The Bulletin of Symbolic Logic
Volume 13, Number 1, March 2007

SET THEORY AND C*-ALGEBRAS

NIK WEAVER

Abstract. We survey the use of extra-set-theoretic hypotheses, mainly the continuum hypothesis, in the C*-algebra literature. The Calkin algebra emerges as a basic object of interest.

Recently Charles Akemann and the author [1] used Jensen's diamond principle to solve an old open problem about the existence of nontrivial C*-algebras with only one irreducible representation. This raises the question: in general, to what extent is extra-set-theoretic reasoning relevant to C*-algebras?

The one extra axiom that has been repeatedly used by C*-algebraists is the continuum hypothesis, although there are a few instances where it was noted that Martin's axiom or some more special weakening of CH would suffice to settle the problem at hand. Interestingly, it appears that C*-algebraists generally tend to regard a problem as solved when it has been answered using CH. This may have to do with the fact that in most cases the other direction of the presumed independence result would involve set theory at a substantially more sophisticated level. There could be an opportunity for set theorists here; it seems likely that most of the theorems we survey below are the easy halves of independence results, the hard—or more set-theoretically sophisticated—directions of which have not yet been proven.

CH has been particularly valuable in the study of the Calkin algebra $\mathcal{C}(l^2) = \mathcal{B}(l^2)/\mathcal{K}(l^2)$ (see Example 1.5 below). This algebra can be viewed as a noncommutative or "quantum" analog of the Stone-Čech remainder $\beta\omega - \omega$ [55]. It should be of basic set-theoretic interest and a rich source of future work.

In the first section below I give a quick introduction to C*-algebras. In the second section I survey a variety of consistency results taken from the C*-algebra literature, and in the third I focus on the Calkin algebra. My concern is consistency results and I do not discuss other relations between C*-algebras and set theory or logic, such as the Glimm-Effros dichotomy [30] which arose out of C*-algebra representation theory, connections with nonstandard analysis [6, 31], Mundici's work on decidability of isomorphism

Received March 21, 2006.

1079-8986/07/1301-0001/$3.00

of AF-algebras [43], or simply the use of basic set-theoretic techniques that do not go beyond ZFC (e.g., my solution of Dixmier's problem using a transfinite recursion of length $2^{\aleph_0}$ [60]).

I wish to thank Ilijas Farah and the referee for providing several suggestions for improving the exposition of this paper.

§**1. C*-algebras.** We begin with a brief introduction to C*-algebras. Some good general references are [21, 37, 59], or, at a more technical level, [46, 50, 54]. Our set-theoretic terminology is standard [33, 40].

1.1. Basic definitions. Let H be a complex Hilbert space. The scalar field will be complex throughout.

There is an inconsistency in the mathematics and physics literature as to whether the inner product $\langle \cdot, \cdot \rangle$ on H should be linear in the first variable and antilinear in the second, or vice versa. We follow the mathematical convention which takes the inner product to be linear in the first variable. Thus, $\mathbf{C}^n$ is a Hilbert space with inner product $\langle v, w \rangle = \sum_1^n v_i \bar{w}_i$ and l^2 is a Hilbert space with inner product $\langle v, w \rangle = \sum_1^\infty v_i \bar{w}_i$.

A *bounded operator* on H is a linear map $A\colon H \to H$ whose *norm*

$$\|A\| = \sup \left\{ \frac{\|Av\|}{\|v\|} : v \neq 0 \right\}$$

is finite, where $\|v\| = \sqrt{\langle v, v \rangle}$. Here we adopt the usual notation according to which $A(v)$ is written Av. This originates in the finite-dimensional case, where $H \cong \mathbf{C}^n$ and A can be identified with an $n \times n$ matrix $\tilde{A}$, with $A(v)$ being the matrix product of $\tilde{A}$ with v.

Let $\mathcal{B}(H)$ be the set of all bounded operators on H. Then $\mathcal{B}(H)$ is a Banach space with the obvious linear structure and the norm defined above. Moreover, it carries two additional algebraic operations, a product and an involution, which all together give it a very robust structure. The product of two operators is defined to be their composition and the involution on $\mathcal{B}(H)$ is the bijection $*\colon \mathcal{B}(H) \to \mathcal{B}(H)$ that takes an operator A to its *adjoint* A^*, which is characterized by the condition

$$\langle Av, w \rangle = \langle v, A^* w \rangle$$

for all $v, w \in H$. When $H = \mathbf{C}^n$ and A is identified with an $n \times n$ matrix (a_{ij}), its adjoint has matrix $(\bar{a}_{ji})$.

An operator A is *self-adjoint* if $A = A^*$. These operators play the role of "real" (as opposed to complex) elements of $\mathcal{B}(H)$, primarily because of the spectral theorem for bounded self-adjoint operators. There are several versions of this result, but the easiest to state and probably the most useful is the following: if $A \in \mathcal{B}(H)$ is self-adjoint then there is a measure space (X, μ), a bounded measurable real-valued function f on X, and a Hilbert space isomorphism $U\colon H \cong L^2(X, \mu)$ taking A to M_f, the operator of multiplication by f. That is, M_f is the operator on $L^2(X, \mu)$ defined by $M_f(g) = fg$, and we have $A = U^{-1}M_f U$. Thus a self-adjoint operator on H "is" a realization of H as some $L^2(X, \mu)$ together with a measurable real-valued function on X.

Any operator $A \in \mathcal{B}(H)$ can be decomposed into its "real" and "imaginary" parts, $A = \mathrm{Re}(A) + i\mathrm{Im}(A)$, where $\mathrm{Re}(A) = (A + A^*)/2$ and $\mathrm{Im}(A) = (A - A^*)/2i$ are both self-adjoint, and this is analogous to the decomposition of a complex-valued function into its real and imaginary parts. Notice that in the finite-dimensional case, when A is identified with a matrix, self-adjointness of A does *not* correspond to its entries being real. If we had used real scalars from the start, we would still have to deal with non self-adjoint operators and the general decomposition into real and imaginary parts would not exist. This shows the intrinsically complex (as opposed to real) character of Hilbert space operators and should help explain why it is natural for us to use complex scalars. (To emphasize this point further, note that there is also a "polar decomposition" of arbitrary bounded operators which is analogous to expressing a complex number in the form $z = re^{i\theta}$.)

A C*-algebra is a subset of $\mathcal{B}(H)$ that is closed under all relevant structure.

Definition 1.1. A *concrete C*-algebra* is a linear subspace $\mathcal{A}$ of $\mathcal{B}(H)$ (for some Hilbert space H) that is closed in the norm topology and satisfies

$$A, B \in \mathcal{A} \Rightarrow AB, A^* \in \mathcal{A}.$$

An *abstract C*-algebra* is a Banach space equipped with a product and an involution that is linearly isometric to some concrete C*-algebra, such that the isomorphism respects multiplicative and involutive as well as linear structure.

