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WEAK DISTRIBUTIVITY, A PROBLEM OF VON NEUMANN AND THE MYSTERY OF MEASURABILITY

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Dedicated to Dorothy Maharam Stone

This article investigates the weak distributivity of Boolean σ-algebras satisfying the countable chain condition. It addresses primarily the question when such algebras carry a σ-additive measure. We use as a starting point the problem of John von Neumann stated in 1937 in the Scottish Book. He asked if the countable chain condition and weak distributivity are sufficient for the existence of such a measure.

Subsequent research has shown that the problem has two aspects: one set theoretic and one combinatorial. Recent results provide a complete solution of both the set theoretic and the combinatorial problems. We shall survey the history of von Neumann's Problem and outline the solution of the set theoretic problem. The technique that we describe owes much to the early work of Dorothy Maharam to whom we dedicate this article.

§**1. Complete Boolean algebras and weak distributivity.** A *Boolean algebra* is a set B with Boolean operations $a \vee b$ (join), $a \wedge b$ (meet) and $-a$ (complement), partial ordering $a \leq b$ defined by $a \wedge b = a$ and the smallest and greatest element, $\mathbf{0}$ and $\mathbf{1}$. By Stone's Representation Theorem, every Boolean algebra is isomorphic to an algebra of subsets of some nonempty set S, under operations $a \cup b$, $a \cap b$, $S - a$, ordered by inclusion, with $\mathbf{0} = \emptyset$ and $\mathbf{1} = S$.

If every subset A of B has a least upper bound $\bigvee A$ (and the greatest lower bound $\bigwedge A$) then B is a *complete* Boolean algebra. An *antichain* in B is a set $A \subseteq B$ such that distinct elements $a, b \in A$ are *disjoint* i.e., $a \wedge b = \mathbf{0}$. B satisfies the *countable chain condition* (ccc) if it has no uncountable antichains.

If B is a ccc Boolean σ-algebra, i.e., $\bigvee_{n\in\omega} a_n$ and $\bigwedge_{n\in\omega} a_n$ exist for countable sets, then B is a complete Boolean algebra. For this and other basic facts on Boolean algebras, we refer the reader to [30], [24].

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The set of all nonzero elements of a Boolean algebra B is denoted B^+. A set $D \subseteq B^+$ is *dense* in B if for each $b \in B^+$ there exists some $d \in D$ with $d \leq b$. For every Boolean algebra A there exists a unique (up to isomorphism) complete Boolean algebra B such that A is a subalgebra of B and A^+ is dense in B. The complete Boolean algebra B is called the *completion* of A.

An *atom* of B is a nonzero $a \in B$ that cannot be split into two disjoint nonzero elements. B is *atomic* if the set of all atoms is dense in B, and *atomless* if it has no atoms.

EXAMPLES.

I. THE POWER SET ALGEBRA $\mathcal{P}(\omega)$. Consider the algebra $\mathcal{P}(\omega)$ of all sets of natural numbers. This algebra is complete, with least upper bounds $\bigcup X$ for $X \subseteq \mathcal{P}(\omega)$, and satisfies ccc. This algebra is *atomic*, where the atoms are the singletons.

II. THE COHEN ALGEBRA Let A be the countable atomless Boolean algebra (this is unique up to isomorphism), and let $\mathcal{C} = \mathcal{C}_\omega$ be the completion of A. The standard representation of A is the algebra of all clopen sets of the Cantor space, the *Cantor algebra*. Since A has a dense set isomorphic to an infinite binary tree (the set of all finite 0-1 sequences under reverse inclusion), the algebra $\mathcal{C}$ is nowadays called the *Cohen algebra*, in recognition of its role in forcing.

The classical description of $\mathcal{C}$ is the quotient algebra of Borel sets (of reals) modulo meager sets. Since meager sets form a σ-ideal, $\mathcal{C}$ is a σ-algebra. As every Borel set is equivalent to an open set (mod meager), the open intervals with rational endpoints form a countable dense set. Since $\mathcal{C}$ has a countable dense set, every antichain in $\mathcal{C}$ is necessarily countable, and so $\mathcal{C}$ satisfies ccc. Thus $\mathcal{C}$ is a complete Boolean algebra, and is atomless.

III. A MEASURE ALGEBRA Consider the quotient algebra $\mathcal{M}$ of Borel sets in the interval [0, 1] modulo null sets, i.e., sets of Lebesgue measure 0. $\mathcal{M}$ is an atomless σ-algebra, and carries a (σ-additive strictly positive) *measure*, a numerical function m with the following properties

$$m(\mathbf{0}) = 0, \ m(a) > 0 \text{ for } a \neq \mathbf{0}, \text{ and } m(\mathbf{1}) = 1 \tag{1.1}$$

$$m\Big(\bigvee_{n\in\omega} a_n\Big) = \sum_{n\in\omega} m(a_n) \text{ whenever the } a_n \text{ are pairwise disjoint.}$$

An atomless σ-algebra that carries a measure is called a *measure algebra*.

If A is an antichain in a measure algebra then for every n, only finitely many $a \in A$ have measure greater than $\frac{1}{n}$; and A is necessarily countable. Hence every measure algebra satisfies ccc (and is complete). By a special case of the classification theorem of Maharam [26], the algebra $\mathcal{M}$ is the unique atomless measure algebra with countably many generators.

The complete Boolean algebras $\mathcal{M}$ and $\mathcal{C}$ are different, as $\mathcal{C}$ does not carry a measure (cf. [35]): Assume m is such a measure. Since $\mathcal{C}$ has a countable dense set $\{d_n : n < \omega\}$, choose for each n some nonzero $x_n \leq d_n$ with $m(x_n) < \frac{1}{2^{n+1}}$. If $x = \bigvee_{n\in\omega} x_n$ and $y = -x$, then $y \neq \mathbf{0}$ but no d_n is below y; a contradiction.

We remark that the atomic algebra $\mathcal{P}(\omega)$ does carry a measure, for instance one such that $m(\{n\}) = \frac{1}{2^{n+1}}$ for each $n \in \omega$. Also, the Cohen algebra carries a *finitely additive measure*, i.e., a function that satisfies $m(a \vee b) = m(a) + m(b)$ for disjoint a, b.

The algebras $\mathcal{M}$ and $\mathcal{C}$ were considered by von Neumann in his 1936-37 lectures [42], [43] on Continuous Geometry at the Institute for Advanced Study. Among others, he introduced the *weak distributivity law*, an algebraic property that distinguishes these two algebras.

Every complete Boolean algebra satisfies the following generalization of the distributive law

$$\bigvee_{x\in X} a_x \wedge \bigvee_{y\in Y} b_y = \bigvee_{\substack{x\in X\\ y\in Y}} (a_x \wedge b_y), \tag{1.2}$$

where X and Y are arbitrary indexed sets.

A general distributive law

$$\bigwedge_{x\in X} \bigvee_{y\in Y} a^x_y = \bigvee_f \bigwedge_x a^x_{f(x)}, \tag{1.3}$$

where f ranges over all functions from X to Y, holds only when the algebra is atomic. In fact, both $\mathcal{M}$ and $\mathcal{C}$ fail to satisfy the simplest case of infinitary distributivity, namely

$$\bigwedge_{n\in\omega} (a^n_0 \vee a^n_1) = \bigvee_{f\colon \omega\to\{0,1\}} \bigwedge_{n\in\omega} a^n_{f(n)}, \tag{1.4}$$

To see this, consider the binary expansions $0.\varepsilon_1\varepsilon_2\varepsilon_3\ldots$ of numbers in $[0, 1]$, and let A^n_i $(i = 0, 1)$ be the set of all reals with $\varepsilon_n = i$. Let a^n_i be A^n_i modulo either null or meager sets. Then the left hand side of (1.4) is $\mathbf{1}$ while for every f, $\bigwedge_{n\in\omega} a^n_{f(n)} = \mathbf{0}$.

To characterize measure algebras, von Neumann formulated the following *weak distributivity law*:

$$\text{if } a^n_0 \leq a^n_1 \leq \ldots \text{ for } n = 1, 2, \ldots, \text{ then} \tag{1.5}$$
$$\bigwedge_n \bigvee_k a^n_k = \bigvee_{f\colon \omega\to\omega} \bigwedge_n a^n_{f(n)}.$$

(The left hand side of (1.5) is always $\geq$ the right hand side, in every complete Boolean algebra.) That a measure algebra satisfies (1.5) is proved as follows (this idea appears earlier in [6] where Banach and Kuratowski proved that under the Continuum Hypothesis, there is no σ-additive extension of Lebesgue measure to all sets of reals).

Let m be a measure on B, and let $a_0^n \leq a_1^n \leq \cdots$ for $n = 0, 1, 2, \ldots$. Without loss of generality we assume that $\bigvee_k a_k^n = \mathbf{1}$ for all n. To verify (1.5) it suffices to find for each $\varepsilon > 0$ some f such that $m(\bigwedge_n a_{f(n)}^n) \geq 1 - \varepsilon$. And that can be done by choosing $f(n)$ for each n so that $m(a_{f(n)}^n) \geq 1 - \frac{\varepsilon}{2^{n+2}}$.

Unlike $\mathcal{M}$, the algebra $\mathcal{C}$ is not weakly distributive. Let $\{d_n : n \in \omega\}$ be a countable dense set in $\mathcal{C}$. For each n we can find a strictly increasing sequence $a_0^n < a_1^n < \cdots$ with $\bigvee_k a_k^n = \mathbf{1}$ such that $d_n \not\leq a_k^n$ for all $k \in \omega$. Now if $f : \omega \to \omega$ is arbitrary, we have $a_f = \bigwedge_n a_{f(n)}^n = \mathbf{0}$ because otherwise there would have to exists some $d_n \leq a_f$, which is impossible.

