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CLOSING THE CIRCLE: AN ANALYSIS OF EMIL POST'S EARLY WORK

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This paper is dedicated to the memory of Emil Leon Post.

Abstract. In 1931 Kurt Gödel published his incompleteness results, and some years later Church and Turing showed that the decision problem for certain systems of symbolic logic has a negative solution. However, already in 1921 the young logician Emil Post worked on similar problems which resulted in what he called an "anticipation" of these results. For several reasons though he did not submit these results to a journal until 1941. This failure 'to be the first', did not discourage him: his contributions to mathematical logic and its foundations should not be underestimated. It is the purpose of this article to show that an interest in the early work of Emil Post should be motivated not only by this historical fact, but also by the fact that Post's approach and method differs substantially from those offered by Gödel, Turing and Church. In this paper it will be shown how this method evolved in his early work and how it finally led him to his results.

(...) *after all it is not ideas but the execution of ideas that constitute a mark of greatness.*[1]

1. Introduction. It is by now an accepted fact [4, 13, 27] that Emil Leon Post "anticipated" the incompleteness and undecidability results of Gödel, Church and Turing [2, 9, 28] in the period from 1920–21 during his stay at the Princeton University as a Procter fellow.[2] It was not until 1941 that he tried to publish these results in his *Absolutely Unsolvable Problems and Relatively Undecidable Propositions—Account of an Anticipation*. In this posthumously published paper [24] Post describes his early research on this

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[1]Emil Post in a letter to Gödel dated October 30, 1938.

[2]He did not prove the unsolvability of the Entscheidungsproblem as originally posed by Hilbert. After having formulated a thesis similar to that of Turing and Church, he showed on the basis of this thesis the unsolvability of the decision problem of a certain class of systems he developed together with a more general incompleteness result of all finitary symbolic logic, relative to this class of systems.

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subject and explains how he arrived at his results, in a rather informal and personal way.[3]. Post had submitted the manuscript to the *American Journal of Mathematics* but it was rejected.[4] In a reply from Hermann Weyl—the editor at that time—one reads:[5]

> [. . .] I have little doubt that twenty years ago your work, partly because of its then revolutionary character, did not find its due recognition. However, we cannot turn the clock back; in the meantime Gödel, Church and others have done what they have done, and the American Journal is no place for historical accounts; . . . (Personally, you may be comforted by the certainty that most of the leading logicians, at least in this country, know in a general way of your anticipation.)

Post himself was very well aware of the fact that Gödel, Turing and Church had already published similar results in the thirties. The question of why he nonetheless wanted his results to be published is thus significant. The supposition that it was self-absorption that led him to this seems to be wrong given the modesty with which he writes, his care in referring to other sources, and not least, the way in which he acknowledged the fact that Gödel fully deserved his credit:

> As for any claims I might make perhaps the best I can say is that I would have *proved* Gödel's Theorem in 1921—had I been Gödel.[6]

In 1936 in a way he had bad luck again. He formulated and formalised a concept of computability [16] which is almost identical to that formulated and published by Turing at about the same time [28]. Post's paper though did not contain the concept of a universal machine, nor the important undecidability theorems in Turing's paper. These misfortunes seem not to have discouraged Post: in the forties he wrote his important paper on recursion theory [19], he published his proof of the unsolvability of the Correspondence Problem [20], and furthermore he proved the unsolvability of the word problem for semi-groups [21]. A further important contribution was not in mathematical logic but in group theory, namely his long paper on polyadic groups [17].[7] In 1954 he died of a heart-attack—possibly as

[3]Thanks to Martin Davis [4], who was a student of Emil Post, this manuscript was made available to the general public. It was republished in his Collected Works also edited by Davis [25].

[4]It should be remarked however that an abbreviated version of this paper was published in this journal, containing among other things, Post's normal form theorem. See [18].

[5]Herman Weyl, in a letter to Post dated March 2, 1942. Quoted from the introduction of [7].

[6]From a postcard addressed to Gödel, dated October 29, 1938. Quoted from the introduction of [7].

[7]Another part of the research of Post that should be mentioned here is his work on provability and definability. He tried to find an absolute and fundamental explication of these notions comparable to those already offered in [28, 16, 2] for the notion of computability.

a consequence of the electro-shock treatment he received for the manic-depressive illness he suffered from throughout his entire career.

The work of Emil Post has had many influences ranging from mathematical logic to computer science. To give some examples, he is known as one of the founders of recursion theory [7, 13, 27], he seems to have influenced John Backus in formulating what came to be called "Backus Normal Form" [6], he had an impact on the study of NP-complete languages in structural complexity theory [7, 26], and is mentioned by Chomsky in his [1]. Post is also one of the people who constructed a bridge between logic and classical mathematics in linking undecidability results with word problems in algebra. As John Stillwell remarks—in one of the rare papers on Emil Post [27, p. 6]:

> Thus Post should be celebrated, not only for his fundamental work in logic, but also for constructing a bridge between logic and classical mathematics.

Important as it is to give a historical account of Post's work, that is not the main motivation for writing this paper. In the introduction to his [24] Post fully acknowledges the fact that there would be little point in publishing his "anticipation ... merely as a claim to unofficial priority" to the results of Gödel, Turing and Church. But he also remarks:

> [...] with the *Principia Mathematica* of Whitehead and Russell as a common starting point, the roads followed towards our common conclusions are so different that much may be gained from a comparison of these parallel evolutions.

