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GÖDEL'S PROGRAM REVISITED
PART I: THE TURN TO PHENOMENOLOGY

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Abstract. Convinced that the classically undecidable problems of mathematics possess determinate truth values, Gödel issued a programmatic call to search for new axioms for their solution. The platonism underlying his belief in the determinateness of those questions in combination with his conception of intuition as a kind of perception have struck many of his readers as highly problematic. Following Gödel's own suggestion, this article explores ideas from phenomenology to specify a meaning for his mathematical realism that allows for a defensible epistemology.

§1. Introduction. Gödel's Program is the proposal to search for new *axioms* of set theory that are strong enough to answer questions left undecided by the standard axioms ZFC (Zermelo-Fraenkel with the axiom of choice). Its inception coincides, arguably, with the publication of a remarkable article in 1947 where Gödel discusses mathematical and philosophical aspects of the most famous undecidable problem of classical mathematics, the continuuum hypothesis (CH).[1] At present CH is still open, but a number of important developments, some of them quite recent, have shed new light on the question. Naturally, one would expect to learn something about the feasibility of both the Program and the underlying philosophy from the mathematical results that have been obtained. The answers, however, are anything but obvious. Under suitable interpretations some of these results lend support to Gödel's view whereas others are difficult to reconcile with the outspoken platonism emerging from a cursory reading of his writings. One option is to evaluate the

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[1][30]. Strong axioms of infinity had already made their debut in the investigation of transfinite order types by Hausdorff [41] and by Mahlo [74], [75]. Two decades later Sierpiński and Tarski [92] and Zermelo [108] used *inaccessible cardinals* for obtaining natural models for the standard axioms. What sets Gödel's article apart from this earlier work is the explicit proposal to employ strong axioms of infinity systematically to solve outstanding problems of classical mathematics. A first step in that direction had been the extension of the Lebesgue measure to all sets of real numbers by Ulam [99].

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current status of Gödel's Program independently of his philosophical ideas. The situation would then be analogous to that of Hilbert's Program after the discovery of the incompleteness theorems where the majority of those who carried on the program in a modified form no longer subscribed to the Kantian perspective from which Hilbert had originally justified his agenda. Similarly, of Gödel's philosophical standpoint only a general inclination toward mathematical realism appears to influence contemporary set theory.

On the other hand, as the ongoing debate about new axioms shows, the disagreements for the most part are philosophical in nature.[2] Here Gödel's ideas about intuition and mathematical objects are actively discussed, but given the terseness and cryptic style of his major statements, there are serious doubts about the viability of his position. The purpose of this article is to remove some of these doubts by reconsidering Gödel's thought in relation to Husserl whom he studied intensively in the latter part of his life.

It may seem that Leibniz would be the natural point of reference for gaining an insight into Gödel's philosophical world view given the fact that Gödel described his own philosophy as "a monadology with a central monad, namely, God".[3] Indeed several of the entries which Gödel included in a list with the title "My philosophical viewpoint" are of a decidedly Leibnizian flavor.[4] Of particular interest with respect to his mathematical program are the following two showing a strong rationalistic optimism about the powers of human reason.

> There are systematic methods for the solution of all problems (also art etc.).
>
> There is incomparably more knowable *a priori* than is currently known.

The expression "systematic methods for the solution of all problems" in the first statement is probably alluding to Leibniz's idea of a *characteristica universalis*: a calculus of reasoning capable of expressing and handling any question in science as well as philosophy. Already in 1944, at the end of his paper on Russell's mathematical logic, Gödel remarked, rather enigmatically, that this is by no means a utopian project, and that it may lead to the systematic solution of mathematical problems via an analysis of the concepts occurring in them.[5] At the same time he was convinced that a *characteristica universalis* as originally conceived by Leibniz was unfeasible since, by the incompleteness theorems, "any systematic procedure for solving problems

[2]See for example [18] and the references given therein.

[3]Wang [102, p. 309] reports that Gödel made this statement in March 1976, i.e., almost two decades after he had begun his systematic study of Husserl.

[4]This list was found among Gödel's papers in an undated bundle of loose pages written in Gabelsberger shorthand, probably around 1960. A transcription by Cheryl Dawson is reproduced in [102, 9.4.17].

[5][28, p. 152f]. According to [102, p. 76, p. 87] Gödel had been studying Leibniz intensively between 1943 and 1946.

of all kinds must be nonmechanical".[6] Thus in order to come to terms with his favored candidate for such a procedure in mathematics, the formation of stronger and stronger axioms of infinity,[7] he was forced to look beyond Leibniz. One reason why Gödel turned to Husserl at this point may be found in his disagreement with Leibniz on the nature of analyticity.[8]

In the *Monadology*[9] Leibniz describes the analysis of concepts as a mechanical breaking down into atomic constituents comparable to the decomposition of a number into its prime factors. Proceeding this way one ultimately arrives at primitive principles, or *identical propositions* as Leibniz calls them, whose opposite involves an express contradiction. They enunciate containment relations among concepts of the form 'AB is A' or 'A is A' and could thus be called *analytic* (even though Leibniz himself never uses that term). Moreover, for Leibniz all mathematical axioms reduce to those that are identical, or immediate consequences of definitions.[10] A rather different view of analyticity is put forward by Gödel in his paper on Russell [28, p. 151]. "A proposition is called analytic if it holds 'owing to the meaning of the concepts occurring in it', where this meaning may perhaps be undefinable (i.e., irreducible to anything more fundamental)".[11] From this Gödel distinguishes another sense of *analytic* which he calls *tautological*. This is to be understood "in the purely formal sense that the terms occurring can be defined (either explicitly or by rules of eliminating them from sentences containing them) in such a way that the axioms and theorems become special cases of the law of identity and disprovable propositions become negations of that law" [op. cit., p. 150f]. Gödel also remarks that the axioms may reasonably be called analytic when that expression is used in the former sense.[12] By contrast their reduction to a special case of the law of identity could be effected only in virtue of the meanings of the terms occurring therein, which can never be completely expressed in a set of formal rules.

The distinction between analytic and tautological with respect to mathematical axioms also appears in the Gibbs Lecture.[13] There Gödel reiterates

[6] [102, 6.3.16], [31, p. 316]. In Leibniz's conception of the *characteristica universalis*, "all composite notions in the whole world are reduced to a few simple ones as their alphabet; and by the combination of such an alphabet, in turn, a way is made of finding in time, by an ordered method, all things with their theorems and whatever is possible to investigate concerning them" [68, vol. 1, p. 57f].

[7] [35, Remark 3, p. 306], [102, 6.3.15].

[8] I am not claiming that this is the only reason. Apparently Gödel regarded certain elements in Husserl's phenomenology as realizations of Leibnizian ideas (cf. the reference in footnote 93). However, the relationship between Leibniz and Husserl is a complex issue falling outside the scope of this article.

[9] [69, §§33–35], [68, vol. 6, p. 612].

[10] [68, vol. 5, p. 92].

[11] It should be emphasized that Gödel speaks of the meaning of concepts, and not of linguistic expressions as Quine does in his criticism of the positivist definition of analyticity.

[12] See also [35, Remark 2, p. 306].

[13] [31, p. 320f].

that the axioms express relations between concepts that are by no means tautologies, but still follow from the meaning of the primitive terms under consideration. As an indication for the inappropriateness of the label 'tautological' he mentions the possibility that an analytic proposition may well be undecidable. Indeed, in connection with the difficulty with the continuum hypothesis and other classically unsolvable problems Gödel had speculated four years earlier that

> there are also deeper reasons behind it and that a complete solution of these problems can be obtained only by a more profound analysis (than mathematics is accustomed to give) of the *meanings* of the terms occurring in them (such as "set", "one-to-one correspondence", etc.) and of the axioms underlying their use. [30, p. 517f], emphasis added. Compare [28, p. 152].

In summary, an ideal solution of the continuum problem for Gödel would consist in the discovery of new *axioms* that are (i) 'implied' by the meanings of the basic terms and (ii) strong enough to decide CH. But what kind of approach should one take for making the basic terms more precise and for gaining secure insight into the fundamental relations obtaining among them?

Gödel offers a suggestion in the shorthand manuscript [33]. Clearly the required procedure cannot amount to explicit definitions for concepts or proofs from axioms, for this would force us into an infinite regress involving other undefined concepts. It must therefore, at least to a large extent, consist in a "clarification of meaning" as opposed to explicit defining. Gödel saw in Husserl's phenomenology the beginning of a science which claims to possess a systematic method for such a clarification of meaning.

> Here clarification of meaning consists in focusing more sharply on the concepts concerned by directing our attention in a certain way, namely onto our own acts in the use of these concepts, onto our powers in carrying out our acts, etc. [33, p. 383].

But he cautions that phenomenology is not a science in the customary sense.

> Rather it is [or in any case should be] a procedure or technique that should produce in us a new state of consciousness in which we describe in detail the basic concepts we use in our thought, or grasp other basic concepts hitherto unknown to us. [ibid.].

Gödel does not explain the phenomenological method in any detail, nor does he give any concrete examples for its successful application. And his presentation leaves unclear *how* an introspective examination of one's own mathematical experiences might yield new insight into the basic concepts employed.

This raises a number of questions. What exactly attracted Gödel to phenomenology? What kind of contribution (if any) can phenomenology make in the search for new axioms? In this article I present an answer to the first question by aligning Gödel's ideas about intuition and mathematical objects with certain themes from phenomenology. More specifically, using a phenomenological account of intuition, I will outline a strategy for supplying a defensible epistemology to Gödel's ontological beliefs. The possibility of such an epistemology hinges on a re-interpretation of Gödel's realism that can accommodate the idealist elements in his thought. It will be argued on both systematic and textual grounds that Gödel regarded Husserl's phenomenology as a suitable framework for that re-interpretation.

The ideas in this paper originate in a series of lectures delivered in 1999 at Delphi, Cracow, Stanford and Berkeley. In preparing them for publication I have relied heavily on Hao Wang's invaluable account of his conversations with Gödel,[14] and on material from Gödel's *Nachlass* that appeared for the first time in the meticulously edited *Collected Works*. I wish to thank the Institute for Advanced Study for granting permission to quote from Gödel's shorthand notes on Husserl, and I am grateful to Charles Parsons for making Cheryl Dawson's transcription of those notes available. The interpretation of Husserl which I found most useful for elucidating Gödel's thought owes a lot to the stimulation and help I received from writings by—among others—Dagfinn Føllesdal, Izchak Miller, Jitendranath Mohanty, Gian Carlo Rota, Robert Sokolowski and Ernst Tugendhat. Husserl's style, verbose and ponderous at times, poses special difficulties for translators, and in that respect Dorion Cairns' *Guide for Translating Husserl*[15] proved more than helpful. In addition his English translation of the *Formal and Transcendental Logic* is used throughout the article.

§**2. Two versions of mathematical realism.** At first sight Gödel seems to embrace a radical form of mathematical realism modelling abstract objects in direct analogy with physical objects. The earliest published expression of this appeared in his 1944 article on *Russell's Mathematical Logic* where classes and concepts are conceived "as real objects": classes as "pluralities of things", and "concepts as the properties and relations of things existing independently of our definitions and constructions".[16] In Gödel's view such

[14][102]. There is of course the serious question about the accuracy of Wang's report. Evidently Wang goes beyond the record of the actual conversations selecting, interpreting, and extrapoliting from some of Gödel's observations "along paths that are congenial to [himself]" (xii). Moreover, he cautions that his notes are "very incomplete" (130), and given that he does "not have a verbatim record of Gödel's own words, there are bound to be misrepresentations" (129)—in particular with respect to Gödel's remarks about Husserl whom Wang, by his own admission (142), has not studied thoroughly. Nevertheless, it is my contention that the reconstructed passages quoted below add up to a coherent picture when put in relation to phenomenology. (For an appraisal of Wang's role as philosopher and as intepreter of Gödel see [83].)

[15][3].

[16][28, p. 137]. In his own words, Gödel had been a mathematical realist since his student days in Vienna (cf. his unsent response to the Grandjean questionaire [16, p. 447] and the

a conception of classes is needed in order to provide the proper interpretation of the resulting theory (for classes as well as sets).

> It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions. [ibid.].

Another trait which is characteristic for Gödel's realism is the move from the objective existence of mathematical objects (and the obtaining of certain primitive relations among them) to the determinacy of truth value of all mathematical assertions about them. Thus Gödel regarded it as an immediate 'corollary' to his realist standpoint that Cantor's continuum hypothesis CH and other recalcitrant problems of set theory must possess determinate truth values regardless of our ability to decide their truth or falsity. In the "well-determined reality" described by the axioms of set theory, "Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality" [30, p. 520].

The reality Gödel had in mind is the *cumulative hierarchy* which is the stratification of the universe of all sets obtained through transfinite iteration of the power set operation, i.e., the operation of forming the set of *all* subsets of a given set.[17] This is closely bound up with the *iterative concept of set* according to which "a set is anything obtainable from the integers (or some other well-defined objects) by iterated application of the operation 'set of', and not something obtained by dividing the totality of all existing things into two categories."[18] Indeed, under the *iterative concept of set* every set occurs

two letters reproduced in [101, p. 8–11]). A more ambivalent attitude, however, surfaces in an invited lecture to the Mathematical Association of America in 1933 where he pointed out (cf. [25, p. 19]) that the axioms of mathematics "if interpreted as meaningful statements, necessarily presuppose a kind of platonism, which cannot satisfy any critical mind". Davis [11] describes this as one of several discontinuities in Gödel's metaphysical stance. The mystery may be solved—if at all—by a study of the *Nachlass* and by probing into Gödel's personal circumstances around the year 1933.

[17]The levels of the c umulative hierarchy are defined as follows: $V_0 = \emptyset$, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, and $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for limit ordinals λ. It is easily verified that for $\alpha < \beta$, $V_\alpha \subset V_\beta$.

[18][30, p. footnote 12] expressly allows transfinite iterations of the operation 'set of'. In footnote 13 he concedes that this operation can only be paraphrased rather than explicitly defined (at least in the present state of knowledge). "But as opposed to the concept of set in general (if considered as primitive) we have a clear notion of this operation." Curiously, this sentence is ommitted in the corresponding footnote 14 of [34] where in its place it is emphasized that the expressions used for paraphrasing the operation "set of x's" are conceived of as something which exists in itself no matter whether we can define it in a finite number of words (so that random sets are not excluded).

in a level of the cumulative hierarchy.[19]

Gödel hoped to obtain a clearer description of this hierarchy by strengthening the ontological commitments of set theory via "new axioms which assert the existence of still further iterations of the operation 'set of'. These axioms can also be formulated as propositions asserting the existence of very great cardinal numbers." [ibid.]. The large cardinals mentioned by Gödel, *inaccessible* and *Mahlo cardinals*, are closure points for set theoretic operations defined from the standard axioms.[20] Their existence is intrinsically plausible on the basis of Cantor's *Absolute*, the doctrine that the universe of all sets V is beyond any determination.[21] The formal counterpart of this is the *reflection principle* which says that any set theoretic proposition that holds in V already holds in a suitable initial segment V_α—for otherwise one would have a characterization of V. Written as a scheme in the language of set theory, this is actually provable in ZF (ZFC without the axiom of choice), and therefore it must itself reflect to initial segments of V. The existence of levels of V allowing reflection with second order parameters is a strong axiom of infinity, and the ordinals indexing such levels are known as *indescribable cardinals*.[22] As the natural continuation of the Mahlo hierarchy [70], and in view of the generalizations of the theory of admissible sets and inductive definability afforded by indescribability, axioms asserting the existence of indescribable cardinals[23]

> show clearly, not only that the axiomatic system of set theory as known today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which are only the natural continuation of the series of those set up so far. [30, p. 520].

In the revised edition of his article on Cantor's Continuum Problem,[24] Gödel considered also stronger axioms of infinity that had been investigated by that time. The most important of these, the axiom of *measurable cardinals*, forms the beginning of a hierarchy of large cardinal concepts that have figured prominently in the modern development of set theory. At some point Gödel's belief in the existence of such cardinals was based on arguments by

[19] Formally, by the axiom of foundation, $V = \bigcup_{\alpha<ORD} V_\alpha$ where $V = \{x : x = x\}$.

[20] An uncountable cardinal κ is inaccessible if it cannot be reached 'from below' by the power set operation and by forming unions of smaller cardinals of length $< \kappa$. P. Mahlo isolated the large cardinals bearing his name while studying thinning operations on the class of inaccessible cardinals [74], [75]. (However, he did not use the power set axiom, but rather cardinal limit processes. His hierarchies are thus all 'weak' hierarchies.)

[21] [4, §5].

[22] [40]. Formally, for a class of formulas Γ in the language of set theory extended by higher-order variables, a cardinal κ is Γ-*indescribable* if for any formula $\varphi(X)$ in Γ and any $A \subset V_\kappa$: if $\varphi(X)$ holds in the structure $\langle V_\kappa, \in, A\rangle$, then there is a $\lambda < \kappa$ such that $\varphi(X)$ already holds in the structure $\langle V_\lambda, \in, A \cap V_\lambda\rangle$.

[23] Cf. [64, §6] for a summary of these results and further references.