§2. The problem of von Neumann. From 1935 until 1941, a group of mathematicians in the (then) Polish city of Lwów met frequently in the Scottish Coffee House, often with visitors, and recorded a number of problems in a large notebook started by Stefan Banach. After World War II a copy of the notebook found its way to the United States, where Stanisław Ulam, one of the original participants, published the collection of almost two hundred problems under the title 'The Scottish Book'. An annotated edition, edited by Daniel Mauldin, appeared as [28].

Problem no. 163, dated July 4, 1937, was entered by John von Neumann. It states the weak distributive law (1.5) and asks if ccc and weak distributivity are sufficient for a complete Boolean algebra to carry a measure. (It offers a prize: A bottle of whiskey of measure > 0.)

The rest of this article investigates von Neumann's question, whether every weakly distributive complete ccc Boolean algebra is a measure algebra. With this in mind, we henceforth consider only complete Boolean algebras that satisfy ccc.

§3. The work of Dorothy Maharam and sequential topology. In order to motivate the technique introduced by Maharam in [27], let B be a measure algebra with measure m. For any $a, b \in B$, let $d(a, b) = m(a \triangle b)$, where $a \triangle b$ is the *symmetric difference* $(a - b) \vee (b - a)$. Since

$$\begin{gathered} d(a,b) = d(b,a), \qquad (3.1) \\ d(a,a) = 0 \text{ and } d(a,b) > 0 \text{ if } a \neq b, \\ d(a,b) + d(b,c) \geq d(a,c), \end{gathered}$$

d is a distance function on B, and, as a consequence of σ-additivity of m, (B, d) is a complete metric space.

We remark that if B is the atomic algebra $\mathcal{P}(\omega)$, then under the identification of $\mathcal{P}(\omega)$ with 2^ω, (B, d) is homeomorphic to the Cantor space and so is a compact Hausdorff space.

The first observation of Maharam was that in order to prove ccc and weak distributivity one does not need a measure on B, but an ostensibly weaker property:

A function m on B is a *continuous submeasure* if (3.2)

(a) $m(\mathbf{0}) = 0,\ m(a) > 0$ for $a \neq \mathbf{0},$ and $m(\mathbf{1}) = 1,$

(b) $m(a) \leq m(b)$ if $a \leq b,$

(c) $m(a \vee b) \leq m(a) + m(b),$

(d) $\lim_n m(a_n) = 0$ for every decreasing sequence a_n with $\bigwedge_n a_n = 0.$

We call a continuous submeasure a *Maharam submeasure*, and complete Boolean algebra B a *Maharam algebra* if it carries a Maharam submeasure. Every measure is Maharam submeasure.

PROPOSITION 3.1. *A Maharam algebra satisfies ccc and is weakly distributive.*

PROOF. For ccc, we claim that for every $\varepsilon > 0$, there exist only finitely many disjoint elements a such that $m(a) \geq \varepsilon$. If there existed an infinite antichain $\{a_n\}$ such that $m(a_n) \geq \varepsilon$ for each n, then letting $b_n = \bigvee_{k \geq n} a_k$, we would get a descending sequence violating the continuity of m.

As for weak distributivity, the proof is the same as for a measure. ⊣

The problem of von Neumann splits naturally into the following two problems:

PROBLEM 1. *Is every Maharam algebra a measure algebra*?

PROBLEM 2. *Is every weakly distributive complete ccc Boolean algebra a Maharam algebra*?

The first problem has been studied in functional analysis and is known as the *Control Measure Problem*, see [21] or [10], vol. 3. We shall address it in the next section. The second problem, the *von Neumann–Maharam Problem*, is the main subject of this article, and we shall outline its solution.

Before we introduce Maharam's method we present another observation from her paper [27].

The ordering of the real line is the unique linear order (up to isomorphism) that is complete, dense, with no endpoints, and has a countable dense subset. As a consequence, it satisfies the countable chain condition (ccc), i.e., every disjoint collection of open intervals is at most countable. A problem of Mikhail Suslin [34] from 1920 asks whether every complete ccc dense linear order without endpoints is isomorphic to the real line. The problem remained unsolved until the 1960's when it was established that it is undecidable: it is both consistent with and independent of the axioms ZFC of set theory. See [17], [37], [33].

A *Suslin line* is a complete ccc dense linear order that does not have a countable dense subset (a counterexample to Suslin's problem). A *Suslin tree* is an ω_1-tree with no uncountable chains or antichains. A *Suslin algebra* is an atomless complete ccc Boolean algebra that satisfies the (ω,ω)-distributive law (i.e., the distributivity law (1.3) with $X = Y = \omega$).

A Suslin line, a Suslin tree and a Suslin algebra can be constructed from each other (see [25], [29] or [19] for details).

Maharam showed that a Suslin algebra does not carry a continuous submeasure. To see this, let B be a Suslin algebra and let m be a continuous submeasure on B. First we claim that for every $\varepsilon > 0$, the set $\{a \in B : m(a) < \varepsilon\}$ is dense in B. Otherwise, one could find a decreasing sequence a_n with $\bigwedge_n a_n = 0$ and $m(a_n) \geq \varepsilon$, contradicting the continuity of m.

Thus for each $n \in \omega$ there exists a maximal antichain A_n in B such that $m(a) < 1/n$ for all $a \in A_n$. By the (ω,ω)-distributive law there exists some $b > \mathbf{0}$ such that for every n, $b \leq a$ for some $a \in A_n$. It follows that $m(b) < 1/n$ for every n, a contradiction.

Thus a Suslin algebra is counterexample to the von Neumann–Maharam Problem, and one has to modify the problem as follows

PROBLEM 3. *Is it consistent that every weakly distributive complete ccc Boolean algebra is a Maharam algebra*?

Now we shall introduce Maharam's method. Let B be a Boolean σ-algebra that carries a Maharam submeasure m. Then $d(x, y) = m(x \bigtriangleup y)$ is a metric on B, (B, d) is a complete metric space, and for each $a \in B$, the mapping $T^a(x) = a \bigtriangleup x$ is an isometry ($d(x \bigtriangleup a, y \bigtriangleup a) = d(x, y)$). As $(B, \bigtriangleup, \mathbf{0})$ is an abelian group, d is an invariant metric on this group. The metric topology on B is determined by neighborhoods of $\mathbf{0}$ and is invariant under the translations T^a. Moreover, the Boolean operations $\vee$, $\wedge$ and $\bigtriangleup$ are continuous and $(B, \bigtriangleup, \mathbf{0})$ is a topological group.

It turns out that this topology τ can be defined algebraically on any Boolean σ-algebra B, and the existence of a Maharam submeasure on B is related to properties of the topological space (B, τ).

Convergence and the sequential topology on B. Let B be a Boolean σ-algebra. An infinite sequence $\{a_n\}_n$ *converges* to a, $\lim_n a_n = a$, if

$$\limsup_n a_n = \liminf_n a_n = a,$$

where

$$\limsup_n a_n = \bigwedge_n \bigvee_{k \geq n} a_k, \quad \liminf_n a_n = \bigvee_n \bigwedge_{k \geq n} a_k.$$

Equivalently, we define $\lim_n a_n = \mathbf{0}$ whenever there exists a decreasing sequence b_n with $\bigwedge_n b_n = \mathbf{0}$ such that $a_n \leq b_n$ for all n. Then we let $\lim_n a_n = a$ if $\lim_n (a_n \bigtriangleup a) = \mathbf{0}$.

We summarize the basic properties of convergence:

(a) If $a_n = a$ for all n then $\lim_n a_n = a$. (3.3)

(b) If $\{a_n\}_n$ converges to a and π is a permutation of ω then $\{a_{\pi(n)}\}_n$ also converges to a.

(c) $\lim_n a_n = \mathbf{0}$ if and only if $\limsup_n a_n = \mathbf{0}$,

(d) if the a_n are pairwise disjoint then $\lim_n a_n = \mathbf{0}$,

(e) $\limsup_n (a_n \vee b_n) = \limsup_n a_n \vee \limsup_n b_n$,

(f) if $\lim_n a_n = a$ and $\lim_n b_n = b$ then $\lim_n -a_n = -a$, $\lim_n (a_n \vee b_n) = a \vee b$ and $\lim_n (a_n \wedge b_n) = a \wedge b$.

For details, see [27] or [41].

A set $F \subseteq B$ is *closed* if $\lim a_n \in F$ whenever $\{a_n\}_n$ is a sequence in F. Let τ denote the topology on B so obtained; it is the *sequential topology* on B. The space (B, τ) is T_1 (every singleton is closed). The closure $\mathrm{cl}(A)$ of a set $A \subseteq B$ is generally obtained by taking limits of convergent sequences and iterating this ω_1 times. Maharam pointed out that the iteration is not necessary if B is ccc and weakly distributive: in this case $\mathrm{cl}(A)$ is the set of all limits of sequences of A. We shall return to this in section 5.

The operations $\vee$, $\wedge$ and $\triangle$ are not necessarily continuous as functions of two variables, even though (by (3.3)) they are continuous separately in each variable. Since each T^a is a continuous translation, (B, τ) is a homogeneous space and τ is determined by neighborhoods of $\mathbf{0}$.

If B is a Maharam algebra and (B, d) is the metric space with the distance function $d(a, b) = m(a \triangle b)$, then the metric topology and the sequential topology τ coincide. Maharam showed that conversely, metrizability of τ is itself sufficient for the existence of a Maharam submeasure.

THEOREM 3.2 (Maharam [27]). *A complete Boolean algebra B is a Maharam algebra if and only if the sequential topology on B is metrizable.*

We shall outline a proof of the theorem. It has two main ingredients: continuity of Boolean operations and a metrization theorem of Kakutani. A topological space is called *first countable* if every point a has a countable system of open neighborhoods $\{U_n\}_n$ such that for every open neighborhood V of a there is some n with $U_n \subseteq V$.

