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CLASSIFICATION FROM A COMPUTABLE VIEWPOINT

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§1. Introduction. Classification is an important goal in many branches of mathematics. The idea is to describe the members of some class of mathematical objects, up to isomorphism or other important equivalence, in terms of relatively simple invariants. Where this is impossible, it is useful to have concrete results saying so. In model theory and descriptive set theory, there is a large body of work showing that certain classes of mathematical structures admit classification while others do not. In the present paper, we describe some recent work on classification in computable structure theory.

Section 1 gives some background from model theory and descriptive set theory. From model theory, we give sample structure and non-structure theorems for classes that include structures of arbitrary cardinality. We also describe the notion of Scott rank, which is useful in the more restricted setting of countable structures. From descriptive set theory, we describe the basic Polish space of structures for a fixed countable language with fixed countable universe. We give sample structure and non-structure theorems based on the complexity of the isomorphism relation, and on Borel embeddings.

Section 2 gives some background on computable structures. We describe three approaches to classification for these structures. The approaches are all equivalent. However, one approach, which involves calculating the complexity of the isomorphism relation, has turned out to be more productive than the others. Section 3 describes results on the isomorphism relation for a number of mathematically interesting classes—various kinds of groups and fields. In Section 4, we consider a setting similar to that in descriptive set theory. We describe an effective analogue of Borel embedding which allows us to make distinctions even among classes of finite structures. Section 5 gives results on computable structures of high Scott rank. Some of these results make use of computable embeddings. Finally, in Section 6, we mention some open problems and possible directions for future work.

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We shall assume some background in computability—m-reducibility, arithmetical and hyperarithmetical hierarchies, Σ^1_1 and Π^1_1 sets and relations. This material may be found in [48], or in [1].

1.1. Background from model theory. In classical model theory, the basic setting is the class K of models for a countable complete elementary first order theory. If the theory has an infinite model, then there are models of all infinite cardinalities. For the theory of vector spaces over the rationals, each model is determined up to isomorphism by its dimension. More generally, for an $\aleph_1$-categorical theory T, each model is determined, up to isomorphism, by the dimension of a "strongly minimal" formula, with parameters satisfying a principal type [41], [2]. This result of Baldwin and Lachlan, building on Morley, is the prototypical structure theorem. For the theory of dense linear orderings without endpoints, or any other unstable theory, there are 2^κ non-isomorphic models of cardinality κ, for all $\kappa \geq \aleph_1$, too many to allow nice invariants [53]. (See [27] for a discussion of further structure and non-structure theorems in the setting of models having arbitrary cardinality.)

Vaught [58] focused attention on countable models. Let T be a countable complete elementary first order theory, and consider the number of countable models, up to isomorphism. There are familiar examples illustrating some possible numbers. For the theory of dense linear orderings without endpoints, the number is 1. For the theory of vector spaces over the rationals, the number is $\aleph_0$. For true arithmetic, the number is $2^{\aleph_0}$. Ehrenfeucht found an example for which the number is 3: the theory of dense linear orderings without endpoints, with a strictly increasing sequence of constants. The example is easily modified to give examples for which the number is n, for all $n \geq 3$. The Ehrenfeucht examples are described in [58]. Vaught showed that for a countable complete theory, the number of isomorphism types of countable models cannot be 2.

Vaught conjectured that for a countable complete theory, the number of countable models (up to isomorphism) must be either $\leq \aleph_0$ or $2^{\aleph_0}$. Vaught's Conjecture has been proved for many special kinds of theories, including theories of linear orderings [49] and trees [56], ω-stable theories [54], and superstable theories of finite rank [6]. In the setting of Vaught's Conjecture, the statement that T has $2^{\aleph_0}$ isomorphism types of countable models is a non-structure theorem. The statement that T has only countably many isomorphism types of countable models is a kind of structure theorem. While it does not provide descriptions for the models, it holds out the possibility that there may be nice descriptions.

1.2. Infinitary formulas. In describing countable structures, certain infinitary (but still first order) sentences are useful. For a language L, the $L_{\omega_1\omega}$ formulas are infinitary formulas in which the disjunctions and conjunctions are over countable sets. For a thorough discussion of $L_{\omega_1\omega}$, see Keisler's beautiful book [30]. Scott [51] proved the following.