He then mentions three reasons why his work can still be of importance although not 'being the first'. First, he focuses on the outward form of symbolic expressions and the possible operations thereon—an approach which resulted in a class of very general and, at first sight, simple systems. Second, he adds another equivalent formulation to the list of general recursiveness, λ-definability and Turing computability. As a third and final reason, he mentions his conclusion that "mathematical thinking is, and must be, essentially creative" from which he concludes that such "developments will result in a reversal of the entire axiomatic trend of the late 19th and early 20th centuries, with a return to meaning and truth".[8]

In this paper emphasis will be on the first and the second reason Post mentions in his introduction. It will be shown how his method of focusing on outward form, was motivated by the idea of studying systems of symbolic logic and their metamathematical properties in their most general form, and

There are only two abstracts published on this research [23, 22], but as mentioned by Martin Davis in his introduction to the Collected Works of Emil Post [7] his notes in bound notebooks on this subject are still available.

[8]It should be noted here that this statement should not be confused with those made by Lucas and Penrose [11, 14] who concluded, on the basis of Gödel's incompleteness result, that mathematical thinking and understanding must be non-computable.

how this resulted in classes of very "simple" systems, that nevertheless are fully equivalent to systems having full generality. Furthermore it will be argued that it was precisely this more mathematical approach that forced him into the mathematician's practice of starting with special cases and studying their properties by considering specific examples of these cases. It will be shown that it was precisely these preliminary "calculations" that led him to a negative answer for the question for which he had hoped to find a positive solution, i.e., the decision problem.[9]

2. Focus on form. After he had given up the idea of becoming an astronomer because he had lost an arm, Post began studying mathematics. While a student at Columbia University, he participated in Cassius J. Keyser's seminar on *Principia Mathematica*. This system was one of the possible candidates at that time for the complete formalization of the whole of mathematics hoped for by the formalists. Also Post was fascinated by its powers: at the time he wrote his Ph. D. thesis he believed that *Principia* would be complete and decidable. After he had finished his Ph. D. his goal even became no less than finding a positive solution for the decision problem for the whole of *Principia*—a problem which he then called the finiteness problem.[10]

He seems to have started this quest with his Ph. D. thesis in which he showed—among other things—that the portion of *Principia* concerning the propositional calculus is consistent, complete and decidable. However this same thesis also marks the beginning of the development of a theoretical framework that would get him further and further away from this original goal. Despite the fact that he believed *Principia* to be decidable, he began criticizing it by stating that it cannot be the one true logic, and that there is a need for a more general approach [15, pp. 163–164]:[11]

> But owing to the particular purpose the authors had in view they decided not to burden their work with more than was absolutely necessary for its achievements, and so gave up the generality of outlook which characterized symbolic logic. [. . .] we might take cognizance of the fact that the system of 'Principia' is but one particular development of the theory [. . .] and so [one] might construct a general theory of such developments.

In effectively constructing these desired general methods and systems he interpreted these merely as "*instruments of generalization*" to study systems of symbolic logic. In his Ph. D. thesis he then used such an "instrument"

[9] Although not for the restricted functional calculus.

[10] In the remainder of this paper, the decision problem will thus be called the finiteness problem.

[11] The paper [8] by Martin Davis and his introduction to the Collected Works [7], gives a further analysis of Post's Ph. D. thesis.

for solving the problem of consistency, decidability and completeness of the propositional calculus: its proofs are based on the truth-table method.[12]

A second important feature that is already apparent in Post's thesis is abstraction from specific meaning. In discussing his truth-table method he remarks [15, p. 166]:

> Let us denote the truth-value of any proposition by $+$ if it is true and by – if it is false. This meaning of $+$ and – is convenient to bear in mind as a guide to thought, but in the actual development that follows they are to be considered merely as symbols which we manipulate in a certain way.

Since a natural approach to finding a more general system than *Principia* is indeed to abstract from particular concepts included therein, this "meaninglessness" is closely connected with generalization. One could of course claim that *Principia* already had these properties, but to Post's mind this was clearly not the case. He thus started to effectively abstract from *Principia* and constructed in his dissertation what he later called *systems in canonical form A*. Within these systems strings can be produced through finitary symbol manipulation and they can thus be regarded as combinatorial systems.[13] These were at first developed with the intention of being generalizations of propositional calculus, but were later shown to be far more general. Without going into the details of systems in canonical form A, it is important to notice the explicit use of the word "form". Instead of constructing one specific system with a specific interpretation, Post constructs a form, which can then be used to generate systems. This construction of a form is then consistent with his goal of studying classes of systems of logic and their general properties. Indeed, starting from these forms, it is possible to generate infinitely many systems of logic—each with its axiom(s) and production rules—which then all share the same form, i.e., formalization is taken literally here. In discussing his canonical form A, this is stated explicitly [24, pp. 345–346]:

> We shall say that we thus solved the finiteness problem for the ($\sim$, $\vee$) system. While this solution was purely formal, nevertheless it was suggested by the intuitive interpretation of "$\sim$" and "$\vee$". For the above generalizations such interpretation are not at hand. Nevertheless (...) the writer solved the finiteness problem for those of the above systems in which the primitive functions are all functions of one variable, the resulting relative simplicity of the systems allowing a direct analysis of the formal processes involved.

After having observed that his solution for the decision problem of propositional calculus was suggested by the interpretation of "$\sim$" and "$\vee$", he

[12] Post explicitly interpreted his two-valued truth table method (which he also generalized to many-valued truth-tables in his dissertation) as an instrument of generalization [15, p. 173].

[13] As Davis remarks in his introduction to [7] the strings produced by such systems can be regarded as being an arbitrary recursively enumerable set of strings on a finite alphabet.

immediately adds that he also searched for systems which share the same form as propositional calculus (relative to his canonical form A) but which do not have such an interpretation and solved their finiteness problem too. These solutions though were easier to reach, because of the formal simplicity of the systems involved. He could study these systems as being pure symbol manipulating systems that generate logical propositions without having to take notice of the content of what precisely is deduced.

