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RANDOMNESS AND COMPUTABILITY: OPEN QUESTIONS

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§1. Introduction. It is time for a new paper about open questions in the currently very active area of randomness and computability. Ambos-Spies and Kučera presented such a paper in 1999 [1]. All the question in it have been solved, except for one: is KL-randomness different from Martin-Löf randomness? This question is discussed in Section 6.

Not all the questions are necessarily hard—some simply have not been tried seriously. When we think a question is a major one, and therefore likely to be hard, we indicate this by the symbol ►, the criterion being that it is of considerable interest and has been tried by a number of researchers. Some questions are close contenders here; these are marked by $\triangleright$. With few exceptions, the questions are precise. They mostly have a yes/no answer. However, there are often more general questions of an intuitive or even philosophical nature behind. We give an outline, indicating the more general questions.

All sets will be sets of natural numbers, unless otherwise stated. These sets are identified with infinite strings over $\{0, 1\}$. Other terms used in the literature are *sequence* and *real*.

Section 2 aims at understanding the relationship between Martin-Löf randomness and Turing reducibility. A major question, which has been around for over a decade, is whether the degrees of Martin-Löf random sets are definable. In Section 3, we look at K-triviality from a purely combinatorial point of view. We ask if there is a characterization that is not directly related to ML-randomness or prefix-free complexity K. In the next section we consider the relationship between K-triviality, Martin-Löf randomness and Turing reducibility. There are multiple (sometimes fascinating) interactions between these at first sight rather disparate concepts. We also ask to what extent all the K-trivial sets are similar; for instance, do the K-trivial sets that have an incomplete ML-random set above form a proper subclass?

In Sections 5 and 6, the general question is what happens when the c.e. test concept defining Martin-Löf randomness is replaced by a computable one. For computable and Schnorr randomness, several results show that the behavior is quite different. On the other hand, KL-randomness has not even been separated from Martin-Löf randomness. In fact, the separation question remains open for apparently much weaker randomness notions, like permutation randomness which we introduce here.

In Section 7, we look at concepts stronger than Martin-Löf randomness. One could argue that Martin-Löf random sets are not "really" random, for instance because there is a Martin-Löf random set Turing above an arbitrary given one, or because a ML-random set can be left-c.e. These nonrandom features disappear when one takes 2-randomness instead, that is, Martin-Löf randomness relative to the halting problem. We also consider randomness *much* higher up, where the test notions are not based on computability theory but on effective descriptive set theory.

In Section 8, we consider Chaitin's halting probability Ω and its relativizations. We will see that sometimes highly random and highly nonrandom sets behave similarly. For instance, both 2-random and K-trivial sets are low for Ω. One could ask what is behind this and other similarities. Maybe, in some appropriate sense, both lack useful information.

The randomness notions encountered so far provide an *absolute* hierarchy, similar to the absolute complexity hierarchy given by classes like computable, computably enumerable, Δ^0_2 and so on. The complexity of a set can also be measured in a *relative* way, by comparing it to other sets, say via Turing reducibility. In Section 9, we ask for randomness analogs of relative complexity. Namely, what does it mean for one set to be more random than another? We also consider to what extent such relative notions of degree of randomness cohere with the absolute ones, as evidenced for instance by the upward closure of absolute notions in relative randomness degree structures. (The analogy to relative complexity comes out clearer if one considers *non*-randomness notions rather than randomness notions.)

In the last section we ask to what extent it is possible to distill randomness out of a somewhat random source.

Facts quoted here without reference can be found in [11], or in the forthcoming books [7, 39]. An updated version of the paper will be kept at `http://www.cs.auckland.ac.nz/~nies/`.

§2. ML-randomness.

A *Martin-Löf test* (ML-test) is a uniform sequence $\{U_i\}_{i\in\mathbb{N}}$ of Σ^0_1-classes such that $\mu(U_i) \leq 2^{-i}$. A set A is *Martin-Löf random*, or 1-*random*, if $A \notin \bigcap_i U_i$ for each ML-test $\{U_i\}_{i\in\mathbb{N}}$. The class of ML-random sets is denoted MLR.

For $\sigma \in 2^{<\omega}$, let $K(\sigma)$ be the prefix-free Kolmogorov complexity of σ.

THEOREM 2.1 (Schnorr). *A set A is ML-random iff there is a constant d such that $K(A \restriction n) > n - d$ for every n.*

One of the most important facts about relative randomness is van Lambalgen's Theorem (1990) [54].

THEOREM 2.2. *$A \oplus B$ is ML-random $\Leftrightarrow$ B is ML-random and A is ML-random relative to B.*

A set Z is *diagonally noncomputable* if there is a total function $f \leq_T Z$ such that $f(n) \neq \Phi_n(n)$ for all n. This class is denoted DNC. Note that $\mathsf{MLR} \subseteq \mathsf{DNC}$, via a finite variant of the function $n \mapsto Z \restriction n$.

2.1. Turing degrees. A Turing degree is called *ML-random* if it contains a ML-random set. Let $\mathbf{ML}$ denote this class of degrees.

The following questions were mainly posed by Kučera (since 1990) in conference talks.

► QUESTION 2.3.

(i) *Is the class $\mathbf{ML}$ first-order definable in the Turing degrees?*
(ii) *Is $\mathbf{ML} \cap [\mathbf{0}, \mathbf{0}']$ first-order definable in $[\mathbf{0}, \mathbf{0}']$, the Δ^0_2 Turing degrees?*

The definability of ML in the Turing degrees would give an alternative first-order definition of $\mathbf{0}'$, since $\mathbf{x} \geq \mathbf{0}'$ iff $(\forall \mathbf{y} \geq \mathbf{x})$ $[\mathbf{y} \in \mathsf{ML}]$. The fact that every degree above $\mathbf{0}'$ is in ML is due to Kučera [21]. The other direction uses Stephan's result that the only PA-complete degrees in ML are the ones above $\mathbf{0}'$ [51]. Kučera has also asked if the PA-complete degrees are definable. As with ML, this would give an alternative first-order definition of $\mathbf{0}'$, since $\mathbf{0}' = \inf\{\mathbf{a} \vee \mathbf{b} \colon \mathbf{a}, \mathbf{b} \text{ PA complete } \ \& \ \mathbf{a} \wedge \mathbf{b} = \mathbf{0}\}$ [23].