THEOREM 3.3 (Kakutani [20], see also [15]). *If $(G, +, \mathbf{0})$ is a topological abelian group and its topology is first countable, then it is metrizable and has an invariant metric.*

PROOF OF THEOREM 3.2. Let τ be the sequential topology on B and assume that it is metrizable. Let d be a metric on B such that its topology coincides with τ. It is easy to see that if $\lim_n a_n = \mathbf{0}$ then $\lim_n d(a_n, \mathbf{0}) = 0$ and if $\lim_n d(a_n, \mathbf{0}) = 0$ then some subsequence of a_n converges to $\mathbf{0}$.

We claim that the function $\triangle$ is continuous. Since the translations T^a are continuous, it suffices to prove continuity at $(\mathbf{0}, \mathbf{0})$. If $\triangle$ is not continuous at $(\mathbf{0}, \mathbf{0})$, then there exist sequences x_n and y_n and some $\varepsilon > 0$ such that $\lim_n d(x_n, \mathbf{0}) = \lim_n d(y_n, \mathbf{0}) = 0$ but $d(x_n \triangle y_n, \mathbf{0}) \geq \varepsilon$ for all n. There exists a subsequence $n(1), n(2), \dots$ such that $\lim_k x_{n(k)} = \lim_k y_{n(k)} = \mathbf{0}$. It follows that $\lim_k (x_{n(k)} \vartriangle y_{n(k)}) = \mathbf{0}$, a contradiction.

Thus $(B, \vartriangle, \mathbf{0})$ is a topological abelian group, and so by Kakutani's theorem 3.3, τ is metrizable by an invariant metric ρ, i.e., $\rho(x, y) = \rho(x \vartriangle y, \mathbf{0})$.

If we define $\nu(x) = \rho(x, \mathbf{0})$ then the function ν satisfies

$$\nu(x \vartriangle y) \leq \nu(x) + \nu(y), \tag{3.4}$$

(by the triangle inequality for ρ) and so, if we let further

$$\mu(x) = \min(\nu(x), 1)$$

and

$$m(a) = \sup\{\mu(x) \colon x \leq a\},$$

we can verify that m is a continuous submeasure on B. ⊣

Investigating the sequential topology on B, Maharam was able to formulate a better sufficient condition for the existence of Maharam submeasure.

THEOREM 3.4 (Maharam [27]). *A complete ccc Boolean algebra B is a Maharam algebra if and only if*

(i) *B is weakly distributive, and*
(ii) *the space (B, τ) is first countable.*

Maharam proved Theorem 3.4 by using the assumptions to show that $\vartriangle$ is continuous, and then applied Kakutani's theorem. In Section 6 we shall introduce a condition weaker than first countability (the G_δ property) and prove its sufficiency.

A final result of Maharam was that the following additional requirement on the countable neighborhood base U_n produces not just a submeasure but a measure.

$$\text{if } x, y \notin U_n \text{ and } x \wedge y = \mathbf{0} \text{ then } x \vee y \notin U_{n+1}. \tag{3.5}$$

On this, no progress has been made to date.

§4. Further progress and the eventual solution of the von Neumann–Maharam problem. There has been a vast number of publications related to von Neumann's problem since 1947. We shall only mention the ones most relevant to our results.

In [16], Alfred Horn and Alfred Tarski investigated systematically measures on Boolean algebras, both σ-additive and finitely additive. They presented in detail the work of von Neumann on Boolean algebras and introduced the terminology that is (with some modifications) used today. Among others, they introduced the following two chain conditions:

σ-bounded cc: (4.1)

There is a decomposition $B^+ = \bigcup\{S_n \colon n \in \omega\}$ such that for every n, S_n contains no antichain of size $n + 2$.

σ-finite cc: (4.2)

There is a decomposition $B^+ = \bigcup\{S_n : n \in \omega\}$ such that for every n, S_n contains no infinite antichain.

If B carries a finitely additive measure m then it satisfies (4.1): let $S_n = \{a \in B : m(a) \geq \frac{1}{n+1}\}$. In particular, (4.1) is a necessary condition for B to be a measure algebra. Every Maharam algebra must satisfy the weaker condition (4.2), and it is still an open problem whether (4.1) is equivalent to (4.2). Clearly, this is related to the Control Measure Problem.

In [23], John Kelley investigated Boolean algebras that carry a finitely additive measure, as well as complete Boolean algebras with a σ-additive measure. He showed that these two properties are related, and gave an algebraic characterization of both. Theorem 4.1 had been previously known to A. G. Pinsker, see [22, pp. 428–430].

THEOREM 4.1 (Pinsker; Kelley [23]). *A complete Boolean algebra B carries a σ-additive measure if and only if*

(i) *B is weakly distributive, and*
(ii) *B carries a finitely additive measure.*

Kelley's main result is the following characterization of (ii):

For each finite sequence $s = \langle a_1, \dots a_n \rangle$ of not necessarily distinct elements of B^+, let $k(s)$ be the maximum size of a subset $E \subseteq \{1, \dots, n\}$ such that $\bigwedge_{j \in E} a_j \neq \mathbf{0}$, and let $i(s) = \frac{k(s)}{n}$. For a nonempty $X \subseteq B^+$, the *intersection number* of X is

$$\inf\{i(s) : s \text{ is a finite sequence in } X\}.$$

If m is a finitely additive measure on B and $S_n = \{a \in B : m(a) \geq \frac{1}{n+1}\}$ then one can verify that the intersection number of S_n is greater than or equal to $\frac{1}{n+1}$.

THEOREM 4.2 (Kelley [23]). *A necessary and sufficient condition for a Boolean algebra to carry a finitely additive measure is that there is a decomposition $B^+ = \bigcup_{n \in \omega} S_n$ such that each S_n has a positive intersection number.*

Consequently, a complete Boolean algebra B is a measure algebra if and only if B is weakly distributive and $B^+ = \bigcup_{n \in \omega} S_n$ such that each S_n has a positive intersection number.

The Control Measure Problem. Let U be a metrizable linear topological space and let B be a σ-algebra of sets. A function $\mu : B \to U$ is a *vector measure* if $\sum_0^\infty a_n = \lim_{n \to \infty} \sum_0^n a_n$ is defined in U and is equal to $\mu(\bigvee_n a_n)$ for every disjoint sequence $\{a_n\}_n$. A σ-additive measure m on B is a *control measure* for μ if $\mu(a) = \mathbf{0}$ if and only if $m(a) = 0$ (see [10]).

The Control Measure Problem is equivalent to the question whether every vector measure has a control measure.

In [21], Nigel Kalton and James Roberts found a significant reformulation of the Control Measure Problem. A *submeasure* on a Boolean algebra is a function m that satisfies 3.2 (a), (b), (c) (without continuity (d)).

DEFINITION 4.3. *A submeasure m on B is* exhaustive *if* $\lim_n m(a_n) = 0$ *for every infinite antichain $A = \{a_n : n \in \omega\}$. It is* uniformly exhaustive *if for every $\varepsilon > 0$ there exists $n \in \omega$ such that there is no sequence of n disjoint elements $a_1, \dots, a_n \in B$ with $m(a_i) \geq \varepsilon$ for all $i = 1, \dots, n$.*

Every Maharam submeasure is exhaustive (see the proof of Proposition 3.1) and every measure is uniformly exhaustive. The main result of [21] is the following. Two submeasures m and μ are *equivalent* if $m(a_n) \to 0$ if and only if $\mu(a_n) \to 0$.

THEOREM 4.4 (Kalton–Roberts [21]). *Every uniformly exhaustive submeasure on a Boolean algebra is equivalent to a finitely additive measure.*

COROLLARY 4.5. *The control measure problem is equivalent to the statement*:

Every exhaustive submeasure on a Boolean algebra is uniformly exhaustive. (4.3)

PROOF OF 4.5 FROM 4.4. First we assume that every Maharam algebra is a measure algebra, and prove (4.3). Let m be an exhaustive submeasure on a Boolean algebra B. B can be embedded to a complete Boolean algebra C so that m extends to a Maharam submeasure μ in C (see [9]; C is the metric completion of B). Hence C is a Maharam algebra, and by the assumption it has a measure λ. Because μ and λ are equivalent (cf. [10]), μ is uniformly exhaustive. So is its restriction to B, and hence m is uniformly exhaustive.

In the other direction, assume that (4.3) holds. If B is a Maharam algebra with Maharam submeasure m, then m is exhaustive, and therefore uniformly exhaustive by (4.3). By the Kalton–Roberts Theorem, B carries a finitely additive measure, and by Kelley's Theorem 4.1, B is a measure algebra. ⊣

It can be seen that if (4.3) holds for all countable Boolean algebras then it holds for all Boolean algebras. The statement (4.3) is therefore equivalent to a Π^1_2 statement and as such is absolute for models of set theory, by the Shoenfield Absoluteness Theorem (see [19]). Therefore, the Control Measure Problem is absolute, and we are justified to call this part of the von Neumann Problem the combinatorial part.

At the time this article goes to press (January 2006) Michel Talagrand announced the solution of the Control Measure Problem. In [36] he constructs a submeasure on the Cantor algebra that is exhaustive but not uniformly exhaustive (and therefore not equivalent to a measure). Thus Problem 1 has a negative answer: there exists a Maharam algebra that is not a measure algebra.