[24] [34, footnote 20].

analogy appealing to the uniformity of the universe of sets: "The universe of sets does not change its character substantially as one goes from smaller to larger sets or cardinals; that is, the same or analogous states of affairs appear again and again (perhaps in more complicated versions)."[25] Therefore, any 'reasonable' property possessed by the smallest infinite cardinal $\aleph_0$ should also be found in cardinals greater than $\aleph_0$. For measurable cardinals κ, the relevant property is the existence of a κ complete ultrafilter on the power set of κ which is provable in ZFC in the case $\kappa = \aleph_0$. The difficulty here is to make precise what one means by 'reasonable property'. Consequently, arguments by analogy are not found as compelling as those invoking Cantor's Absolute.[26]

Nowadays definitions of measurable cardinals and their generalizations are phrased in terms of *elementary embeddings*, i.e., truth preserving transformations of the universe V into a subclass equipped with the standard $\in$ relation. A cardinal κ is measurable if and only if there is an elementary embedding j of V into a transitive target model M with κ being the least ordinal moved by j (the *critical point* of j). Progressively stronger large cardinal notions such as *supercompact cardinals* are obtained by demanding yet more resemblance between M and V. It turns out that through elementary embeddings a connection with the Absolute can be established[27] lending support to Gödel's contention that "in the last analysis every axiom of infinity should be derived from the (extremely plausible) principle that V is undefinable."[28]

[25][102, 8.7.5]. See also Solovay's account of his conversations with Gödel in [14, p. 19], the revised footnote 20 of September 1966 in [14, p. 260], and note 20 added to [27] in August 1965 in [14, p. 97].

[26]Inaccessible cardinals are a case in point. Their existence is usually justified by direct appeal to the reflection principle rather than by pointing out that $\aleph_0$ is closed under the power set operation and the formation of finite unions.

[27]One way to see this is by recalling that the indescribability of an inaccessible cardinal κ is equivalent to the existence of elementary emdeddings mapping transitive models of set theory of size κ into models which are sufficiently close to V [42, §1]. Because indescribable cardinals are themselves reflection points, suitable versions of this should should reflect back to V.

It is also worth noting that in proving the transcendence over predecessors of a large cardinal defined from an embedding one often exploits the amount of reflection inherent in elementary embeddings.

As for the direct way of obtaining a measurable cardinal by switching quantifiers in the elementary embedding characterization of indescribability, it must be pointed out that an analogous switch in the definition of supercompactness leads to the inconsistent principle claiming the existence of an emebdding $j\colon V \longrightarrow V$ [67].

[28][102, 8.7.16]. In this context Gödel mentions work by Reinhardt on generalized reflection principles extending Ackerman's set theory A [87]. According to Wang, Gödel evidently believed that Reinhardt's addition to A makes the existence of measurable cardinals plausible on the basis that every 'structural' property of V applying only to elements of V determines a set. (See [102, sec. 8.7] for more details.) However, this contrasts with the remark in [86, fn. 1]

Soon after Cohen had established the unprovability of CH from the standard axioms, it became clear that the *forcing* method which he had developed for his argument is sufficiently universal so that no large cardinal axiom of the kind discussed so far can resolve the continuum problem.[29] However, large cardinals are not the only means for strengthening the standard axioms. More generally, Gödel envisaged that "an axiom which (similar to Hilbert's completeness axiom in geometry) would state some maximum property of the system of all sets" might resolve CH. Only an axiom of this kind, he maintained, would harmonize with the 'maximal' iterative concept of set that allows 'arbitrary' subsets in the formation of the cumulative hierarchy.[30] Similar ideas are put forward by Gödel in his correspondence with Ulam during the 1950s.[31] There he elaborates on the role of von Neumann's axiom—the assertion that any set which does not, in a certain well-defined way, imply an inconsistency exists—as the kind of principle that might settle the continuum problem. Unfortunately, in this vague and suggestive form, maximality becomes a treacherous issue. For one thing nobody has ever produced an adequate formalization, and arguments appealing to maximality in one form or another may lead to to incompatible conclusions.[32] Nevertheless, there is now a class of axioms, *forcing axioms*, which may be conceived of as generalizations of the Baire category theorem in the spirit of maximality and which imply that the size of the continuum is $\aleph_2$. These principles also have many appealing structural consequences for $P(\omega_1)$. But so far there is no consensus among set theorists whether forcing axioms are intrinsically plausible and in fact 'true'.[33]

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that Gödel found Reinhardt's analysis "rather unsatisfactory". For the crucial ingredient in that analysis which gets one past indescribables to measurable cardinals (and beyond) consists in considering not only classes like $\{V\}, P(V), P(P(V))$ etc. but also 'imaginative projections' of the latter. As Reinhardt admits, that makes the notion of structural property relative to the particular representation of 'imaginary situations', and the reflection principles in the ensuing intensional class theory no longer appear to be clearly implied by Cantor's Absolute. See [86, sec. 5] for the motivation to go beyond Cantor's intentions. Further elaborations within higher-type intensional logics appeared in [88].

[29][71]. This was independently noticed by Cohen.

[30][34, footnote 23]. This is diametrically opposed to the remarks at the end of [26]. There the possibility that all sets belong to the constructible universe L (the minimal model of set theory) is entertained as a candidate for a new axiom on the grounds that it "seems to give a natural completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way." M. Davis (cf. [13, p. 31, footnote 21]) takes this as indicating that Gödel's platonism regarding sets may have evolved more gradually than other statements of his would suggest.

Hilbert's completeness axiom, formulated in second-order logic, characterizes Euclidean geometry as the maximal structure (in terms of containment) satisfying the other axioms in Hilbert's system.

[31]Cf. [17, p. 295f] and [102, 8.3.8].

[32]See [45, sec. 3] for an example of such a scenario involving the axiom of choice.

[33]Arguments for the naturalness of these principles are presented in [1].

The post-Cohen era has witnessed a proliferation of models of set theory made possible by combined applications of forcing (i.e., enlarging a given model by adjoining new sets) with the method originating in [26] for constructing submodels of a given model. To some degree that has undermined the notion of *the* universe of set theory. By contrast, the idea of a standard model for arithmetic remains unchallenged. Interestingly, a different structure has emerged in contemporary set theory which is arguably as robust and canonical as the natural numbers; it consists of the large cardinal axioms *linearly* ordered according to their consistency strength.[34] The linearity of this ordering is surprising given the ostensible disparity of the principles for generating large cardinals.[35] In addition the scale of large cardinals serves as a measuring rod for gauging the logical strength of propositions that are independent of the standard axioms—including propositions that do not belong to set theory proper.[36] This provides a sense in which Gödel's conviction that large cardinal axioms are the key to independence has been vindicated.

A remarkable aspect of this vindication concerns undecidable statements of descriptive set theory about definable point sets in Euclidean spaces and their characteristic properties. Some of the central ideas in this area originate in Cantor's project of an arithmetization of the continuum, but the subject as a whole underwent a decisive transformation once it was realized that the behavior of definable sets of a given logical complexity is controlled by the existence of winning strategies for one of the players in infinite two-person games of (roughly) the same complexity. For example, if in every game with projective pay-off set one of the players has a winning strategy (*projective determinacy*), the elementary theory of the first two levels of the projective hierarchy extends to a structure theory for all levels providing the kind of

[34] Axiom A_1 has greater consistency strength than axiom A_2 if it is provable in arithmetic that the consistency of $ZFC + A_1$ implies the consistency of $ZFC + A_2$. Modulo equiconsistency this defines a partial order on the class of set theoretic principles.

[35] Strictly speaking linearity has not yet been verified in all cases of interest, but there is no reason to doubt that it will be accomplished. It is also worth noting that there are non-contrived examples of incomparable strong axioms of infinity. For instance, working in ZFC + CON(ZFC + "There exists a measurable cardinal"), consider the theories T_1 consisting of $ZFC + \forall k < \omega\,(\neg\,CON_k($ZFC + "There exists a measurable cardinal"$) \longrightarrow \neg\,CON_{2^{2^{2^k}}}($ZFC + "There exists an inaccessible cardinal"$))$ and the theory T_2 consisting of ZFC + "There exists an inaccessible cardinal", where $CON_n(T)$ is a number-theoretic formula expressing that the theory T is consistent for proofs of length less than n. Both theories are intuitively plausible; in the case of T_1 this relies on the intuitions behind the inner model program, cf. [63]. (This example is a derivate of an example by Shelah of two incomparable Π^0_1 sentences. However, Shelah in his example was not concerned with the sentences being non-contrived.)

[36] For example the statement that all sets are Lebesgue measurable (in ZF) is equiconsistent with he existence of an inaccessible cardinal (in ZFC) by results of Solovay and Shelah.

answers most useful to analysts.[37] This *solves* all classically undecidable problems of descriptive set theory stated in terms of regularity properties of projective sets such as their Lebesgue measurability—if projective determinacy, which is unprovable in ZFC, is stipulated as an axiom. Unlike some large cardinal axioms, however, determinacy axioms are not evident by themselves, but are accepted mainly in view of their fruitful consequences. That raised the challenge to derive those consequences and even determinacy axioms themselves from large cardinals, a task that looked difficult given that large cardinals and determinacy seemingly bear no relation to each other.[38] Nevertheless, in the end it has become clear that these two classes of axioms are really one class of axioms.[39] Gödel might have interpreted this as a confirmation of his realistic views, for we would have no explanation for such an unexpected convergence unless these two kinds of axioms describe different aspects of one underlying reality. Even if one accepts that interpretation, the question remains how Gödel arrives at a reality consisting of *objects*. We shall come back to this in Section 5.

In the supplement to the second edition of *What is Cantor's Continuum Problem?* Gödel introduces an interesting modification of his platonism that is epistemological rather than ontological in spirit.

> However, the question of the objective existence of objects of mathematical intuition (which, incidentally, is an exact replica of the question of the objective existence of the outer world) is not decisive for the problem under discussion here. The mere psychological fact of the existence of an intuition sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis. [34, p. 272].

In the ontological version of mathematical realism, the meaningfulness of the continuum problem is a consequence of the existence of a realm of mathematical objects with determinate properties. Here Gödel is suggesting putting ontological concerns aside and regarding meaningfulness as a question of epistemology. On this view, the meaning of the continuum problem is tied to an unfolding of concepts through successive refinements of mathematical intuition. One difficulty with this is why it should lead to a unique

[37] The projective sets are the smallest class of sets containing the Borel sets which are closed under complementation and projection on a coordinate axis. They are arranged in a natural hierarchy according to the number of steps in which they are generated from a Borel set, cf. [80].

[38] Large cardinal axioms are global principles related to the length of the cumulative hierarchy expressing the idea that there is no conceivable limit on transfinite iterations of the operation 'set of'. Determinacy axioms by contrast are local assertions about the thickness of a specific level, e.g., $V_{\omega+1}$ since winning strategies in integer games are coded by real numbers.

[39] Cf. [78] and [106].

resolution of CH, for our intuitions could conceivably evolve into different directions inducing us to formulate axioms with opposite outcomes of CH. Of course this is not an issue for someone who holds that mathematical concepts form a "reality, which exists independently both of the acts and of the dispositions of the human mind"[40] and who believes that our intuitive ideas about this reality imply the correct axioms.[41] At the same time, Gödel conceded that this may not directly settle CH: "even if the concept of set becomes clear, even after satisfactory axioms of infinity are found, there would remain more technical (i.e., mathematical) questions of deciding the continuum hypothesis from the axioms".[42]

According to Gödel concepts themselves are determinate and immutable. Ambiguities are due to us, as in the case of 'continuity' or 'point', "where we mix two or more exact concepts into one intuitive concept".[43] What changes in the process of clarification is our perception of the concept, "similar to our perception of an animal first far away and then nearby".[44] To support this thesis Gödel mentions three examples: Turing's analysis of the concept 'mechanical procedure', the formalization of velocity in calculus and the theory of the Lebesgue measure. If there was nothing sharp to begin with, Gödel finds it inconceivable how one could have arrived at solutions that are "*unquestionably* unique, which here is due to the fact that only they satisfy certain axioms which, on closer inspection, we find to be undeniably implied in the concept we had."[45] With respect to the concept of set, some empirical evidence for this view can be found in the history of descriptive set theory leading up to projective determinacy. Because of the orderly generation of projective sets, there had been an intuitive expectation from the beginning that they are Lebesgue measurable, and this attitude prevailed even after

[40][31, p. 323].

[41]Aside: apparently in Gödel's opinion, this is even true for the notoriously elusive concepts of metaphysics. According to Wang, on several occasions Gödel hinted at the possibility of an exact axiomatization of metaphysics in the style of Newtonian physics [102, 7.3.6, 9.4.17.12], and in a discussion with Carnap in 1940 he remarked that an exact theory using concepts as *God*, *soul* and *ideas* would be as meaningful as theoretical physics, cf. [102, p. 309].

[42][102, 7.3.21]. The continuation of the passage, despite its obscurity, is noteworthy: "This is because epistemology and science (in particular, mathematics) are far apart at present. It need not necessarily remain so. True science in the Leibnizian sense would overcome this apartness. In other words, there may be another way of analyzing concepts (e.g., Hegel's) so that true analysis will lead to the solution of the problem".

[43][102, 7.3.8, 7.3.9].

[44]There are numerous passages where Gödel compares the incompleteness and indistinctness of conceptal knowledge to the inadequacy of perceptual knowledge. In [31, p. 321], for instance, he likens the set-theoretic paradoxes to the contradiction between our visual and tactile perceptions of a stick half immersed in water.

[45][102, 7.3.1, 7.3.7]. See also [102, 7.3.3, 7.3.10, 7.3.16]. Curiously, Gödel did not include the clarification of the intuitive concept 'infinitesimal' that was achieved by nonstandard analysis among his examples.

Gödel had proved the existence of non-measurable projective sets in his model for CH. Nowadays we know that *all* axioms of infinity with sufficient logical strength imply PD and thus the Lebesgue measurability and other desirable regularity properties of the projective sets.[46] This is remarkable given that there are $2^{\aleph_0}$ consistent extensions of the standard axioms.

Coming back to the passage quoted above, what seems strange is Gödel's appeal to a "psychological fact" in an attempt to explain what the meaningfulness of a mathematical hypothesis may amount to. I will postpone until Section 7 the analysis of what I believe he was alluding to.[47] The passage under discussion is also of interest in connection with Gödel's views on the relationship between objectivity and realism. Emerging from Gödel's writings from the 1950s onward,[48] as well as from his conversations with Wang,[49] is a preference for construing the objective content of mathematics in terms of the objectivity of mathematical discourse without appealing to external criteria of existence and truth that are supposedly satisfied by mathematical objects. Often the objectivity of that discourse is understood on the basis of formal constraints which regiment what kind of conceptual relationships are possible, leaving no room for arbitrary convention. Thus in his "real argument for objectivism" about numbers Gödel deduces the existence of objects that are different from physical objects from the observation that there are objective facts in arithmetic.[50]

In a similar vein, Cantor once spoke of the "immanent reality" of numbers requiring nothing more than this: "on the basis of definitions they occupy a completely determinate place in our understanding, are distinguished from all other components of our thought [and] stand to them in determinate relationships."[51] One version or another of this primacy of objectivity over objects is still operative in contemporary discussions about new axioms.[52] For example, consider Woodin's recent work on the continuum problem[53] in which ontological considerations about the kind of entities that the hypothesis apparently is about (arbitrary sets of integers and one-to-one correspondences) no longer play any substantive role. Instead, the focus is on

[46]The reason is that any axiom of infinity which implies the consistency of PD actually implies PD.

[47]Another place where Gödel speaks of the existence of an intuition covering the axioms of classical mathematics as a psychological fact is in [32, p. 338, footnote 12]. Apparently, he granted the same status of objectivity (i.e., independence of our conventions and constructions) to psychological objects and facts that he ascribed to physical and mathematical objects and facts.

[48][31, p. 320ff], [32].

[49][102, 7.1.2, 7.4.1, 8.2.3, 8.3.1, 8.3.2, 8.5.20, 9.2.38–9.2.43].

[50][102, 7.1.2].

[51][4, §8].

[52]See [76] or [43].

[53][105].

the metamathematical obstacles to isolating the 'true' theory of $P(\omega_1)$. Taking the 'complete' axiomatization of projective set theory under projective determinacy as his point of departure,[54] Woodin asks whether something analogous can be achieved for $P(\omega_1)$ when this is construed as the 'next' structure beyond second order number theory. It turns out that there is an affirmative answer provided one switches to a different logic that is 'immune' to forcing. Woodin has formulated an axiom which decides the theory of $P(\omega_1)$ in this logic and which implies that $2^{\aleph_0} = \aleph_2$.[55] In addition he devised a canonical model for this axiom.[56] But the main point of his analysis is a crucial asymmetry to the effect that there can be *no* axiomatization of this kind under CH.[57] Woodin argues for the 'appropriateness' of his axiomatization by referring to the intuitions behind the various abstractions and idealizations that enter into its formulation. It is the progressive refinement of those intuitions—in accordance with the passage quoted above—that gives a certain meaning to the question of the truth or falsity of the continuum hypothesis. At this time it cannot be decided if this is the 'correct' meaning, but that does not rule out that the continuum problem is a legitimate mathematical question.[58] On the contrary, the latest developments with their metamathematical admixtures are a graphic illlustration that

> there may exist, besides the usual axioms [and] the axioms of infinity ... other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying *logic* and mathematics would enable us to recognize as implied by these concepts. [30, p. 520f], emphasis added.

Whether this will lead to a universally accepted *solution* of the continuum problem is *prima facie* an empirical question that only the future course of mathematical research can answer.

With regard to Gödel's realism, however, it should be clear that further investigations into its meaning and feasibility must take his views on epistemology into account. A general point to keep in mind is that conceptual realism is by no means opposed to idealism. It is in fact opposed to nominalism, and it may well be considered itself as a kind of objective idealism as we shall see below.

