THE BULLETIN OF SYMBOLIC LOGIC
Volume 12, Number 4, Dec. 2006

INTERNAL CONSISTENCY AND THE INNER MODEL HYPOTHESIS

SY-DAVID FRIEDMAN

There are two standard ways to establish consistency in set theory. One is to prove consistency using *inner models*, in the way that Gödel proved the consistency of GCH using the inner model L. The other is to prove consistency using *outer models*, in the way that Cohen proved the consistency of the negation of CH by enlarging L to a forcing extension $L[G]$.

But we can demand more from the outer model method, and we illustrate this by examining Easton's strengthening of Cohen's result:

THEOREM 1 (Easton's Theorem). *There is a forcing extension $L[G]$ of L in which GCH fails at every regular cardinal.*

Assume that the universe V of all sets is rich in the sense that it contains inner models with large cardinals. Then what is the relationship between Easton's model $L[G]$ and V? In particular, are these models *compatible*, in the sense that they are inner models of a common third model? If not, then the failure of GCH at every regular cardinal is consistent only in a weak sense, as it can only hold in universes which are incompatible with the universe of all sets. Ideally, we would like $L[G]$ to not only be compatible with V, but to be an inner model of V.

We say that a statement is *internally consistent* iff it holds in some inner model, under the assumption that there are inner models with large cardinals. By specifying what large cardinals are required, we obtain a new type of consistency result. Let $\mathrm{Con}(\mathrm{ZFC} + \varphi)$ stand for "ZFC + φ is consistent" and $\mathrm{Icon}(\mathrm{ZFC} + \varphi)$ stand for "there is an inner model of ZFC + φ". A typical consistency result takes the form

$$\mathrm{Con}(\mathrm{ZFC} + \mathrm{LC}) \rightarrow \mathrm{Con}(\mathrm{ZFC} + \varphi)$$

where LC denotes some large cardinal axiom. An *internal* consistency result takes the form

$$\mathrm{Icon}(\mathrm{ZFC} + \mathrm{LC}) \rightarrow \mathrm{Icon}(\mathrm{ZFC} + \varphi).$$

Received February 4, 2006.
The author wishes to thank the Austrian Science Fund (FWF) for its generous support.

Thus a statement φ is internally consistent iff $\text{Icon}(\text{ZFC} + \varphi)$ follows from $\text{Icon}(\text{ZFC} + \text{LC})$ for some large cardinal axiom LC.

A statement can be consistent without being internally consistent. An example is the statement that there are no transitive models of ZFC, which fails in any inner model, assuming there are inner models with inaccessible cardinals. Another example is:

For each infinite regular cardinal κ there is a nonconstructible subset of κ whose proper initial segments are constructible.

This can be forced over L, but does not hold in any inner model, assuming the existence of $0^{\#}$.

If the consistency of a statement without parameters is shown using set forcing, it is straightforward to prove its internal consistency by choosing a generic over a suitable inner model for a forcing which has only countably many subsets in that model. But this is not the case for statements that contain uncountable parameters or for statements whose consistency is shown through the use of class forcing. In these latter cases, questions of internal consistency and of internal consistency strength can be quite interesting, as we shall now see.

Easton's theorem revisited. Let Reg denote the class of infinite regular cardinals and Card the class of all infinite cardinals. An *Easton function* is a class function $F : \text{Reg} \to \text{Card}$ such that:

For all $\kappa \leq \lambda$ in Reg: $F(\kappa) \leq F(\lambda)$.
For all $\kappa \in$ Reg: $\text{cof}(F(\kappa)) > \kappa$.

Easton showed that if F is an Easton function definable in L, then there is a cofinality-preserving class forcing extension $L[G]$ of L in which $2^\kappa = F(\kappa)$ for all regular κ. We say that the model $L[G]$ *realises* the Easton function F.

Which Easton functions definable in L can be realised in an inner model? The following are some partial results, obtained jointly with Pavel Ondrejović [8].