Demuth [6] proved that if A is ML-random and $B \leq_{tt} A$ is noncomputable, then the Turing degree of B also contains a ML-random set. Everything Turing below a set of hyperimmune-free degrees is truth-table reducible to it. This implies that if $\mathbf{a} \in \mathsf{ML}$ has hyperimmune-free degree and $\mathbf{0} < \mathbf{b} \leq \mathbf{a}$, then $\mathbf{b} \in \mathsf{ML}$. Does the latter property characterize the hyperimmune-free degrees in ML? That is,

QUESTION 2.4. *If* $(\mathbf{0}, \mathbf{a}] \subseteq \mathsf{ML}$, *is* $\mathbf{a}$ *hyperimmune-free*?

Two natural candidates for a counterexample fail, namely noncomputable Δ^0_2 sets and 2-random sets (both are of hyperimmune degree). For the first, if $(\mathbf{0}, \mathbf{a}] \subseteq \mathsf{ML}$, then $\mathbf{a}$ and $\mathbf{0}'$ have infimum $\mathbf{0}$, otherwise some non-zero $\mathbf{b} < \mathbf{a}$ is c.e. [22], and hence not random, since no incomplete c.e. set is DNC. For the second, if $\mathbf{a}$ is of 2-random degree then $\mathbf{a}$ bounds a 1-generic degree [25, 18], and no 1-generic set is Turing above a ML-random set.

It would also be of interest to determine whether there is $\mathbf{b} < \mathbf{0}'$ such that $[\mathbf{b}, \mathbf{0}'] \subseteq \mathsf{ML}$. Such a $\mathbf{b}$ cannot be low, because there is a PA-complete low degree above every low $\mathbf{b}$.

§3. *K*-triviality and combinatorial properties. A lot of recent research is centered on K-triviality, which is at the end of the spectrum opposite to ML-randomness. A set A is *K-trivial* if $(\forall n)\ K(A \upharpoonright n) \leq K(n) + \mathcal{O}(1)$. (We identify a string σ in $2^{<\omega}$ with the natural number n such that the binary representation of $n+1$ is 1σ.) There is a c.e. noncomputable (even promptly simple) K-trivial set, and all K-trivial sets are Δ^0_2 (see [10, 11] for proofs and references). A set A is K-trivial if and only if it is *low for ML-random*, namely each ML-random set is random relative to A [37]. In particular, K-triviality is closed downward under Turing reducibility. The class of K-trivial sets is also closed under $\oplus$ [10], and each K-trivial set is truth-table below a c.e. K-trivial set [37]. Thus the K-trivial sets form an ideal in the Turing degrees that is the downward closure of its c.e. members. Restricted to the c.e. sets, it is the first known example of a natural intermediate Σ^0_3 ideal.

3.1. Traceability. It is desirable to characterize a lowness property of a set A by a "combinatorial" notion of computational weakness, in particular the approximability of certain functions computable in A. A *c.e. trace* is a uniformly c.e. family $T = \{T_0, T_1, \dots\}$ of finite sets of natural numbers. For a function h, the trace T has *bound h* if $(\forall n)\ |T_n| \leq h(n)$. T is a *computable* *trace* if there is a computable r such that $T_n = D_{r(n)}$, i.e., T_n is effectively given by a strong index.

Terwijn and Zambella [53] defined a set A to be *computably traceable* if there is a computable bound p such that for every $f \leq_T A$, there is a computable trace T with bound p that approximates the values of f, in the sense that $f(n) \in T_n$. They showed that A is low for Schnorr tests iff A is computably traceable. Later on, in [20], it was shown that computable traceability is in fact equivalent to being low for Schnorr random (see [11] for details).

The following aims at a characterization of K-triviality that is not related to ML-randomness or prefix-free complexity K.

QUESTION 3.1. *Is there a traceability type characterization of low for ML-random* (*that is, K-triviality*)?

One candidate, albeit not too likely, is strong jump traceability, introduced in [13]. The set A is *jump traceable* (*via a function h*) [38] if there is a c.e. trace T with bound h such that $(\forall e)$ $[\Phi_e^A(e)\downarrow \Rightarrow \Phi_e^A(e) \in T_e]$. In [37] it is proved that each K-trivial set is jump traceable. The set A is *strongly jump traceable* iff it is jump traceable via every order function h (where an *order function* is a computable nondecreasing unbounded function). In [13] a c.e. noncomputable strongly jump traceable set is constructed. Interestingly, this construction resembles the cost function construction (see for instance [10, 11]) of a c.e. noncomputable K-trivial set.

§4. *K*-triviality, ML-randomness and Turing reducibility. Some important known interactions are:

- The Kučera–Gács Theorem: each set is Turing (and even wtt) below a ML-random set [21, 15].
- The downward closure of K-triviality under $\leq_T$ [37]. (In particular, K-triviality is a degree notion, rather than a set notion.)

4.1. Definability, exact pairs and bounds.

QUESTION 4.1. *Is the class of K-trivial sets definable in either the Δ_2^0 degrees or in the c.e. Turing degrees*?

For Δ_2^0, this question is the "far from random" analog of Question 2.3 (ii). If the answer is yes for the c.e. case, this would be the first definable subclass of the class of c.e. low degrees.

QUESTION 4.2. *Does the ideal $\mathcal{K}$ of K-trivial sets have an exact pair* $\mathbf{a}, \mathbf{b}$*—namely $\mathcal{K} = [\mathbf{0}, \mathbf{a}] \cap [\mathbf{0}, \mathbf{b}]$—in the c.e. Turing degrees*?

There is a low_2 c.e. degree bounding $\mathcal{K}$, but no such low c.e. degree [37, 38]. Since there is a bound below $\mathbf{0}'$, the Σ_3^0-ideal of K-trivial sets has an exact pair in the Δ_2^0 degrees. This uses results from [48].

QUESTION 4.3. *Is there a Δ_2^0 low degree bounding $\mathcal{K}$*?

4.2. Having a ML-random set above. The following questions aim at characterizing K-triviality using ML-randomness and Turing reducibility (and perhaps a predicate for the c.e. degrees).

DEFINITION 4.4. *Let $\mathcal{C}$ be a randomness notion. We say B is a* basis for $\mathcal{C}$ randomness *if there is a $Z \geq_T B$ such that Z is $\mathcal{C}$ random relative to B.*

For ML-randomness, the notion of basis was introduced by Kučera [24] (in different terminology). By the Kučera–Gács theorem, each low for ML-random set (see Subsection 5.1 below) is a basis for ML-randomness. In fact, the two notions coincide.