THEOREM 1.1 (Scott Isomorphism Theorem). *Let $\mathcal{A}$ be a countable structure (for a countable language L). Then there is an $L_{\omega_1\omega}$ sentence σ whose countable models are just the isomorphic copies of $\mathcal{A}$.*

A sentence σ with the property of Theorem 1.1 is called a *Scott sentence* for $\mathcal{A}$. In proving Theorem 1.1, Scott assigned countable ordinals to tuples in $\mathcal{A}$, and to $\mathcal{A}$ itself. There are several different definitions of *Scott rank* in use. We begin with a family of equivalence relations.

DEFINITION 1.2. Let $\overline{a}$ and $\overline{b}$ be tuples of the same length in $\mathcal{A}$.

1. $\overline{a} \equiv^0 \overline{b}$ if $\overline{a}$ and $\overline{b}$ satisfy the same quantifier-free formulas,
2. for $\alpha > 0$, $\overline{a} \equiv^\alpha \overline{b}$ if for all $\beta < \alpha$, for each $\overline{c}$, there exists $\overline{d}$, and for each $\overline{d}$, there exists $\overline{c}$, such that $\overline{a}, \overline{c} \equiv^\beta \overline{b}, \overline{d}$.

Our relations $\equiv^\alpha$ differ from Scott's in that we extend by tuples $\overline{c}$ and $\overline{d}$, where Scott extended by only single elements c and d.

DEFINITION 1.3.

1. The *Scott rank* of a tuple $\overline{a}$ in $\mathcal{A}$ is the least countable ordinal β such that for all $\overline{b}$, $\overline{a} \equiv^\beta \overline{b}$ implies that for all countable ordinals γ, $\overline{a} \equiv^\gamma \overline{b}$—it follows that $(\mathcal{A}, \overline{a}) \cong (\mathcal{A}, \overline{b})$.
2. The *Scott rank* of the structure $\mathcal{A}$, denoted by $SR(\mathcal{A})$, is the least ordinal α greater than the ranks of all tuples in $\mathcal{A}$.

EXAMPLE. If $\mathcal{A}$ is an ordering of type ω, then $SR(\mathcal{A}) = 2$.

Morley [42] showed that if T is a counter-example to Vaught's Conjecture, then the number of isomorphism types of countable models must be $\aleph_1$. The idea behind the proof is that each model must have countable Scott rank, and for each countable ordinal α, the number of models of rank α must be countable or $2^{\aleph_0}$. Vaught's Conjecture holds for T iff for some countable ordinal α either there are $2^{\aleph_0}$ non-isomorphic models of rank α, or else all countable models have Scott rank at most α. For more information related to Vaught's Conjecture and Scott rank, see [50].

1.3. Background from descriptive set theory. Next, we turn to descriptive set theory. We mention only a few notions and results. For more information, see [25] or [5]. We describe a version of the basic setting. The structures are countable, with fixed universe ω. The languages are also countable. For a given relational language L, we have a list $(\alpha_n)_{n\in\omega}$ of all atomic sentences involving a relation symbol from L plus constants from ω. We may identify a structure $\mathcal{A}$ with the function $f \in 2^\omega$ such that

$$f(n) = \begin{cases} 1 & \text{if } \mathcal{A} \models \alpha_n, \\ 0 & \text{if } \mathcal{A} \models \neg\alpha_n. \end{cases}$$

The class of all structures for the language, or functions in 2^ω, has a natural metric topology, where the distance between $f, g \in 2^\omega$ is $1/2^n$ if

n is least such that $f(n) \neq g(n)$. The class of structures for the language forms a *Polish space*; i.e., it is separable and complete. The basic open neighborhoods have the form $N_\sigma = \{f \in 2^\omega : f \supset \sigma\}$, where $\sigma \in 2^{<\omega}$. The *Borel* subsets are generated from the basic open neighborhoods, by taking countable unions and complements. We may also consider Borel subsets of a product of classes. We have Borel relations on a given class, and Borel functions from one class to another.