THEOREM 2. *Suppose that $0^{\#}$ exists and F is an Easton function which is L-definable using parameters which are countable in V. Then there exists an inner model with the same cofinalities as L in which $2^\kappa = F(\kappa)$ for each infinite regular κ.*

COROLLARY 3. *The statement*

$$2^\kappa = \kappa^{++} \text{ for all infinite regular } \kappa$$

is internally consistent relative to the existence of $0^{\#}$.

Theorem 2 is proved as follows. Let C be the proper class of L-cardinals closed under F. Let P be the reverse Easton iteration of Easton products, where each Easton product realises the Easton function F on the interval between two adjacent elements of C. The main part of the argument is to show that a generic can be defined in $L[0^{\#}]$ by inductively choosing generics for the individual Easton products so as to cohere under maps obtained by shifting indiscernibles.

Internal consistency can sometimes be obtained when we allow uncountable parameters.

THEOREM 4. *Assume that* $0^{\#}$ *exists and* κ *is a regular uncountable cardinal. Then there is an inner model with the same cofinalities as L in which GCH holds below* κ *but fails at* κ.

Theorem 4 (as well as Theorem 6 below) is proved by first obtaining a generic for adding $(\beta^+)^L$ subsets of β for each L-regular $\beta \leq \kappa$, and then "stretching" the generic at κ for adding $(\kappa^+)^L$ subsets of κ to a generic for adding more than $(\kappa^+)^L$-many.

How badly can GCH fail in an inner model? The proof of the following uses the existence of a gap-1 morass in $L[0^{\#}]$.

THEOREM 5. *Assume that* $0^{\#}$ *exists and* κ *is a regular uncountable cardinal. Then there is an inner model with the same cofinalities as L in which* $2^\kappa = (\kappa^+)^V$.

GCH fails below κ in the inner model of Theorem 5. If we require that GCH hold below κ we obtain a weaker conclusion:

THEOREM 6. *Assume that* $0^{\#}$ *exists,* κ *is a regular uncountable cardinal and* α *is less than* $(\kappa^+)^V$. *Then there is an inner model with the same cofinalities as L in which GCH holds below* κ *and* $2^\kappa > \alpha$.

CONJECTURE. Assume the existence of $0^{\#}$. Then an Easton function F which is L-definable with parameters can be realised in an inner model M (having the same cofinalities as L) iff it satisfies:

$$F(\kappa) < (\kappa^{++})^V \text{ for all } \kappa \in \mathrm{Reg}^L.$$

The singular cardinal hypothesis. The analog of Cohen's result for the singular cardinal hypothesis is:

THEOREM 7 (Gitik [11]). *Suppose that K is an inner model satisfying GCH which contains a* totally measurable *cardinal* κ, *i.e., a cardinal* κ *of Mitchell order* κ^{++}. *Then there is a generic extension* $K[G]$ *of K in which* κ *is a singular strong limit cardinal and GCH fails at* κ.

Gitik also shows that a totally measurable cardinal is necessary. Now consider the following weak analogue of Easton's result for the singular cardinal hypothesis:

(Global Gitik) *GCH fails on a proper class of singular strong limit cardinals.*

The proof of the previous theorem shows:

THEOREM 8. *Suppose that K is an inner model satisfying GCH which contains a proper class of totally measurable cardinals. Then there is a generic extension* $K[G]$ *of K in which Global Gitik holds.*

Is Global Gitik internally consistent relative to large cardinals? In analogy to Easton's theorem, we might expect to show that the generic extension $K[G]$ of Theorem 8 can be obtained as an inner model. This is however not true for the natural choice of K. The following work uses the proof of Mitchell's covering lemma for K and is joint with Tomáš Futáš [7].

THEOREM 9. *Suppose that there is a # for a proper class of totally measurable cardinals and let K be the "natural" inner model with a class of totally measurable cardinals. (K is obtained by taking the least iterable mouse m with a measurable limit of totally measurable cardinals and iterating its top measure out of the universe.) Then there is no inner model of the form $K[G]$, where G is generic over K, in which Global Gitik holds.*

On the other hand, Futáš and I show that it is possible to choose K differently, so as to witness the internal consistency relative to large cardinals of Global Gitik:

THEOREM 10. *Suppose that there is an inner model containing a measurable limit κ of totally measurable cardinals, where κ is countable in V. Then there is an inner model in which Global Gitik holds.*

This is proved as follows. Using the countability of κ in V we obtain a generic over an inner model for making the singular cardinal hypothesis fail cofinally below κ, preserving the measurability of κ. Then the desired inner model is obtained by iterating κ past the ordinals.

