservative over ACA_0 for Π^1_2 formulas [BS75].)

To prove that the ω-models of INDEC are hyperarithmetically closed, we start by considering an ω-model $\mathcal{M}$ of INDEC. Of course, we think of $\mathcal{M}$ as set of subsets of ω. Then, we prove that for every computable increasing sequence of ordinals $\{\alpha_n\}_{n\in\omega}$, converging to an computable ordinal α, we have that if $(\forall n)0^{(\alpha_n)} \in \mathcal{M}$, then $0^{(\alpha)} \in \mathcal{M}$. To prove this we use Ash and Knight's machinery to construct a specific linear ordering such that when we apply INDEC to it, we can deduce that $0^{(\alpha)} \in \mathcal{M}$. Then we relativize and use effective transfinite induction to prove that for every set $X \in \mathcal{M}$ and every X-computable ordinal $\alpha \in \mathcal{M}$, $X^{(\alpha)} \in \mathcal{M}$. The Ash and Knight's machinery to which we refer is related to Ash's $0^{(\alpha)}$-priority arguments (see [AK00]).

§8. Open questions. We now mention open questions and possible directions for further research related to the work mentioned in this paper.

8.1. Equimorphism invariants.

8.1.1. *Identification of the elements of $\mathcal{T}r$.* In relation with the finite equimorphism invariants, the main question we leave open is whether, given a tree with labels in $\mathcal{ON} \times \{+,-\}$, we can tell if it belongs to $\mathcal{T}r$ via a finite manipulation of the symbols in the tree, using some basic operations on ordinals. Using our results, what is left to do is to find a procedure to check that an ideal in $\mathcal{T}r$ has a certain cofinality, the ideal being given by the minimal elements of its complement (see Proposition 3.12).

8.1.2. *Operations on $\mathcal{T}r$.* Then comes the question of which operations on $\mathcal{T}r$ can be done via a finite manipulation of symbols. An interesting operation is the product of linear orderings.

8.1.3. *Invariants for Galvin's class.* The same idea we used to define invariants for $\mathbb{S}$ can be used to define equimorphism invariants for the class of σ-scattered linear orderings.

DEFINITION 8.1. We say that $\mathcal{L}$ is *σ-scattered* if it is a countable union of scattered linear orderings. Let $\mathbb{M}$ be the class of equimorphism types of σ-scattered linear orderings.

This class was first studied by Galvin. The reason why one can define invariants for this class as we did for the class of scattered linear orderings is that versions of Theorems 2.1, 2.4 and 2.7 can be proved for this class. (Each of these theorems is due either to Galvin or to Laver; see [Lav71].) In this case, the labels of the trees should also include information about how the linear ordering is constructed from smaller ones. In other words, the label at the root of $\mathtt{T}(\mathcal{L})$ should now include $\tau(\mathcal{L})$, which now is an element of $\mathcal{ON} \times \{+,-\} \cup \{\eta_{\alpha,\beta} \colon \langle\alpha,\beta\rangle \text{ is admissible}\}$. Defining these invariants in a manner which gives the same properties we had for the case of scattered linear orderings is not that straightforward.

8.1.4. *Sharpness of Laver's theorem.* When Laver proved Fraïssé's conjecture, he not only proved that $\mathbb{S}$ is well-quasi-ordered, but also that $\mathbb{M}$ is well-quasi-ordered. Another interesting question about the σ-scattered linear orderings is whether Laver's theorem is best possible. In other words, is it consistent with ZFC that $\mathbb{M}$ is the well-founded part of the whole class of equimorphism types? (The *well-founded part* of a partial ordering $\mathcal{P}$ is the set of $x \in \mathcal{P}$ such that there is no infinite descending sequence in $\mathcal{P}$ starting at x. Note that Laver's theorem says that $\mathbb{M}$ is included in the well-founded part of the class of equimorphism types.) Using Laver's theorem, we have that $\mathbb{M}$ is the well-founded part of the whole class of equimorphism types if and only if the complement of $\mathbb{M}$ has no minimal equimorphism type. It is known that is it consistent that $\mathbb{M}$ is not equal to this well-founded part. This follows from Baumgartner [Bau73]. Recent results by Moore [Mooa, Moob] might be useful to solve this question.

8.2. Reverse mathematics.

8.2.1. *Fraïssé's conjecture.* The main question that is left open is whether Fraïssé's conjecture is equivalent to ATR_0. This question has been open for more than fifteen years. Our results make this question even more interesting since we know now that Fraïssé's conjecture is equivalent to many other statements regarding linear orderings (or signed trees).