We consider classes K, contained in the Polish space described above, such that K is closed under isomorphism. By results of Scott [51], Vaught [59], and D. Miller [40], K is Borel iff it is the class of models of some $L_{\omega_1\omega}$ sentence. We add the assumption that K is Borel. One approach to classification in this setting is to consider the isomorphism relation on K. A result saying that the isomorphism relation on a class K is Borel is a weak kind of structure theorem, and a result saying that the isomorphism relation is not Borel is a non-structure theorem.

For the class of vector spaces over $\mathbb{Q}$, or the class of algebraically closed fields of a given characteristic, the isomorphism relation is Borel. More generally, for any class K with only countably many isomorphism types, the isomorphism relation is Borel. The converse is not true. For example, if K is the class of Archimedean ordered fields, then there are $2^{\aleph_0}$ different isomorphism types, but the isomorphism relation is still Borel. For the class of linear orderings, the isomorphism relation is not Borel. Perhaps surprisingly, Hjorth [23] showed that for the class of torsion-free Abelian groups, the isomorphism relation is not Borel.

There is a second approach to classification, which involves comparing classes, and provides a great deal of information. This is the approach of Friedman and Stanley [16].

DEFINITION 1.4. Let K, K' be classes of structures. We say that Φ is a *Borel embedding* of K in K' if Φ is a Borel function from K to K', and for $\mathcal{A}, \mathcal{A}' \in K$, $\mathcal{A} \cong \mathcal{A}'$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{A}')$. (More properly, we might say that Φ is an embedding of K in K' *up to isomorphism*, or an embedding of $K/_\cong$ in $K'/_\cong$.)

NOTATION. We write $K \leq_B K'$ if there is a Borel embedding of K in K'.

PROPOSITION 1.5. *The relation $\leq_B$ is a partial order on the set of all classes.*

A Borel embedding gives us a way of transferring descriptions from one class to another. If $K \leq_B K'$, under the Borel embedding Φ, and K' has simple invariants, then we may describe $\mathcal{A} \in K$ by giving the invariants for $\Phi(\mathcal{A})$. We say that K is *Borel complete* if for all K', $K' \leq_B K$; equivalently, $K' \leq_B K$, where K' is the class of all structures for a language with at least one relation symbol of arity at least 2. Our intuition says that K' cannot be classified, so the statement that K is Borel complete is a convincing non-structure theorem. Friedman and Stanley [16] produced Borel embeddings

of undirected graphs in fields of arbitrary characteristic, trees, and linear orderings, showing that all of these classes are "Borel complete". We shall say more later about some of these embeddings.

§2. **Computable structures.** We now turn to computable structures. The languages that we consider are computable; i.e., the set of non-logical symbols is computable, and we can effectively determine the kind (relation or function) and the arity. The universe of each structure is a subset of ω, and we identify a structure $\mathcal{A}$ with its atomic diagram $D(\mathcal{A})$. A structure $\mathcal{A}$ is *computable* if $D(\mathcal{A})$ is computable. A *computable index* for $\mathcal{A}$ is a number e such that φ_e is the characteristic function of $D(\mathcal{A})$, where φ_e denotes the e^{th} partial computable function in the standard list. We write $\mathcal{A}_e$ for the structure with index e (if there is such a structure).

2.1. Computable infinitary formulas. In describing computable structures, it is helpful to use *computable infinitary formulas*. Roughly speaking, these are infinitary formulas in which the disjunctions and conjunctions are over computably enumerable sets. For a more thorough discussion, see [1].

EXAMPLE. There is a natural computable infinitary sentence whose models are just the Abelian p-groups—the conjunction of the usual axioms for Abelian groups, plus

$$\forall x \bigvee_n \underbrace{x + \cdots + x}_{p^n} = 0.$$

All together, computable infinitary formulas have the same expressive power as those in the least admissible fragment of $L_{\omega_1\omega}$. It was Ash who described the computable infinitary formulas and recognized their usefulness, which comes from the classification defined below.

COMPLEXITY OF FORMULAS.

1. A computable Σ_0 and Π_0 formula is a finitary open formula.
2. For $\alpha > 0$, a computable Σ_α formula $\varphi(\overline{x})$ is a c.e. disjunction of formulas of the form $\exists \overline{u}\, \psi(\overline{u}, \overline{x})$, where ψ is computable Π_β for some $\beta < \alpha$.
3. For $\alpha > 0$, a computable Π_α formula is a c.e. conjunction of formulas of the form $\forall \overline{u}\, \psi(\overline{u}, \overline{x})$, where ψ is computable Σ_β for some $\beta < \alpha$.