As was stated earlier, this abstraction process was not intended for its own sake: its motivation is the study of systems of logic and their properties in their most general form. However, the main use (and later motivation) of these generalizations was to find a positive solution of the finiteness problem for the whole of *Principia*. He already had such a solution for its propositional part. His next purpose was to extend it to the restricted functional calculus as formalized in *Principia*. Using his generalizations from his dissertation he now wanted to find such a proof, by reducing this calculus to a system in canonical form A. Finding a positive solution for the finiteness problem for this simpler system would then yield a corresponding solution for the more complicated functional calculus. As is remarked by Davis this program was a very ambitious one [7, p. 4]:

> Since *Principia* was intended to formalize all of existing mathematics, Post was proposing no less than to find a single algorithm for all of mathematics.

The possibilities of using generalized methods for decidability proofs, were thus further exploited. However, instead of directly proving the reducibility of the restricted functional calculus to a system in canonical form A, he reduced it to a system in what he called canonical form B and proved the reducibility of canonical form A to B.[14]

By the time he got these results (October 1920), he was already a Procter fellow at Princeton. He now reoriented his research towards a problem that is closely connected with the finiteness problem, namely the problem of determining for any two expressions of a given system, what substitutions would make those expressions identical, i.e., the unification problem. However a solution for this problem for the whole of *Principia* was not immediately at hand—"[this] *general problem proving intractable*". In trying to solve it, Post then applied a technique which was at that time already rather familiar to him, namely simplifying and abstracting from the original problem. This abstraction process resulted in the problem of "tag".

[14]As he remarks, some slight adjustments of his canonical form A made such a reduction proof for the functional calculus easier. Systems in this adjusted canonical form he called systems in canonical form B. It should be further noticed that about one year later he also proved the reducability of canonical form B to A so the equivalence of these two forms was established.

4. The Problem of "Tag". The method of abstracting from meaning and thus focussing on form led Post to a class of symbol manipulating systems for which he formulated a problem closely connected to the finiteness problem. As was already mentioned, he arrived at this problem by performing several successive simplifications on the above mentioned unification problem for *Principia*. Despite the expected simplicity of the problem in this reduced form, a solution was considered to be fundamental for the further development of his research. It was not only relevant for the unification problem, but also for a solution of the finiteness problem for those systems in canonical form A, which go but a little beyond those which involve only primitive functions with one argument [24, p. 361]:

> The general problem proving intractable, successive simplifications thereof were considered, one of the last being this problem of "tag". Again, after the finiteness problem for systems in canonical form A involving primitive functions of only one argument was solved, an attempt to solve the problem for systems going, it seemed, but a little beyond this one argument case, led once more essentially to the selfsame problem of "tag". The solution of this problem thus appeared as a vital stepping stone in any further progress to be made.

Although a solution for this problem of "tag" was considered to be a vital stepping stone, Post believed that finding a solution for his problem of "tag" would be rather straightforward. However, after about nine months of work he realized that this judgement was seriously in error and he was finally led to a reversal of the direction of his entire program.[15]

A tag system is specified in the following way: Consider an alphabet $A = \{a_0, a_1, \ldots, a_{\mu-1}\}$, and a constant natural number v. With each letter of the alphabet a_i $(i \in \{0, \ldots, \mu-1\})$is associated a sequence of finite length over the alphabet. It is then the μ number of symbols, the integer v and the associated sequences which form the basis of a tag system. Furthermore an initial non-empty string I over the alphabet of arbitrary but finite length is chosen. Depending on the first letter of this initial string, the respective sequence of letters is then tagged at the end of the string, and the first v letters are removed.[16] In this way a new string is produced, and the operation of

[15]Cf. infra.

[16]I am indebted to Martin Davis for calling my attention to the origin of the word "tag". As a noun, a "tag" designates a physical object to label something—like the price of an item in a shop. The respective verb "to tag" then designates this operation of putting such a tag on something. A derived meaning can be found in the children's game in which a pursuer tries to catch the pursued and "tag" her (or him) with a touch. It is this meaning of the verb that gave the problem of tag its name: in an earlier formulation of the problem, every v-th letter of the sequence was merely checked off (and not removed) while at the same time the respective sequence was added to the right end of the sequence. Starting on the left the mark then moves more and more to the right, "pursuing" the right end of the string.

tagging and removing letters can be repeated. Post discusses the following very simple example—a tag system of which the general behaviour is still not known (despite the availability of the computer): $A = \{1, 0\}$; $1 \rightarrow 1101$; $0 \rightarrow 00$; $v = 3$ Then choosing for example the following initial condition "1011000100010101110" the following sequences of strings are produced:

1011000100010101110
10001000101011101101
01000101011101101
0010010101110110100
0100010101110110100000
... ...

The problem of tag now has two forms. In its first form the problem is to obtain for a given basis, a general method to decide for any initial string *I* whether the process of tagging and removing letters will ever produce an empty string and thus come to an end. In its second form—where the initial sequence *I* is considered as being part of the basis—the problem for a given basis is then to find a general method for determining for an arbitrary string and this basis whether it will ever be generated by the given basis. It is the problem posed in its second form—as Post remarks—that arose in connection with the finiteness problem.

The significance of the research of Post on this problem of tag is that the only way to get a grip on these systems was to rely on the mathematician's practice of working out specific cases and varying several parameters in order to be able to infer properties of tag systems and identify the different classes of behaviour. This was necessary since the form in which such systems had to be put was so simple and "meaningless" that no direct information could be immediately inferred from it—such research would nowadays probably be done with the help of a computer.