Back to Maharam submeasures. The line of reasoning started by Maharam was continued in [3], leading to the following improvement of Maharam's Theorem 3.2:

THEOREM 4.6 (Balcar, Główczyński, Jech [3]). *A complete ccc Boolean algebra B is a Maharam algebra if and only if the space (B, τ) is a Hausdorff space.*

In December 2003 (see [4]), we obtained the following result. The space (B, τ) has the *G_δ property* if $\{\mathbf{0}\}$ is a G_δ set, i.e., if there exist open neighborhoods U_n of $\mathbf{0}$ such that $\bigcap_{n\in\omega} U_n = \{\mathbf{0}\}$.

THEOREM 4.7 (Balcar, Jech, Pazák [5]). *A complete ccc Boolean algebra is a Maharam algebra if and only if*

(i) *B is weakly distributive and*
(ii) *(B, τ) has the G_δ property.*

The G_δ property is weaker than first countability, and so Theorem 4.7 implies Maharam's theorem 3.4. We remark that weak distributivity follows from first countability of τ, but is necessary in Theorem 4.7, as the Cohen algebra also has the G_δ property and is not weakly distributive.

Theorem 4.7, combined with a consistency result of Stevo Todorcevic [38], answers the von Neumann–Maharam question (Problem 3):

THEOREM 4.8 (Balcar, Jech, Pazák [5]). *It is consistent that every weakly distributive complete ccc Boolean algebra is a Maharam algebra.*

(Following [4], a manuscript of [40] containing an identical result appeared in February 2004 with no reference to [4]; its author's claim that he presented the result in a seminar at CRM cannot be independently verified.)

In June 2004 Stevo Todorcevic improved Theorem 4.7 as follows.

THEOREM 4.9 (Todorcevic [39]). *A complete Boolean algebra B is a Maharam algebra if and only if*

(i) *B is weakly distributive, and*
(ii) *B satisfies the σ-finite chain condition.*

We shall prove Theorems 4.6–4.9 in sections 6–8.

§5. Equivalents of weak distributivity. In section 6 we shall explore the sequential topology on weakly distributive complete ccc Boolean algebras, with the goal of describing under what additional conditions such algebras carry a Maharam submeasure. We shall obtain various equivalent characterizations of Maharam algebras. In the present section we give several equivalent descriptions of weak distributivity.

For $X, Y \subseteq B$ and $a \in B$, we let $X \vee Y = \{x \vee y : x \in X,\ y \in Y\}$, $a \vee Y = \{a \vee y : y \in Y\}$, $X \bigtriangleup Y = \{x \bigtriangleup y : x \in X,\ y \in Y\}$, $a \bigtriangleup Y = \{a \bigtriangleup y : y \in Y\}$.

THEOREM 5.1. *Let B be a complete ccc Boolean algebra. Each of the following conditions is equivalent to the weak distributivity of B.*

(i) *Let $a_0^n \le a_1^n \le \cdots$ for $n = 0, 1, 2, \ldots$ such that $\bigvee_k a_k^n = \mathbf{1}$. Then*

$$\bigvee_{f\colon \omega\to\omega} \bigwedge_n a^n_{f(n)} = \mathbf{1}.$$

(ii) *Let* $a_0^n \le a_1^n \le \cdots$ *for* $n = 0, 1, 2\dots$ *such that* $\bigvee_k a_k^n = \mathbf{1}$. *Then there exist functions* $f_k\colon \omega \to \omega$, $k = 0, 1, 2, \dots$ *such that*

$$\bigvee_k \bigwedge_n a^n_{f_k(n)} = \mathbf{1}.$$

(iii) *Let* $a_0^n \le a_1^n \le \cdots$ *for* $n = 0, 1, 2, \dots$ *such that* $\bigvee_k a_k^n = \mathbf{1}$. *Then there exists a function* $f\colon \omega \to \omega$ *such that*

$$\lim_n a^n_{f(n)} = \mathbf{1}.$$

PROOF. Property (i) modifies definition (1.5) by assuming that $\bigvee_k a_k^n = \mathbf{1}$ for all n, and is easily seen to be equivalent to the definition of weak distributivity.

That (ii) is equivalent to (i) follows from a general property of algebras that satisfy ccc: for every X there is a countable subset $Y \subseteq X$ such that $\bigvee_{x\in X} a_x = \bigvee_{x\in Y} a_x$.

To see how (iii) follows from (ii), let f_k be as in (ii) and let $f(n) = \max\{f_0(n), \dots, f_n(n)\}$. Then for each k, $\bigwedge_{n\ge k} a^n_{f(n)} \ge \bigwedge_{n\in\omega} a^n_{f_k(n)}$, and so

$$\liminf a^n_{f(n)} = \mathbf{1}.$$

Finally, (iii) implies (ii) by taking $\bigvee$ over all finite modifications of f. ⊣

Increasing sequences converging to $\mathbf{1}$ correspond to maximal antichains: if $\mathbf{0} = a_0 \le a_1 \le a_2 \le \dots$ with $\bigvee_n a_n = \mathbf{1}$, then $\{a_{n+1} - a_n\colon n \in \omega\}$ is a maximal antichain, and if $\{a_n\colon n \in \omega\}$ is a maximal antichain, then $a_0, a_0 \vee a_1, a_0 \vee a_1 \vee a_2, \dots$ is an increasing sequence with supremum $\mathbf{1}$. Thus we can formulate the weak distributivity in terms of maximal antichains.

THEOREM 5.2. *Let* B *be a complete ccc Boolean algebra. Each of the following conditions is equivalent to the weak distributivity of* B:

(i) *Let* $A_n = \{a_k^n\colon k \in \omega\}$, $n \in \omega$, *be maximal antichains. Then*

$$\bigvee_{f\colon\omega\to\omega} \bigwedge_n \bigvee_{k\le f(n)} a_k^n = \mathbf{1}.$$

(ii) *If* $A_0, A_1, A_2, \dots$ *are maximal antichains then there exists a dense set* D *such that each* $d \in D$ *meets only finitely many elements of each* A_n.

(iii) *If* $A_0, A_1, A_2, \dots$ *are maximal antichains then each* A_n *has a finite subset* E_n *such that*

$$\lim_n \bigvee E_n = \mathbf{1}.$$

Property (i) is a reformulation of (i) in Theorem 5.1, and (ii) is clearly equivalent to (i). Property (iii) is a reformulation of Theorem 5.1 (iii).

Diagonal property. Property 5.1 (iii) can be further reformulated to give the following equivalents.

THEOREM 5.3. *Let B be a complete ccc Boolean algebra. Either of the following conditions is equivalent to the weak distributivity of B:*

(i) *If* $\lim_k a^n_k = \mathbf{0}$ *for every n, then there exists an increasing function $f : \omega \to \omega$ such that* $\lim_n a^n_{f(n)} = \mathbf{0}$.

(ii) *If* $\lim_n a_n = a$ *and for each n,* $\lim_k a^n_k = a_n$*, then there exists an increasing function f such that* $\lim_n a^n_{f(n)} = a$.

Property (i) is called the *diagonal property*. It is equivalent to 5.1 (iii). This is immediate for decreasing sequences a^n_k. In general, we use the fact that each sequence converging to $\mathbf{0}$ can be majorized by a decreasing sequence converging to $\mathbf{0}$. Property (ii) follows from (i) by using symmetric differences.

An immediate consequence of 5.3 (ii) is that the closure of a set $X \subseteq B$ in the sequential topology is the set of all limits of convergent sequences in X.

The convergence ideal. Let X be an infinite countable subset of B^+. If $\lim_n a_n = \mathbf{0}$ for some enumeration $\{a_n\}_{n\in\omega}$ of X, then $\lim_n a_n = \mathbf{0}$ for every enumeration of X (see (3.3)(b)). Thus we can write $\lim X = \mathbf{0}$ without ambiguity and talk about sets converging to $\mathbf{0}$.

DEFINITION 5.4. *The* convergence ideal $\mathcal{I}$ *is the collection of all countable sets $X \subseteq B$ that converge to* $\mathbf{0}$.

$\mathcal{I}$ is an ideal on $[B^+]^\omega$, i.e., a collection of countable subsets of B^+ closed under unions and subsets. $\mathcal{I}$ contains all infinite antichains, by (3.3)(d).

The convergence ideal was first considered (for a Suslin algebra B) by Abraham and Todorcevic in [1], and introduced in general in [2] by Balcar, Franěk and Hruška, and in [31] by Quickert.

DEFINITION 5.5. *An ideal I on some $[S]^\omega$ is a* P-ideal *if for every sequence X_n of members of I there is an $X \in I$ such that $X_n - X$ is finite for all n.*

In [1], the authors show that in the special case, the convergence ideal is a P-ideal. In [31], Quickert shows that $\mathcal{I}$ is a P-ideal for every weakly distributive complete ccc Boolean algebra. It turns out that this property is another equivalent of weak distributivity:

THEOREM 5.6 (Quickert). *Let B be a complete ccc Boolean algebra. B is weakly distributive if and only if the convergence ideal $\mathcal{I}$ on B is a P-ideal.*

PROOF. It is easy to see that if $\mathcal{I}$ is a P-ideal then B has the diagonal property. We prove that if B has the diagonal property then $\mathcal{I}$ is a P-ideal. Let X_n, $n \in \omega$, be sets in $\mathcal{I}$, and let $X_n = \{x^n_k\}_{k\in\omega}$ for each n. For each n, let $\{y^n_k\}_{k\in\omega}$ be the sequence defined by $y^n_k = x^0_k \vee x^1_k \vee \cdots \vee x^n_k$. As the sequences $\{y^n_k\}_k$ all converge to $\mathbf{0}$ (by (3.3) e), there exists an increasing function f such that $\lim_n y^n_{f(n)} = \mathbf{0}$, by the diagonal property.