PROPOSITION 2.1. *Satisfaction of computable* Σ_α (*or* Π_α) *formulas in computable structures is* Σ^0_α (*or* Π^0_α), *with all possible uniformity* (*over indices for the formulas, structures, and* α).

There is a version of Compactness for computable infinitary formulas which, unlike the usual Compactness, can be used to produce computable structures. The original result of this kind, for ω-logic, is due to Kreisel (in a footnote in [37]). Barwise [4] gave a result for countable admissible

fragments of $L_{\omega_1\omega}$. The special case below, with its corollaries, may be found in [1].

THEOREM 2.2 (Barwise-Kreisel Compactness). *If Γ is a Π^1_1 set of computable infinitary sentences such that every Δ^1_1 subset of Γ has a model, then Γ has a model.*

COROLLARY 2.3. If Γ is a Π^1_1 set of computable infinitary sentences such that every Δ^1_1 subset of Γ has a computable model, then Γ has a computable model.

The next two corollaries illustrate the power of computable infinitary formulas to describe computable structures.

COROLLARY 2.4. If $\mathcal{A}$ and $\mathcal{B}$ are computable (or hyperarithmetical) structures satisfying the same computable infinitary sentences, then $\mathcal{A} \cong \mathcal{B}$.

COROLLARY 2.5 (Nadel [44]). Let $\mathcal{A}$ be a computable (or hyperarithmetical) structure. If $\overline{a}$ and $\overline{b}$ are tuples of the same length, satisfying the same computable infinitary formulas in $\mathcal{A}$, then there is an automorphism of $\mathcal{A}$ taking $\overline{a}$ to $\overline{b}$.

The final corollary is a special case of a stronger result, due to Ressayre [47].

COROLLARY 2.6 (Ressayre). Let $\mathcal{A}$ be a computable (or hyperarithmetical) structure, and let Γ be a Π^1_1 set of computable infinitary sentences involving finitely many new symbols, in addition to those in the language of $\mathcal{A}$. If every Δ^1_1 subset of Γ is satisfied in an expansion of $\mathcal{A}$, then $\mathcal{A}$ has an expansion satisfying Γ.

2.2. Structure and non-structure theorems. Certain classes of structures arising in computable structure theory can be classified simply. Goncharov [19], [17] showed that the Boolean algebras with a property of "computable categoricity" are the ones with only finitely many atoms, and the linear orderings with this property are the ones with finitely many successor pairs. In a talk in Kazan in 1997, Goncharov stated a large number of further problems, calling for classification of further classes of computable structures. Some of these problems seemed likely to have nice answers, while others did not. At the end of the talk, Shore asked Goncharov what would make him give up. Shore's question was considered in [21]. The aim was to find convincing statements for structure and non-structure theorems, like those in model theory and descriptive set theory.

Let K be a class of structures, closed under isomorphism. Let K^c be the set of computable elements of K. Let $I(K)$ be the set of computable indices for elements of K^c. We suppose that there is a computable infinitary sentence for which K^c is the class of computable models. Equivalently, $I(K)$ is hyperarithmetical. In [21], three different approaches are discussed.

All give intuitively correct answers for some familiar classes (e.g., $\mathbb{Q}$-vector spaces can be classified, linear orderings cannot).

The first approach involves "Friedberg enumerations".

DEFINITION 2.7. An *enumeration* of $K^c/_{\cong}$ is a sequence $(\mathcal{C}_n)_{n\in\omega}$ of elements of K^c such that for all $\mathcal{A} \in K^c$, there exists n such that $\mathcal{C}_n \cong \mathcal{A}$; i.e., the sequence represents all isomorphism types of computable members of K. The enumeration is *Friedberg* if for each $\mathcal{A} \in K^c$, there is a unique n such that $\mathcal{C}_n \cong \mathcal{A}$; i.e., each isomorphism type appears just once.