What is the internal consistency strength of Global Gitik, i.e., what large cardinal hypothesis must hold in some inner model to obtain an inner model of Global Gitik?

Theorem 10 provides an upper bound. In analogy to the proof of the internal consistency relative to $0^{\#}$ of Easton's result, one would expect that a # for a proper class of totally measurables, a weaker assumption, would also suffice.

But unlike with Easton's result, it is possible that the internal consistency strength of Global Gitik is the same as its external consistency strength, i.e., just a proper class of totally measurable cardinals, without its #. The next result is an example of this unexpected phenomenon.

A cardinal κ is *Jonsson* iff every structure of cardinality κ for a countable language has a proper substructure of cardinality κ. By work of Mitchell [13], if there is a singular Jonsson cardinal then there is an inner model with a measurable cardinal. Conversely, if M is an inner model with a measurable cardinal then $M[G]$ has a singular Jonsson cardinal, when G is Prikry generic over M. But in fact an inner model with a singular Jonsson cardinal can be obtained inside M:

THEOREM 11. *Suppose that there is an inner model with a measurable cardinal. Then there is an inner model with a singular Jonsson cardinal.*

This is proved as follows: Let κ be measurable in an inner model M. Iterate M using the measure on κ to $M = M_0 \supseteq M_1 \supseteq \cdots$, and let M^* be M_ω. Then $\langle \kappa_n \mid n \in \omega \rangle$ produces a Prikry generic G over M^* and $M^*[G]$ is an inner model with a singular Jonsson cardinal.

Thus the internal consistency strength of a singular Jonsson cardinal is the same as its external consistency strength, that of one measurable cardinal. Is the situation similar with Gitik's Theorem 7? I.e., can Con be replaced with Icon in the implication Con(ZFC + there exists a totally measurable) $\rightarrow$ Con(ZFC + GCH fails at a singular strong limit)? Equivalently:

QUESTION. Suppose that there is an inner model with a totally measurable cardinal. Then is there an inner model in which the GCH fails at a singular strong limit cardinal?

Two more internal consistency results. Mirna Džamonja, Katherine Thompson and I [4, 9] have studied the global complexity of universal classes for certain types of structures. For a regular cardinal λ, we say that a poset P *omits λ chains* iff P contains no linearly ordered subset of cardinality λ. For a regular cardinal $\kappa \geq \lambda$, let $O(\kappa, \lambda)$ denote the collection of posets of cardinality κ which omit λ chains. Then the complexity of $O(\kappa, \lambda)$, written $K(\kappa, \lambda)$, is the smallest cardinality of a subset S of $O(\kappa, \lambda)$ such that every element of $O(\kappa, \lambda)$ can be embedded into an element of S. Thompson and I obtained the following "high complexity" internal consistency result.

THEOREM 12. *Assume that $0^\#$ exists. Suppose that F is an Easton function in L which is L-definable without parameters. Also suppose that λ is a parameter-free L-definable function which to each L-regular cardinal $\kappa > \omega$ associates a regular L-cardinal $\lambda(\kappa) \leq \kappa$. Then there is an inner model with the same cofinalities as L in which $2^\kappa = F(\kappa) = K(\kappa, \lambda(\kappa))$ for each L-regular $\kappa > \omega$.*

Natasha Dobrinen and I [3] have looked at costationarity of the ground model in the context of internal consistency. For cardinals $\kappa < \lambda$, κ regular and uncountable, let $P_\kappa(\lambda)$ denote the set of subsets of λ of cardinality less than κ. A result of Avraham-Shelah is that if G is generic over V for a ccc forcing that adds a real, then $P_\kappa(\lambda) \setminus V$ is stationary in $V[G]$ for all $\kappa < \lambda$. Dobrinen and I show:

THEOREM 13. *It is consistent relative to the existence of a proper class of ω_1-Erdős cardinals that GCH holds and $P_\kappa(\lambda) \setminus V$ is stationary in $V[G]$ for all regular κ greater than ω_1, where G is generic over V for ω_1-Cohen forcing.*

The property expressed in this theorem is internally consistent relative to a # for a proper class of ω_1-Erdős cardinals provided we restrict κ to be a successor cardinal; otherwise the result can be achieved with a "2-superstrong" cardinal, and appears to require at least the internal consistency of a proper class of Woodin cardinals.