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[54]Here completeness means that all 'natural' statements are decidable. Because of Gödel's second incompleteness theorem, there is no way to decide all statements with a given axiomatization. The completeness of second order number theory under PD in this sense is an empirical fact.

[55]Actually the axiom can be reformulated as a strong form of one of the forcing axioms studied in [1]; thus "seemingly disparate threads are woven into a single tapestry" [105, p. 687].

[56]The model and the sense in which it is canonical are described in [104].

[57]Recall that Gödel [34, p. 270f] uses a similar metamathematical asymmetry in arguing for the existence of inaccessible cardinals.

[58]For a more complete discussion of this and related issues see [12], the contributions to [18] and [44].

§3. Intuition. The first thing to note about Gödel's conception of mathematical intuition is that he never offered a systematic account of what is apparently the main factor in his epistemology.[59] As a general rule, his claims about intuition are strong but not much developed. Gödel maintains "there are objective facts about intuition",[60] and for the correct analysis of concepts there is a "science of intuition". It is, however, "not precise, and people cannot learn it yet. At present, mathematicians are prejudiced against intuition. Set theory is along the way of correct analysis."[61]

Whatever one is to think of this, certainly for Gödel intuition goes beyond Kantian *Anschauung* as a matter of sensuousness with respect to objects in space and time.[62] It is also distinguished from derivatives of this conception which played a role in the foundational debates during the first half of the 20th century (Hilbert and Brouwer). Intuition, as Gödel understands it, yields knowledge also about transfinite sets and abstract concepts.[63]

Our efforts to learn more about exactly *what* is being intuited in mathematical intuition are complicated by Gödel's usage of the term both in relation to objects as well in the sense of propositional attitudes. He expressly told Wang:

> We do not distinguish between intuition *de re* and intuition *de dicto*; the one is contained in the other [102, 9.2.46],

on the grounds that

> [t]o understand a proposition we must have an intuition of the objects referred to. [102, 9.2.45].

But with respect to the question what these objects are his statements are replete with ambiguities. In Gödel's ontological scheme, "among the objects, there are physical objects and mathematical objects. Pure sets are the mathematical objects and make up the world of mathematics."[64] This would suggest that the objects of mathematical intuition are sets.

A somewhat different picture emerges from Gödel's writings from the 1950s. According to [31, p. 320] the terms occurring in mathematical propositions denote concepts, and in [32, version V, p. 359] mathematical intuition

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[59] For a summary of the development of Gödel's thoughts about intuition and mathematical realism see [82].

[60] [102, 7.1.12].

[61] [102, 5.3.29, 9.2.26].

[62] Kant [65, A19/B33] defines *sensuousness* (Sinnlichkeit) as our ability to obtain representations according to the manner in which we are affected by objects.

[63] [102, 7.1.12–7.1.14].

[64] [102, 9.1.27]. The above formulation is contained in a manuscript by Wang, based on Gödel's scattered observations, which was allegedly read and commented on by Gödel in 1972. However, Wang concedes that he may have oversimplified some of the ambiguities in Gödel's utterances. For example, Gödel apparently also takes monads to be objects leaving open their relation to physical objects. Furthermore, according to Wang it is unclear whether Gödel held numbers to be reducible to sets [102, p. 295f].

is likened to a physical sense insofar as it enables one to perceive relationships between concepts. In the corresponding passage in §42 of version III, this additional sense is called reason, and Gödel says that "with mathematical reason we perceive the most general (namely "formal") concepts and their relations."[65] And in §31 of the same version he says that rational foundation for mathematical axioms is "that they (or propositions implying them) can directly be perceived to be true (owing to the meaning of the terms or by an intuition of the objects falling under them)." The matter is complicated further by remarks as the following one where Gödel seems to be drawing a distinction between objects and concepts in epistemological terms along the boundary of sensuousness and understanding.

> Sets are objects but concepts are not objects. We perceive objects and understand concepts. Understanding is a different kind of perception: it is a step in the direction of reduction to the last cause. [102, 7.3.12].[66]

We are not told what this last cause is, but a hint may be contained in [102, 7.3.20, 7.3.21] where Gödel describes the analysis of concepts as the central task of philosophy with the aim to arrive at "the primitive concepts of our thinking" and the "inborn intuitions" that thinking is based on. An example of the latter could be "the fundamental property of the mind to comprehend multitudes into unities" which Gödel called "the main fact of mathematics."[67] Besides reason and sense perception there is also an obscure third element in Gödel's epistemology which "does represent an aspect of, and may be a plan of, objective reality.[68]

In the philosophical tradition intuition has generally been understood as some kind of immediate representation leading to knowledge which is not the outcome of a process of deductive resoning.[69] This aspect is also stressed by Gödel when he is contrasting intuition with proof.

[65] In discussing what he takes to be the true significance of the intensional paradoxes Gödel speaks of "the inadequacy of our perception—*that is, understanding*—of the concepts (such as the concept of concept)" [102, 7.4.5] (emphasis added).

[66] Gödel's remarks on the relationship between sets and concepts as recalled by Wang are not easy to disentangle (cf. footnote 64) and are but one illustration of the difficulties encountered by Wang. Two aspects which do emerge from those remarks are (i) a distinction between sets and concepts in mereological terms as different kinds of wholes [102, 9.1.26], and (ii) a characterization of the difference between sets and concepts as one of extension vs. intension where the former defines the subject matter of mathematics and the latter the subject matter of logic (cf. [102, sec. 8.6]). Limitations of space do not allow a discssuion of (ii), but (i) will be taken up below in Section 5.

[67] [102, 8.2.1–8.2.3]. This is the problem of the One and the Many from Plato's Philebos which Cantor alluded to in his 1883 definition of 'set' (cf. [6, p. 204]).

[68] [102, 4.4.11]. Compare [34, p. 272].

[69] By way of illustration consider the following passage from [72, p. 670]. "Syllogism serves our reason [by showing] the connection of the [premisses with the conclusions in] proofs in any one instance and no more; but in this, it is of no great use, since the mind can perceive such connection where it really is, as easily, nay, perhaps better without [syllogism].

> Intuition is not proof; it is the opposite of proof. We do not analyze intuition to see a proof but by intuition we see something without a proof. We only describe in what we see those components which cannot be analyzed any further. [102, 9.2.46].[70]

But in the supplement to the second edition of the Cantor paper he notes that "mathematical intuition should not be conceived of as a faculty giving an *immediate* knowledge of the objects concerned."[71] Indeed, "in order only to understand the axioms of infinity, one must first have developed set theory to a considerable extent."[72] Almost identical thoughts are expressed in [32, p. 353, footnote 43] where it is pointed out that the explication of the content of the general concept of set "may involve a very great (perhaps even an infinite) number of actually realizable independent rational perceptions". At the same time, Gödel [32, p. 354, footnote 45] mentions the "direct perceptibility of mathematical objects" as one of the reasons for asserting their existence. What is needed, then, is a conception of intuition which can account for the directness of intuitive knowledge without contrasting it with discursive thought.

Two major points where Gödel is at odds with other thinkers concern the epistemic force and the perceptual character that he claims for intuition. With regard to the first point, a number of writers have denied that intuition plays a substantive role for mathematical knowledge. Frege [23, p. 5], for instance, sought to "prevent anything intuitive from penetrating" into his account of arithmetical knowledge on the grounds that the laws of arithmetic express logical truths and are thus known independently of intuition. Another example is Wittgenstein who referred to intuition as "an unnecessary way of talking your way out" in his reflections of rule-following.[73]

Gödel, by contrast, regarded the epistemic ineliminability of mathematical intuition as a corollary of the incompleteness theorems. *Any* consistent formal system with a recursive axiomatization which includes elementary number theory contains logically undecidable statements. Via a suitable coding, those undecidable statements are expressible in the language of arithmetic, and therefore "continued appeals to mathematical intuition are necessary ... for obtaining solutions of the problems of finitary number theory."[74] This has been contested on the grounds that the arithmetical statements in

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If we observe the actings of our own minds, we shall find, that we reason best and clearest, when we observe the connexion of the [ideas], without reducing our thoughts to any Rule of Syllogism."

[70]On similar ground Gödel separated intuition from construction [102, 9.2.28].

[71][34, p. 271]. According to Wang [102, p. 226], while discussing the preceding paragraph of the supplement to the Cantor paper, Gödel insisted on distinguishing between "something like a perception of the objects of set theory" and, "more generally, mathematical intuition".

[72][35, Remark 2, p. 306].

[73][103, §213]. See also [op. cit., §186].

[74][34, p. 272].

question are contrived reformulations of metamathematical pathologies.[75] What is yet to be discovered is an application of large cardinals leading to the solution of a previously formulated problem of number theory, say about the distribution of prime numbers. On the other hand, if we move up from arithmetic by one level to second order number theory, or equivalently the theory of the real numbers, strong hypotheses begin to have a profound effect. With projective determinacy the undecidable questions of classical descriptive set theory have been *solved*. But as we pointed out before, determinacy is not supported by intrinsic evidence.

Besides mathematical intuition Gödel [34, p. 272] mentions another "though only probable" criterion for the truth of mathematical axioms, namely their fruitfulness in terms of consequences. A perfect example of this are the aforementioned determinacy axioms which are not evident in themselves but, in virtue of their role in second order number theory, are

> ... so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems ... that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well-established physical theory. [30, p. 521].[76]

Although intrinsic evidence has higher epistemic weight for Gödel than extrinsic evidence, he emphasizes its fallibility—even when it comes to such elementary beliefs as the consistency of arithmetic.[77] But he does not offer any clues as to how intuition could function as a reliable criterion for truth.

With regard to the perceptual nature of intuition Gödel advances what looks like a rather wild thesis in his conversation with Wang by insisting on a causal basis for the intution of concepts. By way of justification he mentions the Aristotelian distinction between active and passive intellect, adding that our handling of abstract impressions is somehow mediated by images.

> I believe there is a causal connection in the perception of concepts. But at present the theory is like the theory of atoms at the time of Democritus. Already *noûs* in Aristotle is a causal affair: the active intellect works on the passive intellect. The active intellect is, I believe, located in some physical organ. It may even have images.—I am cautious and only make public the less controversial parts of my philosophy. [102, 7.3.14].

The place in Aristotles' writings that Gödel alludes to is the notorious Chapter 5 in Book III of *De Anima* which contains a distinction of two kinds of intellect: *noûs poietikos* and *noûs pathetikos*. What exactly is meant by those

[75] For a different point of view see [24] as well as H. Friedman's contribution to [18].

[76] Notice that Gödel's speculative remark about the possibility of such axioms predates the first proposal of determinacy *axioms* [81] by almost 15 years.

[77] [102, 9.2.31].

terms is the subject of an ongoing debate among the experts. One suggestion[78] is that the former denotes the faculty by which we form general conceptions and grasp universal truths while the latter allows us to recognize the presence of universals in particular things which instantiate them. If this is correct *noûs pathetikos* is called passive not because it is itself passive—an impossibility given that all reasoning is an activity—but because it depends on the awareness of something actually given in sense-perception which is not needed by the active *noûs poietikos*. This interpretation also lends itself to Gödel's contention that passive reason "somehow shadows what [active reason] is doing and helps us as a medium" [102, 6.1.22]. But it leaves open in what sense "the active intellect works on the passive intellect", and on the whole it remains unclear what the alleged causal connection in the perception of concepts consists in. Similar remarks apply to Gödel's contention that

> [s]ome physical organ is necessary to make the handling of abstract impressions possible. Nobody is able to deal effectively with them, except in comparison with or on the occasion of sense impressions. This sensory organ must be closely related to the [neural] center for language. [op. cit., 7.3.15, cf. 7.3.5].

For this to carry any explanatory weight, Gödel would need to be more explicit about the alleged connection between sensory and abstract impressions.

In a famous passage from the revised edition of his paper on the continuum problem Gödel asserts:

> But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them, and, moreover, to believe that a question not decidable now has meaning and may be decided in the future. [34, p. 271].

Against this it has been held that nothing that Gödel says elsewhere in this article about the discovery of new axioms reveals an epistemic function for a perception of mathematical objects which could not be performed by an understanding of the concept of set.[79] On this reading, what it means for the axioms to "force themselves upon us as being true" is simply this: the concept of set 'implies' those axioms because they are part of its meaning, and we possess a sufficient understanding of the concept to grasp this implication.[80]

[78] [89, p. 41ff].

[79] Cf. p. 261 lines 12–17, p. 260 lines 18–19, p. 258 lines 11–18 in the pagination of [14].

[80] This is the position defended in [77] who also argues that the axioms do not describe a concept of set in the straightforward sense, i.e., as an (intensional) property the possession of which makes an object a set. What they really do is impose a constraint on any concept

In the analytic tradition that understanding, although it may involve feelings and schematic pictures, is regarded primarily as a linguistic competence. Tait [96, p. 147], for instance, argues that the meaning and validity of our canons of justification for mathematical axioms are not obtained from perception, but rather that it is the other way around: we 'perceive' sets only when we have learned how to use the word *set*. However, the linguistic turn does not constitute a position free of problems. Whether we apprehend the structure of the mathematical world and mathematical facts only *within* language, or to put it less drastically, whether the ontological categories of mathematics rest ultimately on logico-syntactical ones, is to a certain extent an empirical question, and there is some evidence hinting at a negative answer. Detailed studies by Piaget and others on the cognitive development of children seem to suggest that the capacity to recognize logico-mathematical structures (class-inclusion, order, correspondence etc.) develops prior to language acquisition and, moreover, that this capacity is a prerequisite for language and not vice versa.[81]

But even if one grants that our propositional knowledge in mathematics derives from a non-propositional acquaintance with ideal objects, Gödel's comparison with theory formation in physics remains inconclusive without a systematic account of how the axioms force themselves upon us. In the case of physical objects we have at least an outline of a theory explaining how these objects determine our perceptions. But neurophysiology—which still is in an undeveloped state with respect to mathematical intuition—remains silent about the interaction of our cognitive faculties with the objects of transfinite set theory. For the latter "clearly do not belong to the the physical world and even their indirect connection with physical experience is very loose".[82] Many commentators have focussed on this 'problem of access'—recognized already by Plato[83]—as the Achilles heel of Gödel's realism and his philosophical program, due in part to an influential formulation by Benacerraf [2].

Below I shall outline a strategy for avoiding this dilemma by replacing two cornerstones in Benacerraf's approach—Tarski's formalization of truth in

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for qualifying as a concept of set in the straightforward sense or, in extensional terms, on any structure for counting as a universe of set theory. For an elaboration of mathematical intuition as rational intuition not invoking perception of objects see [84].

[81][85, p. 15f, pp. 50–60]. For a more recent study suggesting that the "possession of true numerical concepts" by infants does not dependent on language see [107]. It goes without saying that these findings match Gödel's well-known aversion to linguistic methods in philosophy, cf. [102, 5.5.7–5.5.9]. For instance, Gödel held that "[l]anguage plays no part in the intensional paradoxes, since they are concerned with concepts as properties and relations of things which exist independently of our definitions and constructions" [102, 8.5.11].

[82][34, p. 271]. The difficulties facing naturalistic accounts of mathematical intuition relying, e.g., on speculative theories about neuronal organization are apparent to anyone who has tried to implement the approach suggested in [73].

[83]Cf. for example *Phaedo* 65d–67b, 74a–77a.

terms of satisfaction and the causal theory of knowledge (and reference)—with ideas from Husserl's phenomenology. Through an analysis of the structural analogies between sense perception and our grasping of mathematical truths, Husserl arrives at a broader notion of evidence leading to knowledge not only about physical objects but also about ideal objects. Underlying this strategy is a point where Gödel is in agreement with Husserl, namely that the empiricists owe us an *argument* for the alleged implication from empiricism to physicalism.[84]

§**4. Gödel and Husserl.** Edmund Husserl (1859–1938) was originally trained in mathematics and the natural sciences. As a student in Berlin—then a world center of mathematics under the leadership of Weierstrass, Kummer and Kronecker—he had the opportunity to witness first hand how the subject was transformed by unprecedented advances in formalization and abstraction. After receiving a doctorate in mathematics from the University of Vienna in 1883, Husserl worked one semester as Weierstrass' personal assistant. But his interests soon shifted to philosophy under the influence of Franz Brentano, motivated in part by the conviction that the basic terms of mathematics, in particular those used in the new discipline of mathematical logic, along with mathematical methodology as a whole, called for an epistemologcial clarification. Building on ideas of Brentano, Lotze and Bolzano, he set out to develop the tools for this clarification in the form of a new philosophical discipline called *phenomenology*. Its starting point was the realization that even though the idealities of logic and mathematics are certainly not just existing in the mind of the mathematician, understanding their true nature requires us to reflect on the mental acts through which they are brought to awareness.

Given this subjectivistic cast, Gödel's interest in phenomenology seems surprising at first glance. It is possible, as Wang pointed out,[85] that the sustained efforts in the 1950s to refute logical empiricism as a philosophy of mathematics, may have convinced Gödel of the difficulties in devising a systematic epistemology for his conceptual realism. While his earlier philosophical writings show the formative influence of Plato and Leibniz, he was also aware of Kant's critique of rationalism. At the same time Gödel could not accept the metaphysical dualism of Kant who postulated an unknowable realm of things in themselves that lie hidden behind the phenomena.[86] Moreover, as we already noted, the Kantian conception of intuition as a matter of sensuousness with respect to objects in space and time was too restrictive for Gödel's purposes. In his opinion phenomenology as a systematic further development of Kant's ideas provided the means for overcoming these

[84] [102, 5.4.1, 5.4.5, 9.3.6], [32, p. 347 and footnote 34]. Compare this with [55, §19].