It is clear from his notes that he tested several cases—varying v and μ for several initial conditions. Furthermore he subdivided his tag systems into several classes of behaviour he observed. He mentions three classes of behaviour: termination, periodicity and divergence. This last possibility of divergent behaviour caused major difficulties Post was unable to resolve. It first led him to the idea that the finiteness problem might perhaps not be solvable. He also classified classes of cases which are certainly solvable and which were not. For the case $\mu = 2$, $v = 2$, the problem was completely solved. The case for $\mu = 2$, $v > 2$ he calls intractable, while he terms the cases where $\mu > 2, v = 2$ as being of "*bewildering complexity*". It was these observations—and the inferred theoretical considerations—that led to a reversal of his entire program: his goal of finding a positive solution for the finiteness problem for *Principia* seemed hopeless at that time.

It is noteworthy that Post's search for the most general form to capture systems of symbolic logic ended up with a form as simple as that of tag systems which forced him into the mathematical practice of beginning with special cases for which he could hope to infer more general properties. It was not the theoretical considerations preceding his work on tag systems but his struggle with what appeared to be easy problems that led him to the idea of the finiteness problem possibly being unsolvable [24, p. 363]:[17]

> For a while the case $\nu = 2$, $\mu > 2$, seemed to be more promising, since it seemed to offer a greater chance of a finely graded series of problems. But when this possibility was explored in the early summer of 1921, it rather led to an overwhelming confusion of classes of cases, with the solution of the corresponding problem depending more and more on problems in ordinary number theory. Since it had been our hope that the known difficulties of number theory would, as it were, be dissolved in the particularities of this more primitive form of mathematics, the solution of the general problem of "tag" appeared hopeless, and with it our entire program of the solution of finiteness problems. This *frustration* [my emphasis], however, was largely based on the assumption that "tag" was but a minor, if essential, stepping stone in this wider program.

As is clear from this quote, these observations really bothered Post—he had not expected difficulties for such "*primitive forms of mathematics*", and it was only when he was able to prove that canonical form A is reducible to a form which is closely connected to these tag systems, that these difficulties no longer seemed surprising.[18]

5. Further reductions: From tag systems to Post's thesis. As was already mentioned, Post had already shown that canonical form A can be reduced to canonical form B and further proved that the restricted functional calculus is reducible to a system in canonical form B. At that time he still believed that the finiteness problem for *Principia* had a positive solution—it was this goal that had led him to these first reduction proofs [24, p. 346]:

> [...] impetus was lent to the work by our formally reducing the subsystem of Principia Mathematica treated in *10 and *11 thereof to a system of the above type [canonical form B]. For thereby a solution of the finiteness problem for all of the above

[17] For further arguments concerning this statement, Cf. infra.

[18] As was pointed out to me by Martin Davis, Post hoped that someone else would prove the recursive unsolvability of tag systems. Post himself did not want to work on them again given his frustrating experience with them. Finally his conjecture of tag systems being unsolvable, was proven by Marvin Minsky in his [12], after the problem was suggested to him by Martin Davis (See [7]).

systems would immediately lead to a solution for this important subsystem of Principia Mathematica.

Having understood after his work on tag systems that very basic production systems also give rise to complex behaviour, Post was now led to a whole series of reductions to systems in certain forms, which are far simpler than those in canonical form A and B.

The first transformation step he made was the reduction of systems in canonical form B to a canonical form C. Systems in this last form are nowadays known as *Post production systems*. Now, given a system of which the formulas or enunciations are all expressed in terms of "*finite sequences of letters, the different letters constituting a once-and-for-all given finite set.*"[24, p. 197], such a system is said to be in canonical form C, when the primitive assertions (i.e., the initial strings) of the system are a finite set of enunciations of the above form, and the operations performable in the system are specified by a finite set of production rules all of the following form:

$$
\begin{gathered}
g_{11}P_{i_1^1}g_{12}P_{i_2^1}\dots g_{1m_1}P_{i_{m_1}^1}g_{1(m_1+1)}\\
g_{21}P_{i_1^2}g_{22}P_{i_2^2}\dots g_{2m_2}P_{i_{m_2}^2}g_{2(m_2+1)}\\
\dots\dots\dots\dots\dots\dots\dots\dots\\
g_{k1}P_{i_1^k}g_{k2}P_{i_2^k}\dots g_{km_k}P_{i_{m_k}^k}g_{k(m_k+1)}\\
\textbf{\textit{produce}}\\
g_1P_{i_1}g_2P_{i_2}\dots g_mP_{i_m}g_{(m+1)}.
\end{gathered}
$$

The g's are given sequences of letters of the once-and-for-all given finite set of letters or alphabet and the P's the operational variables—they can be any combination of letters from this same alphabet. The further restriction is added that each P in the conclusion of the production is present in at least one premise of that production. To understand how such a system in canonical form C works, it may be helpful to give an example. Suppose that the alphabet is given by the set of letters $\{a, b\}$, and for the sake of simplicity, further suppose that there is only one initial assertion, namely:

$$ababbabbabaaaaababbababbbbababbbabbb$$

Furthermore, suppose there are two production rules:

$$
\begin{gathered}
abaP_1abP_2b\\
aP_3bbP_4b\\
\textbf{\textit{produces}}\\
bbP_1aabbaab
\end{gathered}
$$

and

$$bP_{1'}baP_{2'}b$$
$$bbP_{3'}ab$$
produces
$$abaP_{3'}b.$$

Thus the basis of a system in canonical form C has been defined. To be able to produce a new string, one has to check whether the initial assertion "fits" into one of the forms given by the production rules. In this case the first production rule can be applied, but not the second, resulting in the following production:

$$\boldsymbol{aba}bb\boldsymbol{ab}babaaaaababbababbbbababbbabb\boldsymbol{b}$$
$$\boldsymbol{a}ba\boldsymbol{bb}abbabaaaaababbababbbbababbbabb\boldsymbol{b}$$
produces
$$bb\boldsymbol{bb}aabbaab.$$