Let $X = \{x^n_k : n \in \omega \text{ and } k \geq f(n)\}$. Clearly, $X_n - X$ is finite for each n, and we claim that $\lim X = \mathbf{0}$. For each n, let E_n be the finite set

$\{x^i_k : i \leq n \text{ and } f(i) \leq k \leq f(n)\}$. Since $\bigvee(X - E_n) = \bigvee_{m \geq n} y^m_{f(m)}$, we have $\limsup X = \bigwedge_{n\in\omega} \bigvee(X - E_n) = \limsup_n y^n_{f(n)} = \mathbf{0}$. ⊣

A Baire Category Theorem. The next equivalent of weak distributivity is reminiscent of the Baire Category Theorem.

Definition 5.7. *A set $X \subseteq B$ is* downward closed *if $a \leq b \in X$ implies $a \in X$. For $a \neq \mathbf{0}$, $B \restriction a$ is the algebra $\{x \in B : x \leq a\}$.*

Note that $B \restriction a$ is a closed subspace of (B, τ).

Theorem 5.8. *Let B be a complete ccc Boolean algebra. Either of following conditions is equivalent to the weak distributivity of B:*

(i) *If $U_0, U_1, U_2, \dots$ are downward closed and $\mathrm{cl}(U_n) = B$ for all n, then $\bigcap_n U_n$ is dense in B.*
(ii) *If $U_0, U_1, U_2, \dots$ are downward closed and $\bigcap_n U_n = \{\mathbf{0}\}$ then $\bigcap_n \mathrm{cl}(U_n) = \{\mathbf{0}\}$.*

Proof. First assume that B is weakly distributive and prove (ii). Let U_n, $n \in \omega$, be as in (ii). Toward a contradiction, assume that some $a \neq \mathbf{0}$ is in $\bigcap_n \mathrm{cl}(U_n)$. For each n there exists a sequence $\{a^n_k\}_{k\in\omega}$ in U_n with a limit a. Using the diagonal property we obtain a sequence $\{b_n\}_{n\in\omega}$ such that $b_n \in U_n$ and $\lim_n b_n = a$. Since $\liminf b_n \neq \mathbf{0}$ there exists some $b \neq \mathbf{0}$ such that for eventually all n, $b_n \geq b$. Now if n is such that $b \notin U_n$, then $b_n \notin U_n$ because U_n is downward closed; a contradiction.

Next we prove that (ii) implies (i). Let U_n be downward closed with $\mathrm{cl}(U_n) = B$ and assume that $\bigcap_n U_n$ is not dense in B. Let $a > \mathbf{0}$ be such that $\bigcap_n U_n \cap B \restriction a = \{\mathbf{0}\}$.

For each n, let $V_n = \{a \wedge x : x \in U_n\}$. As the U_n are downward closed, we have $V_n = U_n \cap B \restriction a$, the V_n are downward closed, and $\bigcap_n V_n = \{\mathbf{0}\}$. By (ii), $\bigcap_n \mathrm{cl}(V_n) = \{\mathbf{0}\}$. However, for each n we have $a \in \mathrm{cl}(U_n)$ and so (by definition of closure), $a = a \wedge a \in a \wedge \mathrm{cl}(U_n) = \mathrm{cl}(a \wedge U_n) = \mathrm{cl}(V_n)$.

Finally, let us assume (i) and show that B is weakly distributive. Let A_n, $n \in \omega$, be maximal antichains. For each n, let U_n be the set of all x that meet only finitely many elements of A_n. U_n is downward closed and $\mathrm{cl}(U_n) = B$. Thus $\bigcap_n U_n$ is dense in B, proving 5.2 (ii). ⊣

Bounding forcing. It is a well known fact in the theory of forcing that weakly distributive complete Boolean algebras yield generic models that have a bounding property.

A function $f : \omega \to \omega$ is *bounded* by $g : \omega \to \omega$ if $f(n) < g(n)$ for all n; f is *eventually bounded* by g, $f < g$, if for some n, $f(k) < g(k)$ for all $k \geq n$.

Theorem 5.9. *Let B be a complete ccc Boolean algebra. Either of the following properties is equivalent to the weak distributivity of B:*

(i) *In $V[G]$, every $f : \omega \to \omega$ is bounded by a function in V.*
(ii) *If $\dot{f}$ is a B-name for a function from ω to ω then there exists a function g such that $\|\dot{f} < g\| = \mathbf{1}$.* ⊣

Property (ii) is an analog of 5.2 (iii): A name $\dot{f}$ corresponds to a sequence of maximal antichains $\{\|\dot{f}(n) = k\| : k \in \omega\}$, and

$$\|\dot{f} < g\| = \mathbf{1} \quad \text{if and only if} \quad \lim_n \|\dot{f}(n) < g(n)\| = \mathbf{1}.$$

§6. **The Decomposition Theorem.** Our work on sequential topology in [3] and [5] led to the following theorem that we shall use in the next section to analyze Maharam algebras.

THEOREM 6.1 (Balcar, Jech, Pazák [5]). *Let B be a complete ccc Boolean algebra. There exist disjoint elements m and d such that $m \vee d = \mathbf{1}$, and*

(i) *the algebra $B \restriction m$ carries a Maharam submeasure;*
(ii) *in $B \restriction d$ every nonempty open set is topologically dense.*

In the theorem, either m or d can be $\mathbf{0}$. For $a \neq \mathbf{0}$, $B \restriction a$ is the algebra $\{x \in B : x \leq a\}$ and $B \restriction \mathbf{0} = \{\mathbf{0}\}$. We remark that for the Cohen algebra, $d = \mathbf{1}$.

Property (ii) means that the algebra is very non-Hausdorff: any two nonempty open set have nonempty intersection. As a consequence, if (B, τ) is a Hausdorff space then B is a Maharam algebra (Theorem 4.6). We shall now prove the following theorem, which implies Theorems 4.6, 4.7 and 3.4.

THEOREM 6.2. *Let B be a complete ccc Boolean algebra.*

(a) *If (B, τ) is a Hausdorff space, then*
  (i) *B is a weakly distributive, and*
  (ii) *(B, τ) has the G_δ property.*
(b) *If B is weakly distributive and (B, τ) has the G_δ property, then*
  (i) *$\vee$ is continuous, and*
  (ii) *(B, τ) is a first countable space.*
(c) (Maharam) *If $\vee$ is continuous and (B, τ) is first countable then B is a Maharam algebra.*

Toward the proof of Theorem 6.2, we start with the following observation:

LEMMA 6.3. *Let B be a complete Boolean algebra. If (B, τ) is a Hausdorff space, then B is weakly distributive.*

PROOF. Let $a_0^n \leq a_1^n \leq \cdots$ for $n = 0, 1, 2, \ldots$ such that $\bigvee_k a_k^n = \mathbf{1}$. We assume that B is Hausdorff and show that for each $a \neq \mathbf{0}$ there exists function $f : \omega \to \omega$ such that $a \wedge \bigwedge_n a_{f(n)}^n \neq \mathbf{0}$.

Let $a \neq \mathbf{0}$. There exists an open neighborhood U of a such that $\mathbf{0} \notin \mathrm{cl}(U)$. Since $\lim_k (a \wedge a_k^0) = a$, there exists some $k = f(0)$ such that $b_0 = a \wedge a_k^0 \in U$. Inductively, we assume that $b_n = a \wedge a_{f(0)}^0 \wedge \cdots \wedge a_{f(n)}^n \in U$ and since $\lim_k (b_n \wedge a_k^{n+1}) = b_n$, we find a $k = f(n+1)$ such that $b_{n+1} = b_n \wedge a_k^{n+1} \in U$. We have $a \wedge \bigwedge_n a_{f(n)}^n = \lim_n b_n \neq \mathbf{0}$, because $\mathbf{0} \notin \mathrm{cl}(U)$. ⊣

We say that B is *nowhere weakly distributive* if $B \restriction a$ is not weakly distributive for each $a \neq \mathbf{0}$. The same argument as in Lemma 6.3 proves that if

B is nowhere weakly distributive, then $\mathbf{0} \in \mathrm{cl}(U)$ for every nonempty open set U. Then $a \in \mathrm{cl}(U)$ for every $a \in B$ (because $a \in \mathrm{cl}(U)$ if and only if $\mathbf{0} \in \mathrm{cl}(U \vartriangle a)$), and we have the following:

COROLLARY 6.4. *If B is a nowhere weakly distributive complete ccc Boolean algebra then* $\mathrm{cl}(U) = B$ *for every nonempty open set* U. $\dashv$

The next lemma summarizes additional properties of the sequential topology under the weak distributivity. Note that if A is downward closed then $A \vee A = A \vartriangle A$.

LEMMA 6.5. *Let B be a weakly distributive complete ccc Boolean algebra.*

(a) *For every $A \subseteq B$, the closure of A is the set of all limits of convergent sequences in A.*

(b) *For every open neighborhood V of $\mathbf{0}$ there exists a nonempty open $U \subseteq V$ that is downward closed. Hence*

$$\mathcal{N} = \{U : U \text{ is nonempty, open and downward closed set}\}$$

is a neighborhood base of $\mathbf{0}$.

(c) *If $U \in \mathcal{N}$ then* $\mathrm{cl}(U) = \bigcap\{U \vee V : V \in \mathcal{N}\} \subseteq U \vee U$ *and* $\mathrm{cl}(U)$ *is downward closed.*

(d) *If $U \in \mathcal{N}$ then* $\mathrm{cl}(U) \subseteq U \vee U$ *and $U \vee U$ is open and downward closed.*

PROOF. (a) follows from 5.3 (ii)

(b) Let $U = B - \mathrm{cl}(A)$ where $A = \{a : (\exists\, b \le a)\, b \notin V\}$. Using (a), one verifies that U is downward closed and $\mathbf{0} \in U$.