THEOREM 4.5. [16] *Every basis for ML-randomness is K-trivial, and hence $Low(\mathsf{MLR})$.*

If A is c.e., $Z \geq_T A$ is ML-random and $Z \not\geq \emptyset'$, then Z is ML-random relative to A [16]. Thus, by the Theorem, any c.e. A for which such a Z exists is K-trivial. Is every c.e. K-trivial of this kind? Recall that each K-trivial set is Turing below a c.e. K-trivial set, so there is no need to assume that A is c.e. when formulating this question.

► QUESTION 4.6 (Stephan, Feb. 2004). *If A is K-trivial, must there be a ML-random $Z \geq_T A$ such that $\emptyset' \not\leq_T Z$? How about $Z <_T \emptyset'$? Z low?*

Each K-trivial set is low [37], so we cannot rule out the last possibility. However, the known proof is complicated, and any proof is necessarily nonuniform (see the proof Theorem 5.10 in [38]). So one can expect a proof of an affirmative answer to the last part of Question 4.6 to be at least as hard.

If $X = X_0 \oplus X_1$ is a ML-random, then X_0 is ML-random relative to X_1. So if $A \leq_T X_0$ and $A \leq_T X_1$, then X_1 is ML-random relative to A and hence A is K-trivial by Theorem 4.5. It is possible for such an A to be noncomputable [21]. Can it be an arbitrary K-trivial set?

QUESTION 4.7. *Let A be K-trivial. Is there a ML-random set $X = X_0 \oplus X_1$ such that $A \leq_T X_0$ and $A \leq_T X_1$? Can X be taken to be Δ^0_2?*

A positive answer to the first part would also answer the first part of Question 4.6 in the affirmative, since at least one of the "halves" X_0, X_1 is not above $\emptyset'$ (in fact, neither if A is noncomputable). Because each part of a Δ^0_2 ML-random is low [9, Thm. 3.4], a positive answer to the second part of the question would also provide an affirmative answer to the last part of Question 4.6.

4.3. Cupping above $\emptyset'$ with a ML-random set. Let us say A is *weakly ML-cuppable* if $A \oplus Z \geq_T \emptyset'$ for some ML-random $Z \not\geq \emptyset'$. A is *ML-cuppable* if one can choose $Z <_T \emptyset'$.

▷ QUESTION 4.8 (Kučera, 9/2004). *Which Δ^0_2 sets are (weakly) ML-cuppable? Is one of the notions equivalent to not being K-trivial?*

Quite a bit is known already. If $A \in \Delta^0_2$ is not K-trivial, then A is weakly ML-cuppable, via $Z = \Omega^A$ ($Z \not\geq_T \emptyset'$ by Theorem 4.5, and $\emptyset' \leq_T A' \equiv_T$

$A \oplus \Omega^A$). If A is low then in fact $\Omega^A <_T \emptyset'$. Thus, each Δ^0_2 set A with a low non-K-trivial set below is ML-cuppable. This includes:

(1) any ML-random set A, since $A \cap 2\mathbb{N}$ is low [9, Thm. 3.4]
(2) any non-low$_2$ and any c.e.a. non-K-trivial set, as those are the supremum of a pair of 1-generic sets (see [26, Ex. IV.3.15] for the first)
(3) any c.e. non-K-trivial set A, because A is a disjoint union of c.e. low sets A_0, A_1, and at least one of them is not K-trivial.

An interesting case of a Δ^0_2 set that is not proved to be ML-cuppable by any of the reasons above is a set of minimal degree which is properly low$_2$ [26, Ex. IX.2.7].

Nies [41] has shown that there is a (necessarily K-trivial) c.e. noncomputable set that is not even weakly ML-cuppable. In fact, if Y is Δ^0_2 ML-random, then there is a promptly simple set A such that, for each ML-random set Z, $Y \leq_T A \oplus Z$ implies $Y \leq_T Z$. If one lets $Y = \Omega$, then one obtains A which is not weakly ML-cuppable, since $\Omega \equiv_T \emptyset'$. Hirschfeldt and Miller have recently given a simpler proof of Nies' Theorem. See [40].

4.4. Almost deep degrees. The following is a strong lowness property within the c.e. degrees. We say that the c.e. set A is *almost deep* if $A \oplus W$ is low for each low c.e. set W. Cholak, Groszek and Slaman [4] built a non-computable almost deep set.

QUESTION 4.9 (Downey, 2004). *Is each almost deep set K-trivial*?

A K-trivial set can be promptly simple, while an almost deep set is not low cuppable, and hence not of promptly simple degree. Thus there is a K-trivial set that is not almost deep. *Note* (*Feb.* 2006): *P. Cholak has announced a negative answer.*

§5. Notions weaker than ML-randomness. One obtains notions weaker than ML-randomness by replacing the ML-tests, which are based on c.e. open sets, by computable test concepts. The main ones explored so far are computable randomness and the even weaker notion of Schnorr randomness.

For the first, a test is a computable martingale, where (for our purposes) a *martingale* is a function $M \colon \{0,1\}^* \to \mathbb{R}^+$ that satisfies for every $\sigma \in \{0,1\}^*$ the averaging condition $2M(\sigma) = M(\sigma 0) + M(\sigma 1)$. A martingale M *succeeds on* Z if $\limsup_{n\to\infty} M(Z \upharpoonright n) = \infty$. A set Z is *computably random* if there is no computable martingale M such that M succeeds on Z. We denote the class of such Z by CR.

For the second, a test is a pair (M, r) consisting of a computable martingale and a computable nondecreasing and unbounded function r. The test succeeds on a set Z if $M(Z \upharpoonright n) > r(n)$ for infinitely many n, and Z is *Schnorr random* if no such test succeeds on Z. Equivalently, one can use ML-tests $\{U_i\}_{i\in\mathbb{N}}$ such that $\mu(U_i) = 2^{-i}$. We denote the class of Schnorr random sets by SR. See [11, §10] for details.

Call Z *weakly* 1-*random* (or Kurtz random) if Z is not in any Π^0_1-class of measure zero (also called a *Kurtz test*). This class is denoted WR. Since every weakly 1-generic set is weakly 1-random, the law of large numbers, namely $\lim_n |\{i < n : Z(i) = 0\}|/n = 1/2$, can fail for such a set, so WR is not a randomness notion. However, we include WR in the discussion because of the naturalness of its definition.

A little explored notion is partial computable randomness: PCR is the class of sets on which not even a partial computable martingale succeeds (i.e., a martingale that may be undefined on strings off the set).