From this new string, another string can be produced by applying the second production rule:

$$\boldsymbol{b}bb\boldsymbol{ba}abbaa\boldsymbol{b}$$
$$\boldsymbol{bb}bbaabba\boldsymbol{ab}$$
produces
$$aba\boldsymbol{bbaabba}b.$$

Constructing systems in this way, makes it possible to generate a variety of systems, which, depending on the production rules and the initial assertion(s), each have their own properties: systems which are monogenic or not[19], systems which produce an infinite or a finite number of strings, systems whose results are periodic or not, ... To finish this example, the first strings produced by the basis given here, are shown:

bbbbaabbaab
ababbaabbab
bbbbaaabbaab
ababbaaabbab
bbbbaaaabbaab
ababbaaaabbab
bbbbaaaaabbab
. . . .

[19] A monogenic system is a system for which there is for each assertion in the system one and only one applicable production.

Although systems in canonical form C can be interpreted as functional systems, it is more straightforward to regard them as pure string rewriting systems. This is partly due to the fact that "the boxes within a box symbolic form of the parenthesis notation is replaced merely by finite sequences of letters [...]"[20]. This results in a more flexible mechanism which becomes even clearer when actually constructing such a system and making it work: the arbitrariness with which such systems can be constructed—manually or programmed—is significant.

A special class of systems in canonical form C consists of systems in normal form. Systems in normal form have only one initial assertion and the finite set of production rules are all of the following form:

$$gP$$
produces
$$Pg'.$$

These systems are the most simple and general systems Post ever developed. However, from Post's normal form theorem it follows that they are no weaker than, for example, systems in canonical form C. This theorem states that for every set of assertions generated by a system in canonical form C over a given alphabet Σ, a system in normal form over an alphabet including Σ can be set up, such that the assertions from the system in canonical form are exactly those assertions of the system in normal form which consist of no other letters than those contained in Σ.[21]

As was already stated earlier, the construction of systems in canonical form C and normal form and the respective reduction proofs, was preceded by Post's work on tag systems.[22] In the limited existing literature on Emil Post's early work [7, 27, 13] these systems of tag are always discussed but no mention is made of their fundamental role in the further development of this work. There are however several arguments which show that systems of tag were fundamental for the construction of Post's canonical form C and normal form and the respective reduction proofs, and thus also the essential step in the reversal of Post's research program.

First of all, it was through his work on tag systems that he realized that such primitive forms of mathematics can indeed give rise to such bewildering complexity, a fact which he clearly had not expected. On the one hand, this made the positive solution of the finiteness problem appear hopeless (See section 4). On the other hand the realization that his normal form could be "as complex" as systems in canonical form, seems to have been necessary

[20] [24, p. 363].

[21] The normal form theorem was first proved in the summer of 1921 and published in his [24]. The proof also appeared in his [18].

[22] As is clear from the chronology of the sequence of events that led to his results Post sketches in his introduction of [24, p. 341].

for the idea of the possibility of reducing this relatively involved form to systems in normal form via reduction to canonical form C. The fact that Post discusses his tag systems in his [18] in which the main purpose is the proof of his normal form theorem, adds strength to the significance of his work on systems of tag for the further reductions made. More convincing are some quotes in which he literally states the dependence of normal form on tag systems. In the last footnote of his [18] Post states:

> In the summer of 1921, the intervening work on the problem of tag suggested the reduction of canonical form C to the canonical form of the present paper, and this reduction was followed by the successive reductions to normal form essentially as given in section 2.

Although this quote seems ambiguous in that it suggests that there was an intermediary canonical form between canonical form C and normal form, the canonical form of the present paper seems to be a reference to his normal form. This interpretation relies on the fact that the two canonical forms of the paper are canonical form C and normal form.[23] If this interpretation is correct, then this quote indeed shows that tag systems led Post to his normal form. The only other sensible interpretation here could be that Post made a typing error: it is possible that it should not be canonical form C, but B. Although this seems less obvious given the fact that the reduction of canonical form B to C is mentioned in this same footnote preceding this quote, the quote interpreted in this way also gives full support to the above mentioned statement: given the fact that canonical form C differs significantly from canonical form B, the fact that tag systems suggested the possibility of this reduction is then fundamental to the formulation of normal form. A second less ambiguous quote from Post's [24] on p. 382 is the following:

> In fact, at one point later in the work on "tag", it seemed that the regularity induced by always removing μ elements from the beginning of a sequence was responsible for the intrusion of number theory in the development, so that it was tentatively suggested that "tag" be generalized to a form, which, indeed, is exactly that of the later derived normal form.

Here Post states that due to a property of tag systems which links them to number theory, a form was suggested which was exactly that of the later normal form. The fact that it was due to a property of tag systems that normal form was suggested, is a clear illustration of the fact that they led to normal form. Moreover, it was a property that showed the intrusion of number theory in his development. This fact not only adds support

[23] Which he indeed identifies as a special case of canonical form C and which is furthermore the central formalism of the paper, the main proof being Post's normal form theorem.

to the significance of tag systems to normal form, but furthermore to their significance to his undecidability result.[24] A last quote which almost literally states the fact that systems of tag were fundamental to the further reductions from canonical form B to C to normal form is the following [24, pp. 202–203]:

> Before turning to the proof of our basic theorem given in the next section [i.e., Post's normal form theorem] we wish to [...] state a problem *which largely determined the direction taken by the reductions of the next section*, and may offer further opportunities for unsolvability proofs. [...] The problem referred to above takes two related forms. Both forms employ [...] "tag" operations as we shall call them.