There is also an elegant abstract Banach algebra characterization of C*-algebras. It can be used to show that the quotient of a C*-algebra by any closed two-sided ideal is again a C*-algebra, which is not obvious from the concrete characterization. (Any such ideal is automatically stable under involution.)

(We shall also make occasional reference to von Neumann algebras. Concretely, a *von Neumann algebra* is a C*-algebra that is closed in the weak operator topology, the weakest topology on $\mathcal{B}(H)$ which makes the function $A \mapsto \langle Av, w\rangle$ continuous for every $v, w \in H$. It is *separably acting* if H is separable. Abstractly, a von Neumann algebra is a C*-algebra that is linearly isometric to the dual of some Banach space, and the abstract version of "separably acting" is that its predual is separable.)

C*-algebras play a basic role in mathematical physics, especially quantum field theory [12, 18, 58], and they have also had major applications to other areas of mathematics, including index theory for foliations [20], the theory of group representations [24], knot theory [35], and the Novikov conjecture in algebraic topology [39].

1.2. Basic examples.

EXAMPLE 1.2. $\mathcal{B}(H)$ is itself a C*-algebra. If $H = \mathbf{C}^n$ then we can identify $\mathcal{B}(H)$ with $M_n(\mathbf{C})$, the algebra of $n \times n$ complex matrices.

Direct sums of C*-algebras are again C*-algebras. Thus $M_{n_1}(\mathbf{C}) \oplus \cdots \oplus M_{n_k}(\mathbf{C})$—the set of block diagonal matrices with blocks of size $n_1, \ldots, n_k$—is a C*-algebra for any $k \in \omega$ and $n_1, \ldots, n_k \in \omega$. Theorem: every finite-dimensional C*-algebra is isomorphic to one of this form [59, Theorem 11.1.2].

EXAMPLE 1.3. l^∞, the space of all bounded sequences of complex numbers, is a C*-algebra. Sums and products of sequences are defined pointwise, the involution is pointwise complex conjugation, and the norm of a sequence is its supremum norm. There is an isometric embedding of l^∞ into $\mathcal{B}(l^2)$ that realizes elements of l^∞ as multiplication operators on l^2. Namely, $a = (a_i) \in l^\infty$ acts on $v = (v_i) \in l^2$ by $av = (a_i v_i)$. This embedding preserves both the norm and all algebraic structure, so l^∞ is a C*-algebra. I will call it the *diagonal embedding* because if we represent operators on l^2 by infinite matrices, then the multiplication operators just mentioned appear as diagonal matrices.

EXAMPLE 1.4. Let X be a compact metric space. Then $C(X)$, the space of all continuous complex-valued functions on X, is a C*-algebra. Just as with l^∞, its algebraic operations are defined pointwise and its norm is the supremum norm. We can embed $C(X)$ into l^∞ by fixing a dense sequence (x_n) in X and taking a function $f \in C(X)$ to the sequence $(f(x_n))$. Since we have already seen that l^∞ is a C*-algebra, this shows that $C(X)$ is also a C*-algebra.

More generally, if X is any compact Hausdorff space then $C(X)$ is a C*-algebra, but embedding it in $\mathcal{B}(H)$ might require that H be nonseparable. Theorem: every unital abelian C*-algebra is isomorphic to one of this form [59, Theorem 5.3.5]. (A C*-algebra is *unital* if it contains a multiplicative unit.)

Notice that l^∞ is a unital abelian C*-algebra, so according to the theorem just mentioned it must be isomorphic to $C(X)$ for some compact Hausdorff space X. Indeed, $l^\infty \cong C(\beta\omega)$.

I mentioned earlier that the quotient of a C*-algebra by any closed two-sided ideal is always a C*-algebra. For example, c_0, the space of all complex sequences which converge to zero, is a closed two-sided ideal of l^∞, so the quotient space l^∞/c_0 is also a (unital abelian) C*-algebra. Indeed, $l^\infty/c_0 \cong C(\beta\omega - \omega)$.

EXAMPLE 1.5. An operator $A \in \mathcal{B}(H)$ is said to be of *finite rank* if its range is finite-dimensional. The set of all finite rank operators on H is stable under sums, products, and adjoints, but if H is infinite-dimensional it fails to be a

C*-algebra because it is not closed in norm. However, its closure, denoted $\mathcal{K}(H)$, is a C*-algebra. The operators in $\mathcal{K}(H)$ are precisely those bounded operators that can be approximated in norm by finite rank operators; they are called *compact* operators because they are also characterized by the fact that the image of the unit ball of H under such an operator has compact closure.

$\mathcal{K}(H)$ is a C*-algebra and it is also a closed two-sided ideal of $\mathcal{B}(H)$. When $H = l^2$ the quotient C*-algebra $\mathcal{C}(l^2) = \mathcal{B}(l^2)/\mathcal{K}(l^2)$ is called the *Calkin algebra*.

Recall the diagonal embedding of l^∞ into the bounded operators on l^2 (Example 1.3). The diagonal operators which are compact are precisely those whose diagonal entries converge to zero, and hence they correspond to elements of $c_0 \subseteq l^\infty$. In other words, identifying l^∞ with its image in $\mathcal{B}(l^2)$, we have $l^\infty \cap \mathcal{K}(l^2) = c_0$.

I suggested earlier that self-adjoint operators are analogous to real-valued functions. This is one link in an extended chain of analogies which compare closed subsets of a set with closed subspaces of a Hilbert space, spaces of continuous functions on compact Hausdorff spaces with C*-algebras, L^∞ spaces with von Neumann algebras, etc. This is all motivated at a fundamental level by the transition from classical to quantum mechanics, so that one sometimes hears C*-algebras referred to as "quantum topological spaces", and so on. A readable account of this motivation is given in Chapter 1 of [59].

In particular, $\mathcal{B}(l^2)$ can be regarded as a noncommutative analog of $l^\infty \cong C(\beta\omega)$, $\mathcal{K}(l^2)$ as an analog of $c_0 = C_0(\omega)$ (the continuous functions on ω that vanish at infinity), and $\mathcal{C}(l^2)$ as an analog of $l^\infty/c_0 \cong C(\beta\omega - \omega)$. Because of the correspondence between topological spaces and abelian C*-algebras one might simply say that $\mathcal{B}(l^2)$, $\mathcal{K}(l^2)$, and $\mathcal{C}(l^2)$ are analogs of $\beta\omega$, ω, and $\beta\omega - \omega$. I will return to this point in Section 3.