The inclusions are

$$\mathsf{MLR} \subset \mathsf{PCR} \subset \mathsf{CR} \subset \mathsf{SR} \subset \mathsf{WR}.$$

They are known to be proper. The classes can even be separated on the left-c.e. sets, with the possible exception of PCR and CR. (A set Z is *left-c.e.* if the set of finite strings lexicographically preceding Z is a c.e. set. They are also called c.e. reals.)

5.1. Lowness and bases. Let $\mathcal{C}$ be a randomness notion. A set A is *low for* $\mathcal{C}$*-random* if every $\mathcal{C}$-random set is $\mathcal{C}$-random relative to A. We denote this class by $\mathrm{Low}(\mathcal{C})$. All the lowness notions for the classes above have been characterized in terms not referring to those randomness notions, with the exception of WR.

- $\mathrm{Low}(\mathsf{MLR})$ equals K-trivial,
- $\mathrm{Low}(\mathsf{CR}) = \mathrm{Low}(\mathsf{PCR})$ equals computable (see [37] for CR, [7, 42] for PCR),
- $\mathrm{Low}(\mathsf{SR})$ equals computably traceable, as mentioned in Section 3.1.

It makes sense to study the degree class $\mathrm{Low}(\mathcal{C}) = \{A : \mathcal{C}^A = \mathcal{C}\}$ not only for randomness notions, but in fact for any relativizable class $\mathcal{C}$ where $\mathcal{C}^X$ only depends on the Turing degree of X. (However, $\mathrm{Low}(\mathcal{C})$ is downward closed only if the stronger condition $X \le_T Y \Rightarrow \mathcal{C}^X \supseteq \mathcal{C}^Y$ holds, otherwise it is not necessarily a lowness property.) For instance, in [55] it is shown that the only low for 1-generic sets are the computable ones, and that low for weakly 1-generic is somewhere between computably traceable and hyperimmune-free (joint with J. Miller). For the latter inclusion one uses the result of Kurtz [25] that the degrees of weakly 1-generic and hyperimmune sets coincide. The class $\mathrm{Low}(\mathsf{WR})$ behaves similarly. If A computably traceable then A is low for Kurtz tests and hence A is $\mathrm{Low}(\mathsf{WR})$. Since each weakly 1-generic set is in WR, each set in $\mathrm{Low}(\mathsf{WR})$ is hyperimmune-free.

QUESTION 5.1 (Downey, Yu, Kjos-Hanssen). *Characterize the lowness notions for the classes of weakly* 1-*random*, *weakly* 1-*generic and diagonally noncomputable* (DNC) *sets.*

Here Z is DNC^A if there is a total function $f \le_T Z \oplus A$ such that $(\forall n)\ f(n) \neq \Phi^A_n(n)$.

Recall Definition 4.4. By the Kučera–Gács Theorem, if ML-randomness implies $\mathcal{C}$-randomness, then every set that is low for $\mathcal{C}$-random is a basis for $\mathcal{C}$-randomness.

QUESTION 5.2. *For each of the weak randomness notions $\mathcal{C}$ defined above, characterize the bases for $\mathcal{C}$-randomness.*

Unlike in the case of ML-randomness, the base notion can be strictly larger than the lowness notion, for instance for CR: each Δ^0_2 set that is not DNC is a base for CR. On the other hand, no PA-degree is a base for CR (this uses Theorem 4.5). See [16]. It is not known if a ML-random set can be a base for CR.

Note (*Dec.* 2005): *Stephan and Yu* [52] *have shown that A is low for weakly* 1-*generic iff A is of hyper-immune free and non*-DNC *degree. This class strictly includes computably traceable. They also show that each low for weakly* 1-*generic is low for weakly random*. (*May* 2006): *Stephan and Yu* [52] *have also shown that only the computable sets are low for diagonally noncomputable.*

§6. KL-randomness. We discuss a major open question: whether KL-randomness, a notion based on a computable yet very powerful test concept, can be separated from ML-randomness. This question has been asked in several places, using different terminology: in [36, Open question 8.11], where ML-random sets are called *chaotic sequences* and KL-random sets are called *unpredictable sequences*. Then again in [1, Open problem 2.9], where ML-random sets are called Σ^0_1 *random sequences*, and KL-random sets are called *nonmonotone computably random sequences*. Although it is believed that ML-randomness is strictly stronger than KL-randomness, major effort by various researchers has not been sufficient to settle the question. However, they did achieve results implying that KL-randomness is at least much closer to ML-randomness than the notions encountered previously.

We give the formal definition, but refer to [31] for more details. An *assignment* is a (finite or infinite) sequence

$$x = (r_0, b_0), \dots, (r_n, b_n), \dots$$

of pairs of consisting of a natural number and a bit. The corresponding *bit sequence* is $b_0, b_1, \dots$. The set of all finite assignments is denoted by $FinA$. A *scan rule* is a partial computable function $S\colon FinA \to \mathbb{N}$ such that $(\forall w \in FinA)\ S(w) \notin \mathrm{dom}(w)$. The assignment σ^Z_S given by the scan rule S is defined as follows: let $\sigma^Z_S(0) = \emptyset$, and if $x_n = \sigma^Z_S(n)$ and $S(x_n)$ are defined, let

$$\sigma^Z_S(n+1) = x_n \,{}^\wedge\, (S(x_n), Z(S(x_n))).$$

($\sigma^Z_S(n+1)$ is undefined otherwise. The symbol $^\wedge$ denotes concatenation.)

DEFINITION 6.1. *Z is* KL-random *if for each scan rule S, if the assignment σ_S^Z is infinite, then the corresponding bit sequence is partial computably random.*

Thus the tests are pairs (S, M), where S is a scan rule and M is a partial computable martingale; (S, M) *succeeds* on Z if M succeeds on the bit sequence of σ_S^Z. (Equivalently, one may replace M by a *stake function* defined on the domain of S rather than on strings, see [31].) Given a test (S, M), there are two tests where the scan rule and the martingale are total (even primitive recursive) [30], so that one of the two succeeds wherever (S, M) succeeds. Thus KL-randomness is given by a computable test concept, called a *nonmonotonic betting strategy*.

By [31, Remark 9], a nonmonotonic betting strategy that for any Z scans all places can by replaced by a computable martingale. If it scans all places only for almost all sets Z, then this is still true, though one also needs a further test consisting of a Π_1^0-class of measure 0. Thus to exploit the full strength of nonmonotonicity, it is essential to avoid scanning places for a class of sets Z of nonzero measure. Even for computable Z, the sets of scanned places is in general only c.e., so some may question that the test concept actually deserves to be called computable.