Although being less direct, maybe the most remarkable thing Post notices in this context, is the fact that once he had constructed the reduction from canonical form A to normal form, these reductions functioned as a kind of explanation for the difficulties he had experienced with tag systems [24, p. 386]:

> We have observed [...] how the seemingly simple problem of "tag" in fact proved intractable for $\mu = 2, v > 2$, of bewildering complexity for $\mu > 2, v = 2$. In view of our reduction of canonical form A to a form as close to that of "tag" as the normal form, the difficulty of "tag" is no longer surprising.

His work on tag systems was probably the most frustrating research Post did—struggling on the edge of unsolvability. However, they showed him that the simplicity of the basis of a system does not necessarily imply simplicity of the behaviour of the system once it is "run".

Once he had proven these reductions, he completed his work by "closing the circle"—he showed that systems in normal form are reducible to canonical form A. All the forms discussed here (except for the form of "tag") were shown to be completely equivalent to each other in this manner. The fact that he had now shown that systems in a simple form could produce the same assertions deducible within a more complicated system was now clearly understood, and he even used it as an argument for the possible reduction of the whole of *Principia* to the more simple canonical form B [24, p. 384]:

> In the absence of the forbidding amount of work needed to actually carry out this reduction, added strength is lent to the above conclusion by the further reductions carried though (...); for if the meager formal apparatus of our final normal systems can wipe out all of the additional vastly greater complexities of canonical form B, the more complicated machinery of the latter should clearly

[24] The significance of tag systems to the idea of the finiteness problem not having a negative solution is also explicitly stated in the last part of the second quote of section 4.

> be able to handle formulations correspondingly more complicated than itself.

Having at his disposal this whole sequence of reductions, he knew that if the finiteness problem would be solved for one of these forms—be it a negative or a positive solution—it would be solved for all of these forms, a fact which satisfies his initial claim for generality. However, no such solution was yet found.

Notwithstanding the fact that his tag systems had shown him the possibility of certain decision problems being unsolvable—it is significant that he still must have had some slight hope for proving the solvability of the finiteness problem of systems in normal form. In reducing such systems to still another form, such a solution again seemed within reach [24, p. 382]:

> While [...] special cases of "tag" might well be worth consideration as major problems in themselves, the [...] further reduction of the normal form seemed more promising.

Despite first successes in using this new form for solving the finiteness problem, he was only able to prove this for a subclass of this form, namely for those systems of which the operations consisted of three of the four operations allowed for in this form. He concluded [24, p. 283]:

> The resulting methods held out the possibility of an attack on the finiteness problem for systems having all four of the above types of operations, though certain of the difficulties of "tag" even then seemed glimmering in the distance. And just when hope was thus renewed for a solution of the general finiteness problem, a fuller realization of the significance of the previous reductions led to a reversal of our entire program.

This "fuller realization" points at Post's thesis, namely that each generated set of sequences, can be generated by a system in normal form.[25] The fact that he had been able on the one hand to reduce canonical form A to a normal form—through his intermediary work on tag systems—together with the fact that he believed the whole of *Principia* to be reducible to canonical form B, led him to this thesis. It is significant to notice that to Post the notion of a generated set—and its generalization to normal form—was rather concrete: the validity of the unsolvability or solvability of the finiteness problem for systems in normal form should not be based on the definition of generated set, but depends on the possibility of real implementation of an algorithm for generating such a set [24, p. 386]:

> [...] we must show how to *generate* that set, i.e., set up a system of "combinatory iteration"

where "combinatory iteration" is described as follows [24, p. 386]:

[25] Martin Davis was the first to speak in this context of Post's thesis in his [5].

> [...] the method of combinatory iteration [...] eschews all interpretation, and studies the system merely as a formal system. The operations of the system are then described as "combinatory" since they largely involve but a reshuffling of symbols; and it is through the "iteration", i.e., continued reapplication, of these combinatory operations that the entire system is obtained.

In understanding the computable character of the notion of generated set, and in identifying it with every possible system of symbolic logic, *Principia* included, the fuller realization of the power of his normal form finally led him to apply Cantor's diagonal method to conclude that the finiteness problem for the entire class of systems in normal form is unsolvable [24, p. 386]:[26]

> We [...] conclude that the finiteness problem for the class of all normal systems is unsolvable, that is that there is no finite method which would uniformly enable us to tell of an arbitrary normal system and arbitrary sequence on the letters thereof whether that sequence is or is not generated by the operations of the system from the primitive sequence of the system.

Of course the truth of this assertion really depended on his thesis: it was only the assumption that every generated set of sequences can be obtained from a normal system that enabled this negative conclusion.[27] It is significant to note that Post's understanding that only a *real* implemented example could disprove this thesis, later led him to the interpretation of such theses as being *working hypotheses* which should be *continually verified* [16], and to his criticizing Church for calling his "thesis" a definition [2] thus implying that no further verification was needed.

Post further realized that these results implied the incompleteness of all systems of finitary symbolic logic reducible to a system in normal form [24, p. 216]:[28]

> Having noted the identity of canonical systems and normal sets [...] our last conclusion was transformed into the generalization that every generated set of sequences on a finite set of letters was a normal set. [...] In the early fall of 1921, the formal proof of this unsolvability [...] was outlined, and led to the further conclusion that not only was every *finitary* symbolic logic incomplete relative to a certain fixed class of propositions (those stating that a given

[26] For a more detailed explanation of the diagonal argument Post uses see [7, 27].