(c) It is easy to see that $\mathrm{cl}(U) \subseteq U \vartriangle V$ for every open neighborhood V of $\mathbf{0}$ and in fact $\mathrm{cl}(U) = \bigcap\{U \vartriangle V : V \in \mathcal{N}\}$. In particular $\mathrm{cl}(U) \subseteq U \vartriangle U$. Since U and V are downward closed, we have $U \vartriangle V = U \vee V$.

(d) Easy. $\dashv$

The main technical lemma is this:

LEMMA 6.6. [3] *Let B be a weakly distributive complete ccc Boolean algebra. For every $U \in \mathcal{N}$ there exists a $V \in \mathcal{N}$ such that $V \vee V \vee V \subseteq U \vee U$.*

PROOF. Assume that $U \in \mathcal{N}$ is such that the statement fails. We construct sequences V_n, x_n, y_n, z_n as follows: Let $V_0 = U$. For each n, let $x_n, y_n, z_n \in V_n$ be such that $x_n \vee y_n \vee z_n \notin U \vee U$, and let $V_{n+1} \subseteq V_n$ be in $\mathcal{N}$ such that $x_n \vee V_{n+1} \subseteq V_n$, $y_n \vee V_{n+1} \subseteq V_n$ and $z_n \vee V_{n+1} \subseteq V_n$; V_{n+1} exists by the one-sided continuity of $\vee$.

Let $X = \bigcap_n \mathrm{cl}(V_n)$ and $x = \limsup x_n$, $y = \limsup y_n$, $z = \limsup z_n$. X is closed, downward closed and $X \subseteq \mathrm{cl}(U) \subseteq U \vee U$. For each n and each k we have $x_n \vee x_{n+1} \vee \cdots \vee x_{n+k} \in V_n$, hence $\bigvee_{i \ge n} x_i \in \mathrm{cl}(V_n)$ and therefore $x \in \mathrm{cl}(V_n)$. It follows that $x \in X$, and similarly $y \in X$ and $z \in X$. A similar argument shows that for each n and each k, $x_n \vee \cdots \vee x_{n+k} \vee X \subseteq \mathrm{cl}(V_n)$, and so $\bigvee_{i \ge n} x_i \vee X \subseteq \mathrm{cl}(V_n)$. As $\mathrm{cl}(V_n)$ is downward closed and $x \le \bigvee_{i \ge n} x_i$, we have $x \vee X \subseteq \mathrm{cl}(V_n)$. It follows that $x \vee X \subseteq X$, and similarly $y \vee X \subseteq X$, $z \vee X \subseteq X$.

Hence $x \vee y \vee z$ is in X and hence in $U \vee U$. But $x \vee y \vee z = \limsup(x_n \vee y_n \vee z_n)$. For each n, $\bigvee_{i \geq n}(x_i \vee y_i \vee z_i) \notin U \vee U$ because $U \vee U$ is downward closed, and since $U \vee U$ is open, we have $x \vee y \vee z = \limsup(x_n \vee y_n \vee z_n) \notin U \vee U$. A contradiction. ⊣

Proof of Theorem 6.2 (a). We first show that for every $b \neq \mathbf{0}$ there is a nonzero $c < b$ and a sequence $\{V_n\}_n$ in $\mathcal{N}$ such that $c \wedge \bigvee(\bigcap_n V_n) = \mathbf{0}$. As the space is Hausdorff, there exists a $V_0 \in \mathcal{N}$ such that $b \notin V_0 \vee V_0$. For each n let $V_{n+1} \in \mathcal{N}$ be such that $V_{n+1} \subseteq V_n$ and $V_{n+1} \vee V_{n+1} \vee V_{n+1} \subseteq V_n \vee V_n$. Let $A = \bigcap_n V_n$ and $a = \bigvee A$.

For each n we have $V_2 \vee V_2 \supset V_{n+2} \vee \cdots \vee V_{n+2}$ ($n+2$ times) and hence $V_2 \vee V_2 \supset A \vee \cdots \vee A$ (n times). As $a = \lim_n(a_1 \vee \cdots \vee a_n)$ for some sequence $\{a_n\}_n$ in A, we have $a \in \mathrm{cl}(V_2 \vee V_2)$. Therefore $a \in V_2 \vee V_2 \vee V_2 \vee V_2 \subseteq V_0 \vee V_0$, and since $V_0 \vee V_0$ is downward closed, $b \nleq a$. Thus $c \wedge \bigvee A = \mathbf{0}$, where $c = b - a$.

Now let C be a maximal antichain such that for each $c \in C$ there exists a sequence $\{V_n\}_n$ in $\mathcal{N}$ with $c \wedge \bigvee(\bigcap_n V_n) = \mathbf{0}$.

Then $\{V_n^c : c \in C,\ n \in \omega\}$ is a countable set of open neighborhoods of $\mathbf{0}$, and $\bigcap_c \bigcap_n V_n^c = \{\mathbf{0}\}$. ⊣

Proof of Theorem 6.2 (b). Let $\{U_n\}_n$ be a sequence of open neighborhoods of $\mathbf{0}$ such that $\bigcap_n U_n = \{\mathbf{0}\}$. Since B is weakly distributive, we find such U_n that are downward closed, and by Theorem 5.8 (ii), we have $\bigcap_n \mathrm{cl}(U_n) = \{\mathbf{0}\}$. We shall show first that the operation $\vee$ is continuous.

Thus assume that $\vee$ is not continuous at $\mathbf{0}$. There exists a $U \in \mathcal{N}$ such that for every $V \in \mathcal{N}$ there exists $x, y \in V$ with $x \vee y \notin U$.

We construct sequences V_n, x_n, y_n as follows: Let $V_0 = U_0 \cap U$. For each n, let $x_n, y_n \in V_n$ such that $x_n \vee y_n \notin U$. By one-sided continuity of $\vee$ there exists a $V_{n+1} \subseteq U_{n+1}$ such that $x_n \vee V_{n+1} \subseteq V_n$ and $y_n \vee V_{n+1} \subseteq V_n$. Let $x = \limsup x_n$ and $y = \limsup y_n$.

For each n and each k we have $x_n \vee x_{n+1} \vee \cdots \vee x_{n+k} \in V_n$, because $x_i \vee V_{i+1} \subseteq V_i$ for all i. Hence $\bigvee_{i \geq n} x_i \in \mathrm{cl}(V_n)$, and therefore $x \in \mathrm{cl}(V_n)$. It follows that $x = \mathbf{0}$, and similarly, $y = \mathbf{0}$, hence $x \vee y = \mathbf{0}$.

But $x \vee y = \limsup x_n \vee \limsup y_n = \limsup(x_n \vee y_n)$. Since U is downward closed and $x_n \vee y_n \notin U$, we have $\bigvee_{k \geq n}(x_n \vee y_n) \notin U$, and because U is open, we get $\mathbf{0} = x \vee y \notin U$. A contradiction.

Now, we prove that (B, τ) is first countable. By the continuity of $\vee$ and by the G_δ property there exist $U_n \in \mathcal{N}$ such that $\bigcap_n \mathrm{cl}(U_n) = \{\mathbf{0}\}$ and $U_{n+1} \vee U_{n+1} \subseteq U_n$ for every n. We claim that $\{U_n\}_{n \in \omega}$ is a neighborhood base. Assume not. Then there exists a $V \in \mathcal{N}$ such that for every n, $U_n \nsubseteq V$. For each n let x_n be such that $x_n \in U_n - V$.

It follows by induction on k that for each n and each k, $x_{n+1} \vee x_{n+2} \vee \cdots \vee x_{n+k} \in U_n$. Thus $\bigvee_k x_{n+k} \in \mathrm{cl}(U_n)$ and it follows that $\limsup x_n \in \mathrm{cl}(U_m)$ for each m; hence $\lim x_n = \mathbf{0}$. This is a contradiction because V is an open neighborhood of $\mathbf{0}$. ⊣

Proof of Theorem 6.2 (c). $(B, \triangle, \mathbf{0})$ is a topological group and a first countable space. By Kakutani's Theorem, $(B, \triangle, \mathbf{0})$ has an invariant metric, and so B is Maharam, as in the proof of Theorem 3.2. ⊣

Remark. In [3] we proved that continuity of $\vee$ is itself sufficient for B to be a Maharam algebra. The condition can be stated as follows: For every $U \in \mathcal{N}$ there exists a $V \in \mathcal{N}$ such that $V \vee V \subseteq U$. Compare this with the condition in Lemma 6.6 which holds for every weakly distributive B.

Proof of theorem 6.1. In view of Corollary 6.4 it suffices to prove the theorem under the assumption that B is weakly distributive; the proof of the general case combines the proof below with the proof of Corollary 6.4.

Thus assume that B is weakly distributive. Let $D = \bigcap\{\mathrm{cl}(U) : U \in \mathcal{N}\}$, $d = \bigvee D$ and $m = -d$. D is closed and downward closed, and $D = \{U \vee U : U \in \mathcal{N}\}$. Now if $a \notin D$, in particular if $a \leq m$, then $a \notin U \vee U$ for some $U \in \mathcal{N}$. Hence U and $a \triangle U$ are disjoint, and so $(B \restriction m, \tau)$ is a Hausdorff space. Hence $B \restriction m$ is a Maharam algebra.

We now show that in $B \restriction d$, every nonempty open set is topologically dense. We claim that D is closed under $\vee$. By Lemma 6.6, for every $U \in \mathcal{N}$ there is a $V \in \mathcal{N}$ such that $V \vee V \vee V \subseteq U \vee U$, and therefore there is a $V \in \mathcal{N}$ such that $V \vee V \vee V \vee V \subseteq U \vee U$. Hence $D = \bigcap\{U \vee U : U \in \mathcal{N}\} = \bigcap\{V \vee V \vee V \vee V : V \in \mathcal{N}\}$ and so $D \vee D = D$.