► QUESTION 6.2. *Is there a KL-random set that is not ML-random?*

This is unknown even for left-c.e. sets, though the answer is more likely to be negative here. Thus we separately ask

QUESTION 6.3. *Is each left-c.e. KL-random set ML-random?*

A single nonmonotonic betting strategy fails on some c.e. set. However, even the interaction of two is already beyond our present understanding. We cannot rule out that there are two nonmonotonic betting strategies such that one of them succeeds on any non-ML random set.

The following randomness notions may be helpful in approaching the questions. In general, the next position chosen by a scan rule depends on the values of Z at the previous places. We consider a special class of scan rules S where this is not the case: S is given by a computable injection $h\colon \mathbb{N} \to \mathbb{N}$, so the nth position chosen is $h(n)$, and $\sigma_S^Z(n)$ is simply $(h(n), Z(h(n)))$. We say Z is *injective random* if $Z \circ h$ is partial computably random for each computable injection h, and Z is *permutation random* if this is so for each computable permutation h of $\mathbb{N}$. The following should then be easier to answer than Question 6.2.

QUESTION 6.4. *Is there a permutation random set that is not ML-random? An injective random set? What if one requires the sets to be left-c.e.?*

A caveat: permutation randomness is quite strong already. If there is an unbounded, nondecreasing computable function g such that $(\forall m)\ K(Z \upharpoonright g(m)) \leq g(m) - m$, then Z is not KL-random by Muchnik [36]. He provides a pair of permutation betting strategies (one in fact monotonic, but both with partial martingales) so that one of them succeeds on Z. Hence Z is not even permutation random. On the other hand, it can be shown that there are partial computably random sets with small initial segment complexity (say $K(Z \upharpoonright n) \leq \log^2 n + \mathcal{O}(1)$, see for instance [30]). So by Muchnik's result, PCR is not closed under computable permutations, and hence different from permutation random.

A further result showing that KL-randomness is close to ML-randomness is that if $Z = Z_0 \oplus Z_1$ is KL-random, then at least one of the "parts" Z_0, Z_1 is ML-random [31]. (It is unknown if both are, unless we also assume that Z is Δ^0_2.) The technique, a variant of Muchnik's technique, can be refined in order to show that each KL-random has effective dimension 1 (see Section 10 for a Definition). Using a much more complex argument, this has been extended to the larger class of KL-stochastic sets in [31].

A set A is K-trivial iff each ML-random set is computably random relative to A [37]. This implies that each low for KL-random (injective random, permutation random) set is K-trivial.

QUESTION 6.5. *Determine the class of low for KL-random sets. Does it coincide with the low for ML-random sets? Do the same for injective and permutation randomness.*

§7. Notions stronger than ML-randomness.

7.1. Strong Chaitin randomness. Z is *n-random* if Z is ML-random relative to $\emptyset^{(n-1)}$. So, 1-randomness is the same as ML-randomness. In [32] and [44] it is proved independently that if Z is 2-random then $(\exists^\infty n)\ C(Z \upharpoonright n) \geq n - \mathcal{O}(1)$, that is, within a fixed constant, the plain Kolmogorov complexity of $Z \upharpoonright n$ infinitely often achieves its maximum value n. The converse holds as well [44].

Is there an analogous result for prefix-free complexity? The maximum value of $K(x)$ is $|x| + K(|x|) + \mathcal{O}(1)$. Solovay [49] proved that if $K(x)$ is maximal then so is $C(x)$ (up to a fixed constant). A set Z is called *strongly Chaitin random* if $(\exists^\infty n)\ K(Z \upharpoonright n) \geq n + K(n) - \mathcal{O}(1)$. Each 3-random set is strongly Chaitin random, and (by the above) each strongly Chaitin random set is 2-random.

$\triangleright$ QUESTION 7.1. *Does strongly Chaitin random equal* 3*-random, or* 2*-random, or neither?*

7.2. Weak 2-randomness. Z is weakly 2-random if Z is in no Π^0_2-class of measure 0. We have the inclusions:

$$\text{2-random} \Rightarrow \text{weakly 2-random} \Rightarrow \text{ML-random},$$

none of which can be reversed, even for the Turing degrees of the sets. If Z is of hyper-immune free degree, then Z is weakly 1-random iff Z is weakly 2-random (an observation of Yu Liang extending [44, Theorem 4.3]). Since there is a ML-random set of hyper-immune free degree, but no such 2-random set, we have that the first inclusion is proper for degrees. No weakly 2-random set is Δ^0_2, so also the second inclusion is proper.

Since ML-randomness and 2-randomness can be characterized by the complexity of initial segments, the following question is obvious.

QUESTION 7.2. *Is there a characterization of weak* 2*-randomness via a growth condition on the initial segment complexity*?

There is a non-computable c.e. low for weakly 2-random set, and each low for weakly 2-random set is K-trivial [12].

QUESTION 7.3. *Is low for weakly* 2*-random the same as* K*-trivial*? *Note*: (*Mar.* 2006): *The authors have independently given an affirmative answer.*

7.3. Effective descriptive set theory. In a little known paper [28], Martin-Löf considered a randomness notion based on effective descriptive set theory. He suggested the (lightface) Δ^1_1-classes of measure 0 as tests. Thus, Z is Δ^1_1-random if Z is in no null Δ^1_1-class.

An analog of ML-randomness in the effective descriptive set theory setting is studied in [17], where the (open) Σ^0_1-classes in the definition of ML-tests are replaced by open Π^1_1-classes. Thus a Π^1_1-ML-test is a uniform sequence $\{U_i\}_{i\in\mathbb{N}}$ of Π^1_1 open sets such that $\mu(U_i) \le 2^{-i}$, and Z is Π^1_1-*ML-random* if it passes each such test. One could define Δ^1_1-ML-randomness in a similar way. However, by an observation of Yu Liang involving [47, Lemma 1.8.III], for each null Δ^1_1-class S one can find a Δ^1_1-ML-test $\{U_i\}_{i\in\mathbb{N}}$ such that $S \subseteq \bigcap_i U_i$, so this is the same as Martin-Löf's notion from [28]. In particular, Π^1_1-ML-random implies Δ^1_1-random.

In the Π^1_1-case things go differently. Let us say a set Z is Π^1_1-*random* if Z is in no null Π^1_1 subset of 2^ω. This notion is first mentioned in an exercise in [47, Ex. 2.5.IV] (but called Σ^1_1-random there). The notion is strictly stronger than Π^1_1-ML randomness, for instance because any Π^1_1-random set Z satisfies $\omega_1^Z = \omega_1^{\mathrm{CK}}$, while an analog of Ω is Π^1_1-ML random and fails to have that property. Recently it was shown that there is a greatest Π^1_1 null set, that is, a universal test. See [17].