THE INNER MODEL HYPOTHESIS

Recall that a statement is internally consistent iff it holds in some inner model. Therefore the meaning of internal consistency depends on what inner models exist. If we enlarge the universe, it is possible that more statements become internally consistent.

The *inner model hypothesis* (IMH) asserts that the universe has been maximised with respect to internal consistency in the following sense: If a statement φ without parameters holds in an inner model of some outer model of V (i.e., in some model compatible with V), then it already holds in some inner model of V. Equivalently: If φ is internally consistent in some outer model of V then it is already internally consistent in V.

This is formalised as follows. Regard V as a model of Gödel-Bernays class theory, endowed with countably many sets and classes. Suppose that V^* is another such model, with the same ordinals as V. Then V^* is an *outer model of* V (V is an *inner model of* V^*) iff the sets of V^* include the sets of V and the classes of V^* include the classes of V. V^* is *compatible with* V iff V and V^* have a common outer model.

The strong inner model hypothesis, introduced later, has considerable large cardinal strength. It also implies the negation of CH. This shows that a considerable part of our basic assumption, that of the internal consistency of large cardinals, as well as a solution to the continuum problem can be derived from the hypothesis that internal consistency has been maximised.

Other strong absoluteness principles are discussed in [1], [12] and [14]. All of these principles are restricted to set-generic extensions of V. One of the motivations for the inner model hypothesis is to obtain a strong absoluteness principle which applies to arbitrary extensions of V. For other strong absoluteness principles of this type we refer the reader to [6].

We first observe that the inner model hypothesis can be regarded as a second-order generalisation of:

Parameter-free Lévy-Shoenfield absoluteness. Suppose that φ is a Σ_1 sentence true in an extension of V. Then φ is true in V.

Recall that Σ_1 formulas are *persistent* in the sense that if such a formula is true in a transitive set, it is also true in all larger transitive sets. We consider persistent *second-order* formulas: A formula is *persistent* Σ^1_1 iff it is of the form

$$\exists M (M \text{ is a transitive class and } M \vDash \psi),$$

where ψ is first-order. Clearly if V satisfies a persistent Σ^1_1 formula then so do all of its outer models.

Theorem 14. *The following are equivalent*:

(a) (Parameter-free persistent Σ^1_1 absoluteness). *If a parameter-free persistent Σ^1_1 formula is true in an outer model of V then it is true in V.*
(b) (Inner model hypothesis). *If a first-order sentence is true in some model compatible with V then it is true in some inner model of V.*

PROOF. (a) $\rightarrow$ (b): Suppose that the first-order sentence φ is true in some model M compatible with V. Then M is an inner model of some outer model V^* of V. The existence of M can be expressed as a parameter-free persistent Σ^1_1 formula, and therefore by (a), φ holds in an inner model of V.

(b) $\rightarrow$ (a): Suppose that the parameter-free persistent Σ^1_1 formula $\varphi \equiv \exists M(M$ is a transitive class and $M \vDash \psi)$ holds in some outer model V^* of V. Then ψ is true in a definable transitive class of some model compatible with V and therefore by hypothesis of some inner model of V. It follows that φ holds in V. ⊣

We now state some consequences of the inner model hypothesis and summarise what is known about its consistency strength.