[85] [102, sec. 5.3].

[86] [102, 4.4.10, 5.3.37].

limitations in a way "which, entirely as intended by Kant, avoids both the same death-defying leaps of idealism into a new metaphysics as well as the positivistic rejection of all metaphysics."[87] Both the *Nachlass* and the conversations with Wang contain numerous indications that from 1959 onward Gödel immersed himself in the study of Husserl's major writings hoping perhaps they might offer a general philosophical framework to accomodate his own views.[88] Indeed his philosophical program of a meaning-clarification of the basic concepts underlying logic and mathematics is modeled to a certain extent on the Husserlian ideal of *Philosophy as a Rigorous Science*,[89] a self-warranting *philosophia prima* that relies on introspective intuition into the *a priori*, meaning-bestowing structures of consciousness. Phenomenology was seen by Gödel not as a systematic theory, but as a technique for arriving at a philosophical understanding—as opposed to a verification—of our fundamental assumptions concerning objectivity, truth, being etc.[90] Implementing this technique requires a reflective change of attitude in the form form of two 'reductions', an *eidetic* one and a *transcendental* one, both of which, it must be stressed, are reductions of the components of experience, not of the constituents of reality.

The eidetic reduction, from the Greek word *eidos* for essence, is a methodological device for ensuring that phenomenology is "a science which aims exclusively at establishing 'knowledge of essences' and *absolutely no facts*".[91]

> For instance, the phenomenology of perception of bodies will not be (simply) a report on the factually occurring perceptions or those to be expected; rather it will be the presentation of invariant structural systems without which perception of a body and a synthetically concordant multiplicity of perceptions of one and the same body as such would be unthinkable. [57, §4].

From the viewpoint of epistemology, the rationale for concentrating on essences is clear: of the actual acts of consciousness one can only come up with contingent truths pertaining, for example, to causal or cultural contexts. Our only hope to arrive at necessary (in a sense to be further explicated) truths is by focussing on the essences that those acts instantiate. What one discovers in this way are among other things essential correlations between temporal acts and atemporal meanings. These findings remain

[87][33, p. 384f]. See also [102, 5.3.36]. Husserl [52, p. 200] describes his extension of intuition to ideal objects as a decisive advantage over Kant's epistemology.

It should also be noted that Gödel regarded phenomenology as a further development of certain ideas of Leibniz (for example, see the reference in footnote 93 below). However, a detailed examination of this will have to wait for another occasion.

[88]See the introductory note by D. Føllesdal to [33] and [102, p. 164f]. According to [90, p. 175], Gödel regarded Husserl as the greatest philosopher since Leibniz.

[89][54].

[90]Cf. [102, 9.3.10, 5.3.9] and the remarks from [33] cited in the introduction.

[91][55, Introduction, p. 6].

valid no matter who is the real individual to whom these acts belong or in what kind of bodily organism—possibly different from ours—they are realized. At first glance, the eidetic reduction may amount to no more than the widely held belief that all philosophical truths are conceptual rather than factual. A distinctive feature of phenomenology, however, is the doctrine that essences are (higher-order) *objects* that can be 'perceived' via a faculty of intuition.[92] In many ways this parallels what Gödel said about concepts and our epistemic access to them, and it is worth noting that he thought of the intuition of essences as an affirmation of the Leibnizian "ideal of seeing the primitive concepts clearly and distinctly".[93]

Husserl's essentialism stems from a general opposition, also prevalent in Gödel's writings,[94] against naturalistic moves in philosophy, especially with respect to mind, logic and mathematics. A reduction of logic, for instance, to psychology is mistaken because it is incompatible with the very essence of logical truth. And a reduction of consciousness to patterns of neuronal activation disregards the very essence of conscious experiences that they possess meanings. Husserl takes every scientific discipline to be about a specific category of objects, maintaining that its methodology is determined exclusively by the possible kinds of experiences in which objects of that category may be encountered.[95] This is where his concept of an *ontology* comes in. Every individual falls under a highest 'contentual' essence. A physical object, for instance, with all its essential interconnections falls under the highest contentual essence 'material thing'. Numbers, sets and proofs are subordinated to the essence 'mathematical object'. Each such essence defines a *region*.[96] A regional ontology is the *a priori* science that marks out a 'frame' prescribing the form—-as opposed to content—of possible experiences of objects belonging in this region.[97] Thus, by upholding the integrity of scientific domains against reductionist attempts, the regional ontologies studied in phenomenology preserve the classical role of philosophy of providing a conceptual foundation for the sciences. In the case of empirical psychology, for example, phenomenology plays a similar role in the uncovering of structural principles of

[92] [55, §3].

[93] [102, 5.3.19], cf. also the reference in footnote 42. The reader needs to be cautioned, though, that—contrary to the claim made in [102, 5.3.15]—it is not always clear whether Gödel means 'essence' when he says 'concept'. According to Husserl, in the intuition of an essence one is focussed on things whereas understanding a concept requires a reflection on meaning. The difference is for Husserl a manifestation of the fundamental cleavage in logic under the two correlative aspects of *formal ontology* and *formal apophantics* [50, §67, p. 244], [58, §27b].

[94] [31], [32]. See also [102, Chapter 6].

[95] [60, §5f].

[96] [55, §§9, 16].

[97] [60, §7, p. 33f].

experience as geometry does in the natural sciences for the study of material objects.[98]

But for Husserl phenomenology retains the philosophical naïvité of a positive science as long as it takes the entities in its domain to exist in a straightforward sense, as observer-independent entities having determinate properties. This is why he introduces the *transcendental reduction*,[99] a.k.a *epoché*, which abstracts from the worldly (or ideal) existence of the entities that are objects of intentional acts, and focusses on how they are given in those acts. Husserl emphasizes that the epoché amounts to much more than an ordinary move to the introspective stance. The point is that one is not just redirecting attention from objects to acts within the natural context of experience, but abandoning that context altogether. To this end we cannot regard the mind as a part of the world, but on the contrary must assign to it a formative or foundational role with respect to the world as a whole. Inevitably this places some functions of the mind—in particular those enabling reference to and knowledge of mathematical entities—beyond the explanatory reach of the natural sciences. In order to account for those functions Husserl turns to the special realm of entities revealed by the epoché which he calls *transcendental subjectivity*.[100] It encompasses the total system of the interpretive structures of consciousness that are the precondition of meaningful experience together with our possible reflective awareness of them.

> Every imaginable sense, every imaginable being ... falls within the domain of transcendental subjectivity, as the subjectivity that constitutes sense and being ... Trying to capture the universe of true being as something that stands outside of the universe of potential consciousness, potential cognition, potential evidence, both related to one another merely externally by a rigid law, is nonsensical. [59, §41, p. 86].

Because of its seemingly idealistic implications the transcendental reduction was rejected by many phenomenologists who wished to uphold the robust realism about universals permeating the *Logische Untersuchungen*. Nevertheless, Gödel's sympathies evidently were with the latter phases of Husserl's thought in which the descriptive study of the essential structures of consciousness in the *Logische Untersuchungen* had been raised to the transcendental level.[101] Wang reports that Gödel considered the *Ideas* and the

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[98][60, §11, p.69f]. On the other hand, Husserl cautions against construing phenomenology as a geometry of the mind. For one thing, phenomenological truths cannot be deduced from a small number of axioms ([60, §11, p. 61f]). Moreover, the essences studied by phenomenologists are of a fundamentally different kind than mathematical essences ([55, §§72–75], [59, §20, p. 49f]).

[99][55, §31]. The attribute *transcendental* is used in conformity with [65, A12/B25] as a reference to the conditions for the possibility of objective knowledge [55, §97, p. 245].

[100][59, §12, p. 29; §8].

[101]On the back of the white flyleaf in Gödel's copy of the *Ideen*, there is a shorthand note to the effect that Husserl's failure to acknowledge transcendental subjectivity ("transc.

Cartesian Meditations (both of which were written after Husserl's transcendental turn) the most important of Husserl's published works, the latter on the grounds that it "is closest to real phenomenology—investigating how we arrive at the idea of self".[102] On another occasion Gödel remarked:

> Husserl's is a very important method as an entrance into philosophy, so as finally to arrive at some metaphysics. Transcendental phenomenology with *epoché* as its methodology is the investigation (without knowledge of scientific facts) of the cognitive process, so as to find out what really appears to be—to find the objective concepts ... [102, 5.3.6].[103]

There are numerous indications that Gödel was attracted to idealistic philosophy, and especially post-Kantian German idealism, well before he began his systematic study of Husserl.[104] On hint of this can be found in his correspondence, dating from the 1950s, with G. Günther, a philosopher of Hegelian background with whom he shared an interest in disassociating mathematical logic from positivism.[105] Commenting on Günther's views on consciousness, its relation to objects, and self-consciousness, Gödel wrote to him on June 30, 1954:

> The reflection on the subject treated in idealistic philosophy ... , the distinction of levels of reflection, etc., seem to me very interesting and important. I even consider it entirely possible that this is "the" way to the correct metaphysics. [16, p. 503f].[106]

Subj.") was one of the two cardinal mistakes of the *Logical Investigations* [38, p. 5]. See also footnote 115 below.

[102][102, 5.3, p. 164]. Presumably, Gödel refers to the "idea of self" in Husserl's *Fifth Cartesian Meditation*: a *monad* belonging to a community of conscious beings who stand in reciprocal relations whereby the internal states of any one is mirrored and registered by those of any other [59, §60]. This presents interesting points of contact with Leibniz the investigation of which, however, cannot be pursued any further in this article.

[103]But in the quotation immediately following this one Wang reports that Gödel cautioned "[p]henomenology is not the only approach. Another approach is to find a list of the main categories (e.g., causation, substance, action) and their interrelations, which, however, are to be arrived at phenomenologically. The task must be done in the right manner" [102, 5.3.7]. It is unclear what other possible approaches Gödel was thinking of. A candidate could have been Hegel's theory of concepts the improvement of which was seen by Gödel as a worthwhile project [102, 9.4.11], see also footnote 42. However, he did not regard Hegel's dialectic as useful on the grounds that it is logically flawed [102, 8.4.8, 5.3.17]. He also critisized Hegel for moving away from the Leibnizian "ideal of seeing the primitive concepts clearly and distinctly" [102, 5.3.19].

[104]According to K. Menger, Gödel studied post-Kantian German idealist metaphysics in the early 1930s until he began to concentrate on Leibniz [102, 2.3.1].

[105][102, 5.4.7–5.4.9].

[106]The phenomenological reflection, too, can be iterated in the sense that I may reflect phenomenologically upon my phenomenological reflection upon say a perception. However, the modification of neutrality brought about by the epoché—unlike imaginative modification—is not iterable [55, §§107, 109, 111, 112].

Gödel was quick to add though that a correct metaphysics must allow for the objective meaning of thought as well as the objective existence of material things and abstract entities.

> When I say that one can (or should) develop a theory of classes as objectively existing entities, I do indeed mean by that existence in the sense of ontological metaphysics, by which, however, I do not want to say that abstract entities are present in nature. They seem rather to form a second plane of reality, which confronts us just as objectively and independently of our thinking as nature. [loc. cit.].

Transcendental phenomenology, as we shall see, allows one to interpret Gödel's realism in a manner that does justice to the idealistic elements of his thought. It should be kept in mind, though, that Husserl was careful to distance himself from his precursors in the idealistic tradition. For example, his defense of the objective existence of universals against nominalistic reinterpretations in the *Second Logical Investigation* begins with the statement that "idealism is not a metaphysical doctrine", but rather "the only possibility of a coherent epistemology".[107] From Gödel's notes on the *Logical Investigations* it is apparent that Gödel approved of this characterization to which he added that "ideal objects are the precondition for the possibility of objective knowledge".[108] The next section contains an exposition of some ideas from phenomenology that will be used in the sequel to substantiate this claim.

§5. Intentionality, noema and constitution. The central notion of phenomenology, its *Generalthema*, is *intentionality*.[109] This refers to the directedness or 'aboutness' of our mental acts as their characteristic feature: to be conscious always means to be conscious *of*.[110] In a tradition going back to Aristotle intentionality had been construed as a relation with an object into which relation the subject enters by way of a conscious act. This picture runs into problems in situations where the intended object does not exist but the corresponding act is nevertheless directed.[111] Husserl's important insight was that the *actual* existence of an object is not decisive for the directedness of an act. What matters for directedness is that our consciousness has a certain structure when we perform the act. Husserl calls this structure *noema*.[112] It is an ideal entity, distinct from both the act and its

[107] [51, p. 107f].

[108] [39, p. 2].

[109] [55, §84].

[110] When talking of acts "all thought of activity must plainly be excluded" [51, §13, p. 379]. In Husserlian terminology *acts* are mental experiences in which one is directed to something objective (e.g., perceiving, judging, believing etc.) in contrast to other mental states (such as pain) that lack intentionality.

[111] Examples are acts directed at contradictory concepts such as a round square, the set of all sets or an elementary embedding of the universe into itself together with AC.

[112] [55, part 3, chap. 3].

object, that can be construed as a generalization of the concept of 'meaning' from the sphere of linguistic acts to all acts of consciousness, in rough analogy to the Fregean doctrine that 'sense' determines reference.[113] Under this interpretation the relation between noema and the intended object is to be construed in terms of satisfaction in the model theoretic sense,[114] while noema and act—or more precisely, noema and what Husserl calls *noesis* or meaning giving aspect of the act—stand in a relation of correspondence.

That correspondence lies at the heart of Gödel's strategy for clarifying the meanings of basic concepts by examining our acts in the use of these concepts.[115] To give a simple example, the sense of being beyond space and time which is inseparable from the noema of any mathematical entity has its noetic correlate in the characteristic feature of our mathematical experiences that they are about something which isn't individuated in consequence of a temporality or location belonging to it originally: the moment we encounter a mathematical item, we realize it is, so to speak, laid out 'forever', so that we can in principle always go back to it, and so that others can refer to it as well as ourselves.

A noema is a complex structure, and under the noetic-noematic parallelism each of its components is in fact correlated with a certain aspect of the act. One of these components, the *noematic sense*, is what gives an act its directedness by specifying the attributes and the particular point of view with which the act's object is intended. In an act of perceiving, the noematic sense is a description of a physical object (an individual concept in the sense of [9, p. 6f]). In an act of judging, the noematic sense is a proposition (and the intended 'object' a state of affairs). Another noematic component corresponds to the *mode of givenness* of the act's object and determines the degree of concreteness or 'fullness' of an intentional experience. This mode of givenness is, in turn, itself complex. Its main constituent expresses a noetic quality called the *thetic* (or *positing*) *character* of the act which is a measure for its 'reality character' depending on whether the intentional object is perceived or remembered or merely imagined with a given noematic sense.[116]

[113] This interpretation of the noema and the ensuing reading of Husserl are due to [20]. Although not uncontested, it is supported by systematic and textual considerations.

[114] A formalization of the underlying intensional logic is still outstanding. (Ostensibly both existential generalization and universal substitutivity must fail.) The logic of sense and denotation outlined in [7], [8], [10] can be regarded as an attempt in that direction.

[115] Cf. the passages from [33] quoted in the introduction. That Gödel attached considerable relevance to this noetic-noematic paralellism is also evident from his shorthand notes on the *Ideas*. See, e.g., [36, p. 3] and his markings in the margins of pp. 240, 330–331 in his copy of the *Ideas*, cf. [38, p. 10, p. 12]. On the back of the white flyleaf in that book, Gödel notes that Husserl's disregard for the noetic-noematic parallelism was the second cardinal mistake of the *Logical Investigations* [38, p. 5]. Cf. also footnote 101.

[116] More precisely, the thetic character not only allows one to distinguish between those different species of acts, but it also encompasses, among other things, *doxic modalities* (of believing, assuming, doubting etc.) and *modalities of attention* [55, §§91f, 99, 132f].

The function of the noesis is to organize the various elements of an experience in such a way that the general concept of object finds application to them, a process which Husserl calls *constitution*. Typically this is achieved by combining a multiplicity of experiences so that they come to possess a certain sense, e.g., when they are apprehended as different presentations of one and the same object.[117]

> But in a wider sense an object 'constitutes' itself—'whether it is actual or not'—in certain concatenations of consciousness bearing in themselves a discernible unity in so far as they carry with themselves by virtue of their essence the consciousness of an identical X. [55, §135, p. 332].[118]

Constitution has both a noetic and a noematic aspect. Roughly, the former concerns the sense-bestowing activity of consciousness, while the latter refers to the sense bestowed. In other words, the noetic aspect deals with the emergence of our mental acts 'within' the stream of consciousness,[119] while the noematic aspect unravels how the meanings carried by the objects we experience arise from logically prior meanings. To Gödel "phenomenological investigations of the constitution of mathematical objects is [sic] of fundamental importance for the foundations of mathematics."[120]

For Husserl the term *object* does not carry any suggestion of materiality, nor is it used to distinguish a specific ontological category that stands over against other ontological categories such as *class, function, property, process, state of affairs*. Rather *object* denotes the broadest, least specific ontological category there is.[121] In Husserl's ontological scheme there are just two kinds of objects: *real* (or sensory) objects and *categorial* (or ideal) objects. The distinction is phenomenological in that it rests on the difference in the type of acts in which these objects are brought to awareness. Sensory objects are there all at once for us in one act-stage, whereas categorial objects constitute themselves in higher-order acts which are *founded* on more basic acts.[122]

The concept of foundation with regard to acts and their components is crucial in Husserl's theory of intentionality. More precisely,

[117]The corresponding noematic sense is composed of *attribute-meanings* (which Husserl somewhat carelessly calls *predicate senses* even though they do not involve judgement) and the *determinable X*, the formal denotation of the purported object in abstraction from its attributes in analogy to Kant's *transcendental object* as the "concept of anything-whatever". Cf. [55, §131], [65, A250–251].