These notions behave to some extent similar to algorithmic randomness notions. For example, by Gandy's basis theorem, there is a Π^1_1-random Z such that $\mathcal{O}^Z \le_h \mathcal{O}$. This is analogous to the existence of a low ML-random set. One can study analogs of most of the questions asked for the case of algorithmic randomness. For instance,

QUESTION 7.4. *Is there a low for* Π^1_1*-random set that is not hyperarithmetical*? *Is there a low for* Δ^1_1*-random set that is not hyperarithmetical*?

The only low for Π^1_1-ML-random sets are the hyperarithmetical ones [17]. Any low for Π^1_1-random set A satisfies $\omega_1^A = \omega_1^{\mathrm{CK}}$.

One can also look at versions of KL-randomness in the new setting. Modifying Definition 6.1, we say that Z is Π^1_1-KL random if the definition of KL-random applies when the functions involved (scan rule and partial martingale) have a Π^1_1 graph. If those functions are required to be total then the success set is Δ^1_1, so that version is the same as Δ^1_1 randomness (in contrast to the algorithmic case where one can require totality by Merkle's observation [30] mentioned after Definition 6.1). Clearly Π^1_1-ML-random implies Π^1_1-KL-random.

QUESTION 7.5. *Is there a Π^1_1-KL-random set that is not Π^1_1-ML-random?*

§8. Chaitin's halting probability.

8.1. Low for Ω and weakly low for K. A set A is low for Ω if Ω is ML-random relative to A [44]. This property does not depend on the choice of the universal prefix-free machine. By Theorem 4.5 and [37], the Δ^0_2 low for Ω sets are just the K-trivial sets. Also 2-random sets are Low(Ω). In fact, by van Lambalgen's Theorem 2.2, $\mathsf{MLR} \cap \mathrm{Low}(\Omega) = 2$-random. There are still other low for Ω sets, since each Π^0_1-class contains a low for Ω set [9].

Each 2-random set is of hyperimmune degree [25, 18], as is each noncomputable K-trivial set (because they are Δ^0_2). An affirmative answer to the following question would generalize these facts.

QUESTION 8.1. *Is each noncomputable low for Ω set hyperimmune?*

Recall that A is *low for K* if $(\forall y)\ K^A(y) \geq K(y) - \mathcal{O}(1)$ (this notion coincides with K-triviality [37]). J. Miller [33] introduced the following weaker form: A is *weakly low for K* (WLK) if $(\exists^\infty y)\ K^A(y) \geq K(y) - \mathcal{O}(1)$. He proved that each 3-random set is WLK, and each WLK set is low for Ω. By the latter result, for Δ^0_2 sets, WLK equals low for K.

QUESTION 8.2. *Is there a low for Ω set that is not weakly low for K?*

A negative answer would show that strongly Chaitin random is equivalent to 2-random (see Question 7.1), since Miller also proved that ML-random and WLK together imply strongly Chaitin random.

One can study the analog of low for Ω for randomness notions weaker than ML-random. For instance, for c.e. A, Ω is Schnorr random relative to A iff A is c.e. traceable [20].

QUESTION 8.3. *Characterize the (c.e.) sets A such that Ω is computably random relative to A. Does this depend on the version of Ω used?*

If Ω is computably random relative to a Δ^0_2-set A, then A is low_2. To see this, use the result that any left-c.e. computably random set is high [44, Prop. 4.1], relativized to A.

8.2. Relativizing Chaitin's halting probability. The questions in this subsection are from [9]. A partial computable functional $M^A\colon 2^{<\omega} \to 2^{<\omega}$ is a *prefix-free oracle machine* if M^A is prefix-free for every $A \in 2^\omega$. Such a machine N is *weakly universal* if for each A, the prefix-free machine N^A is universal among the prefix-free partial A-computable machines.

QUESTION 8.4. *Is there a weakly universal prefix-free oracle machine N such that Ω_N is degree invariant?*

In [9], the previous question is answered in the negative under a reasonable uniformity assumption on U. A prefix-free oracle machine U is *universal* if for every prefix-free oracle machine M there is a prefix $\rho_M \in 2^{<\omega}$ such that

$$(\forall A \in 2^\omega)(\forall \sigma \in 2^{<\omega})\ U^A(\rho_M \sigma) = M^A(\sigma).$$

In other words, U can simulate any prefix-free oracle machine (for all oracles) by prepending an appropriate string to the input.

In [9, Theorem 4.3] it is shown that each A-random A-left-c.e. set is of the form Ω_U^A for some universal prefix-free oracle machine U. The following asks if some version of Ω^A can be above $\emptyset'$ without actually being left-c.e.

QUESTION 8.5. *If $X \geq_T \emptyset'$ is an A-random A-left-c.e. set for some $A \in 2^\omega$, is X necessarily left-c.e.?*

A Δ^0_2 set A does not serve here, because if Ω^A is above $\emptyset'$ then A is K-trivial, and hence Ω^A is left-c.e.

In [9] it is shown that if U is a universal prefix-free oracle machine, then Ω_U is not degree invariant. It is also proved that if A is K-trivial, then Ω_U^A is a left-c.e. set, hence $\Omega_U^A \equiv_T \emptyset'$. Therefore, Ω_U must be degree invariant at least on the K-trivial sets. It is possible for Ω_U to be degree invariant only for the K-trivial degrees. Is this always the case?

QUESTION 8.6. *For a universal prefix-free oracle machine U and a set $A \in 2^\omega$ that is not K-trivial, is there a $B \equiv_T A$ such that $\Omega_U^B \not\equiv_T \Omega_U^A$?*

It is possible to construct a universal prefix-free oracle machine U for which range(Ω_U) is not closed [9].

QUESTION 8.7. *Is it true for every universal prefix-free oracle machine U that range(Ω_U) is not closed?*

The range of Ω_U is a (lightface) Σ^1_1-class. However, no better upper bound on the complexity of the range of Ω_U is known. The measure of the range is a left-Σ^1_1 real. In [9] it is shown that this measure is strictly between 0 and 1.