There is a sequence $\{a_n\}_n$ in D such that $d = \lim(a_0 \vee \cdots \vee a_n)$ and since $D \vee D = D$ and D is closed, we have $d \in D$. It follows that $B \restriction d = D$.

We finish the proof by showing that $\mathrm{cl}(G) \supset D$ for every nonempty open set G in $B \restriction d$. There exist an $a \in D$ and some $U \in \mathcal{N}$ such that $G \supset (a \triangle U) \cap D$. Since $\mathrm{cl}(U) \supset D$, we have $\mathrm{cl}(a \triangle U) \supset a \triangle D = D$ and so $\mathrm{cl}(G) \supset D$. ⊣

§7. **Maharam algebras.** In this section we present a number of additional necessary and sufficient conditions for a complete ccc Boolean algebra B to carry a Maharam submeasure. It turns out that some of the properties are natural generalizations of the conditions for weak distributivity presented in section 5.

By the results of sections 3 and 6, each of the following is equivalent to being Maharam.

(B, τ) is metrizable (Maharam [27]), (7.1)

(B, τ) is Hausdorff (Balcar, Główczyński, Jech [3]), (7.2)

B is weakly distributive and (B, τ) is first countable (Maharam [27]), (7.3)

B is weakly distributive and (B, τ) has the G_δ property (Balcar, Jech, Pazák [5]). (7.4)

We remark that the assumption of weak distributivity in (7.3) is not necessary (but is necessary in (7.4)). Todorcevic's paper [39] presents (7.3) and (7.4) from a different point of view.

Uniform weak distributivity and uniformly bounding forcing. The following two condition are uniform versions of the corresponding conditions 5.1(ii) and 5.9(ii) for weak distributivity.

THEOREM 7.1. *Let B be a complete ccc Boolean algebra. Either of the following is equivalent to B being Maharam*:

(i) *There exists a sequence of functions $\{F_n\}_n$ such that for each maximal antichain A, $F_n(A)$ is a finite subset of A, and if $A_0, A_1, A_2, \dots$ are maximal antichains then* $\lim_n \bigvee F_n(A_n) = \mathbf{1}$.

(ii) *There exists a sequence of functions $\{F_n\}_n$ such that for each B-name $\dot{a}$ for a natural number, $F_n(\dot{a})$ is a natural number, and if $\dot{f}$ is a B-name for a function from ω to ω, then, letting $g(n) = F_n(\dot{f}(n))$, we have* $\|\dot{f} < g\| = \mathbf{1}$.

PROOF. As (ii) is reformulation of (i) in terms of Boolean-valued models, let us consider (i). If m is Maharam submeasure, let $F_n(A)$ be a finite $E \subseteq A$ so that $m(\bigvee E) > 1 - \frac{1}{2^n}$.

Conversely, if B satisfies (i), let U_n be for each n, the set of all $\bigvee F_n(A)$, for all maximal antichains A. One verifies that $\mathbf{0}$ is an interior point of U_n, and that $\bigcap U_n = \{\mathbf{0}\}$. Hence B has the G_δ property and is weakly distributive since (i) implies 5.1(ii). ⊣

Strong diagonal property.

THEOREM 7.2. *Let B be a complete ccc Boolean algebra. B is a Maharam algebra if and only if there exists a family $\mathcal{S}$ of sequences converging to $\mathbf{0}$ such that every sequence with limit $\mathbf{0}$ has a subsequence in $\mathcal{S}$, and if $\{a^n_k\}_{k\in\omega}$, $n = 0, 1, 2, \dots$ are members of $\mathcal{S}$, then* $\lim_n a^n_n = \mathbf{0}$.

PROOF. If m is a Maharam submeasure, let $\mathcal{S}$ be the set of all sequences $\{a_n\}_n$ such that $m(a_n) < \frac{1}{2^n}$, for each n.

Conversely, if B has the strong diagonal property, let U_n be, for each n, the set of all x such that x is below the n^{th} term of some sequence $s \in \mathcal{S}$. One verifies that $\mathbf{0}$ is an interior point of each U_n and that $\bigcap U_n = \{\mathbf{0}\}$. Hence B has the G_δ property (and is weakly distributive because it has the diagonal property). ⊣

Hausdorff variations. We consider three related properties of the space (B, τ). The first one is a reformulation of Hausdorffness.

THEOREM 7.3. *Let B be a complete ccc Boolean algebra. Each of the following properties is equivalent to B carrying a Maharam submeasure.*

(i) *Every $a \neq \mathbf{0}$ has an open neighborhood G such that* $\mathbf{0} \notin \mathrm{cl}(G)$.

(ii) *Every $a \neq \mathbf{0}$ has an open neighborhood G such that $G \cap X$ is finite, for every X with limit* $\mathbf{0}$ (*i.e.*, $X \in \mathcal{I}$ *see Definition* 5.4).

(iii) *Every $a \neq \mathbf{0}$ has an open neighborhood G such that every antichain $A \subseteq G$ is finite.*

Proof. (i) states that (B, τ) is a Hausdorff space, and (i) implies (ii) implies (iii). It suffices to show that (iii) implies (i).

Thus assume that the space is not Hausdorff. By the Decomposition Theorem 6.1 there exists a $d \neq \mathbf{0}$ such that in $B \restriction d$, any two nonempty open sets intersect. We claim that every open neighborhood of d includes an infinite antichain.

Let G be any open neighborhood of d. Since $\{x \leq d : x \in G\}$ and $\{x \leq d : d - x \in G\}$ intersect, we find disjoint nonzero $a_1 \in G$ and $a_2 \in G$ below d. Continuing this, we produce an infinite antichain in G. ⊣

Fragmentations. The next three properties involve a decomposition of B^+ into countably many parts (*fragmentations*) and are related to the three preceding conditions. The first one is a reformulation of the G_δ property, while the third one is the σ-finite chain condition of Horn and Tarski (thus Theorem 7.4 (iii) proves Todorcevic's Theorem 4.9).

Theorem 7.4. *Let B be a complete ccc Boolean algebra. If B is weakly distributive then each of the following conditions is equivalent to B being a Maharam algebra*:

(i) $B^+ = \bigcup_n S_n$ *such that for each n, S_n is closed.*
(ii) $B^+ = \bigcup_n S_n$ *such that for each n, $S_n \cap X$ is finite for every $X \in \mathcal{I}$.*
(iii) $B^+ = \bigcup_n S_n$ *such that for each n, $S_n \cap A$ is finite for every antichain A.*

Proof. Every Maharam algebra has these properties (let $S_n = \{a : m(a) \geq \frac{1}{n}\}$) and clearly, (i) implies (ii) implies (iii). It suffices to show that (iii) implies 7.3(iii).

Let $\{S_n\}_n$ be as in (iii); we may assume that each S_n is upward closed. Letting $U_n = B - S_n$, we have $\bigcap_n U_n = \{\mathbf{0}\}$, and so $\bigcap_n \mathrm{cl}(U_n) = \{\mathbf{0}\}$ by 5.8(ii), because B is weakly distributive. It follows that every $a \neq \mathbf{0}$ is an interior point of some S_n, and 7.3(iii) follows. ⊣

Infinite games. Finally, we show that strategic versions of weak distributivity are equivalent to carrying a Maharam submeasure. We consider three games, one for each of the properties 5.2(iii), 5.3(i) and 5.9(ii) from section 5. The weak distributive game is Fremlin's modification of a game introduced by Jech in 1980 (cf. [18]) and by Charles Gray in his dissertation [14]. The diagonal game and the bounding game are our reformulations. In December 2004 Fremlin proved Theorem 7.5 below for the weak distributivity game, cf. [12].

Each of the following game is an infinite game of two players. Players I and II take turns to successively produce two infinite sequences of moves. We consider the properties of B stated in terms of winning strategies.

Weak distributivity game. I plays maximal antichains $A_0, A_1, A_2, \ldots$, and II plays finite subsets $E_n \subseteq A_n$. II wins if and only if $\lim_n \bigvee E_n = \mathbf{1}$.

Diagonal game. I plays sequences $\{a^n_k\}_{k\in\omega}$, $n = 0, 1, 2, \dots$ each converging to $\mathbf{0}$, and II plays integers $k(0), k(1), k(2), \dots$. II wins if and only if $\lim_n a^n_{k(n)} = \mathbf{0}$.

Bounding game. I plays B-names $\dot f(0), \dot f(1), \dot f(2), \dots$ for integers, and II plays integers $g(0), g(1), g(2), \dots$. II wins if and only if $\|\dot f < g\| = \mathbf{1}$.

These games correspond to properties in Theorems 5.2(iii), 5.3(i) and 5.9(ii), and for complete ccc algebras are mutually equivalent, i.e., I (respectively II) has a winning strategy in one game if and only if I (respectively II) has a winning strategy in either game. One can show that if B is a complete ccc algebra then B is weakly distributive if and only if player I does not have a winning strategy, cf. [18].

THEOREM 7.5 (Fremlin [12]). *Let B be a complete ccc Boolean algebra. B is a Maharam algebra if and only if Player II has a winning strategy in either of the three games.*

PROOF. Let us consider the diagonal game. If m is a Maharam submeasure, then II has the following winning strategy: at move n, II plays $k(n)$ so that $m(a^n_{k(n)}) < \frac{1}{2^n}$.

Conversely if II has a winning strategy, then B has the diagonal property and so is weakly distributive. We show that (B, τ) is a Hausdorff space. If not, then by Decomposition Theorem 6.1 there exists some $d \neq \mathbf{0}$ such that in $B \restriction d$, any two nonempty open sets intersect. Without loss of generality we assume that $d = \mathbf{1}$ (because II has a winning strategy in $B \restriction d$).