[118]In his copy of the *Ideas* Gödel wrote "important" in the margin of this page [38, p. 12].

[119]This is linked to inner time consciousness, our awareness of the immanent temporal ordering of our mental acts (cf. [59, §18, p. 45]). Gödel thought highly of Husserl's work on time, acknowledging that "time is the only natural frame of reference" for the mind [102, 9.5.1].

[120][102, 8.2.8].

[121][50, §61, p. 228f].

[122][52, §§46ff].

> a content[123] of the species α is founded in a content of the species β if an α by its essence ... cannot exist, unless also a β exists; where it is left open whether also the co-existence of certain γ, δ is required or not. [51, Investigation III, §21].

Some comments on this definition are in order. First, it is purely formal in that it does not take any contentual or contingent constraints into account. In particular 'species' is not restricted to natural kinds, and 'co-existence' has no temporal connotations (e.g., simultaneity). Second, the relation of foundation holds in the first place between essences and only derivatively between the individuals in which they are instantiated. Finally, depending on whether constitution is studied under its noetic or its noematic aspect, the metaphor of foundation is applied to acts or to objects (or rather object-meanings).

Every whole is founded on its parts, for it would not be that whole if it were not composed of just those parts. More interestingly, however, foundation also runs in the opposite direction, between parts and the whole that they comprise, and also between the parts of a given whole.[124] Guided by the idea that dependence and independence is always governed by an *a priori* law "having its conceptual roots in what is universal in the part and the whole",[125] Husserl outlines the beginnings of a formal theory of part-whole in the *Third Logical Investigation*. It serves as his main analytic tool in the study of the dependence relations subsisting among mental acts and their components, enabling him to shed new light on the nature of intuition and to clarify the primacy of transcendental subjectivity. The crucial role which mereological concepts perform in our thinking was also emphasized by Gödel.

> Whole and part—partly concrete parts and partly abstract parts—are at the bottom of everything. They are most fundamental in our conceptual system. ... [102, 9.1.24].[126]

[123]The term *content* is applied with a variety of meanings by Husserl. For our purposes it is best to think of a 'content' as something which is the object or intentional correlate of a presentation (cf. [51, Investigation III, §2, p. 228]). Under this interpretation it is clear that 'foundation' and 'constitution' pertain to the specific manner in which objects are given to consciousness and not to the conditions for or nature of their existence.

[124][51, Investigation III, §4] analyzes a classic example from [95, p. 109ff]: the two moments of *quality* and *extension* within the whole comprised by a visual perception, e.g., the hue of a color and its spatial extension. In Husserlian terminology 'moment' refers to a dependent, or abstract (cf. footnote 126 below), content whose existence, as a phenomenon, is necessarily bound up with the existence of some other phenomenon forming a more encompassing whole. Examples of moments are pitch (of a tone), assertoric force (of a judgement) or unity (of a complex act). We become aware of a moment by *abstraction*, i.e., by singling out and attending to certain parts of a composite whole while disregarding and neglecting others. A different kind of abstraction, the one leading to the formation of general concepts, will become important in Section 5. For the difference between these two kinds of abstraction see [51, Inv. II, §§40f], [52, §52, p. 162].

[125][51, Investigation III, §10, p. 249].

[126]Gödel's formulation "partly concrete parts and partly abstract parts" indicates that he was indeed alluding to the *Third Logical Investigation*. In Husserl's mereology *abstract*

That an act of sensory perception is unfounded does not mean that it is without a complex internal structure, but rather that it does not depend for its occurrence on the performance of more 'primitive' kinds of acts having their own intentional objects. In a perception the object is thematically present from the very beginning and it remains the constant theme of our experience as we live through the act.[127] What this means is that in perception the whole is prior to the parts: in each phase of the act where a new feature displays itself, it is not this feature but the full-fledged object that we are directed to. This object remains one and the same throughout the entire act "not merely in the physical sense, but according to the meaning of the perceptions themselves".[128]

By contrast the intellect must be at work when we register a state of affairs involving a sensory object. For example, while perceiving a physical object W our attention may spontaneously be focussed on some proper part P of W. As a result, there are now two separate perceptions, capable of serving as founding acts for a synthesis through which the part-whole relationship between P and W is articulated. (The details of this synthesis which proceeds in two stages and involves an act of reflection are described in [61, §50a].) The state of affairs constituted in this way can serve as a foundation for acts of still higher order, for instance acts assigning a value to the fact that P is a part of W. As a general rule the relation of foundation may be iterated any number of times, and this gives rise to an open-ended hierarchy of constitutional syntheses.

Similarly sets constitute themselves as objects in multiple stages. First distinct elements $A, B, C, \ldots$ (or more precisely their individual presentations in perception, thought or imagination) are mentally collected and set side by side.[129] At this point the set of these elements is *preconstituted* in the sense that we are collectively directed to A and to B and to C etc. in a 'multi-rayed' act, but the unity obtaining in this collection has not yet been explicated. That unity, and thereby the set $\{A, B, C, \ldots\}$, presents itself as a new object by means of a 'single-rayed' act in which the outcome of the preceding act of collecting is reflectively 're-grasped'.

> Accordingly a set is an objectivity that is originally preconstituted through a collective activity binding together disjoint objects and whose active comprehension consists in a simple re-grasping or grasping of that which has just been constituted. [61, §61, p. 295].

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and *concrete* are synonymous with *dependent* and *independent* (cf. the opening sentence in the introduction to [51, Investigation III, p. 225]). Husserl's terminology differs therefore from the common distinction between those entitities that can be perceived with the senses, presumably in virtue of their causal properties, and those that cannot. (To distinguish between those kinds of entities Husserl uses the terms *real* and *ideal* respectively.)

[127] [61, §63, p. 302f].

[128] [51, §47, p. 149].

[129] The implications of this process are studied in immense detail in [61, chap. II].

That the unity obtaining among the elements of a set is, for us, of a different kind than the unity we experience among the constituents of a physical object was apparently of relevance to Gödel. In his view sets are "the limit case of spatiotemporal objects" since for a set the interconnections of the "parts" plays no role.[130] Similarly, Husserl stresses the "syntactical" character of the unity found in a set. Here the mode of combination of the elements is completely independent of any constraints on their homogeneity, and what the elements have in common may simply be that they belong to the same set.[131] Nevertheless, for Husserl that unity is something objective which resists reductionist moves seeking to explain it via the simultaneous perceptions of the elements of the set. Rather,

> here a *unitarily intentional* relation is given and corresponding to it a unitary object which can constitute itself only in this act-concatenation, much in the same way as a state of affairs can only constitute itself when presentations are *relatingly* combined. [52, §51].

To put it another way: in a categorial synthesis we are not combining objects, but rather presentations of objects, and thereby we encounter an entity that can only be experienced in a higher-order act.[132] Thus the idea of sets constituting themselves 'within' the stream of consciousness is compatible with the view of mathematical objects existing independently of our mental acts and dispositions that lies at the heart of Gödel's mathematical realism.[133]

Acts of collection have figured prominently in the accounts of the concept of set by many commentators, including Cantor.[134] An immediate difficulty is how those analyses can be extended to infinite sets. In fact, already in the case of large finite sets it becomes practically impossible that our awareness of a Many as a One is a function of the collective combination of individual elements. This has led some philosophers to consider idealized acts of collection to be performed by a subject whose capabilities exceed those of

[130] [102, 8.2.4].

[131] [61, §62].

[132] Already in his *Habilitationsschrift*, Husserl distinguishes sharply between sets as "objectively-real things" and our presentations of them (cf. [47, p. 298]), and in the *Philosophie der Arithmetik* he insists that "our mental activity does not *create* the relations; they are simply there and are noticed when the interest is appropriately directed just as well as any other contents" [48, p. 42].

[133] [31, p. 311, p. 323], [32, p. 337].

[134] See for example his 'definition' of 'set' [5, p. 282]. Notice the similarity with the 'definitions' in Husserl's *Habilitationsschrift* [47, p. 333] and the opening paragraph of *Zur Lehre vom Inbegriff*, a manuscript dating from 1891 (cf. [48, p. 384]). By contrast, Cantor's 1883 definition (cf. [6, p. 204]), which predates Husserl's arrival in Halle by three years, does not explicitly mention mental acts of collection and has a decidedly metaphysical flavor given its modal connotations. (Husserl and Cantor were collegues in Halle from 1886–1900).

the human mind. Husserl, however, follows a different route to arrive at the *concept of set*.

> A set [Inbegriff] is given and can only be given in the actual comprehending-as-one, that is in an act which is brought to expression in the form of conjunctive combination A and B and C .... But the concept of set is not acquired by reflection on this act; rather than focussing on the giving act, we must focus instead on what it gives, on the set that it brings to *concrete* appearance, and we have to raise its general form to the level of general conceptual consciousness. [52, §44].

This "general conceptual consciousness" is reached via another kind of higher-order syntheses where "the objects of the founding acts do not enter into the intentions of the founded acts."[135] Such syntheses underlie the constitution of the logical forms (the copula 'S is p', $\wedge$, $\forall$ etc.), and the *general* concept of 'set', 'number', 'function' etc. These universals are brought to consciousness not as the species of this or that particular instance, but of *all possible* instances of that same specific content. Husserl's method of *free variation*[136] is an attempt to master this infinite space of possibilities. One starts with an arbitrary example of the class of entities one wants to discover. The features of this example are then varied in imagination up to a point where one realizes that any further transformation is incompatible with membership in the class under consideration. According to Husserl, we have discovered the essence when we have reached the limits of imaginative variability. Note that the shift to imaginative consciousness is crucial for escaping the contingency of empirical generalization. For only in imagination do we encounter test cases that turn out to be impossible on inspection and so mark the essential boundaries of the universal we are trying to reach. Husserl argues that there must be something that remains invariant throughout the process of variation, otherwise the variants would appear merely as random individuals and not as examples of the same 'type'. This invariant is

> the eidos, the *idéa* in the Platonic sense, but conceived purely, and free of all metaphysical interpretations, hence apprehended exactly as it comes for us to its intuitive giveness in the seeing of ideas that arises in such a way. [61, §87a, p. 411].[137]

[135][52, §52]. In similar vein [34, p. 262] declared that the concept of set is intuitively clear to us independently of our possessing intuitions individually of the objects to which this concept applies.

[136]The idea of free variation in imaginative consciousness does not occur (at least not explicitly) in the *Logical Investigations*, but is formulated only later on in the mid 1920s in [56, §9] and in the lectures underlying [61, §§86ff].

[137]Recall Husserl's characterization of idealism not as a metaphysical doctrine but as the only possibility of a coherent epistemology (cf. footnote 107).

It is not at all easy to understand or to paraphrase the passages where Husserl explains that the singular entity which is before the mind in the "seeing of an idea" is not an individual in the ordinary sense "but a concrete hermaphroditic unity of mutually eliminative, co-existentially exclusive individuals"[138]—the kind of entity, we might add, which is customarily described in mathematics with the help of expressions like '*arbitrary* set of integers'.

Husserl conceived of free variation as a phenomenological description of the method by which the mathematician reaches *a priori* generality and necessity, hoping that it could be adapted for investigating the essential structures of consciousness.[139] This has been contested from the very beginning—even within the phenomenological movement. An immediate objection is that the account given above of how we become acquainted with an eidos is straighforwardly circular: being able to implement free variation seems to presuppose that we already have a grasp of what is and what is not essential to an eidos. However, with respect to a meaning clarification as envisaged by Gödel such criticism is misplaced. For 'clarification' in fact means that we already possess a tacit, pre-theoretical grasp which can subsequently be refined.

But even when this is granted (and nominalistic suspicions about essences as special kinds of objects are ignored), it remains unclear why our awareness of universals should be assimilated with perception, as both Husserl and Gödel insisted.[140] Those who disagree may argue that intuition of an essence is a kind of theoretical understanding as opposed to an acquaintance with an object or object-like entity. Moreover, in their eyes, Husserl's model of free variation is inappropriate for explaining how such understanding is brought about. For example our understanding of 'arbitrary set of integers' includes complicated instances like $0^\sharp$ which are not discovered by simply modifying a given set of integers in imagination. Furthermore this understanding involves mastering sophisticated techniques and the capacity to evaluate different kinds of evidence, to test hypotheses and make predictions; in short, an ability to apply a *theory* in explaining problematic phenomena.[141]

The view deriving from Plato that the two orders of being—ideas and physical objects—are correlated with separate faculties of thought and sense perception runs through a major line of the philosophical tradition. This includes also Kant who differentiated between sensuousness and understanding as the two fundamental faculties of the mind from which all knowledge

[138] [61, §87d, p. 417].

[139] For example, 'variant' is an analogue of 'variable' in the mathematical sense [56, §10, p. 92]. On the whole, however, mathematics and phenomenology, are 'eidetic sciences' of a fundamentally different kind, cf. the references in footnote 98.

[140] [55, §3], [102, 9.1.12]. The latter should also be compared with the "principle of all principles" in [55, §24].

[141] $0^\sharp$ is a set of integers encoding a truth definition for L with a proper class of indiscernibles. Under large cardinals there is in fact a systematic hierarchy of sharps for inner models, cf. [63].

arises.[142] According to Kant, each of these faculties relies on its own type of mental representation. Intuitions, the instruments of sensuousness, are singular representations which refer directly to just one object, whereas concepts, the instruments of understanding, are mental representations that potentially apply to many objects by virtue of their possession of a common feature.[143] On this view it seems outright impossible that there could be such a thing as intuition of categorial objects.

§**6. Fulfillment, evidence and truth.** Husserl escapes from this impasse by replacing the two-faculty theory with distinctions of a rather different sort. The notion of foundation is introduced as a technical device for separating acts of understanding from acts of sensuousness. To characterize the intuitive Husserl invokes the basic feature of consciousness, first isolated by Brentano, that things can be meant 'in their absence'. For example one may refer to the house on Händel Street Nr. 13 in the city of Halle without ever having been there, or one can single out a complex number with the algebraic expression $-e^{i\pi}$ whilst failing to realize that it is in fact the number 1. In the *Logische Untersuchungen*[144] such acts are called *significative* because through their performance one is merely pointing to an object without being familiar with the object itself. In an *intuitive act*, on the other hand, the object is experienced *in propria persona*. The distinction as such was already familar to Brentano who differentiated between symbolic and authentic presentations. What Husserl added was the realization that, strictly speaking, an intuitive act is not an independent whole, but always a dependent part of a larger whole that includes in addition a significative act with the same noematic sense.[145] These two acts are bound together in virtue of the reciprocal relationship between the respective manners in which their common intentional object is presented. A significative act succeeds in meaning an object in its absence because it provides a set of instructions whose execution in an intuitive act results in the experience of the object as itself-given. Conversely, whether an act qualifies as intuitive cannot be read off directly from the act, but is determined by the meaning-intention of a corresponding significative

[142] Cf. footnote 62.

[143] [65, A320/B376–377]. It should be added though that, for Kant, intuitions and concepts cooperate in constituting objects [A258/B314, A51/B75-76]. Every object of intuition is in fact formed by categorial determinations. Moreover, we have an intuitive grasp of quantity, of size, and even of causality, although, strictly speaking, it is only the schemes of the categories intuition has access to.

[144] [52, §8, §21f].

[145] The term '(in)dependent' carries the same meaning here as in the whole-part theory in [51, Investigation III]. This is already foreshadowed in [49, p. 111]. And nearly three decades later, in a research manuscript dated between 1918 and 1921, Husserl writes: "We cannot set side by side two components, sense and fullness, in intuition as such. Only by contrasting empty and filled sense can we obtain the difference, that is, through a synthesis of intuition and empty consciousness" [62, Beilage VI, p. 363].

act. The upshot of this is that it allows one to characterize intuition in purely functional terms as the *fulfillment* of an 'empty' intention expressed in a significative act.[146]

> We experience how in intuition [Anschauung] *the same* objective something is intuitively made present which was "merely thought" in the symbolic act, and that it becomes intuitable [anschaulich] exactly as something determined in such a way as it was initially merely thought (merely meant). [52, §8].

With the discovery of the noema, Husserl came to see that for characterizing the dichotomy of signification and intuition the reference to two acts could be avoided by focussing instead on the difference in the noematic correlates of the modes of givenness of the intentional object. Indeed, most phases of our ordinary thinking do not exhibit a strict separation into acts that are either purely significative or purely intuitive. In the realm of sensuousness, for instance, even the paradigmatic intuitive acts, namely sensory perceptions, contain significative elements. This is because in every perceptual situation the object is experienced perspectivally. Certain aspects of the object are presented to us, whereas other aspects—for example the back of a house or the continuation of a melody—are only intended as 'co-present' without being directly perceived. However, these significative components of the perception are not fulfilled in separate intuitive act but in a continuous synthesis with further perceptions: "the object affirms its presence by means of 'itself', by presenting itself from different sides while continuously remaining one and the same".[147] Accordingly, fulfillment is to be regarded as a graded notion with the ideal and unattainable limit that all co-intended significative components of the perceptual noema are filled.