QUESTION 8.8. *Is range(Ω_U) an arithmetical class (or even Borel) for a universal prefix-free oracle machine U? What can be said about the complexity of the measure?*

8.3. Further questions. While Ω_U is wtt-complete for each universal machine U, it is never tt-complete [3].

QUESTION 8.9. *Are there universal machines U_0, U_1 such that $\Omega_{U_0} \not\equiv_{tt} \Omega_{U_1}$?*

Fix a universal machine U. For a non-empty set $S \subseteq 2^{<\omega}$, let $\Omega[S] = \mu\{\sigma\colon U(\sigma) \in S\}$, the probability that U halts with output in S. Grigorieff asked when $\Omega[S]$ is ML-random. While this is true for c.e. sets, Becher, Figueira, Grigorieff and Miller [2] constructed a Δ^0_2 set S for which $\Omega[S]$ is not ML-random.

QUESTION 8.10. *Is there a* Π^0_1 *set* S *such that* $\Omega[S]$ *is not ML-random? What about a* Π^0_1*-complete set?*

We can also consider more complicated sets. Becher, et al., proved that if S is Σ^0_n-complete, then $\Omega[S]$ is 1-random.

QUESTION 8.11. *If* $S \subseteq 2^{<\omega}$ *is* Σ^0_n*-complete for* $n \geq 2$, *then must* $\Omega[S]$ *be computably* n*-random? Alternately, can it ever be weakly* 2*-random?*

It is known that for $n \geq 2$, if S is a Σ^0_n set of strings, then $\Omega[S]$ is not n-random [2]. Also, there is a Σ^0_n set S such that $\Omega[S]$ is computably n-random (and from above, another for which $\Omega[S]$ is not random).

Note (*May* 2005): *Figueira, Stephan and Wu* [14] *have answered Question* 8.9 *in the affirmative after seeing a preliminary version of this paper. They also made progress on Question* 8.1, *showing that each low for* Ω *set that is in a* Π^0_1*-class without computable members is hyperimmune. Finally they have shown that* 8.10 *is true for some machine satisfying a weak form of universality.*

§9. Degrees of randomness. Various notions have been introduced to compare the "degree of randomness" of sets. The questions below focus on the K-degrees, the C-degrees and the van Lambalgen degrees. The definition of the *K-degrees* is motivated by Schnorr's Theorem 2.1, and draws on the intuition that "more random" sets have more complex initial segments. We write $A \leq_K B$ if $(\forall n)\ K(A \upharpoonright n) \leq K(B \upharpoonright n) + \mathcal{O}(1)$. The equivalence classes under $\equiv_K$ are called K-degrees. The C-degrees are defined in the same way. Both were implicit in Solovay's manuscript [49], and studied in more depth by Downey, Hirschfeldt, and LaForte [8]. Note that MLR is closed upward under $\leq_K$, and that the least K-degree consists of the K-trivial sets.

The *van Lambalgen degrees* were introduced by Miller and Yu [34] while studying the K-degrees. We write $A \leq_{vL} B$ if

$$(\forall Z \in 2^{\omega})[A \oplus Z \text{ ML-random} \Rightarrow B \oplus Z \text{ ML-random}].$$

This reducibility is the weakest of those that have been considered; both $\leq_K$ and $\leq_C$ imply $\leq_{vL}$ [34]. It is known that there is no join in the vL-degrees because almost every pair of sets has no $\leq_{vL}$ upper bound) [34]. Hence, the same is true for the K-degrees and C-degrees. Is there a meet operation?

QUESTION 9.1. *Do meets always exist in the vL-degrees* (K*-degrees,* C*-degrees*)? *Is there a non-zero branching degree?*

In [5] it is proved that there is a minimal pair of K-degrees, i.e., a pair of non-zero K-degrees with infimum zero.

9.1. Coherence with natural randomness classes. The various degree notions are intended to measure the randomness content of sets, but do they? One way to test would be to determine if they respect natural randomness classes. It is known that the n-random sets are closed upward in the vL-degrees for every $n \geq 1$, hence also in the K-degrees and C-degrees [34]. On the other hand, coherence breaks down for weak randomness notions. For instance, there are sets $A <_K B$ such that A is computably random but B is not even weakly 1-random (this follows from [30]). Coherence is open for classes stronger than ML-randomness in all of the degree structures. Most of the randomness notions introduced in Section 5 relativize in the expected way. For instance, Z is computably n-random if no martingale computable in $\emptyset^{(n-1)}$ succeeds on Z. Slight care must be taken when defining weak n-randomness: Z is weakly n-random if it is in no null Π^0_n-class (or equivalently for $n \geq 2$, if it is in no null $\Pi^0_2[\emptyset^{(n-2)}]$-class).

QUESTION 9.2. *For every $n \geq 2$, are the weakly n-random (Schnorr n-random, computably n-random) sets closed upwards in the vL-degrees (K-degrees, C-degrees)?*

Related to these questions are the manifold problems of finding natural initial segment and oracle characterizations for the various randomness classes. (See below for the relationship between the vL-degrees and relative randomness.) One example was already given in Question 7.2. Other examples:

QUESTION 9.3. *Is there a natural characterization of the computably 2-random sets in terms of initial segment K-complexity?*

QUESTION 9.4. *Can the weakly 2-random sets be characterized in terms of which (ML-random) oracles they are random relative to?*

The second example could be particularly interesting. Recall that a set X is GL_2 if $X'' \equiv_T (X \oplus \emptyset')'$. It is not hard to prove that if a set A is ML-random relative to a non-GL_2 oracle X, then A is weakly 2-random (J. Miller). Does the converse hold?

9.2. The *K*-degrees. Recall that ML-random and 1-random are synonyms. Miller and Yu [35] have shown that there are 1-random sets $A <_K B$. To do so, they characterized the functions f for which there is a 1-random set A such that $(\forall^\infty n)\ K(A \restriction n) \leq n + f(n)$. Such an f needs to be somewhat large.

THEOREM 9.5. *$\sum_{n\in\omega} 2^{-f(n)}$ converges iff there is a ML-random $A \in 2^\omega$ such that $(\forall^\infty n)\ K(A \restriction n) \leq n + f(n)$.*

The question remains how random A, B can be in the absolute hierarchy. For instance,

QUESTION 9.6. *Are there comparable K-degrees of 2-random sets?*

The relativization of Theorem 9.5 is not helpful for solving this question because it involves $K^{\emptyset'}$.

QUESTION 9.7. *Are there maximal K-degrees? Are there maximal 1-random K-degrees?*

While Question 9.6 is unsolved, it even remains open whether *almost every* set has maximal K-degree. To approach the second subquestion, one might hope to characterize the functions g for which there is a ML-random set A such that $(\forall^\infty n)\ K(A \upharpoonright n) \geq n + g(n)$.