Let σ_0 be a winning strategy for player II in the diagonal game, and let σ_1 be a winning strategy for II in a related game where I plays sequences converging to $\mathbf{1}$ and II tries to construct a diagonal sequence with limit $\mathbf{1}$. Let X_0 and Y_0 be, respectively, the sets of all elements of B given by the first moves of player II using σ_0 (respectively using σ_1). More precisely, $X_0 = \{x_{k(0)} : k(0) = \sigma_0(\{x_i\}_{i\in\omega})$ where $\lim_i x_i = \mathbf{0}\}$ and $Y_0 = \{y_{k(0)} : k(0) = \sigma_1(\{y_i\}_{i\in\omega})$ and $\lim_i y_i = \mathbf{1}\}$. We claim that $\mathbf{0}$ is an interior point of X_0 (and $\mathbf{1}$ is an interior point of Y_0). If not, there is a sequence $\{a_n\}_n$ outside X_0 that converges to $\mathbf{0}$. But if $k(0) = \sigma_0(\{a_n\}_n)$, then $a_{k(0)} \in X_0$, a contradiction.

Hence $X_0 \cap Y_0$ is nonempty and let $a_0 \in X_0 \cap Y_0$. There exists sequences $\{x^0_n\}_n$ and $\{y^0_n\}_n$ such that $a_0 = x^0_{k(0)} = y^0_{l(0)}$, where $k(0) = \sigma_0(\{x^0_n\}_n)$ and $l(0) = \sigma_1(\{y^0_n\}_n)$. Let X_1 and Y_1 be, respectively, the set of all elements of B given by the second moves of player II, using σ_0 and σ_1 respectively, with the first move of I being $\{x^0_n\}_n$ and $\{y^0_n\}_n$, respectively. Again, $\mathbf{0}$ and $\mathbf{1}$ are, respectively, interior points of X_1 and Y_1; therefore $X_1 \cap Y_1 \neq \emptyset$ and so there exists sequences $\{x^1_n\}_n$ and $\{y^1_n\}_n$ such that $a_1 = x^1_{k(1)} = y^1_{l(1)}$, where $k(1) = \sigma_0(\{x^0_n\}_n, \{x^1_n\}_n)$ and $l(1) = \sigma_1(\{y^0_n\}_n, \{y^1_n\}_n)$. We continue in this fashion and construct a sequence $a_0, a_1, a_2, \dots$. Since both σ_0 and σ_1 are winning strategies, we have simultaneously $\lim_n a_n = \mathbf{0}$ and $\lim_n a_n = \mathbf{1}$, a contradiction. ⊣

§8. Consistency of the von Neumann–Maharam problem. We prove Theorem 4.8 and then discuss some examples. The proof uses a general consistency result of S. Todorcevic and one of the equivalences presented in section 7.

P-IDEAL DICHOTOMY (Todorcevic). *Let S be an infinite set. Then for every P-ideal $I \subseteq [S]^{\omega}$, either*

(i) $\exists Y \subseteq S$ *uncountable such that* $[Y]^{\omega} \subseteq I$, *or*
(ii) $S = \bigcup_{n\in\omega} S_n$ *such that for each n, $S_n \cap X$ is finite, for every $X \in I$.*

THEOREM 8.1 (Todorcevic [38]). *The P-ideal dichotomy is consistent with ZFC.*

In [38] it is shown that the P-ideal dichotomy follows from the Proper Forcing Axiom (PFA), but is also consistent with GCH. For PFA and its consistency, we refer the reader to [32], [7] or [19].

Theorem 4.8 is now a consequence of the following.

THEOREM 8.2 (Balcar, Jech, Pazák [5]). *Assuming the P-ideal dichotomy, every weakly distributive complete ccc Boolean algebra carries a Maharam submeasure.*

PROOF. Let B be a weakly distributive complete ccc Boolean algebra, and let $\mathcal{I}$ be the convergence ideal for B. By Theorem 5.6, $\mathcal{I}$ is a P-ideal. Note that condition (ii) for $\mathcal{I}$ in the P-ideal Dichotomy is exactly the condition (ii) in Theorem 7.4, which implies that B is a Maharam algebra.

Thus it is enough to show that condition (i) in P-ideal dichotomy fails for $\mathcal{I}$. This is proved in the following lemma, completing the proof of Theorem 8.2.

LEMMA 8.3. *Let B be a complete ccc Boolean algebra. For every uncountable $Y \subseteq B^+$ there exists a countable $X \subseteq Y$ such that* $\limsup X > \mathbf{0}$.

PROOF. It suffices to prove this for Y of cardinality $\aleph_1$, so let $Y = \{a_\alpha : \alpha < \omega_1\}$. For each $\alpha < \omega_1$, let $b_\alpha = \bigvee_{\xi\geq\alpha} a_\xi$. Since the b_α are decreasing and B satisfies ccc, there exists an α_0 such that $b_\alpha = b_{\alpha_0}$ for all $\alpha \geq \alpha_0$. Let $b = b_{\alpha_0}$.

For each $\alpha \geq \alpha_0$ there exists an $f(\alpha) > \alpha$ such that $b = \bigvee_{\xi\geq\alpha} a_\xi = \bigvee_{\alpha\leq\xi<f(\alpha)} a_\xi$ (again using ccc). Let $\alpha_{n+1} = f(\alpha_n)$ for each n, and $\alpha_\omega = \lim_n \alpha_n$. Letting $X = \{a_\xi : \alpha_0 \leq \xi < \alpha_\omega\}$ we have $\limsup X = b$. ⊣

We shall conclude the article with a discussion on the independence of the von Neumann–Maharam problem. In Section 3 we presented one counterexample, the Suslin algebra. It is an ω-distributive complete ccc Boolean algebra that is not a Maharam algebra, and it is consistent that one exists. We present two other consistent examples of weakly distributive complete ccc algebras that are not Maharam.

THEOREM 8.4 (Główczyński [13]). *Let κ be a regular uncountable cardinal that carries a σ-saturated σ-ideal I containing all singletons and assume that*

$2^{\aleph_0} > \kappa$ and that Martin's Axiom holds. Then $\mathcal{P}(\kappa)/I$ is a weakly distributive complete ccc Boolean algebra and is not a Maharam algebra.

The assumptions in Theorem 8.4 are consistent: they can be obtained by forcing over a model with a measurable cardinal. Note also that this example shows that the assumption of PFA in Theorem 8.1 cannot be weakened to MA.

Let $B = \mathcal{P}(\kappa)/I$. B is an atomless complete ccc algebra; to show that it has the desired properties, we look first at the atomic algebra $\mathcal{P}(\kappa)$. It is well known that $\mathcal{P}(\kappa)$ has the diagonal property if $\kappa < \mathfrak{b}$, and $(\mathcal{P}(\kappa), \tau)$ is sequentially compact if $\kappa < \mathfrak{s}$ (cf. [3] for details as well definitions of the cardinal invariants $\mathfrak{b}$ and $\mathfrak{s}$). As a consequence of MA we have $\mathfrak{b} = \mathfrak{s} = 2^{\aleph_0} > \kappa$, and so $\mathcal{P}(\kappa)$ has the diagonal property and is sequentially compact.

It is easy to see that these two properties are preserved under quotients by a σ-ideal. Thus B has the diagonal property and (B, τ) is sequentially compact. Thus B is weakly distributive, and we claim that B is not a Maharam algebra.

Assume that B is a Maharam algebra. Then (B, τ) is metrizable, and because it is sequentialy compact, it is a compact metric space. By [5], B is isomorphic to $\mathcal{P}(\omega)$, a contradiction.

In his Problem List [11] D. Fremlin asked the following question:

'Suppose that every completely countably generated subalgebra of a given complete Boolean algebra B is measurable. Must B be measurable?'

A consequence of Theorem 8.2 is that the P-ideal dichotomy implies the affirmative answer to Fremlin's question:

Assume that every weakly distributive complete ccc Boolean algebra is Maharam, and let B be a complete Boolean algebra such that every complete subalgebra is a measure algebra. B is ccc and weakly distributive because every countably generated subalgebra is. Hence B is a Maharam algebra. Then B is a measure algebra, by the following observation of Fremlin:

LEMMA 8.5. *If B is a Maharam algebra such that every countably generated complete subalgebra is a measure algebra then B is a measure algebra.*

PROOF. Let m be a Maharam submeasure on B. For every countably generated subalgebra C, the restriction of m to C is an exhaustive submeasure, and since C is a measure algebra, m is uniformly exhaustive, by [21]. It follows that m is uniformly exhaustive, and by [21] again, B is a measure algebra. $\dashv$

The following recent example proves the consistency of the negative answer to Fremlin's question:

THEOREM 8.6 (Farah, Velickovic [8]). *Assume there is a cardinal λ such that $\lambda^{\aleph_0} = \lambda$, $2^\lambda = \lambda^+$ and $\square_\lambda$ holds. Then there is a complete Boolean algebra B of size λ^+ such that B is not a Maharam algebra but every subalgebra of size $\leq \lambda$ is a measure algebra.*

Under the given assumptions, one constructs B as the union of a chain $\{B_\alpha : \alpha < \lambda^+\}$ of measure algebras, each of size at most λ, in such a way that B is not a measure algebra. Since B is the union of such a chain, it satisfies ccc and is weakly distributive. By Lemma 8.5, B is not a Maharam algebra.

The assumptions of Theorem 8.6 hold unless there exists an inner model with a measurable cardinal κ of Mitchell order κ^{++}. It follows that the consistency of the von Neumann–Maharam Problem (Theorem 4.8) implies the consistency of a measurable cardinal κ of Mitchell order κ^{++}.

A final comment: the consistency of the von Neumann-Maharam problem for small algebras does not require large cardinals. This is because for complete Boolean algebras of cardinality at most $2^{\aleph_0}$ the P-ideal dichotomy is consistent with ZFC alone, cf. [1].

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