The complexity of the enumeration is that of the sequence of computable indices for elements of K. Saying that $K^c/_{\cong}$ has a hyperarithmetical Friedberg enumeration is an assertion of classifiability. Lack of repetition in the list reflects understanding of the different isomorphism types. This approach has considerable appeal. For one thing, in mathematical practice, a classification is often given by a list—think of the classification of finite simple groups. Another attractive feature of this approach is that it fits with an established body of work on enumerations, in computability and computable structure theory [15].

The second approach to classification considered in [21] is related to Scott ranks and Scott sentences. The result below, a consequence of Corollary 2.5, says which Scott ranks are possible for computable structures. We have already mentioned computable ordinals. These are the order types of computable well orderings of ω, and they form a countable initial segment of the ordinals. The first non-computable ordinal is denoted by ω_1^{CK}. Nadel used a different definition of Scott rank, but his work in [44] shows the following.

THEOREM 2.8 (Nadel). *For a computable structure $\mathcal{A}$, $SR(\mathcal{A}) \leq \omega_1^{CK} + 1$.*

There are computable structures taking the maximum possible rank. Harrison [22] proved the following.

PROPOSITION 2.9 (Harrison). *There is a computable ordering of type $\omega_1^{CK}(1+\eta)$ (with a first interval of type ω_1^{CK}, followed by densely many more).*

PROOF. Kleene showed that there is a computable tree $T \subseteq \omega^{<\omega}$ with a path but no hyperarithmetical path. Let $<$ be the Kleene-Brouwer ordering on T. Under this ordering, $\sigma < \tau$ iff either σ properly extends τ or else there is some first n on which σ and τ differ, and for this n, $\sigma(n) < \tau(n)$. Harrison showed that $(T, <)$ has order type $\omega_1^{CK}(1+\eta)+\alpha$, where α is a computable ordinal. There is an initial segment of type $\omega_1^{CK}(1+\eta)$. ⊣

The Harrison ordering has Scott rank $\omega_1^{CK}+1$. If a is an element outside the initial interval of type ω_1^{CK}, then a has Scott rank ω_1^{CK}.

THEOREM 2.10 (folklore). *For a computable structure $\mathcal{A}$,*

1. *$SR(\mathcal{A}) < \omega_1^{CK}$ if there is some computable ordinal β such that the orbits of all tuples are defined by computable Π_β formulas.*

2. $SR(\mathcal{A}) = \omega_1^{CK}$ *if the orbits of all tuples are defined by computable infinitary formulas, but there is no bound on the complexity of these formulas.*
3. $SR(\mathcal{A}) = \omega_1^{CK} + 1$ *if there is some tuple whose orbit is not defined by any computable infinitary formula.*

If a computable structure has low Scott rank, then we expect it to have a simple Scott sentence. Nadel [44], [45] showed the following.

THEOREM 2.11 (Nadel). *A computable (or hyperarithmetical) structure has computable Scott rank iff it has a computable infinitary Scott sentence.*

PROPOSITION 2.12. *Suppose K is a class of structures closed under isomorphism. If there is a computable bound on the Scott ranks of elements of K^c, then there is a computable ordinal α such that the elements of K^c all have computable Π_α Scott sentences.*

If we understand that the goal of computable classification is to describe each computable member of K, so as to distinguish its computable copies from other *computable* structures, then something weaker than a Scott sentence may suffice. A *pseudo-Scott sentence for* $\mathcal{A}$ is a sentence whose computable models are just the computable copies of $\mathcal{A}$. We do not know whether there is a computable structure with a computable infinitary pseudo-Scott sentence but no computable infinitary Scott sentence. What we can prove is the following.

PROPOSITION 2.13. *Let K be a class of structures closed under isomorphism. Then the following are equivalent.*

1. *There is a computable ordinal α such that any elements of K^c satisfying the same computable Π_α sentences are isomorphic.*
2. *Each $\mathcal{A} \in K^c$ has a computable infinitary pseudo-Scott sentence.*

The statements in Proposition 2.13 are structure theorems.

We turn to the third approach.

DEFINITION 2.14. The *isomorphism problem* for K is

$$E(K) = \{(a, b) : a, b \in I(K) \ \& \ \mathcal{A}_a \cong \mathcal{A}_b\} .$$

The statement that $E(K)$ is hyperarithmetical is one more example of a structure theorem. For the class of computable models of a computable infinitary sentence, the three approaches are equivalent.