In the realm of understanding, objects constitute themselves in synthetic, higher-order acts as we explained in the preceding section. Consequently, a significative act points to a categorial object by specifying a certain synthesis which, when carried out, will engender an explicit awareness of this object. The corresponding intuitive act through which the 'empty' naming intention is fulfilled consists in the conscious performance of the required synthesis. For the purpose of illustration, consider the unordered pair $\{A, B\}$ where A and B are sensory objects. What intuitively corresponds to this two-element set can of course not be apprehended by the senses, nor can it be represented in an image.

> I can paint A and I can paint B, I can also paint them both on the same canvas; I cannot, however, paint the *both*, nor paint the *A and B*. [52, part 2, §51, p. 160].

[146] In Husserl's opinion Kant's failure to recognize the significance of this was the main reason why the latter was unable to develop a more general conception of intuition [52, §66, p. 202f].

[147] [52, §14a, p. 56].

To register the set "*A and B*" intuitively

> we have only the one possibility which is always open to us: to perform the new act of conjunction (collection) on the basis the two single acts of [sensory] intuition. [loc. cit.].

More precisely, the needed synthesis for bringing about an intuitive awareness of this set consists in the reflective re-grasping of the act of collecting (cf. Section 5). For multiplicities whose elements are too numerous to be placed side by side in an act of collecting, it is the insight into the ideal possibility of such an act that comprises an intuition of the corresponding set. To see how the dichotomy between signification and intuition continues to apply in this context, take the transitive closure of a given set M which is defined as the smallest transitive set containing M as a subset.[148] Here the purely significative character of the definition is underlined by the fact that the existence of such an object is by no means automatic. The transitive closure of M is brought to intuitive givenness once we comprehend the ideal possibility of the synthesis leading to $M \cup \bigcup M \cup \bigcup\bigcup M \ldots$, a possibility which is guaranteed by the axioms of union, infinity and replacement.

In what sense are those axioms deliverances of intuition? If we agree with Gödel that the axioms are 'implied' by the concept of set, the answer must reside with the significative intentions which determine the meaning of 'set'. Those intentions, as we saw in the previous section, point to a free variation, and consequently the correlative intuitive act must consist in the deliberate performance of that variation. It is true that the free play of imagination can never exhaust all the possible instances of the concept of set. Nevertheless it can produce the insight that we could go on indefinitely in a certain direction without overstepping the essential boundaries of the concept. This consciousness of being able to go on, even without actually going on, is indispensable for intuiting an essence.[149]

Consider the power set axiom for example.[150] In general it is not immediately clear that the collection of all subsets of a given set should again be a set. For an infinite set, the size of this collection can consistently be assumed to be arbitrarily big.[151] According to Gödel, "[w]e can form the power set of a set, because we understand the selection process (of singling out any subset from the given set) *intuitively*, not *blindly*.[152] From what Gödel says

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[148] A set M is *transitive* if it is downward closed under $\in$, i.e., if every element of M is also a subset of M.

[149] [56, §9c]. More precisely: it characterizes 'intuiting an essence' under the noetic aspect.

[150] For a phenomenological treatment of the axiom of replacement (and its difference in evidential character from the axiom of choice) see [45].

[151] [94]. Here 'arbitrarily big' has a definite sense attached to it, i.e., on the scale of the $\aleph$s.

[152] [102, 8.2.17], emphasis added. Gödel's formulation is reminiscent of Leibniz' distinction between *intuitive* and *blind* cognition in the *Meditationes de Cognitione, Veritate et Ideis* [68, vol. 4, p. 423] where the term 'blind' carries roughly the same meaning as 'symbolic' in [51, Investigation II, §14, p. 141]. Compare also [102, 9.2.9].

elsewhere it is clear that 'intuitively' means "realization of a possibility",[153] namely, "[t]aking all possible ways of leaving out members of the set is a kind of 'method' for producing all its subsets. We then feel that we can overview the collection of these subsets as well."[154] Indeed, for Gödel, mathematical objects in general "don't concern actualities about which the sense (the inner senses included) inform us but possibilities and impossibilities".[155]

Remarks like these resonate with Husserl's understanding of *eidos* as prescribing a 'rule' of synthesis of possible experiences in order that a concept or form may be validly applied. What an intuition of an eidos amounts to, then, is our grasping of that 'rule'.[156] Two points are worth stressing. First, in conformity with Gödel's opinion on the overemphasis of language in philosophy,[157] there is no *a priori* reason why that rule needs to be sententially structured. The aforementioned eidos 'arbitrary set of integers' is a case in point: the first two things one wants to say about this entity are outright inconsistent. Likewise, the eidetic insights underlying Cantorian set theory generated a fair amount of mathematics long before they were propositionally articulated in Zermelo's system. Second, for most essences, and certainly for the eidos 'set', the grasping of the 'rule' is inescapably partial. Thus the power set axiom may be intuitively evident for somebody who has not yet discovered $0^\sharp$ or other complicated sets of integers. Such a person understands nevertheless that any subset of the integers, including the ones he will never encounter, *can* be generated by deleting certain members from the set of all integers.

If we accept the above interpretation that eidetic intuition consists in the "realization of a possibility", where does the purported analogy with perception lie? In connection with the power set axiom Gödel simply declares that "[w]e recognize possibilities in our minds in the same way as we see objects with our senses".[158] Husserl, on the other hand, explains:

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[153] [102, 7.1.19].

[154] [102, 8.2.13]. Compare with [102, 7.1.18].

[155] And for this very reason "they can be known (in principle) without using the senses (that is, by means of reason alone)" [31, p. 312, footnote 18].

[156] [52, §62, p. 189f], [61, §86, p. 410]. Viewed in this way, Husserl's doctrine of *Wesensschau* bears a strong resemblance with Kant's *Schematismus der reinen Verstandesbegriffe* (see [65], in particular A140–A142/B179–B181). For Kant, a scheme is the presentation of a general method for supplying an image to a concept. This image, however, is not a singular representation of an existing thing but rather the mental image of a rule predelineating possible concrete intuitions (*Anschauungen*) of objects falling under the concept. The *Schema-Bild*, as it is called in [46, §§20, 21], derives its image character from the intention to map the universality of that rule from the sphere of understanding into the sphere of sensuousness. To that end it must 'display' no more than those features which are to be found in arbitrary instances of the concept and is thus necessarily distinct from any image of a specific object falling under the concept.

[157] Cf. the corresponding references in footnote 81.

[158] [102, 7.1.19].

> This is of course not a sensuous perceiving. The universal Red cannot be seen like one sees an individual singular Red; but the extension of seeing as a figure of speech which is customary in the general use of language, is inevitable. With it one expresses that something which is common and general for arbitrarily many individually seen examples has become ours in entirely analogous *direct* manner and *as itself*. [61, §88, p. 421]. (Compare [51, §45, p. 142f].)

The reason why Husserl speaks of intuition of essences as a form of perception is that its function in the fulfillment of significative intentions of categorial objects is entirely analogous to that of sensory perception in the fulfillment of significative intentions of physical objects. In other words, the appeal to perception, correctly understood, does not posit any extraordinary faculty on the part of the phenomenologist, but amounts to no more than distinguishing, with regard to essences, between 'empty' thinking and thinking accompanied by insight. Moreover, like ordinary perceptions, intuitions give rise to the direct awareness of an object not involving intermediaries such as sense data, images, percepts and the like.[159] Therefore, if we follow Husserl's usage of 'perception' as a generic label for those kinds of mental acts in which the intentional object is directly experienced as itself-given, then categorial intuitions are automatically perceptions. It is simply unthinkable that their "objects become ours as themselves" in any other way. However, the reader should keep in mind that this does not entail a claim that 'perception' provides us with an epistemologically secure access to an object.

The main theme of the phenomenological theory of truth is again the opposition of signification and intuition. Husserl starts from the traditional formula that truth consists in the *adaequatio rei et intellectus*. Unlike his predecessors, however, he does not construe the *adaequatio* in metaphysical terms, as agreement between subject and object or as coincidence of thought with reality, but in noematic terms, as the coinciding of the noematic correlates of two modes of givenness.

> The *intellectus* is here the intellectual intention, that of meaning. And the *adaequatio* is realized when the objectivity that is meant is *given* in intuition [*Anschauung*] in the strict sense and given as exactly that as which it is thought and meant. [52, §37, p. 118].

The experience we undergo when we notice the coinciding of the meant with the given is what Husserl calls *evidence*. *Truth* is the objective correlate of this act, i.e., that coincidence itself.[160] An important point here is that the coincidence, although brought to consciousness by an act of identification, transcends this act in the sense that it entails the possibility

[159] [52, §14a, p. 56; §37, p. 116], [55, §§43, 90].

[160] [52, §39]. Actually several conceptions of truth are distinguished there. The differences among them need not concern us here.

of re-identification in other acts of a pre-delineated kind.[161] Thus, strictly speaking, truth is to be understood as the correlate not of a singular act, but of an act taken *in specie*. This allows Husserl to maintain that there can be truths holding independently of our knowledge. In these cases the ideal possibility of instantiating the corresponding act species exists but hasn't been realized.[162]

The relationship between evidence and truth will be discussed in the next section. But already with what has been said so far, Gödel's contention that

> we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. [34, p. 271].

looses its mysterious overtones and becomes almost tautological. That something is forced upon us *as* being true means nothing more than that we register an agreement between signification and intuition. By definition, intuition is the consciousness of an object as itself-given where 'object' is to be taken in the broad sense of an ontological *summum genus*. In the case of a set theoretic axiom, the intuited objects (i.e., all *possible* sets or, equivalently, the concept of set)[163] and hence also the intuition are categorial rather than sensorial. In other words, in an intuition of the objects of set theory the intellect is at work, and so "the sphere of 'sensuousness' has been left, and that of 'understanding' entered."[164] Nevertheless, the phenomenologist regards this non-sensorial intuition as something *like* a perception on the grounds that, in functional terms, its role in the fulfillment of a significative intention is entirely analogous to the role of sense perception in ordinary experience, and because it gives rise to the direct, unmediated experience of an object.[165] Notice that this sense of directness is not contraposited to discursive thought (cf. the remark in Section 3).

To those who reject this analogy because, in their eyes, it would amount to admitting a kind of supersensory perception, we may reply with Husserl: categorial intuition is indeed *super*sensory (*über*sinnlich) in so far as it is

[161] In general, an intentional reference to a synthesis of recognition is included in the evidential consciousness of any objectivity whatsoever [58, §60, p. 143].

[162] [52, §39, p. 123]. See also [50, §62, p. 229f, §51, p. 190f]. In his shorthand notes to the First Volume of the *Logische Untersuchungen*, Gödel called the passage at the end of p. 190 a "very clear explanation of truth and evidence (not confined to judgement, but encompassing e.g., the intuitive conceived as being)" [39, p. 7].

[163] Or, if one prefers, all *concepts* that could possibly count as the 'concept of set' as understood by Martin (cf. footnote 80).

[164] [52, §47, p. 152].

[165] What makes mathematical intuition analogous to sense perception for Gödel is also the fact that the correctness of an intuitive judgement is not arbitrary and, like the correctness of a judgement based on sensory perception, answers to something objective. (See for example [102, 7.4.5].) This point will be taken up from a phenomenological perspective in the next section.

founded or built *upon* (sich erbauend *über*) sensuousness, but certainly not in the sense of a kind of viewing in abstraction from all sensuousness.[166] The latter, to be sure, would be outright impossible for Husserl given his insistence that every categorial act is ultimately (but perhaps very indirectly) founded on sensuousness.[167]

There is no textual evidence, as far as I know, that Gödel explicitly endorsed the account of intuition given above. However, according to [102, p. 80], in the 1960s he did recommend to several logicians Husserl's treatment of categorial intuition in the *Sixth Logical Investigation* which is where that account originates.

§7. The problem of access. One respect in which sensory perception does differ from mathematical intuition is that in the former case the object is experienced as impinging on one's sensory surfaces. Thus an act of perceiving involves the continuous (albeit passive) awareness of our embodiment in the surrounding world with which we causally interact subject to its constraints. In a way perception *is* the experience of that causal interaction, and the problem with a Gödelian view of intuition is apparently that in mathematical experience there is no counterpart to that.

However, what we perceive on a given occasion is certainly not impingements on a sensory surface, but determined by the noematic sense of the perception itself. Aristotle wrote that seeing (*opsis*) is as much a doing as it is a being-affected,[168] and in modern theories of vision it has become commonplace that perception, as experience in general, is always the filling-in of a scheme, and never the mere addition of novel part to part. Our perceptual faculties are highly interpretative and idiosyncratic: in the first place perceiving is perceiving *as*. Ambigous shapes are a case in point. For instance, in the well-known example due to Jastrow[169] the physical stimuli reaching us from outside are the same no matter whether we see the image of a duck or a rabbit in the drawing, so the difference must be due to us. The task of phenomenology is to account for that difference.

From a phenomenological point of view, our senses primarily play the role of providing boundary conditions that eliminate certain noemata, although they cannot narrow down the possibilities to a unique one. Husserl uses the term *hyle* (from the Aristotelian word for *matter*) for these sensory constaints which strictly speaking are not objects that we perceive, or features of such objects, but experiences that we undergo in the perception of an object.[170] Under this interpretation it may be argued that there is an analogue of *hyle*

[166] [52, §45, p. 143].
[167] [52, §60, p. 183]; cf. [65, A19/B33, lines 19–24].
[168] *On Dreams*, 459b–460a.
[169] Cf. [103, part II, sec. XI, p. 309].
[170] For a lucid discussion on these issues see [21].

in mathematical experience, namely the objective constraints we encounter as working mathematicians (e.g., logical consistency). They, too, may be characterized as something ego-foreign that enters our acts and limits what can be intuitively present to consciousness.[171]

In order to give us an act with the appropriate object, the hyletic elements have to be structured in a certain way by the noesis, for example by apprehending them as co-instantiated attributes of an identical something.[172] But this process does not only depend on hyle that one is explicitly aware of. No perception is a static, self-contained unit of experience but carries within it, as an essential part of its sense, more or less explicit allusions to past experiences as well as adumbrations of possible future ones. A perceptual object

> is, so to speak, an object determined in advance, before it has actually been determined by a specific perception. It has an empty horizon of yet unknown features that may be made visible by future or possible sequences of appearances. [56, §35, p. 181].

However, this aiming-beyond is not only the anticipation of determinations insofar as they pertain to this object of experience; in another respect it is also the aiming-beyond the thing itself to the surrounding world. This means that every object given in experience has not only a boundless *inner horizon*, comprised of the object's parts and aspects, or its known and unknown functional features, but also an indefinite *outer horizon* of co-given contents. These are contents toward which I am not now actually turned but toward which I can turn at any time. They also include implicit references to other conscious beings who experience the *same* object, but from a different perspective or orientation, and to whom I must adapt through communication and interaction.

> This preknowledge is indeterminate as to content or not completely determined, but it is never completely empty; and were it not already co-accepted, the experience would not at all be experience of this one, this particular thing. [61, §8, p. 27].

In short, perception, rather than a passive registering of information, is more of an active search, goal-directed and context-dependent, for anticipated facts. Likewise, the determinations entering into our mathematical intuitions "point to complementary determinations which themselves might

[171] Recall that these objective constraints play a central role in Gödel's arguments for mathematical objectivism (e.g., the second argument in [31, p. 314] or [102, 9.2.38], see also the references in footnotes 50, 133 and 165).

[172] In the *Ideas*, Husserl speaks of *hyle* as the "formless matter" of experience which is 'animated' by the noesis (the meaning-giving moment of the act) whereby our experience is *intentional* (cf. [55, §85, p. 208f, §97, p. 244]). This *hyle-noesis* scheme has come under criticism that found its most forceful expression in [79]. Roughly, the objection is that even so-called sensations are already endowed with a meaning and, as a consequence, a theory depending on a radical distinction between intrinsically meaningless material which is simply present in the mind and the meaning-bestowing activities of consciousness becomes senseless.

appear in other possible [intuitions], definitely or more or less indefinitely, according to our 'experiental acquaintance' with the object"[173]—a phenomenon that is also evident in the successive unfolding of strong axioms of infinity which are "the natural continuation of the series of those set up before".[174] Moreover, by virtue of the 'horizonedness' of experience, even direct perceptual judgements stand to some extent in inferential relations to the rest of our knowledge.[175] Therefore Husserl's notion of categorial intuition, although it lacks the immediacy normally associated with an intutive act, is not as dissimilar to perception as one might first think. And this may be another reason why Gödel insisted that "[t]here are more similarities than differences between sense perception and the perception of concepts".[176]

Finally, it should to be mentioned that the abstract entities for which there is allegedly no feasible epistemology under a realist interpretation are indispensible for even the most mundane forms of perception. In the day to day experiencing of things "everyone sees 'ideas', 'essences', and sees them, so to speak, continuously; one operates with essences in one's thinking and one also makes judgements concerning essences. It is only from the epistemological 'standpoint' that people interpret them away."[177]

> The factual world of experience is experienced as a typified world ... When we see a dog we immediately anticipate its additional modes of behavior: its typical way of eating, playing, running, jumping and so on. We do not see its teeth at this moment; but even though we had never seen this dog, we know in advance how its teeth will look—not in their individual determination but according to type ... [61, §83a, p. 389f].