EXAMPLE 1.6. For each $n \in \omega$ define an embedding $\phi_n \colon M_{2^n}(\mathbf{C}) \to M_{2^{n+1}}(\mathbf{C})$ by

$$\phi_n(A) = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}.$$

This is a linear isometry from $M_{2^n}(\mathbf{C})$ into $M_{2^{n+1}}(\mathbf{C})$ which respects both multiplicative and involutive structure, so with a slight abuse of notation we can define $\mathcal{F}_0 = \bigcup_1^\infty M_{2^n}(\mathbf{C})$. Taking the Cauchy completion then yields a C*-algebra $\mathcal{F}$, the *Fermion* or *CAR* algebra. ("CAR" stands for "canonical anticommutation relations".) Concretely, $\mathcal{F}$ can be realized as acting on a Hilbert space H that has an orthonormal basis indexed by the finite subsets of ω. To see this, observe that the standard basis of $\mathbf{C}^{2^n}$ can be indexed by the subsets of n. Thus H naturally decomposes as a direct sum of copies of $\mathbf{C}^{2^n}$, one for each finite subset of $\omega - n$, and we can let $M_{2^n}(\mathbf{C})$ act on each copy. These actions match up with the embeddings ϕ_n, so $\mathcal{F}$ can be defined to be the closure of the union over $n \in \omega$ of the images in $\mathcal{B}(H)$ of $M_{2^n}(\mathbf{C})$.

The Fermion algebra might be considered an analog of (the continuous functions on) the Cantor set K. The intuition here comes from associating a diagonal $2^n \times 2^n$ matrix $A = \mathrm{diag}(a_1, \ldots, a_{2^n})$ with the function $f \in C(K)$ which takes the value a_i on the ith subinterval of length 3^{-n}. In this way we obtain a natural isomorphism of the "diagonal" of $\mathcal{F}$, i.e., the closure of the union over $n \in \omega$ of the diagonal matrices in $M_{2^n}(\mathbf{C}) \subset \mathcal{F}$, with the continuous functions on K.

1.3. States and representations. Although in practice most C*-algebras have some simple favored realization in some special $\mathcal{B}(H)$ (the Calkin algebra is a notable exception), the theory of representations of C*-algebras is nonetheless of great intrinsic interest and of fundamental importance, for example, in applications of C*-algebras to physics [18], group representations [24], and wavelets [36].

DEFINITION 1.7. Let $\mathcal{A}$ be a C*-algebra and let H be a Hilbert space. A *representation* of $\mathcal{A}$ on H is a linear map $\phi\colon \mathcal{A} \to \mathcal{B}(H)$ which respects the product and involution on $\mathcal{A}$. It is *unital* if $\mathcal{A}$ is unital and ϕ takes the unit of $\mathcal{A}$ to the identity operator on H. It is *reducible* if H can be decomposed as a direct sum of two orthogonal proper subspaces each of which is invariant for the action of $\mathcal{A}$ (i.e., $\phi(A)$ takes each of the two subspaces into itself, for every $A \in \mathcal{A}$); otherwise it is *irreducible*.

Every representation is automatically continuous (in fact, contractive).

Irreducible representations are the "building blocks" of representation theory, and morally we think of all representations as being in some sense decomposable into irreducible representations, although this sentiment can be made precise only for separable C*-algebras and even then the exact statement is somewhat complicated. Also, if one approaches C*-algebras from the perspective of their being a noncommutative generalization of the spaces $C(X)$, then the irreducible representations are for some purposes a good generalization of the notion "points of the underlying space X". This is because the irreducible representations of $C(X)$, up to the natural notion of equivalence, canonically correspond to the points of X. Namely, for each $x \in X$ we have an evaluation functional $\phi\colon C(X) \to \mathbf{C} \cong M_1(\mathbf{C})$ given by $\phi(f) = f(x)$, and this is an irreducible representation of $C(X)$ on a one-dimensional Hilbert space, and every irreducible representation of $C(X)$ is equivalent to one of these. (Two representations $\phi_i\colon \mathcal{A} \to \mathcal{B}(H_i)$, $i = 1, 2$, are *equivalent* if there is a Hilbert space isomorphism $U\colon H_1 \cong H_2$ such that $\phi_1(A) = U^{-1}\phi_2(A)U$ for all $A \in \mathcal{A}$.)

A very useful tool for dealing with representations is provided by the concept of a state on a C*-algebra. Before explaining this notion we first need to define positivity. Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a C*-algebra and let $A \in \mathcal{A}$.

Then we say that A is *positive*, and we write $A \geq 0$, if any of the following equivalent conditions is satisfied: (1) $\langle Av, v\rangle \geq 0$ for all $v \in H$; (2) $A = B^*B$ for some $B \in \mathcal{A}$; (3) there is a measure space (X, μ), a bounded measurable function $f: X \to [0, \infty)$, and a Hilbert space isomorphism from H to $L^2(X, \mu)$ taking A to M_f. Further, for $A, B \in \mathcal{A}$ self-adjoint we set $A \leq B$ if $B - A$ is positive. This is a partial order on the self-adjoint elements of $\mathcal{A}$.

DEFINITION 1.8. Let $\mathcal{A}$ be a unital C*-algebra. A linear functional ρ: $\mathcal{A} \to \mathbf{C}$ is a *state* if (1) $\rho(I_{\mathcal{A}}) = 1$, where $I_{\mathcal{A}}$ is the unit of $\mathcal{A}$; (2) $A \geq 0$ implies $\rho(A) \geq 0$; and (3) $\|\rho\| = 1$, where $\|\rho\|$ is defined to be $\sup\{|\rho(A)|/\|A\|$: $A \neq 0\}$. It is *pure* if it is an extreme point of the set of states, i.e., it cannot be expressed as a convex combination of two other states.

Actually any two of the conditions in the definition of a state imply the third. For simplicity we omit discussion of states on non-unital C*-algebras, although generalization to that case is reasonably straightforward. (For example, every non-unital C*-algebra can be trivially embedded in a unital C*-algebra, so that one can effectively reduce to the unital case.)

The states on $C(X)$ correspond to Borel probability measures on X. For any such measure μ, the map $f \mapsto \int f\, d\mu$ is a state on $C(X)$, and all states arise in this way. The pure states are the evaluation functionals, which correspond to probability measures concentrated at a single point.

The prototypical example of a state on a general unital C*-algebra arises in the following way. Let $\phi: \mathcal{A} \to \mathcal{B}(H)$ be a unital representation and let $v \in H$ be a unit vector. Then the map $\rho: \mathcal{A} \to \mathbf{C}$ defined by $\rho: A \mapsto \langle \phi(A)v, v\rangle$ is a state on $\mathcal{A}$, called a *vector state* (for that representation). Thus, representations give rise to states.