THEOREM 2.15 (Goncharov-Knight). *If $I(K)$ is hyperarithmetical, then the following are equivalent:*

1. *$K^c/_\cong$ has a hyperarithmetical Friedberg enumeration,*
2. *there is a computable ordinal α such that any two members of K^c satisfying the same computable Π_α sentences are isomorphic,*
3. *$E(K)$ is hyperarithmetical.*

PROOF. It is not difficult to show that 2 implies 3 and 3 implies 1. To see that 1 implies 2, we form a hyperarithmetical structure $\mathcal{A}^*$ including (disjoint copies of) all structures in K^c. We also include a set corresponding to $I(K)$ and a binary relation associating each index $a \in I(K)$ with the elements of the corresponding structure $\mathcal{A}_a$. Let I be a hyperarithmetical subset of $I(K)$, representing indices in the Friedberg enumeration. We add this to $\mathcal{A}^*$. Let a, b be new constants, and let $\Gamma(a, b)$ be a Π^1_1 set of computable infinitary formulas saying $a, b \in I$, $a \neq b$, and for all computable ordinals α, $\mathcal{A}_a$ and $\mathcal{A}_b$ satisfy the same computable Π_α sentences. If 2 fails, then every Δ^1_1 subset of $\Gamma(a, b)$ is satisfiable in $\mathcal{A}^*$. Then using Corollary 2.6, we can show that the whole set is satisfiable. By Corollary 2.6, $\mathcal{A}_a$ and $\mathcal{A}_b$ are isomorphic. Since the elements of I represent nonisomorphic structures, this is a contradiction. ⊣

§3. Isomorphism problems. Theorem 2.15 says that the three approaches to classification for computable structures are equivalent. The first approach, involving Friedberg enumerations, is certainly appealing. However, the third approach, involving isomorphism problems, has proved to be more productive. When we attempt to determine, for various specific classes K, whether there is a hyperarithmetical Friedberg enumeration of $K^c/_{\cong}$, it seems that we must first calculate the complexity of $E(K)$. In this section, we give a number of these calculations, taken mainly from [7] and [9].

Of course, if $I(K)$ is hyperarithmetical, then $E(K)$ must be Σ^1_1, since the isomorphism relation may be defined by a statement asserting the existence of a function that is total and bijective and preserves the basic relations and operations. For some classes of structures, it is well-known that the isomorphism problem is "m-complete" Σ^1_1. For a complexity class Γ (Σ^1_1, Π^0_3, etc.), we say that A is m-complete Γ if A is in Γ, and for all S in Γ, $S \leq_m A$.

THEOREM 3.1 (folklore). *For each of the following classes K, $E(K)$ is m-complete Σ^1_1:*

1. *linear orderings,*
2. *undirected graphs,*
3. *Boolean algebras,*
4. *Abelian p-groups,*
5. *arbitrary structures for language with at least one relation symbol of arity at least* 2.

PROOF. The ideas go back to Kleene and the fact that the class of indices for computable trees without paths and the class of indices for well orderings are m-complete Π^1_1 (see [48]). For 1, we show that for any Σ^1_1 set S, there is a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ of linear orderings such that if $n \in S$, then $\mathcal{A}_n$ is a Harrison ordering, and if $n \notin S$, then $\mathcal{A}_n$ is a well ordering. We have an m-reduction of S to the set of indices for the Harrison

ordering. For 2-5, the proofs are similar. In each case, the class K contains a structure $\mathcal{A}$ such that for any Σ^1_1 set S, there is a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ of structures in K such that $\mathcal{A}_n \cong \mathcal{A}$ iff $n \in S$. For complete proofs of all parts of the theorem, see [21]. ⊣

Parts 1, 2, and 5 of Theorem 3.1 match our intuition. Whatever classification means, it should be impossible for these classes. Parts 3 and 4 may be less clear. There are well-known results describing the structure of countable Abelian p-groups [29] in terms of the Ulm sequence and the dimension of the divisible part. Similarly, there are results of Ketonen [31], [46] describing the structure of countable Boolean algebras. Theorem 3.1 says that the invariants for computable Abelian p-groups, and computable Boolean algebras are not uniformly simple. For the classes in the following theorem, which is proved in [7], our intuition may not say whether there should be a structure theorem, but the proofs make it clear that these are as bad as linear orderings or undirected graphs.