[27] It should be noted that Post understood that his analysis of the notion of generated set was only fragmentary, since for "full generality a complete analysis would have to be given of all the possible ways in which the human mind could set up finite processes for generating sequences." (See [24, p. 387]). In his [16] he then gave a complete analysis of the notion of an algorithm.

[28] For a more detailed exposition of how Post proved incompleteness on the basis of these results—in comparing this approach to Gödel's—see [27].

> sequence was or was not an assertion in a given normal system) but that every such logic was extendible relative to that class of propositions.

As was clearly stated, it was the understanding of the generality of his normal form in identifying it with the notion of a generated set that seems to have played a vital role in his formulation of these undecidability and incompleteness results. However, he considered this generalization of such a primitive form to generated sets only after his work on tag systems. On the one hand they suggested to him the possibility of the existence of unsolvable decision problems, and on the other hand suggested that complicated forms could perhaps be reduced to simpler ones because of the complexity to which these simple forms could give rise. Only on one occasion was his hope for a proof of the solvability of the finiteness problem renewed. However, in realizing that normal forms, which were generalized from tag systems, seem to be able to generate everything that can possibly be generated, in that it seemed impossible to find an actual algorithm that can "outrun" systems in normal form, this initial hope was completely destroyed. It thus seems clear that his investigations of tag systems, making inferences on the basis of the analysis of concrete cases, played a vital role in the development of Post's further results. This approach though did not appear out of the blue. It followed from his generalizing method in approaching systems of symbolic logic and their finiteness problems—a method which is not an arbitrary amalgam of different approaches, but forms a coherent whole.

5. Closing the circle: Extracting Post's method. In sketching the evolution of Post's ideas starting with the publication of his Ph. D. thesis in 1920 and ending in the fall of 1921, it was shown how the direction of this development was motivated by a method of generalization and detachment from meaning. In trying to understand why he developed this method it is important to notice that what Post was interested in was not the development of one unique system of symbolic logic which formalizes certain concepts, but rather in the structure and the possibilities of symbolic logic in the most general sense—he tried to grasp what precisely characterizes all systems of symbolic logic. This is already clear in his Ph. D. thesis where he literally states that he wants to develop general methods to be used as instruments to study systems of symbolic logic. Generalization and detachment from interpretation are then the necessary tools to obtain this most general form which captures logic. In discussing the differences of his work with that of Gödel, Church and Turing he is explicit in recognizing the advantages of this approach over theirs [24, pp. 340–341]:

> Perhaps the chief difference in method between the present development and its more complete successors is its preoccupation with

> the outward forms of symbolic expressions, and possible operations thereon, rather than with logical concepts as clothed in, or reflected by, correspondingly particularized symbolic expressions, and operations thereon. While this in part is perhaps responsible for the fragmentary nature of our development, it also allows greater freedom of method and technique. In particular, it reveals the "Gödel Representation" to be merely a case of resymbolization [. . .].

It is indeed because he abstracted from meaning, and extracted in this way a form, that he achieved this greater freedom—a process which is very clearly exemplified in his reduction proofs. There one sees how step-by-step all meaningfulness is removed. To abstract from the specific meaning of an assertion he first eliminates the meaningful symbols like "$\vee$", and constructs a form which is already then characterized by its arbitrariness and combinatorial character. This became more explicit once he began his work on tag systems. For example, there is no syntax or order of a specific assertion because of the removal of delimiters (brackets). In that way the concept of a well-formed formula becomes rather trivial since there are only sequences of letters. Every arbitrary combination of letters or symbols from the predefined set of letters or symbols, is well-formed—a property which is not shared by the systems developed by Gödel and Church. Furthermore, the concept of an axiom becomes almost trivial since during his investigations of systems of tag, it was only in varying these "axioms" as parameters, together with the parameters of the bases, that Post identified different classes of behaviour and different classes of cases.

Because of his understanding of this abstraction process, Post indeed obtained a more flexible method, resymbolization. Unlike for example the Gödel numbering developed for its specific purposes, he does not take into account any specific interpretation and uses this resymbolization as a general scheme behind his research. This is already implicit in his first reduction proofs, and becomes more explicit in his reductions after his work on the problem of "tag": following these reduction proofs one sees how by successive steps of resymbolization, one form is transformed into another. Post himself clearly understood this by sometimes referring to these reductions as string rewritings or transformations. To him these different forms were indeed all abstractly the same [19, p. 292]—"only" a reshuffling of symbols makes them identical.[29]

[29] A fact which was of course also noticed by Church, Gödel and Turing since they themselves published equivalence proofs. Turing and Church used these proven equivalences as arguments for their notion of computability. However, whereas in Post's work resymbolization is a general technique of which reduction proofs are just one application, this is not so explicitly stated by Turing, Church nor Gödel. This is also indirectly pointed out by Post, in terming Gödel's numbering as being merely *a* case of resymbolization.

Resymbolization also played an important role in the development of his problem of "tag": this problem arose through successive simplifications of the unification problem—which in itself is a resymbolization problem. As he identifies the concept of a generated set with his normal form, and since he interprets the method of generation as the "continued reapplication of the reshuffling of symbols", resymbolization is furthermore an explicit characteristic of the forms themselves he developed—especially after and during his work on tag.

In trying to focus on form the systems themselves also become simpler. This was a consequence of his method of which he was very well aware, since it was one of the motivations for performing his abstractions: in reducing a system to a simpler one, a solution for the finiteness problem might be more easily obtained. Indeed this turned out to be the case, although it was a solution that was not expected.