Conversely, states give rise to representations via the *GNS* (*Gelfand-Naimark-Segal*) *construction*, executed as follows. Let $\mathcal{A}$ be a unital C*-algebra and let ρ be a state on $\mathcal{A}$. For any $A, B \in \mathcal{A}$ define $\langle A, B\rangle = \rho(B^*A)$. In general this is only a "pseudo inner product" because nonzero vectors might have norm zero. But by factoring out null vectors and completing we obtain a Hilbert space H_ρ, and $\mathcal{A}$ acts on this Hilbert space by left multiplication. The resulting representation is denoted $\phi_\rho: \mathcal{A} \to \mathcal{B}(H_\rho)$.

Let $v \in H_\rho$ be the (unit) vector that comes from the unit of $\mathcal{A}$. Then $\langle \phi_\rho(A)v, v\rangle = \rho(A)$ for all $A \in \mathcal{A}$. Thus, we recover the original state ρ as a vector state. So we can pass back and forth between states and representations. Theorem: The state ρ is pure if and only if the representation ϕ_ρ is irreducible, and every irreducible representation is equivalent to ϕ_ρ for some pure state ρ [46, Theorem 3.13.12].

An operator $U \in \mathcal{B}(H)$ is *unitary* if $U^*U = UU^* = I$; equivalently, $U: H \to H$ is a Hilbert space isomorphism, i.e., a bijection that preserves the inner product. We say that two pure states ρ_1 and ρ_2 on a unital C*-algebra $\mathcal{A}$ are *equivalent* if there is a unitary $U \in \mathcal{A}$ such that $\rho_1(A) =$

$\rho_2(U^*AU)$ for all $A \in \mathcal{A}$. Theorem: ρ_1 and ρ_2 are equivalent pure states if and only if ϕ_{ρ_1} and ϕ_{ρ_2} are equivalent irreducible representations [46, Proposition 3.13.4].

Let $\mathcal{A}$ be a unital C*-algebra and let $\mathcal{B} \subseteq \mathcal{A}$ be a C*-algebra contained in $\mathcal{A}$ that contains the unit of $\mathcal{A}$. Then any state on $\mathcal{A}$ restricts to a state on $\mathcal{B}$ (this is obvious), and by the Hahn-Banach theorem any state on $\mathcal{B}$ extends to at least one state on $\mathcal{A}$. Moreover, using convexity theory it is not too hard to show that any pure state on $\mathcal{B}$ can be extended to at least one pure state on $\mathcal{A}$. However, a pure state on $\mathcal{A}$ might not restrict to a pure state on $\mathcal{B}$. For example, let $\mathcal{A} = \mathcal{B}(L^2[0,1])$ and let $\mathcal{B} = C[0,1]$, embedded in $\mathcal{A}$ as multiplication operators. Then for any unit vector $\psi \in L^2[0,1]$ the corresponding vector state on $\mathcal{A}$ is pure since every vector state on any $\mathcal{B}(H)$ is pure, but its restriction to $\mathcal{B}$ is the state $f \mapsto \int f|\psi|^2\,dx$, which is not pure.

§2. Consistency results for C*-algebras. I will now outline a number of consistency results appearing in or near the C*-algebra literature. In some cases they are explicitly formulated as consistency statements, but often they are simply presented as theorems proven with the aid of CH.

Exactly what constitutes "the C*-algebra literature" is debatable. The first three topics discussed below are slightly questionable in this respect, though all three should be intelligible and of interest to C*-algebraists. The last three topics in this section deal with von Neumann algebras.

2.1. Automatic continuity. Probably the best known independence result in functional analysis states that if X is an infinite compact Hausdorff space then it is independent of ZFC whether there exists a discontinuous homomorphism (= linear multiplicative map) from $C(X)$ into a Banach algebra [22]. Although $C(X)$ is a C*-algebra, this is more of a Banach algebra result than a C*-algebra result because the involution on $C(X)$ is not in play and the range algebra is well out of the realm of C*-algebras.

There is a substantial literature on automatic continuity, some of which specifically considers noncommutative C*-algebras (being mapped into Banach algebras). Assuming CH, Laursen [41] showed that a separable C*-algebra has a discontinuous homomorphism into a commutative Banach algebra if and only if the (non-closed) two-sided ideal generated by all commutators has infinite codimension. Runde [49] used CH to construct a discontinuous homomorphism from a noncommutative C*-algebra which has a pathological property, namely, that it fails to decompose in a way that is possible for any discontinuous homomorphism from a commutative Banach algebra.

2.2. The second dual of $C[0,1]$. The Banach space dual of $C[0,1]$ is $M[0,1]$, the space of complex Borel measures on $[0,1]$. The dual of $M[0,1]$ is the "enveloping von Neumann algebra" of $C[0,1]$ and is fairly intractable.

However, assuming CH, R. D. Mauldin [42] proved the following representation theorem for bounded linear functionals on $M[0,1]$: for every such functional T there is a bounded function ψ from the set of Borel subsets of $[0,1]$ to $\mathbf{C}$ such that

$$T(\mu) = \int \psi \, d\mu$$

for all $\mu \in M[0,1]$. Here the integral notation signifies the limit over the directed set of Borel partitions $\{B_1, \ldots, B_n\}$ of $[0,1]$ of the quantity $\sum \psi(B_i)\mu(B_i)$, and part of the assertion is that ψ can be chosen so that this limit always exists. The proof begins by choosing a maximal family of mutually singular Borel measures on $[0,1]$; using CH, this family is then indexed by countable ordinals and ψ is then defined by transfinite recursion.

2.3. Sum decomposition of operator ideals. The compact operators on l^2 constitute a two-sided ideal of $\mathcal{B}(l^2)$, and this is the only nontrivial closed two-sided ideal. However, there are many nontrivial non-closed ideals, each contained in $\mathcal{K}(l^2)$ and containing all finite rank operators. Of special interest are the *Schatten p-ideals* $\mathcal{S}_p(l^2)$ defined for $1 \leq p < \infty$ as the set of compact operators the eigenvalues of whose real and imaginary parts are both p-summable; these are regarded as Hilbert space analogs of the Banach spaces l^p.

The (not necessarily closed) ideals of $\mathcal{B}(l^2)$ form a complete lattice under inclusion. Blass and Weiss [17] proved that CH implies that every proper ideal that properly contains the finite rank operators is the sum (join) of two smaller ideals. They also noted that MA is actually sufficient for their proof. Subsequently Blass and Shelah [15, 16] established that the failure of $\mathcal{K}(l^2)$ to be the join of two smaller ideals is equivalent to the assertion of near coherence of ultrafilters (NCF), i.e., any two free ultrafilters over ω are related (in both directions) by a finite-to-one map from ω to itself, and that the latter condition is consistent with ZFC.