In similar vein Gödel declares that for learning how to perceive concepts one needs to examine everyday, prescientific thinking. He apparently saw phenomenology as a method for removing certain theoretical prejudices so that one may clearly recognize what one all along has been acquainted with.[178]

Although the history of science, to put it mildly, has not been kind to the simple intuitions of common sense, the above observations together with the earlier remarks suggest that—contrary to Benacerraf's contention—sense perception is certainly not one of "the better understood means of human cognition".[179] And, even when there is a causal connection between the perceiver and the object of his perception, that connection does not always play

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[173] [52, §10, p. 40].
[174] [30, p. 520].
[175] The 'horizonedness' of our experiences poses a major difficulty for artificial intelligence, the notorious *frame problem*.
[176] [102, 7.3.2].
[177] [55, §22, p. 49].
[178] [102, 5.3.3, 5.3.4, 9.2.3, 9.3.23].
[179] [2, Sec. II].

the role which is commonly assumed. The moral of this is that the alleged problem of our epistemic access to mathematical objects ought to be stated in more general terms, applying to all forms of knowledge about an external domain, as the question how consciousness is able to transcend the orbit of one's own mind and come into genuine contact with objects in the world.

> In all its manifestations cognition is a psychic process: cognition of the cognitive subject. Cognition stands opposite its objects. How can cognition now be certain of its agreement with the objects of cognition; how can it reach beyond itself and reliably meet its objects? The givenness in cognition of the objects of cognition, which is unproblematic for the natural way of thinking, becomes an enigma. [53, p. 20].

As far as the second pillar of Benacerraf's argument is concerned, it is hard to see how one could justify the claim that Tarski's is the "only viable systematic general account we have of truth".[180] The underlying conception of true as corresponding with reality contains the unanalyzed terms 'correspondence' and 'reality' that are themselves in need of clarification. But Tarski's formal treatment of those terms works only *within* mathematics and is demonstrably incapable of providing a semantics for mathematics as a whole.[181] Most importantly, however, those who regard 'correspondence with reality' as the proper explication of the concept of truth seem to believe that the meaning of *experience* consists in the taking of something alien to consciousnes into consciousness—a view that, according to Husserl, is bound to lead into absurdity.

> For how could I make a rational statement to that effect, without seeing such a state of affairs and therefore seeing not only consciousness but also something alien to consciousness—that is: *experiencing* the alien affair? [58, §94, p. 206].

On the contrary, Husserl insists that, in a certain way, consciousness already 'contains' its object.

> The object of consciousness in its identity with itself during a flowing mental process does not enter into that process from outside, but it has been included [in the process] itself as sense, and that is as intentional production of synthetic consciousness. [59, §18, p. 44].

For the intentional object of a perception this sense is that of an identical something which is thematically present from the very beginning and which appears under more and more aspects as we live through the act. That

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[180] [2, Sec. III].

[181] Some of the logical difficulties involved are laid out in [97, §§5ff]. Here it is worth recalling the decisive point of the phenomenological explication of the *adaequatio* in noematic terms, i.e., as the concretely realizable coinciding of the noematic correlates of two different modes of givenness (cf. Sec. 6).

something must persist throughout this manifold of appearances is obvious, for otherwise we would have to say that in each phase of the act a different object is perceived. Likewise each objective feature of the object is itself an invariant in a continuum of appearances.

> The *same* color appears "in" continuous manifolds of Farb*abschattungen* [color-*adumbrations*]. Something similar holds for sensory quality and likewise for every spatial figure. The one and the same figure (given in person *as* the same) appears over and over "in a different manner", in ever different Gestaltabschattungen [figural profiles]. This is a necessary state of affairs and obviously of more general validity. [55, §41, p. 93].

Strictly speaking *Abschattung* has a two-fold meaning in Husserl. At the hyletic level it denotes an impressional episode in sensuousness, while at the noetic level it expresses the presenting of an objective aspect or feature. (Later on it will be important that this presentation is immediate and not a *re*-presentation by virtue of similarity.) Looked upon from the natural standpoint, variations in an Abschattung occur because of changes in the medium of perception or changes in our relative position etc. Their phenomenological significance is that they provide us with a criterion for distinguishing contents that transcend consciousness from those which are immanent to consciousness, for the latter are not presented from varying perspectives or under different orientations.[182] Notice that transcendence under this construal does not entail a complete separation of object and subject, or of being and consciousness. On the contrary, whether or not an object is transcendent to consciousness and essentially surpasses our ability to be aware of it in its entirety, is something we *can* know on the basis of our subjective experience.

With this last observation—which, under suitable modifications, applies equally well to mathematical objects[183]—we are given a hint how the misconceptions underlying the problem of access can be avoided. The reason why this is a problem in the first place is that one is aiming for a universal critique of cognition from a level where no such universality obtains. By conceiving of one's intuitions as episodes in the mental life of an embodied

[182] [55, §42, p. 96f].

[183] Gödel held that "[t]he analog of perceiving sense objects from different angles is the perception of different logically equivalent concepts" [102, 7.3.2]. That mathematical objects are invariants in our experience can hardly be contested. (See also [91] or [98, p. 447ff] on this point.) Think of the continuum in its various axiomatizations or of the concept of a group as the invariant structure in disparate manifestations of symmetry. Likewise, in the theory of large cardinals identity through differences determines a sense of objective existence e.g., for weakly compact cardinals. Arguably, one could also construe say the various aspects of say measurability (the connection with the classical meaure problem, the transcendence over smaller cardinals, ultrapower constructions, sharps etc.) as *Abschattungen*. As in the case of physical objects we are confronted with an infinite totality of such aspects of which we only know finitely many at any given time.

self and questioning their validity with respect to an external reality, one is in effect presupposing the very notion of objectivity the possibility of which is at issue.[184]

According to Husserl, the meanings of 'world', 'reality', 'objective' etc. become transparent only when we abandon the natural attitude and regard them as intentional achievements, not of a psycho-physical subjectivity that forms a part of the world, but of a transcendental subjectivity which is epistemically antecedent to the world.[185] To put it another way, questions of principle about the possibility of objective knowledge are legitimate only in so far as they pertain to conceptions of truth and objectivity which can be expressed in terms of the intentional structures of consciousness. These structures, Husserl insists, are accessible to us through *phenomenological reflection*, that is in the introspective mode of consciousness which is reached via the reductions (cf. Section 4). Here the crucial step in effecting the *epoché* becomes important: we may no longer construe our own mental life as possessing psychological reality. Even the seemingly self-evident belief must be suspended that consciousness is a stratum of the world, belonging to a community of other conscious beings, and that all this is founded upon the organic and inorganic strata of nature. For Husserl it is this reduction that uncovers the intentional life of transcendental subjectivity, with its noetic-noematic structure and its internal temporality, from where the phenomenologist seeks to unravel the constitution of the sense of the world and the objects in it. The base assumption is of course that intentionality is an *intrinsic* feature of our conscious experiences that they possess independently of whether or not what they intend does in fact exist.[186] Obviously that implies limitations with respect to what can be accomplished by phenomenological analyses. In particular, facts about the ontological relationship between consciousness and the world, since they are not reflectively observable, are irrelevant to the phenomenological account of intentionality. Nevertheless

> ... by the universal performance of the epoché with respect to the being or non-being of the world, we have not lost the world for phenomenology; we retain it, after all, *qua cogitatum*. ... Thus when the phenomenological epoché is consistently executed, there is left for us, on the noetic side, the openly endless life of our pure consciousness, and as its correlate on the noematic side, the meant world purely as meant. [59, §15, p. 38f].

What may be discovered in this realm of 'pure meaning', as we have seen, is for example a characterization of the transcendence of an object—its being

[184] [59, §41, p. 86].

[185] [55, §50, p. 118], [58, §99].

[186] The ensuing phenomenological theory of cognition anticipates, in different ways, modern trends in cognitive psychology such as the computational theory of mind [93], [19].

'outside' consciousness—in terms of contents of consciousness. And by reflectively analyzing the horizon-structure of experience, with its system of inherent and tacit cross-referencing, we also come to understand the sense in which perception, and likewise intuition, can "reach beyond itself" so to speak and seize its object.

> We could also say, what we are conscious of in transcendent perception as an object, which is present in person, is itself a meaning with an intention that is partially actualized, yet at the same time carrying in itself an anticipated certainty that, for the unactualized components of the intention, further actualizations are open *in infinitum*. [56, §36, p. 183].

Accordingly, a phenomenological analysis of intentionality must reach beyond the isolated subjective processes.

> By explicating their correlative horizons it brings the highly diverse mental processes into the thematic field comprised of those that function 'constitutively' with respect to the objective sense of [the object as meant]: i.e., not only the actual, but also the potential [mental processes], which, as such, are implicit [and] 'predelineated' in the sense-bestowing intentionality of the actual ones and which, when exhibited, have the evident character of [processes] that explicate the implicit sense. Thus alone can the the phenomenologist make understandable to himself how, within the immanency of conscious life, and in what sort of modes of consciousness of this perpetual stream of consciousness, anything like fixed and abiding objects can be brought to awareness. [59, §20, p. 50f]; cf. [52, §10, p. 40f].

Recall that Gödel argued along such lines for the meaningfulness of the classically unsolvable problems of set theory, pointing out that it is not the objective existence of mathematical objects that is decisive, but "the mere psychological fact of an intuition sufficiently clear to produce the axioms of set theory and an open series of extensions of them".[187] Although in that passage, he did not mention explicitly the transcendental reduction and its indispensability for overcoming the epistemological misconceptions of the natural standpoint, in his conversations with Wang he was more outspoken and carefully distanced phenomenology from an introspective psychology whose interests are tied to persons and objects in the real world.[188]

[187] [34, p. 272].

[188] Admittedly the above analysis of the meaning of 'transcendent object' fails to exhibit the full impact of the transcendental reduction since it may be argued that the conclusion can be reached by just shifting the focus of attention from objects to the specific manners in which they are brought to consciousness without suspending *all* claims concerning the psycho-physical reality of those acts. An analysis for which the full transcendental reduction is indispensable is Husserl's account of the constitution of intersubjectivity in the *Fifth Cartesian Meditation*. (Cf. the reference in footnote 102.) However, restrictions on space

> Philosophical thinking differs from thinking in general. It leaves out attention to objects but directs attention to inner experiences. (It is not so hard if one *also* directs attention to objects.) [102, 9.2.4].
>
> Introspection is an important component of thinking; today it has a bad reputation. ... *Epoché* concerns how introspection should be used ... [102, 5.3.27].

From the naturalistic standpoint it may be objected that Husserl's analysis of the meaning of our acts directed at transcendent object fails to provide any guarantees whether an act actually makes contact with reality. If our awareness of transcendent objects is indeed the awareness of a possible synthesis of partial actualized intentions, it must still be the case that those intentions are capable of being combined. Otherwise, as Kant argued, "we would have only a play of presentations that does not refer to any object at all".[189] Husserl's strategy for dealing with this issue is to construe 'objective existence' as the intentional correlate of the possibility of evidential verification.[190]

To a phenomenologist "*evidence* designates that performance on the part of intentionality that consists in the giving of something-itself", i.e., an object is directly present to consciousness in exactly the way in which it is symbollically intended.[191] Consequently, for transcendent objects, the inexhaustibility of their horizons combined with the reference to presently unknown future actualizations of those horizons renders all evidence inherently incomplete and provisional. On the other hand, this implies that "the single evidence, by its intentionality, can implicitly 'demand' further givings of the object itself, it can 'refer one' to them for a supplementation of its objectivating effect".[192] In fact, it is only in virtue of this implicit anticipation of additional evidence that one comes to know that a given piece of evidence with regard to a transcendent object may be defective. Thus while there can be no 'absolute' evidence for transcendent objects, the relativity of our evidence for them must always obtain with respect to other evidence.

> The *possibility of deception* is inherent in the evidence of experience and does not annul either its fundamental character or its effect; though becoming evidently aware of [actual] deception "annuls" the deceptive experience or evidence itself. The evidence of a new experience is what makes the previously uncontested experience undergo

make it impossible to incorporate into this article that crucial dimension of experience, and neither the shift of perspectives in Husserl's thought brought about by the move from transcendental subjectivity to transcendental intersubjectivity. (Recall that even rudimentary experiences like for example the perception of a material thing are delimited by horizons in which reference is made to the possible perceptions by other minds.)

[189] [65, A194/B239].

[190] [59, §§25–28], [55, §138].

[191] [58, §59, p. 157f], see also [59, §24].

[192] [58, §60, p. 159f].

> that modification of believing called "annulment" or "cancellation"; and it alone can do so. Evidence of experience is therefore always presupposed by the process. [58, §58, p. 139].

By re-interpreting the inherent imperfection of evidence as the ideal possibility of its perfection through additional evidence, Husserl claims to avoid the pitfalls of dogmatic theories insisting on an apodictic finality of evidence.[193] According to Wang, Gödel regarded the incorporation of fallibility as a valuable asset of the phenomenological theory of evidence.[194] Independent confirmation for this is provided through Gödel's disapproval of the view, put forward by the early Husserl in the first volume of the *Logische Untersuchungen*, that the experiencing of evidence can essentially have no other precondition but truth.[195] In his shorthand notes, Gödel objected: "That is false" , adding "moreover (1.) connectness of truth with us, (2.) not too much complication."[196]

In the final section of the *Ideas*, entitled *Reason and Reality*, the issue of justification is subsumed under the new theme of a *phenomenology of reason*, a phenomenological account of what makes our acts rational. Husserl takes the rationality of an act to be a function of its thetic character and the availability of evidence. Surely an act, before it can be assessed as rational or irrational, must involve some kind of ontological commitment or express a judgement. On the other hand, a *positing* act is rational only in so far as it refers to possibilities of evidential confirmation.[197] Husserl maintained that these possibilities are predelineated by a 'rule' that is contained in the noema of the positing act. In the case of a physical object, for example, the spatial determinations must be unfolded according to the laws of 'pure geometry'.[198] (Actually, as we shall see below, Husserl insisted on an essential correlation between types of possible fulfilling experiences and the regional essence instantiated by the intentional object of the positing act.) The task of a phenomenology of reason is to describe the sequences of possible experiences through which evidence for all kinds of object—conceived as invariants in those experiences—is acquired. Conversely it studies the conditions under which conflicting experiences or experiences that disappoint our anticipations enforce a modification of the thetic character of a given act or even lead to the cancellation of the positing altogether.[199]

[193] [58, §60, p. 143, §106].

[194] See the section entitled *Fallible Knowledge* in [102, p. 302].

[195] [50, §51, p. 191].

[196] [39, p. 8].

[197] [55, §136, p. 335], [59, §23, p. 58].

[198] [55, §142, p. 350, §149].

[199] [55, §138, p. 339f]. The discovery of the set-theoretic antinomies provides a good example for the latter. An example of the former is given by our shifting perceptions of the relative strengths of determinacy axioms vs. large cardinal axioms.

The two paragraphs at the end of the final section of the *Ideas* where Husserl equates the solution of these constitutional problems with the implementation of a comprehensive phenomenology of reason, and indeed of phenomenology altogether, were regarded by Gödel as the most important part of the entire book.[200] One reason for this may well have been the emphasis he placed on a theory that can account for the fallibility of intuition.[201] In Husserlian terms, any fulfilling act, no matter how partial or provisional the fulfillment may be, provides rational justification for an ontological commitment.

> The positing of a thing on the basis of its appearance in person is certainly a rational act, but the appearance is always a merely one-sided, "imperfect" appearance. [55, §138, p. 338].

Rational grounds for sustaining an ontological commitment exist as long as our experiences take a 'harmonious' course in the progressive fulfillment of other significative intentions, and as long as there is no evidence which is undermining or overturning the noematic sense of the original positing. The phenomenology of reason thus entails a theory of justification which—despite its ostensibly solipsistic character—is entirely fallibilistic and holistic rather than Cartesian, emphasizing the importance of the notion of coherence.[202] All of this is in line with contemporary views on the foundations of set theory according to which "what 'forces itself upon us as true' is more likely to be a theory as a whole" rather than a self-evident principle. Moreover, by virtue of the teleological structure of evidence outlined above "the process is more likely to be gradual".[203] What this also means is that even though our mathematical intuitions are 'direct' (i.e., the object is experienced as 'itself-given'), mathematical axioms need not be evident from the beginning.[204]

Another important aspect of the phenomenology of reason resides in the underlying conception of reason itself. Throughout the modern age, rationalism by and large had been guided by an epistemological ideal according to which one can know the world without having experienced it. Actually, what thinkers in the tradition of Leibniz and Spinoza aimed for was the state of knowledge attributed to God in the creation of the world. Husserl is diametrically opposed to that ideal by placing the essential characteristics of perception at the center of his theory of rationality. The potential

[200] [38, p. 1]. Ch. Dawson notes that the transcription as "the most important" has some doubtful aspect to it. On the other hand, Gödel observed that on p. 353 of the *Ideas* the entire *Sixth Logical Investigation* (of which he thought highly) is classified as a warm-up for the phenomenology of reason [op. cit., p. 5].

[201] Cf. the reference given in footnote 194.

[202] [22] gives an informative account on Husserl's theory of justification.

[203] See p. 422 in Steel's contribution to [18].

[204] Cf. the discussion in Section 3 and the reference in footnote 72.

implications of this for the meaning of Gödel's realism will be examined below.