As ML-random sets have high K-complexity, one might expect an affirmative answer to the following. If so, the two subquestions of Question 9.7 coincide.

QUESTION 9.8. *For every $A \in 2^\omega$, is there a 1-random set $X \geq_K A$?*

Some progress has been made in understanding the cones above random sets in the K-degrees. There is a ML-random with uncountably much K-above it [35], while every 3-random set has a countable upper cone [33].

We also know that every ML-random set has a countable K-degree. In fact, if $X \in 2^\omega$ is ML-random, then $X \equiv_K Y$ implies $X' \equiv_{tt} Y'$ [33]. It is not known if this can be improved.

QUESTION 9.9. *For ML-random sets X, Y, does $X \equiv_K Y$ imply $X \equiv_T Y$?*

This is known to fail for nonrandom sets. For example, there are noncomputable K-trivial sets. However, for K-trivial sets it is at least true that $X \equiv_K Y$ implies $X' \equiv_{tt} Y'$, since $X' \equiv_{tt} \emptyset'$ when X is K-trivial [37]. There is no general bound on the complexity of K-equivalent reals because there are uncountable K-degrees [33].

9.3. The C-degrees. All of the questions asked for $\leq_K$ can also be asked for $\leq_C$. In fact, less is known about the C-degrees. For example:

QUESTION 9.10. *Are there ML-random sets $A, B \in 2^\omega$ such that $A <_C B$?*

Furthermore, very little is known about the relationship between $\leq_K$ and $\leq_C$. It is known that $X \equiv_K Y$ does not, in general, imply $X \equiv_C Y$ (because there are noncomputable K-trivial sets). Other basic questions remain open.

QUESTION 9.11. *Does $X \leq_C Y$ imply $X \leq_K Y$? Do $\leq_K$ and $\leq_C$ differ on the ML-random sets?*

9.4. The vL and LR-degrees. Most of the questions that were asked for the other degrees structures are easily answered for the vL-degrees (see [34]). Instead, let us look at two simple reformulations of $\leq_{vL}$. If $A, B \in 2^\omega$ are ML-random, then by van Lambalgen's theorem $A \leq_{vL} B$ iff

$$(\forall Z \in 2^\omega)[Z \text{ is ML}^A\text{-random implies } Z \text{ is ML}^B\text{-random}].$$

In this form, the reducibility was introduced independently by Nies [37], who used the notation $B \leq_{LR} A$ (where LR stands for *low for random*). He proved that there is a c.e. incomplete A such that $A \equiv_{LR} \emptyset'$. Not much is known about the degree structure on c.e. sets.

QUESTION 9.12. *Are the LR-degrees of c.e. sets dense?*

It is also not known how close sets with the same LR-degree are, from a computability theoretic perspective, although there are partial results. Nies proved that if $A \leq_{LR} B$ are c.e., then $A' \leq_{tt} B'$. Furthermore, for any sets A, B, if $A \oplus B \leq_{LR} B$, then A is K-trivial relative to B, and so again $A' \leq_{tt} B'$ [37].

QUESTION 9.13. *Does $A \leq_{LR} B$ imply $A' \leq_T B'$, or even $A' \leq_{tt} B'$?*

Another reformulation, which works for all $A, B \in 2^\omega$, also follows from van Lambalgen's theorem: $A \leq_{vL} B$ iff

$$(\forall \text{ ML-random } Z \in 2^\omega)[A \text{ is } \mathrm{ML}^Z\text{-random implies } B \text{ is } \mathrm{ML}^Z\text{-random}].$$

Informally, B is at least as random as A since it is ML-random relative to at least as many ML-random oracles as A. Is it necessary to require that Z is ML-random? We could define an apparently stronger pre-order $A \leq_{SvL} B$ by dropping the restriction that Z ranges only over ML-random sets.

QUESTION 9.14. *Are $\leq_{vL}$ and $\leq_{SvL}$ actually different?*

§**10. Effective dimension.** Lutz [27] introduced an effective version of Hausdorff dimension. Although the Hausdorff dimension of the singleton $\{X\}$ is zero, for a set X, the *effective (Hausdorff) dimension* need not be. In fact, Ryabko [46] and Mayordomo [29] proved that the effective dimension of $\{X\}$ is

$$\dim(X) = \liminf_{n\to\infty} \frac{K(X \upharpoonright n)}{n}.$$

See [11, 15.2] for details. L. Staiger [50] shows that this equality can be derived from the existence of an optimal lower semi-computable semimeasure.

► QUESTION 10.1 (J. Reimann, S. Terwijn, 2003). *Does every set of positive effective dimension Turing compute a Martin-Löf random set?*

It cannot always be the case that the ML-random set has the *same* Turing degree as the given set of positive dimension, because the degrees of sets of positive dimension are closed upward while the degrees of ML-random sets are not. For a specific example, take $Y \leq_T D$ where D is a low set of PA-complete degree and Y is ML-random. Then the dimension of $Y \oplus D$ is $1/2$, but there is no ML-random set in its degree by [51].

The answer is negative when one uses many-one reducibility in place of Turing reducibility (Reimann and Terwijn, see [45, Cor 3.11]). This has been extended to wtt-reducibility. For each rational $\alpha \in (0, 1)$, there is a Δ^0_2 set A of effective dimension α such that each $Z \leq_{\mathrm{wtt}} A$ has effective dimension at most α [43]. For Turing reducibility, it is known that there is a set Y and an unbounded, nondecreasing computable function h such that $(\forall n)\ K(Y \upharpoonright n) > h(n)$ and Y does not compute a Martin-Löf random set (Reimann and Slaman, see [45, Thm 4.17], and independently [19, Cor. 7]).

A possible approach toward a negative answer would be to find a lowness property that admits a set of non-zero dimension, but not a ML-random set. Question 10.1 can be broken down into the following subquestions, each of which remains open. Of course, a negative answer to any of the following would provide a negative answer to the previous question.

QUESTION 10.2. *Consider* $X \in 2^\omega$.

1. *If* $\dim(X) \in (0,1)$, *does* X *compute a set of higher effective dimension*? *Arbitrarily close to* 1? *Equal to* 1?
2. *If* $\dim(X) = 1$, *does* X *compute a Martin-Löf random set*?

These are fundamental questions about our ability to effectively compress random content. Given a set whose initial segments are guaranteed to have a high Kolmogorov complexity, can we get our hands on that complexity? Can we distill it out to produce a random set? Can we at least get a set with higher information density?

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