2.4. Ultrapowers of C*-algebras. Ultraproduct constructions are useful for studying approximation properties in C*-algebras and have been heavily used, especially in von Neumann algebra theory. This motivated Ge and Hadwin to initiate a general study of C*-algebra ultraproducts [27], which are straightforwardly obtained from set-theoretic ultraproducts by removing elements of infinite norm and factoring out elements of norm zero. They showed that CH is equivalent to the assertion that all nontrivial ultrapowers of any separable C*-algebra are isomorphic. In slightly more detail, CH implies that if $\mathcal{A}$ is a separable C*-algebra, then any two ultrapowers of $\mathcal{A}$ with respect to free ultrafilters over ω are isomorphic by a map which fixes the elements of $\mathcal{A}$; conversely, $\neg$CH implies that any separable C*-algebra whose set of projections satisfies a simple condition has non-isomorphic ultrapowers with respect to free ultrafilters over ω. The second statement follows easily from a theorem of Dow and Shelah [25, 52] that ¬CH implies that any partially ordered set containing an infinite chain has nontrivial ultrapowers which are not order isomorphic.

A related result of Ge and Hadwin states that CH implies that all nontrivial tracial ultrapowers (a variation on the C*-algebra ultrapower construction) of a separably acting finite von Neumann algebra are isomorphic. This answered a thirty year old question of McDuff.

Ge and Hadwin also proved that the existence of a measurable cardinal is equivalent to the existence of a nontrivial separable ultrapower of a separable C*-algebra.

2.5. Naimark's problem. Naimark [44] showed that every irreducible representation of $\mathcal{K}(H)$ is equivalent to the identity representation on H, and he asked [45] whether there are any other C*-algebras with only one irreducible representation up to equivalence. Rosenberg [48] soon showed that there are no other separable C*-algebras with this property. In subsequent work of Glimm [28] and others this result was subsumed in a general theory of representations of separable C*-algebras. General C*-algebras were classified into two categories, type I and not type I; the irreducible representations of type I C*-algebras were in some sense completely described, whereas separable non-type I C*-algebras were shown to have a uniformly bad representation theory. (The simplest definition of "type I" is: every irreducible representation on a Hilbert space H contains $\mathcal{K}(H)$. However, there are many equivalent conditions framed in very different terms; see Theorem 6.8.7 of [46].) In particular, Glimm showed that any non-type I C*-algebra contains a C*-algebra that has the Fermion algebra $\mathcal{F}$ (Example 1.6) as a quotient. Since $\mathcal{F}$ has $2^{\aleph_0}$ inequivalent irreducible representations it easily follows that the same is true of any separable non-type I C*-algebra.

Naimark's question for nonseparable C*-algebras remained open for fifty years. This seemed odd since it was known that there could be no type I counterexamples, whereas one might assume that the space of representations of any nonseparable non-type I C*-algebra should be, if anything, even less tractable as a topological space than the representation spaces of separable non-type I C*-algebras. (The set of all representations up to equivalence carries a natural topology.) However, this expectation is naive because even though nonseparable non-type I C*-algebras ought to have "many" pure states, they might also contain "many" unitaries making these pure states equivalent. There is therefore an evident possibility of constructing a nonseparable counterexample to Naimark's problem by, loosely speaking, repeatedly adding unitaries until all pure states become equivalent.

This was done in [1] using diamond to get around the problem that as unitaries are added the set of pure states grows. The key technical step is a construction which enables one to add a unitary to any simple, separable, unital C*-algebra in such a way that two chosen pure states extend uniquely to the larger algebra and are made equivalent. A recursive construction of length ω_1 can then be carried out where at each step some new pure state, selected using diamond, is made equivalent to a pure state that is fixed throughout the construction. Ultimately all pure states are equivalent since any pure state on the resulting C*-algebra restricts to a pure state on a closed unbounded family of C*-algebras appearing in the construction; diamond then ensures that every pure state will have been made equivalent to the fixed pure state at some stage.

By Glimm's result mentioned earlier, any counterexample to Naimark's problem cannot be generated by fewer than $2^{\aleph_0}$ elements. As the counterexample constructed using diamond is generated by $\aleph_1$ elements, it follows that the existence of an $\aleph_1$-generated counterexample is independent of ZFC. However, it is presumably the case that the nonexistence of any counterexample is consistent with ZFC. This question remains open. Even better would be the consistency of every nonseparable non-type I C*-algebra having uncountably many inequivalent irreducible representations.

2.6. Pure states on von Neumann algebras. Akemann, Anderson, and Pedersen [3, 2] used CH to investigate the behavior of pure states on von Neumann algebras. In [3] they showed that any sequence of pure states on a separably acting von Neumann algebra that is supported by a sequence of mutually orthogonal, positive, norm-one elements has at least one limit point which is a pure state. This was done by using CH to construct an ultrafilter with special properties in order to identify a limiting pure state. The authors remark that their proof "could proceed with an axiom which is strictly weaker than the Continuum Hypothesis," probably a reference to MA. What is actually needed for the proof is the nonexistence of an inextendible κ-tower in $P(\omega)/\mathrm{fin}$ for any $\kappa < 2^{\aleph_0}$.

The result in [2] is related to the concept of a *perfect* C*-algebra [5]. It is shown in [2] that a separable C*-algebra is perfect if and only if it has only trivial diffuse sequences, where a sequence (A_n) in a unital C*-algebra $\mathcal{A}$ is *diffuse* if every net of pure states (ρ_α) on $\mathcal{A}$ that converges weak* to a pure state satisfies $\lim_{\alpha,n} \rho_\alpha(A_n^*A_n + A_nA_n^*) = 0$, and a diffuse sequence is *trivial* if $A_n \to 0$. It is easier for nonseparable C*-algebras to be perfect [13]. The authors conjecture that every von Neumann algebra is perfect, and as evidence for this conjecture prove that assuming CH, every separably acting von Neumann algebra has only trivial diffuse sequences. This result is proven by first showing that the existence of a nontrivial diffuse sequence implies the existence of a mutually orthogonal, positive, norm one diffuse sequence, and then invoking the theorem in [3] described above to reach a contradiction.

2.7. Quotients of finite von Neumann algebras. A von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ is *finite* if it does not contain an isometry from H onto a proper subspace of H. It is *properly infinite* if $\mathcal{M} \cong \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\mathcal{M}_1$ finite implies $\mathcal{M}_1 = \{0\}$. Every von Neumann algebra is the direct sum of a finite von Neumann algebra and a properly infinite von Neumann algebra.