THEOREM 15. *The inner model hypothesis implies that for some real R, ZFC fails in $L_\alpha[R]$ for all ordinals α. In particular, there are no inaccessible cardinals, the reals are not closed under # and the singular cardinal hypothesis holds.*

PROOF. By a theorem of Beller-David (see [2]) there is an outer model V^* of V containing a real R such that $L_\alpha[R]$ fails to satisfy ZFC for each ordinal α. By the inner model hypothesis there is such a real R in V, as this property of R is absolute to any inner model containing R. ⊣

THEOREM 16. [10]

(a) *The inner model hypothesis implies the existence of an inner model with measurable cardinals of arbitrarily large Mitchell order.*
(b) *The consistency of the inner model hypothesis follows from the consistency of a Woodin cardinal with an inaccessible above.*

Theorem 16 (a) is proved as follows. If the conclusion fails then any ordinal α of sufficiently large cofinality whose cofinality is less than its cardinality is singular in Mitchell's core model K. We use this property as in [5] to show that for each n, one can force a CUB class of ordinals all of whose elements of sufficiently large cofinality "drop" at least n times along a canonical square sequence in K. Applying the IMH, there are inner models M_n where this happens relative to the K of M_n. By comparing these ω-many different K's, we get a CUB class of ordinals which "drop" infinitely many times, a contradiction. A key technical point is to show that there are many fixed-points of this comparison, which requires replacing K by its iterate K', obtained by applying each order 0 measure of K exactly once.

Theorem 16 (b) is proved by using the Woodin cardinal to get a model in which the theory of $(L(d), \in, d)$ is constant on a cone of Turing degrees d. The desired model of the IMH is the minimal model of set theory containing a real x whose Turing degree serves as the base for such a cone.

Absolute parameters and the strong inner model hypothesis. How can we introduce parameters into the inner model hypothesis? The following result shows that inconsistencies arise without strong restrictions on the type of parameters allowed.

PROPOSITION 17. *The inner model hypothesis with arbitrary ordinal parameters or with arbitrary real parameters is inconsistent.*

PROOF. With arbitrary ordinal parameters, inconsistency results from the fact that $\aleph_1$ can be countable in an outer model. To obtain an inconsistency with arbitrary real parameters, argue as follows. By Theorem 15, even the parameter-free version implies the existence of a real R such that ω_1 equals ω_1 of $L[R]$. Then the statement "ω_1 of $L[R]$ is countable" (with parameter R) holds in some outer model but not in any inner model. $\dashv$

So instead we consider *absolute parameters*, as in [6]. For any set x, the *hereditary cardinality* of x, denoted $\mathrm{hcard}(x)$, is the cardinality of the transitive closure of x. If V^* is an outer model of V, then a parameter p is *absolute between V and V^** iff V and V^* have the same cardinals $\leq \mathrm{hcard}(p)$ and some parameter-free formula has p as its unique solution in both V and V^*.

Inner model hypothesis with locally absolute parameters. Suppose that p is absolute between V and V^* and φ is a first-order sentence with parameter p which holds in an inner model of V^*. Then φ holds in an inner model of V.

THEOREM 18. [10] *The inner model hypothesis with locally absolute parameters is inconsistent.*

This is proved by considering the "intersection" $\langle C_\alpha \mid \alpha < \beth_\omega^+\rangle$ of the canonical $\square_{\beth_\omega}$-sequences of the $L[x]$ for reals x such that $x^\#$ does not exist. For each n, this parameter is absolute between V and an outer model containing a CUB subset of $\beth_\omega^+$ all of whose elements α of sufficiently large cofinality have the property that C_α has ordertype at least $\beth_n$. Applying the inner model hypothesis with absolute parameters there are such CUB sets in V, which of course have a nonempty intersection. This gives a contradiction.

To obtain the strong inner model hypothesis, we impose more parameter absoluteness. We say that the parameter p is (*globally*) *absolute* iff there is a parameter-free formula which has p as its unique solution in all outer models of V with the same cardinals $\leq \mathrm{hcard}(p)$ as V.

STRONG INNER MODEL HYPOTHESIS (SIMH). Suppose that p is absolute, V^* is an outer model of V with the same cardinals $\leq \mathrm{hcard}(p)$ as V and φ is a first-order sentence with parameter p which holds in an inner model of V^*. Then φ holds in an inner model of V.

THEOREM 19. *Assume the SIMH. Then CH is false. In fact, $2^{\aleph_0}$ cannot be absolute and therefore cannot be $\aleph_\alpha$ for any ordinal α which is countable in L.*

PROOF. Suppose that $\kappa = 2^{\aleph_0}$ were absolute. Then the same would hold for κ^+. In a cardinal-preserving forcing extension of V there are κ^+ reals.