It is significant that the simple normal form is equivalent to λ-definability, Turing machines and general recursiveness. Also Post himself was amazed by the complexity his primitive forms of mathematics gave and give rise to. The ease with which one can construct systems in normal form, canonical form C or tag, is then a result of this primitiveness. The only thing one has to take into account in constructing such systems is the form: choose a finite alphabet of arbitrary symbols; write down an axiom consisting of the symbols of the chosen alphabet; and construct one or more production rules which have the form prescribed by the form you are dealing with.[30] The concept of "form" indeed works in this way: it is more a boundary condition under which a system can work rather than being that to which the working system can be reduced, i.e., the form becomes rather a method or a mold into which the systems generated by it fit, even if they can differ significantly from each other in comparing their behaviour and properties.

One could object of course that this is also the case for Turing machines and λ-definability, but this is only true from a theoretical point of view. One can develop arbitrary Turing machines, and perform conversions on arbitrary λ-expressions. However first of all, the machines must satisfy certain syntax to be workable, as is the case with λ-defined expressions. Furthermore, working in the λ-calculus or with Turing machines, one almost always has a specific purpose in mind in developing specific machines or λ-conversions, which is due to the fact that they were developed for such specific purposes. Church for example wanted to develop a formalism for manipulating functions and the expressions defining them, and already in the beginning of his [2, 3] he starts to add concepts and meaningful conversions, like the fact that numbers are introduced as being abbreviations of certain formulas, and basic arithmetical operations are λ-defined. Turing in his [28] on the other

[30]Of course, if one starts performing research on systems in one of these forms a more clever approach becomes necessary.

hand clearly computes numbers, and constructs machines with the purpose of a specific output.

Post's construction of general forms of systems of symbolic logic however, seems to have been no specific purpose. As was mentioned, he developed these general forms and methods to be able to *study* systems of symbolic logic. Central to this study was solving the finiteness problem for the whole of symbolic logic as formalized in *Principia*. It was the reason for developing his canonical form A—as providing a more general form for studying and solving the finiteness problem. It motivated the construction of his form B, to be able to reduce the restricted functional calculus to this simpler system. Moreover, he was confronted with the problem of "tag" in trying to find this solution. His further reductions were then directed by this frustrating problem which then finally led him to the unsolvability of this finiteness problem.

Significant in this development, is that through further abstraction, motivated by a difficult theoretical problem, the only "meaningful" method that could be implemented to further study of these abstractions is a typical mathematical one. In Post's work his technique of studying systems by working out special cases seems to be a consequence of precisely this abstraction process. Since the result of this process were systems in a very simple and "meaningless" form, the information about a system one normally deduces through the interpretation of its axioms, production rules and syntax was not available. There is no specific system, but an infinite group of systems sharing the same form. The only way to then study and understand the differing properties of such a class of systems is to test them, by checking for different kinds of initial conditions, different values of v keeping for example other parameters constant, testing tag systems with a different number of symbols, ... Only then was it possible to further theorize. This approach then indeed advanced Post's research in a direction he would probably not have taken without it. In this way the development of Post's theoretical results embodies another kind of closing the circle than the one mentioned by him: through theorization, practice became necessary, which then led to a reversal of the ideas put forward by the theory.

The further research of Post after he suffered from a manic-depressive attack, and after Gödel had published his results, will not be discussed here in detail. However, it should be remarked that Post clearly kept in line with this earlier research: he still searched for general approaches as being methods to study the formal and structural aspects of logic and mathematics and their boundaries. He for example remarked that his normal form—because of its formal simplicity on the one hand and its power on the other hand (Post's thesis)—could be used as a method to solve further removed decision problems [18, p. 200]:

> And it may be that because of its formal simplicity, the normal form may lend itself more readily to representation in specialized

> mathematical developments, and the unsolvability of its decision problem. thus lead to the unsolvability of various hitherto unsolved decision problems of classical mathematics.

He proved for two more decision problems the unsolvability [20, 21] through reduction methods, and finally he deduced a general mathematical theory out of the concept of general recursiveness which led to recursion theory. This is now a well-established branch of mathematics of which the language and style used is still influenced by Post's claim for informality in [19].

7. Conclusion. In discussing the results and methods which led to Post's anticipation of Gödel, Turing and Church, the question now seems to be settled of why he still wanted these results to be published. It seems improbable that this was only with the intention of letting the world know—10 years after Gödel's paper was published—that he was actually the first, but rather in wanting to communicate *his* thoughts about and methods for studying systems of symbolic logic. By investigating Emil Post's [24] not simply as a historical account but as the work of a logician who developed original methods and ideas, it was shown how this approach *did* lead him to a conclusion which at that time would have been revolutionary. Central in this analysis was Post's work on systems of tag. These simple systems suggested to him the possibility of the finiteness problem not being solvable. They led to systems in normal form and the formulation of his thesis, which almost directly implied this unsolvability.

Emphasis was put on the fact that Post in his work on tag systems was forced into the more concrete task of testing specific cases by varying parameters. Trivial as this emphasis may seem to the working mathematician, these facts are not trivial at all from an epistemological point of view. The fact that such practice is often necessary in mathematics and logic is often obscured in published results, while much progress in these branches would not have been possible without these preliminary "calculations". Post's [24] is one of the exceptions here. It is the more significant because this practice led to results he had not expected at all and would maybe not have been reached without it. This report from Post of his "anticipation" thus beautifully illustrates the fact that logic and mathematics are not as perfect and abstract as is often believed. Maybe the work of Post is indeed but a historical account as related to the pure undecidability and incompleteness results. However, the method he used to deduce these results is not: it shows that there are always different roads that lead to Rome, each having its own interesting side-effects.

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