THEOREM 3.2 (Calvert). *For the following classes K, $E(K)$ is m-complete Σ^1_1:*

1. *ordered real closed fields (not necessarily Archimedean),*
2. *arbitrary fields of any fixed characteristic.*

PROOF. The proof for 1 uses a lemma of van den Dries. For an ordered real closed field F, we define an equivalence relation $\sim$, where $a \sim b$ if there are polynomials $p(x)$ and $q(x)$, with rational coefficients, such that $p(a) \geq b$ and $q(b) \geq a$. The equivalence classes are intervals, so we have an induced ordering on $F/_\sim$. We can pass from a computable linear ordering L to a computable real closed field $F(L)$ such that $\{x \in F : x \geq 0\}/\sim$ has order type L.

For 2, we get the characteristic zero case from 1. An alternate proof of that case, which extends to arbitrary characteristic, uses the Friedman and Stanley embedding of undirected graphs in fields (see Section 4). ⊣

The torsion-free Abelian groups are, essentially, subgroups of vector spaces over the rationals. Friedman and Stanley conjectured (in 1989) that the class of countable torsion-free Abelian groups is Borel complete [16]. The conjecture remains open, although Hjorth [23] succeeded in showing that the isomorphism relation on this class is not Borel. In [8], Hjorth's proof is adapted to give the following result.

PROPOSITION 3.3. *Let K be the class of torsion-free Abelian groups. Then $E(K)$ is not hyperarithmetical.*

The proof involves coding trees in torsion-free Abelian groups (Hjorth used slightly different structures). For each α, we can show that $E(K)$ is Δ^0_α hard (i.e., for any Δ^0_α set S, $S \leq_m E(K)$). However, the coding method varies with α, so we do not get the fact that the isomorphism problem is m-complete Σ^1_1.

So far, our calculations of the complexity of the isomorphism problem $E(K)$ have involved classes K for which it is complicated. Now, we turn to classes where it should be less complicated. We consider some classes with simple invariants. Perhaps the nicest examples are vector spaces over a fixed field and algebraically closed fields of fixed characteristic. For these classes, results on the isomorphism problems match our intuition. For the vector spaces, we assume that the relevant field is infinite, so that the class has infinite members of finite as well as infinite dimension. We also suppose that the field is computable. The following result is in [7].

PROPOSITION 3.4. *For the following classes K, $E(K)$ is m-complete Π^0_3:*

1. *vector spaces over a fixed infinite computable field,*
2. *algebraically closed fields of fixed characteristic.*

PROOF. In each case, we show that $E(K)$ is Π^0_3 by noting that $I(K)$ is Π^0_2 and the following relation, on pairs (a, b) in $I(K)$, is Π^0_3:

> "For all n, the structure $\mathcal{A}_a$ has at least n independent elements if and only if $\mathcal{A}_b$ does."

For completeness, we show that for any Π^0_3 set S, there is a uniformly computable sequence of structures $(\mathcal{A}_n)_{n\in\omega}$ in K such that $n \in S$ iff $\mathcal{A}_n$ has infinite dimension. ⊣

In [8], Proposition 3.4 is generalized to the class K of models of a strongly minimal theory satisfying some extra conditions. The models are classified by dimension. We add a model-theoretic condition ($acl(\emptyset)$ is infinite) to guarantee that there are infinite models of finite as well as infinite dimension. We also need some effectiveness conditions (e.g., T is decidable, with effective elimination of quantifiers).

It is also relatively simple to classify the computable Archimedean real closed fields, since each is determined, up to isomorphism, by the Dedekind cuts that are filled. The following result is in [7].

PROPOSITION 3.5. *If K is the class of Archimedean real closed fields, then $E(K)$ is m-complete Π^0_3.*

At this point, we return to the class of Abelian p-groups. Although we have an algebraic classification of countable Abelian p-groups, we have seen that the isomorphism problem is m-complete Σ^1_1. In fact, there is a single computable Abelian p-group whose inclusion in the class guarantees this. Below, we consider subclasses of Abelian p-groups that are "reduced" and have Ulm sequences of bounded length. For these subclasses the isomorphism problem is hyperarithmetical, and the complexity increases with the bound on the length.