By the SIMH, there is an inner model of V with κ^+ reals. But this is a contradiction, as an inner model of V cannot have more reals than V.

The last part of the conclusion follows as the existence of $0^\#$ implies that any L-countable ordinal is absolute. ⊣

THEOREM 20. [10] *The strong inner model hypothesis implies the existence of an inner model with a strong cardinal.*

The proof of this theorem is similar to that of Theorem 18, using the canonical $\square_\kappa$ sequence of the core model K for a strong cardinal, where κ is the ω-th limit cardinal which is a "cut point" of K.

REMARKS. It is conjectured that core model theory can be extended from strong cardinals to Woodin cardinals, without any large cardinal assumptions. If this is the case, then the strong inner model hypothesis very likely implies the existence of an inner model with a Woodin cardinal. David Asperó and I observed that the consistency of the SIMH for the parameter ω_1 follows as in Theorem 16 (b) from that of a Woodin cardinal with an inaccessible above. I showed that for any finite set of absolute parameters, the version of the SIMH where V^* is required to be a *set-generic* extension of V is consistent for those parameters.

QUESTIONS. 1. What is the exact consistency strength of the inner model hypothesis? 2. Is the strong inner model hypothesis consistent relative to large cardinals?

REFERENCES

[1] J. BAGARIA, *Axioms of generic absoluteness*, ***Colloquium Logicum '02*** (Z. Chatzidakis, P. Koepke, and W. Pohlers, editors), Lecture Notes in Logic, vol. 27, ASL and AK Peters, 2006, pp. 28–47.

[2] A. BELLER, R. JENSEN, and P. WELCH, ***Coding the universe***, London Mathematical Society, Lecture Note Series, vol. 47, Cambridge University Press, 1982.

[3] N. DOBRINEN and S. FRIEDMAN, *Costationarity of the ground model*, ***The Journal of Symbolic Logic***, vol. 71 (2006), no. 3, pp. 1029–1043.

[4] M. DŽAMONJA, S. FRIEDMAN, and K. THOMPSON, *Global complexity results*, ***Set theory and its applications*** (Alessandro Andretta, editor), Quaderni di Matematica, Napoli, to appear.

[5] S. FRIEDMAN, *New Σ^1_3 facts*, ***Proceedings of the American Mathematical Society***, vol. 127 (1999), pp. 3707–3709.

[6] ———, *Stable axioms of set theory*, ***Set theory, Centre de recerca Matemàtica, Barcelona, 2003–2004*** (Joan Bagaria and Stevo Todorcevic, editors), Trends in Mathematics, Birkhäuser Verlag, 2006.

[7] S. FRIEDMAN and T. FUTÁŠ, *The internal consistency of the singular cardinal hypothesis*, in preparation.

[8] S. FRIEDMAN and P. ONDREJOVIĆ, *The internal consistency of Easton's theorem*, in preparation.

[9] S. FRIEDMAN and K. THOMPSON, *Internal consistency and embedding complexity*, in preparation.

[10] S. Friedman, P. Welch, and W. H. Woodin, *On the consistency strength of the inner model hypothesis*, to appear.

[11] M. Gitik, *The negation of SCH from* $o(\kappa) = \kappa^{++}$, ***Annals of Pure and Applied Logic***, vol. 43 (1989), no. 3, pp. 209–234.

[12] J. Hamkins, *A simple maximality principle*, ***The Journal of Symbolic Logic***, vol. 68 (2003), no. 2, pp. 527–550.

[13] W. Mitchell, *An introduction to inner models and large cardinals*, a chapter for the ***Handbook of Set Theory***, to appear.

[14] W. H. Woodin, ***The axiom of determinacy, forcing axioms, and the nonstationary ideal***, Walter de Gruyter, 1999.

KURT GÖDEL RESEARCH CENTER
UNIVERSITY OF VIENNA
WÄHRINGER STRASSE 25
A-1090 VIENNA, AUSTRIA
E-mail: sdf@logic.univie.ac.at