§8. A possible meaning of Gödel's realism. The phenomenological account of intuition and the possibility of mathematical knowledge given above is internalist in the sense that all of the factors needed for an intuitive belief to be epistemically justified were to be found within the cognitive sphere of the subject entertaining that belief. The test for the truth of intuitions is coherence with our experiences as working mathematicians and not correspondence with an external reality. And what is commonly referred to as the mind-independence of mathematical objects turns out upon analysis to be a "transcendence in immanence"[205]—objects, whether real or ideal, are more than what is grasped in any single mental act, but are nothing over and above an indefinite system of actual and possible experiences. Indeed much of what has been said in the preceding section reads like the account that an idealist might offer, a reading that is corroborated by passages like the following.

> Consequently I, the transcendental phenomenologist, have objects . . . *solely as the intentional correlates of modes of consciousness of them.* [59, §15, p. 39].
>
> . . . every sort of existent itself, real or ideal, becomes understandable as a "product" of transcendental subjectivity . . . [op. cit. §41, p. 87].
>
> As a matter of principle (in the *a priori* of absolute eidetic generality) to *every "truly existing" object* there corresponds the *idea of a possible consciousness* in which the object itself may be grasped *originarily* and at the same time *completely adequately*. Conversely, when this possibility is guaranteed, then *eo ipso* the object truly exists. [55, §142, p. 349].

Certainly Gödel, as much as he was inclined towards idealism, could not have subscribed to any view ruling out the existence of mind-independent entities. But this does not disqualify transcendental idealism as a viable option. The fact of the matter is that Husserl's brand of idealism has nothing whatsoever to do with reductive metaphysical positions. Properly understood, it offers a philosophical framework for incorporating a reflection on the subject, yet does not carry an obligation to deny the objective meaning of thought—very much like Gödel insisted in his letter to G. Günther of June 30, 1954. Indeed, as Gödel emphasized, "phemomenology does not make any ontological judgements".[206] Thus by stipulating, for example, that objects are to be studied exclusively as "intentional correlates of modes of consciousness", Husserl does not claim that what the ordinary person thinks of as external

[205] [53, p. 34ff].

[206] [38, p. 1]. Gödel is quoting from [55, §153, p. 379].

reality turns out upon philosophical analysis to be nothing but a collection of ideal meanings. Rather, what he means to say is that once we abandon the natural attitude, as we must do as philosophers, and attend to the noematic structures which are the precondition of any meaningful experience, those structures become the sole subject matter of our investigation. The total system that they comprise together with our possible reflective awareness of them (transcendental subjectivity) is epistemically closed and complete in the sense that for any objectivity which enters into our experience there must exist within this system an equivalent noema that is accessible to us via phenomenological reduction. Transcendental subjectivity is thus "absolutely independent" in the sense of that phrase elucidated in the mereology of the Third Logical Investigation.[207] It is the final whole which is not founded on anything beyond itself, whereas 'world' and 'reality', as phenomena, exist only in relation to, or as moments of, transcendental subjectivity.[208]

As for Husserl's contention in the second passage that every sort of existent can be understood as a "product" of transcendental subjectivity, again one has to keep in mind that he is operating strictly at the level of meaning, and there can be no question of consciousness literally producing things. One indication of this is the enclosing of the word 'product' in inverted commas to signal a shift from the ordinary denotation to noematic sense.[209] Similarly, moving now from 'object' to 'existence', the correspondence between true existence and complete accessibility by a possible consciousness that is put forward in the last passage cannot be taken literally. Phenomenologically, i.e., within the epoché, one can only speak about acts that posit their objects and acts that do not. Thus "truly existing" (in inverted commas!) is not to be understood as a specification of a distinguished form of being in ontological terms, but rather as refering to a state of affairs that can only be described phenomenologically. This description essentially has three constituents. First, an act A directed to something objective in virtue of its noematic sense. Second, A must be positing, i.e., of the appropriate thetic character. Third, there exists an infinite system of further acts of possible confirmation or disconfirmation whose types are predelineated by the full noema of A.[210]

What is at issue in the theory of constitution are the conditions under which objectivities from distinct ontological categories become accessible to consciousness and achieve the status of being itself-given; or in other words,

[207]Cf. the first reference given in footnote 145.

[208][55, §49, p. 115, p. 117, §55, p. 134], [59, §41, p. 86].

[209]See the general convention concerning the use of inverted commas enunciated in [55, §89, p. 222].

[210][59, §27]. Recall (cf. Section 6) that, for Husserl, the complete adequacy of evidence is an unattainable ideal. Thus if "truly existing" were taken literally in the above passage as opposed to referring to an infinite horizon of potentialities, then strictly speaking *no* object could exist. In the sequel it will become clear that in the above passage Husserl is really trying to distance himself from Kant's representative realism.

what is meant by 'evidence' for such objects. And here the recurrent theme is that itself-givenness is not a mere 'datum' in consciousness, but something that has to be accomplished in synthetically structured acts. These structures, Husserl insists, are prescribed with essential necessity by the category to which the object in question belongs.

> *Category of objectivity and category of evidence are perfect correlates. To every fundamental species of objectivities*—as intentional unities maintainable throughout an intentional synthesis and, ultimately, as unities belonging to a possible "experience"—*a fundamental species of "experience", of evidence, corresponds*, and likewise a fundamental species of intentionally indicated evidential style in the possible enhancement of the perfection of the having of an objectivity itself. [58, §60, p. 144].

This means that the theory of constitution, in the first place, is concerned with act-schemes rather than acts. And an act-scheme employed in the constitution of an object, although its validity can only be recognized when one instantiates it in an act, transcends each of these instantiations. Husserl now takes this transcendence a step further by insisting that it holds not only with respect to each individual act but also with respect to any possible mind. Analyzing, for instance, the eidetic correlation between 'thing' and 'perception', he concludes that the inadequacy of perceptual evidence has nothing to do with factual limitations of our cognitive faculties. On the contrary, by virtue of what it means to be extended in space, material objects, as a matter of principle, must be experienced from various perspectives and under different orientations—"even by God as the ideal representative of absolute knowledge".[211] Similary, to the *a priori* of pure logic and mathematics there corresponds an *a priori* of a "psychic kind", a system for specifying the essential features that the experiences by any possible mind must possess in order to yield mathematical knowledge.[212] Unlike the synthetic *a priori* in Kant which resides in the structure of our cognitive faculties, this phenomenological *a priori* expresses necessary correlations between noematic essences (of object-meanings) and noetic essences (of acts in which an object of the corresponding ontological category is brought to itself-givenness).[213]

How important these correlations were to Gödel can be seen from his markings of related passages in the *Ideas*.[214] In view of this Gödel's remark

[211][55, §150, p. 371]. See also [op. cit., §42].

[212][56, §3e, p. 38]. Gödel [39, p. 7] also took notice of the point made in Husserl's refutation of psychologism that the laws of pure logic, as ideal conditions of judicative evidence, must hold for any possible consciousness [50, §50, p. 187].

[213][52, §66, p. 203].

[214]Cf. the references provided in footnote 115. At the same time, Gödel (cf. [37, p. 7]) is aware of an apparent circularity: On the one hand, the regional essences of ontological categories provide a *Leitfaden* for finding the relevant constitutional operations ([55, §149f],

about abstract essentialities (Wesensheiten) forming "a second plane of reality"[215] gains in significance: not only are abstract essentialities experienced as objective, mind-independent entities, they are also indispensible for understanding how intentionality arises in the first place. Looked upon in this way, ideal objects are indeed the precondition of the possibility of objective cognition.[216]

Other reasons for Gödel's endorsement of Husserl's transcendental idealism come to light by drawing a comparison with Kant. What lies at the heart of Kant's transcendental idealism is an attempt to account for the *a priori* character of mathematics and the natural sciences not in terms of the traits of objects but in terms of the structure of human cognition. Comparing his own strategy to that of Copernicus, Kant explains:

> If intuition must conform to objects, I do not see how we could know anything about the latter *a priori*; but if the object (as object of the senses) must conform to our faculty of intuition, I have no difficulty in conceiving such a possibility. [Bxvii].

But why should objects that exist independently of the mind have to conform to the forms of intuition? The only possible answer, according to Kant, is that the objects that we experience in mathematics (and in general) cannot be things in themselves.

> If the object [the triangle] were something in itself, apart from any relation to you, the subject, how could you say that what necessarily exist in you as subjective conditions for the construction [i.e., exhibition in pure intuition] of a triangle must of necessity belong to the triangle itself? [B65].

Thus Kant is led to his fundamental dichotomy between *things in themselves* (*noumena*) that exist independently of human cognition and *appearances* (*phenomena*) that exist only as contents or objects of sensible presentations,[217] and he concludes that human knowledge is restricted to the latter whereas the former are unknowable to human beings.

In Gödel's eyes, this tends to exaggerate the distinction between subjective and objective elements in our knowledge "which is so impressively suggested

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[59, §§21, 29]). On the other hand, one argues for the objective validity of those categories in terms of the constitution of objects.

[215] See his letter to G. Günther of June 30, 1954 ([16, p. 504]).

[216] Cf. the quotation from [39, p. 2] at the end of Section 4.

[217] [65, A249/B306–A253/B309, A491/B519–A494/B522]. The terms 'noumenon' and 'thing in itself' are coextensive yet not exactly synonymous, even if they refer to the same entity. For Kant, the epistemological device we have of a thing in itself is not that of a noumenon in the positive (and etymological) sense of an entity knowable by the pure intellect (noûs), but rather that of a noumenon in the negative sense of an entity not knowable by sensuous means.

by Kant's comparison with the Copernican system" into a "boundless subjectivism".[218] With regard to physical reality Gödel wrote in the 1940s that it should be possible for scientific knowledge, at least partially and step by step, to go beyond the appearances and approach the world of things. He expected that the means for doing so can be found in the successive levels of objectivation devised by modern physics which do not retain the forms of our sense perception and therefore, unlike Newtonian Physics, can do more than setting up relations between appearances. In his opinion, quantum physics in particular seems to indicate that physical reality is something still more different from the appearances than even the four-dimensional Einstein-Minkowski world.[219]

These passages portray Gödel as advocating a kind of representative realism which holds that the real thing is not as it appears to us. Access to the real thing is by reasoning and inference, and not by direct perception. However, a different picture emerges from Gödel's conversations with Wang in the 1970s. There he emphatically declares that "things are as they appear" and that "the ideas we form of sensations refer to the object itself."[220] Incidentally, this is precisely the position of Husserl whom Gödel contrasts with Kant in this context.

For Husserl, the relation between appearance and thing in perception cannot be one of representation. An appearance is, according to him, an *Abschattung* of the thing itself. In other words, in an appearance we do encounter the thing 'in person', albeit from a particular perspective or under one of its aspects. The noumenon-phenomenon dichotomy applies only in the sense that the thing itself is one-sidedly and therefore *inadequately* given through an *Abschattung*. This interpretation enables Husserl to uphold the Kantian doctrine of the unknowability of noumena by conceiving of the thing in itself as the infinite totality of its possible appearances, and by construing *adequate* knowledge of this infinite totality as a regulative idea.[221] Thus the phrase 'thing in itself' is no longer understood as referring to an entity which is totally disconnected from consciousness, but as a way of distinguishing an ideal mode of givenness. In the case of perception, this amounts to the contrast between intending an omnilateral continuum of *Abschattungen* and the actual intuiting of a one-sided profile.[222]

[218] [29, footnote 24].

[219] [29, p. 240, p. 244].

[220] [102, 5.3.37, 4.4.10].

[221] [55, §143].

[222] Incidentally, Kant stated that a noumenon must be accessible via some faculty of intuition in order to count as a real object, only to postulate to that end a special form of intuition which is non-sensorial (cf. [65, A252/B307–A254/B309]). However, Kant's contention, that this non-sensorial or intellectual intuition lies completely outside our cognitive faculties is rejected by Husserl. For the latter the term 'possible consciousness' must denote an eidos under which all forms of consciousness, whether factually occurring or imaginatively

Our discussions have reached a point where it is worthwhile re-examining one of the most notorious and enigmatic passages in Gödel's writings.

> It should be noted that mathematical intuition need not be conceived of as a faculty giving *immediate* knowledge of the objects concerned. Rather it seems that, as in the case of physical experience, we *form* our ideas also of those objects on the basis of something else which *is* immediately given. Only this something else here is *not*, or not primarily, the sensations. That something besides the sensations actually is immediately given follows (independently of mathematics) from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, e.g., the idea of object itself, whereas, on the other hand, by our thinking we cannot create any qualitatively new elements, but only reproduce and combine those that are given. Evidently the "given" underlying mathematics is closely related to the abstract elements contained in our empirical ideas. [Added by Gödel in footnote: Note that there is a close relationship between the concept of set explained in [34, footnote 14] and the categories of pure understanding in Kant's sense. Namely, the function of both is "synthesis", i.e., the generating of unities out of manifolds (e.g., in Kant, of the idea of *one* object out of its various aspects.)] It by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certain things upon our sense organs, are something purely subjective, as Kant asserted. Rather they, too, may represent an aspect of objective reality, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality. [34, p. 271f].

Rather than attempting a systematic interpretation, which would exceed the scope of this paper, I shall concentrate on a few points where I believe the invocation of Husserl is helpful for understanding how Gödel sought to overcome certain limitations of Kant's transcendental idealism. One example of this is the remark in the opening sentence about the non-immediacy of mathematical intuition. This can be clarified by pointing out that mathematical intuition is a founded act (cf. Section 5) and by adverting to the 'horizonedness' of experience and the teleological nature of evidence (cf. Section 7).

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contrived, are subsumed. Only then can valid claims be made about what is possible and what is impossible. But this implies that all these different manifestation are in some sense 'isomorphic' for otherwise they could not instantiate the same eidos. Therefore it is possible in principle (although not always factually so) for beings at lower level of consciousness to advance to a higher level of consciousness. (Cf. the passage from the reasearch manuscript K II 4, p. 109a/b as quoted in [66, p. 130].) Gödel, speaking of "rational beings of a different and higher kind" who "are connected to the others by analogy" ([102, 9.4.17]), may have held a similar view.

In addition, Gödel claims that the *forming* of our ideas about mathematical objects depends on something besides intuition that is immediately given and closely related to the abstract elements contained in our ideas about empirical objects, e.g., the idea of object itself. The only place where Gödel offers something of a positive characterization of the 'given' underlying mathematics is in the suggestive remark in the footnote when he compares the cognitive function of the operation 'set of x's' to Kant's categories of pure understanding. However, at first glance this comparison seems puzzling since for Kant the categories—which include 'unity' and 'multiplicity'—possess objective validity in so far as they provide necessary conditions for objective reasoning,[223] whereas Gödel attributes the view to Kant that "the data of this second kind" are purely subjective. What Gödel may have in mind is that "the idea of *one* object out of its various aspects" is subjective for Kant who maintained that "among all presentations *synthesis* is not given through objects, but can only be performed by the subject itself."[224] By contrast, in Gödel's thinking "a [set] is conceived of as something which exists in itself"[225] independently of our mental acts. And here Husserl's account of the constitution of sets (cf. Section 5) fits much better than what Kant has to say about synthesis.

On a Husserlian reading, arguably, "data of this second kind" would pertain to the determinations of the scheme that is implemented in the constitution of a particular object. Included among those determinations are regional essences together with certain noetic-noematic correlations. What evidence do we have that Gödel was thinking along such lines? For one thing, Gödel used the verb 'to form', which he italicized in the above passage, as an explicit reference to *constitution* (in the Husserlian sense) on at least one other occasion in explaining to Wang that "the way of how we form mathematical objects from what is given—the question of *constitution*—requires a phenomenological analysis."[226] Moreover, Gödel's reasons for believing that our ideas about mathematical objects contain constituents which are qualitatively different from anything obtainable by thinking resemble Husserl's argument [60, §7] for distinguishing regional essences from generic concepts. According to Husserl, a regional essence, by setting a style for intuitions, prescribes a rule for the possible course of experience. This is radically different from the manner in which generic concepts operate in

[223][65, A93/B126]. In B167f Kant expressly states that the categories must not be conceived of as subjective dispositions of thought, for otherwise they would lack the crucial attribute of necessity.

[224][65, B130]. See also B134f.

[225][34, footnote 14].

[226][102, 9.2.27]. In similar vein, Gödel described the aim of phenomenology as going back "to the process of how we form ... knowledge, and to uncovering what is given to us from inside" [102, 5.3.16]. The phrase at the end of this remark could be a glossing-over the "data of this second kind" which Gödel takes to be present in us.

our thought. Thinking, he says, does not anticipate ways of experiencing; it presupposes them. Once a particular form of experiencing is granted, our thinking structures the contents of possible experiences of that form by bringing generic concept to bear. But all this is done without creating any qualitatively new elements; we only reproduce and combine those that are given within the chosen frame of experience. A regional essence, by contrast, defines this frame itself.

In a certain sense the abstract determinations of a constitution-scheme are also "immediately given". Referring to the scheme for set-constitution, for example, Gödel speaks of a "fundamental property of mind", a primitive idea of our thinking that cannot be reduced any further.[227] Another point worth noting is that for Husserl a way of experiencing is a fundamental way of seizing objects. Thus if regional essences are included among what Gödel calls our abstract ideas about objects, then indeed "the ideas we form of sensations refer to the object itself".[228] In contrast, Kant's categories of pure understanding do not apply to things in themselves, but guarantee that certain relations among our presentations are not merely subjective.[229]

For somebody who accepts the above interpretation it is then a short step to recognize what Gödel might mean by "other kind of relationship between ourselves and reality". It could well be *intentionality*—construed not as a natural phenomenon, but in its inmost essence as the correlation between noesis and noema that not only provides access to the world but is the very presenting of the world.

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