J. Vesterstrøm [57] investigated the question of when the quotient of a finite von Neumann algebra by a norm closed two-sided ideal is a von Neumann algebra. The problem had previously been answered by Takemoto for separably acting properly infinite von Neumann algebras; in that case the quotient is a von Neumann algebra if and only if the ideal is weak operator closed. The finite case is more complicated, but assuming CH Vesterstrøm gave reasonable necessary and sufficient conditions for the quotient of a separably acting finite von Neumann algebra by a norm closed two-sided ideal to be a von Neumann algebra. He used CH in proving the necessity of the condition that the quotient of the center by its intersection with the ideal in question must be of the form $L^\infty(Y, \nu)$ with (Y, ν) a σ-finite measure space. This comes down to showing that if $L^\infty(Y, \nu)$ is a quotient of $L^\infty(X, \mu)$, $L^\infty(X, \mu) \cong L^1(X, \mu)^*$ and $L^\infty(Y, \nu) \cong L^1(Y, \nu)^*$, and $L^1(X, \mu)$ is separable, then (Y, ν) is σ-finite, which follows from CH by a simple cardinality argument. Actually, all that is needed for this argument is $2^{\aleph_0} < 2^{\aleph_1}$.

2.8. Weak expectations. Blackadar [14] used CH to construct a noninjective separably acting factor von Neumann algebra which has a weak expectation. The technique involves the construction of a nuclear C*-algebra by a transfinite recursion of length ω_1, with CH being invoked to ensure that there are only $\aleph_1$ simple separable C*-algebras to be handled by the construction. See [14] for explanation of the terms used here. As Blackadar notes, it is likely that the assumption of CH can be dropped.

§3. The Calkin algebra.

Recall from Section 1 that the Calkin algebra is the quotient of the bounded operators on l^2 by the compact operators, $\mathcal{C}(l^2) = \mathcal{B}(l^2)/\mathcal{K}(l^2)$. It will become apparent below that $\mathcal{C}(l^2)$ can fruitfully be thought of as a noncommutative analog of (the continuous functions on) $\beta\omega - \omega$. For example, an old open problem asks whether every automorphism of $\mathcal{C}(l^2)$ is inner, i.e., implemented by a unitary $u \in \mathcal{C}(l^2)$. In more detail: let $\phi\colon \mathcal{C}(l^2) \to \mathcal{C}(l^2)$ be a linear bijection which respects the product and involution; must there exist $u \in \mathcal{C}(l^2)$ such that $\phi(a) = u^*au$ for all $a \in \mathcal{C}(l^2)$? This problem is a natural analog of the question whether there exist nontrivial self-homeomorphisms of $\beta\omega - \omega$, which suggests that it may be independent of ZFC and that the techniques developed for proving independence of the classical question [51, 53, 55, 56] would be useful in showing this.

3.1. Projections in $\mathcal{C}(l^2)$. For any closed subspace of a Hilbert space H there is a bounded operator on H which orthogonally projects vectors onto the subspace. Such an operator is called an *orthogonal projection*, or often simply a *projection*. Projections are algebraically characterized by the fact that they are self-adjoint idempotents, i.e., they satisfy $P = P^2 = P^*$.

In a unital abelian C*-algebra $C(X) \subset \mathcal{B}(H)$, this algebraic characterization shows that the projections are identified with the continuous functions on X whose range is contained in $\{0, 1\}$. These are precisely the characteristic functions of clopen subsets of X. Thus the set of projections in $C(X)$ is a Boolean algebra that is naturally isomorphic to the Boolean algebra of clopen subsets of X. If X is a Stone space (i.e., is totally disconnected compact Hausdorff) then the linear span of the set of projections in $C(X)$ is dense in $C(X)$, and working with the Boolean algebra of clopen subsets of X is more or less equivalent to working with the C*-algebra $C(X)$. For general compact Hausdorff spaces, Stone duality with Boolean algebras is therefore, broadly speaking, generalized by Gelfand duality with unital abelian C*-algebras; see, e.g., Section 5.1 of [59].

Now $\beta\omega$ and $\beta\omega - \omega$ are Stone spaces, and their dual Boolean algebras are $P(\omega)$ and $P(\omega)/\mathrm{fin}$. Thus, we may regard the lattices of projections in $\mathcal{B}(l^2)$ and $\mathcal{C}(l^2)$, which we denote $\mathcal{P}(\mathcal{B}(l^2))$ and $\mathcal{P}(\mathcal{C}(l^2))$, as Hilbert space analogs of $P(\omega)$ and $P(\omega)/\mathrm{fin}$.

There is a fairly substantial literature on chains of projections in $\mathcal{B}(l^2)$ (they are called "nests"; see [23]). The study of chains of projections in $\mathcal{C}(l^2)$ was initiated by Hadwin in [29], where he proved that CH implies that all maximal chains are order-isomorphic. The proof goes by showing that any maximal chain is $\aleph_0$-saturated and then invoking a classical result of Hausdorff. Hadwin conjectured that isomorphism of all maximal chains of projections in $\mathcal{C}(l^2)$ is equivalent to CH, which seems unlikely. My student Eric Wofsey [62] has recently shown that the standard notion of forcing used to force inextendible towers in $P(\omega)/\mathrm{fin}$ also forces inextendible towers in $\mathcal{P}(\mathcal{C}(l^2))$, and he used this to prove the consistency of the existence of nonisomorphic maximal chains in $\mathcal{P}(\mathcal{C}(l^2))$. Wofsey is also studying maximal families of orthogonal projections in $\mathcal{C}(l^2)$, i.e., families of projections the product of any two of which is zero. He calls such a family a *maximal almost orthogonal family* or *maof*, and he has shown that it is consistent with ZFC that there is an maof of cardinality strictly less than $2^{\aleph_0}$.

In studying the lattice of projections in $\mathcal{C}(l^2)$ it is possible to largely bypass the machinery of C*-algebras. That is because this lattice can be identified with the lattice of projections in $\mathcal{B}(l^2)$, i.e., the closed subspaces of l^2, modulo a natural equivalence relation. For the reader's benefit we prove this folklore result. We use the following easily verified fact: suppose (X, μ) is a measure space and $f \in L^\infty(X, \mu)$, and let $M_f \in \mathcal{B}(L^2(X, \mu))$ be the operator of multiplication by f. Then M_f is compact if and only if for every $\epsilon > 0$ the space $L^2(X_\epsilon, \mu|_{X_\epsilon})$ is finite-dimensional, where $X_\epsilon = \{x \in X : |f(x)| \geq \epsilon\}$.

Let $\pi : \mathcal{B}(l^2) \to \mathcal{C}(l^2)$ be the natural quotient map.

Proposition 3.1. *Let $p \in \mathcal{C}(l^2)$ be a projection. Then there exists a projection $P \in \mathcal{B}(l^2)$ such that $\pi(P) = p$.*

PROOF. We have $p = p^* = p^2$. Choose $A \in \mathcal{B}(l^2)$ such that $\pi(A) = p$. Then

$$\pi(A^*) = \pi(A)^* = p^* = p$$

so $\pi(B) = p$ where $B = (A + A^*)/2$. Now B is self-adjoint so we can replace $\mathcal{B}(l^2)$ by $\mathcal{B}(L^2(X, \mu))$ for some measure space (X, μ) and B by M_f for some real-valued $f \in L^\infty(X, \mu)$.