8.2.2. *The coloring Theorem.* In [Lav73], Laver used his previous work on FRA to prove some partition results about scattered linear ordering. The theorem we are interested in, when restricted to the class countable linear orderings, says the following.

THEOREM 8.2. [Lav73] *For every countable linear ordering $\mathcal{L}$ there exists a natural number $n_{\mathcal{L}}$ such that for every coloring of the elements of $\mathcal{L}$ with finitely many colors, there exists a subset of $\mathcal{L}$ which is equimorphic to $\mathcal{L}$ and is colored with at most $n_{\mathcal{L}}$ many colors.*

In [Mon05a, Section 6.7] we show that Laver's theorem above implies FRA over RCA_0, but we do not know whether the other implication holds or not. A short discussion about this reversal can also be found in [Mon05a, Section 6.7]. We think that it is very likely that these two statements are equivalent.

Analyzing the proof-theoretic strength of the other partition results in [Lav73] could be interesting too.

8.2.3. *Extendibility.* An interesting question left open in [DHLS03] is whether the extendibility of ω is equivalent to ACA_0, or is strictly in between WKL_0 and ACA_0. Recall that the extendibility of ω, is equivalent to the statement that say that every well-founded partial ordering has a well-ordered linearization.

With respect to the extendibility of η, putting together results of J. Miller (unpublished) and of [Mon06], we get that it is equivalent to ATR_0 over $\Sigma^1_1\text{-}\mathsf{AC}_0 + \Sigma^1_1$-induction. This is not a very satisfactory answer since the base system is too strong. So the question here is whether this equivalence can be proved over a weaker system like RCA_0.

The equivalence between Jullien's Theorem and Fraïssé's conjecture was also not proved over RCA_0. It was proved over RCA_*, which is a very weak system, but it would still be better to have this equivalence over RCA_0.

8.3. Maximal order type of $\mathbb{H}_\alpha$. An important invariant related to well-quasi-orderings is the maximal order type. If $\mathcal{W} = (W, \leqslant_W)$ is a well-quasi-ordering we let the *length*, or *maximal order type* of $\mathcal{W}$ be

$$o(\mathcal{W}) = \sup\{(W, \leqslant_L)\colon \text{ where } \leqslant_L \text{ is a linearization of } \mathcal{W}\}.$$

Note that every linearization of a well-quasi-ordering is a well-ordering, so the supremum above is taken over a set of ordinals. It was shown by de Jongh and Parikh [dJP77] that this supremum is actually attained by a linearization of $\mathcal{W}$, and this is why $o(\mathcal{W})$ is referred as the maximum order type of $\mathcal{W}$.

Another way of computing $o(\mathcal{W})$ is by taking the well-founded-rank of $\mathbb{B}\mathrm{ad}(\mathcal{W})$, the tree of *finite bad sequences* of $\mathcal{W}$:

$$\mathbb{B}\mathrm{ad}(\mathcal{W}) = \{\langle x_0, \dots, x_{n-1}\rangle \in W^{<\omega} : \forall i < j < n\ (x_i \not\leq_W x_j)\}.$$

Note that $\mathcal{W}$ is well-quasi-ordered if and only if $\mathbb{B}\mathrm{ad}(\mathcal{W})$ is a well-founded tree. For a study on computable maximal linearizations of well-quasi-orderings see [Mona].

Finding the maximal order type of $\mathbb{H}_\alpha$ for each countable ordinal α is not only interesting in itself, but it could also be very useful for the study of the proof-theoretic strength of FRA. For instance, this is the way Rathjen and Weiermann [RW93] found the exact proof theoretic strength of Kruskal's theorem.

8.4. Hyperarithmetic analysis. A question left open in [Monc] is whether INDEC (defined in Section 7) is equivalent to one of the already known systems of hyperarithmetic analysis. The systems it could be equivalent to are Δ^1_1-CA_0 and weak-Σ^1_1-AC_0.

Four other statements of hyperarithmetic analysis involving clopen games are introduced in [Monc], but only some implications among them are proved. For example we look at the *Determined-Game Choice Axiom*, which says that given a sequence of determined clopen games, there exists a sequence of winning strategies for them. We prove that it is in between Σ^1_1-AC_0 and Δ^1_1-CA_0, but we do not know if it is equivalent to either of those or if it is strictly in between.

Probably the most interesting question coming out of [Monc] is whether there are more natural examples of statements of hyperarithmetic analysis.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHICAGO
CHICAGO, IL, USA
E-mail: antonio@math.uchicago.edu
URL: www.math.uchicago.edu/~antonio