We have $\pi(B) = p = p^2 = \pi(B^2)$, so $B - B^2$ is compact. That is, M_g is a compact operator where $g = f - f^2$. Define

$$h(x) = \begin{cases} 1 & \text{if } f(x) \geq 1/2 \\ 0 & \text{if } f(x) < 1/2. \end{cases}$$

We claim that $M_f - M_h = M_{f-h}$ is compact, so that $P = M_h$ is the desired projection such that $\pi(P) = p$. Thus, fix $\epsilon > 0$ and let $X_\epsilon = \{x \in X\colon |f(x) - h(x)| \geq \epsilon\}$; we must show that $L^2(X_\epsilon, \mu|_{X_\epsilon})$ is finite-dimensional. But assuming $\epsilon \leq 1/2$, we have

$$X_\epsilon \subseteq \{x \in X\colon |g(x)| \geq \epsilon - \epsilon^2\}$$

and therefore compactness of M_g implies that $L^2(X_\epsilon, \mu|_{X_\epsilon})$ is finite-dimensional. $\dashv$

(For more results along these lines see Theorem 4.3 of [4] and the references cited there.)

DEFINITION 3.2. Let E and F be closed subspaces of l^2. Say that E is *essentially contained in* F if for every $\epsilon > 0$ there exists a cofinite-dimensional subspace E_0 of E such that every unit vector v in E_0 satisfies $d(v, F) \leq \epsilon$.

Here $d(v, F)$ is the distance between the vector v and the subspace F. Equivalently, we could ask that for every $\epsilon > 0$ there is a cofinite-dimensional subspace E_0 of E such that $\|P_F v\| \geq 1 - \epsilon$ for every unit vector $v \in E_0$, where P_F is the orthogonal projection onto F.

PROPOSITION 3.3. *Let P and Q be projections in $\mathcal{B}(l^2)$. Then $\pi(P) \leq \pi(Q)$ if and only if* $\mathrm{ran}(P)$ *is essentially contained in* $\mathrm{ran}(Q)$.

PROOF. ($\Rightarrow$) Suppose $E = \mathrm{ran}(P)$ is not essentially contained in $F = \mathrm{ran}(Q)$. Then we can find $\epsilon > 0$ and an orthogonal sequence of unit vectors (v_n) in E such that $d(v_n, F) \geq \epsilon$ for all n. If $\pi(P) \leq \pi(Q)$ then $r = \pi(Q) - \pi(P)$ is a projection and there exists a projection $R \in \mathcal{B}(l^2)$ such that $\pi(R) = r$. Then $\pi(P) + \pi(R) = \pi(Q)$ so $P + R - Q$ is compact. But

$$\begin{aligned}\langle (P + R - Q)(v_n), v_n\rangle &= \langle v_n, v_n\rangle + \langle Rv_n, v_n\rangle - \langle Qv_n, v_n\rangle \\ &\geq 1 + 0 - (1 - \epsilon^2) = \epsilon^2,\end{aligned}$$

so we cannot have $(P + R - Q)(v_n) \to 0$. This contradicts compactness of $P + R - Q$, so we must have $\pi(P) \not\leq \pi(Q)$.

($\Leftarrow$) Suppose E is essentially contained in F. Let E' be the closure of $Q[E] \subseteq F$ and let P' be the orthogonal projection onto E'. We claim that $P - P'$ is compact, i.e., $\pi(P) = \pi(P')$; since $P' \leq Q$ this will imply $\pi(P) \leq \pi(Q)$.

(Note that π preserves order because $A \geq 0 \Rightarrow A = B^*B$ for some $B \Rightarrow \pi(A) = \pi(B^*B) = \pi(B)^*\pi(B) \geq 0$.)

Let $\epsilon > 0$. It will suffice to find a finite rank operator A such that $\|P - P' - A\| \leq \delta = \epsilon + 2\sqrt{\epsilon}$. To do this let E_0 be a cofinite-dimensional subspace of E such that $d(v, F) \leq \epsilon$ for every unit vector v in E_0. Let P_0 be the orthogonal projection onto E_0 and let P_0' be the orthogonal projection onto E_0' = the closure of $Q[E_0] \subseteq F$. Then $A = (P - P_0) - (P' - P_0')$ is a finite rank operator, so we must show that $\|P_0 - P_0'\| \leq \delta$. Let $w \in l^2$ be any unit vector. Then $w_0 = P_0(w)$ is the closest vector to w in E_0 and similarly for $w_0' = P_0'(w)$. But there exists $w_0'' \in E_0'$ such that $\|w_0 - w_0''\| \leq \epsilon$. So

$$\|w - w_0'\| \leq \|w - w_0''\| \leq \|w - w_0\| + \epsilon,$$

and similarly $\|w - w_0\| \leq \|w - w_0'\| + \epsilon$. Since $w - w_0'$ is perpendicular to $w_0' - w_0''$ we then have

$$\begin{aligned}\|w_0' - w_0''\|^2 &= \|w - w_0''\|^2 - \|w - w_0'\|^2 \\ &\leq (\|w - w_0\| + \epsilon)^2 - (\|w - w_0\| - \epsilon)^2 \\ &\leq 4\epsilon.\end{aligned}$$

So

$$\|(P_0 - P_0')(w)\| = \|w_0 - w_0'\| \leq \|w_0 - w_0''\| + \|w_0' - w_0''\| \leq \epsilon + 2\sqrt{\epsilon} = \delta,$$

and as w was arbitrary this shows that $\|P_0 - P_0'\| \leq \delta$. ⊣

COROLLARY 3.4. *The lattice of projections in $\mathcal{C}(l^2)$ is isomorphic to the lattice of closed subspaces of l^2 modulo equivalence, where E and F are equivalent if each is essentially contained in the other.*

It should be emphasized that the equivalence relation appearing here is strictly weaker than the more obvious relation according to which two closed subspaces of l^2 are equivalent if each is contained in a finite-dimensional extension of the other. (For example, letting (e_n) be the standard basis of l^2, the closed subspace generated by the vectors e_{2n} is equivalent in the weaker sense, but not the stronger sense, to the closed subspace generated by the vectors $e_{2n} + \frac{1}{n}e_{2n+1}$.) The latter also gives rise to an analog of $P(\omega)/\mathrm{fin}$, but the C*-algebraic motivation for the former indicates that it is of greater intrinsic interest.

3.2. Masas and pure states on $\mathcal{C}(l^2)$. A *masa* ("maximal abelian self-adjoint subalgebra") of a C*-algebra is a maximal abelian C*-subalgebra. In $\mathcal{B}(l^2)$ we have the following characterization: a C*-algebra $\mathcal{A} \subseteq \mathcal{B}(l^2)$ is a masa if and only if there is a σ-finite measure space (X, μ) and a Hilbert space isomorphism $U: l^2 \cong L^2(X,\mu)$ taking $\mathcal{A}$ to $L^\infty(X,\mu)$ (acting as multiplication operators on $L^2(X,\mu)$). That is, $\mathcal{A} = \{U^{-1}M_f U : f \in L^\infty(X,\mu)\}$.

Johnson and Parrott [34] showed that if $\mathcal{A}$ is a masa of $\mathcal{B}(l^2)$ then $\pi(\mathcal{A})$ is a masa of $\mathcal{C}(l^2)$, where as before $\pi: \mathcal{B}(l^2) \to \mathcal{C}(l^2)$ is the natural quotient map. In particular, the canonical masa of $\mathcal{B}(l^2)$, namely l^∞ in its diagonal embedding, projects to a masa of $\mathcal{C}(l^2)$ which is isomorphic to l^∞/c_0. Call this the *standard masa* of $\mathcal{C}(l^2)$.

Assuming CH, Huruya [32] proved that there exists a bounded linear map from some $C(X)$ into $l^\infty/c_0 \cong C(\beta\omega - \omega)$ which cannot be expressed as a linear combination of positive linear maps; moreover, this is still true if l^∞/c_0 is identified with the standard masa of $\mathcal{C}(l^2)$ and the positive linear maps are allowed to go from $C(X)$ to $\mathcal{C}(l^2)$. It seems likely that other independence phenomena involving Banach space properties of l^∞/c_0 (e.g., see [26]) should have analogs for $\mathcal{C}(l^2)$.

Anderson [10] showed that CH implies there are other masas of $\mathcal{C}(l^2)$ besides those of the form $\pi(\mathcal{A})$ for $\mathcal{A}$ a masa of $\mathcal{B}(l^2)$. He did this by proving that there is a masa in $\mathcal{C}(l^2)$ which contains an uncountable family of commuting projections $\{p_\alpha\}$ such that every element of $\mathcal{C}(l^2)$ commutes with all but countably many p_α. He also proved that assuming CH, for every countable set of states on $\mathcal{C}(l^2)$ there is a masa $\mathcal{A}$ such that the restriction of each state to $\mathcal{A}$ is pure.

Anderson [7] also used CH to construct a masa $\mathcal{A}$ of $\mathcal{C}(l^2)$ with the property that every pure state on $\mathcal{A}$ has only one (necessarily pure) state extension to $\mathcal{C}(l^2)$; it is unclear whether this masa lifts to a masa of $\mathcal{B}(l^2)$. This relates to the problem of classifying the pure states on $\mathcal{C}(l^2)$, which is a major unsolved problem in C*-algebra theory.

The centrality of this pure state problem arises from the fact that every pure state on $\mathcal{C}(l^2)$ is also, after composition with π, a pure state on $\mathcal{B}(l^2)$. As any pure state on $\mathcal{B}(l^2)$ is either a vector state or else it factors through $\mathcal{C}(l^2)$, this means that the pure states on $\mathcal{C}(l^2)$ correspond precisely to the nontrivial pure states on $\mathcal{B}(l^2)$ (much as the points of $\beta\omega - \omega$ correspond precisely to the nontrivial ultrafilters over ω).

Let $\mathcal{U}$ be a free ultrafilter over ω and let (e_n) be the standard orthonormal basis of l^2. Then the map $A \mapsto \lim_{\mathcal{U}}\langle Ae_n, e_n\rangle$ is a pure state on $\mathcal{B}(l^2)$ [9] which annihilates the compact operators and therefore also defines a pure state on $\mathcal{C}(l^2)$. Anderson [11] conjectured that every nontrivial pure state on $\mathcal{B}(l^2)$ is of this form with respect to some orthonormal basis of l^2.

A more famous question called the *Kadison-Singer problem* [38] asks whether for every free ultrafilter $\mathcal{U}$ the pure state $a \mapsto \lim_{\mathcal{U}} a_n$ on l^∞ (i.e., evaluation of the sequence $a = (a_n) \in l^\infty \cong C(\beta\omega)$ at $\mathcal{U} \in \beta\omega$) has only one extension to a pure state of $\mathcal{B}(l^2)$ when l^∞ is diagonally embedded in $\mathcal{B}(l^2)$. This problem has attracted a great deal of attention because it has important implications in a variety of disparate areas [19]. It seems unlikely to be independent of ZFC because it can be restated in simple combinatorial terms involving partitioning finite sets of vectors in $\mathbf{C}^n$ [61] (and this formulation is arithmetical). Anderson's conjecture mentioned above is a more likely candidate for independence. However, partial results on the Kadison-Singer problem have used CH: Reid [47] showed that every rare ultrafilter over ω has the unique extension property (and noted that CH implies the existence of rare ultrafilters), and Anderson [8] directly used CH to construct an ultrafilter that is not rare but still has the unique extension property.

If projections in $\mathcal{C}(l^2)$ are analogous to clopen subsets of $\beta\omega - \omega$, then pure states on $\mathcal{C}(l^2)$ (or equivalently, nontrivial pure states on $\mathcal{B}(l^2)$) may be seen as analogous to free ultrafilters over ω. There should be many interesting issues to pursue here, besides those mentioned above.

Note added in proof. Nathanial Brown has informed me that Blackadar's weak expectation problem discussed in Section 2.8 does not require CH (as Blackadar predicted). In fact, using no special set-theoretic hypotheses Brown has shown that there are many separably acting factors with weak expectations (*Connes' embedding problem and Lance's WEP*, ***International Mathematics Research Notices***, vol. 10 (2004), pp. 501–510).

Since this paper was written two problems mentioned in Section 3 were partially solved. N. Christopher Phillips and the author showed that under CH the Calkin algebra has $2^{\aleph_1}$ outer automorphisms (*The Calkin algebra has outer automorphisms*, to appear in ***Duke Mathematical Journal***), and Charles Akemann and the author showed that under CH Anderson's conjecture discussed in Section 3.2 fails (*$B(H)$ has a pure state that is not multiplicative on any MASA*, to appear in ***Proceedings of the National Academy of Sciences of the United States of America***). In both cases the consistency of a different answer under $\neg$CH remains open.

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DEPARTMENT OF MATHEMATICS
WASHINGTON UNIVERSITY IN SAINT LOUIS
SAINT LOUIS, MO 63130, USA
E-mail: nweaver@math.wustl.edu