We summarize the definitions leading to the Ulm sequence. For more information, see [29]. An Abelian p-group is *reduced* if it has no nontrivial divisible subgroup. Let G be a countable reduced Abelian p-group. We begin with the sequence of subgroups G_α, where $G_0 = G$, $G_{\alpha+1} = pG_\alpha$,

and for a limit ordinal α, $G_\alpha = \cap_{\beta<\alpha} G_\beta$. Let P be the set of elements of G of order p, and let $P_\alpha = P \cap G_\alpha$. Since G is reduced, there is a countable ordinal α such that $G_\alpha = \{0\}$. The least such α is the *length* $\lambda(G)$. For each $\beta < \alpha$, $P_\beta/P_{\beta+1}$ is a vector space over the field with p elements. We write $u_\beta(G)$ for the dimension, which is either finite or ∞. Finally, the *Ulm sequence* of G is the sequence $(u_\beta(G))_{\beta<\lambda(G)}$.

Ulm's Theorem says that a countable reduced Abelian p-group is determined, up to isomorphism, by its Ulm sequence. Previous papers, including [3] and [9] have given formulas, for any α and k, of complexity (depending on α), expressing that the α^{th} Ulm invariant is at least k. Let K_α be the class of reduced Abelian p-groups of length at most α. In [9], there are results for these classes. It is shown that $E(K_\omega)$ is Π^0_3, $E(K_{\omega\cdot 2})$ is Π^0_5, and in general, $E(K_{\omega\cdot\alpha})$ is $\Pi^0_{\hat{\alpha}}$, where $\hat{\alpha} = \sup_{\gamma<\alpha}(2\gamma+3)$.

We have a problem expressing a meaningful completeness result. Until now, we have considered classes K for which $I(K)$ was simpler than $E(K)$. However, $I(K_\omega)$ is Π^0_3, the same complexity as $E(K_\omega)$ (and, in general, $I(K_{\omega\cdot\alpha})$ is $\Pi^0_{\hat{\alpha}}$, the same as $E(K_{\omega\cdot\alpha})$). It is conceivable that when we decide whether $(a,b) \in E(K_\omega)$, the main difficulty is determining whether $a, b \in I(K)$. The following definition, stated in [9], separates the isomorphism problem from the problem of determining membership in the class.

DEFINITION 3.6. Suppose $A \subseteq B$, and let Γ be a complexity class (e.g., Π^0_3).

1. We say that A is Γ *within* B if there is some $R \in \Gamma$ such that $A = R \cap B$.
2. We say that $S \leq_m A$ *within* B if there is a computable $f : \omega \to B$ such that for all n, $n \in S$ iff $f(n) \in A$.
3. We say that A is *m-complete* Γ *within* B if A is Γ within B and for all $S \in \Gamma$, we have $S \leq_m A$ within B.

The relevant example is where $B{=}I(K)^2$ for some class K, and $A{=}E(K)$. Note that the condition of being m-complete Γ within B is stronger that simply being m-complete Γ, since the "negative" examples produced by the reduction must be members of B. We will often write "Within K" to mean "Within $I(K)^2$." All previous theorems in this section remain true when "Within K" is added to the statement. Using these definitions, we state the result from [9], with the appropriate completeness.

THEOREM 3.7 (Calvert). *Let $K_{\omega\cdot\alpha}$ be the class of reduced Abelian p-groups of length at most $\omega\cdot\alpha$, and let $\hat{\alpha} = \sup_{\gamma<\alpha}(2\gamma+3)$. Then $E(K_{\omega\cdot\alpha})$ is m-complete $\Pi^0_{\hat{\alpha}}$ within K_α.*

Now, we return to torsion-free Abelian groups, to consider a simpler subclass of those. The *rank* of a torsion-free Abelian group is the dimension of the least $\mathbb{Q}$-vector space in which the group can be embedded. For groups of finite rank, the isomorphism problem is simple because there are simple isomorphisms.

DEFINITION 3.8. A structure $\mathcal{A}$ is *computably categorical* if for any $\mathcal{B} \simeq \mathcal{A}$, there is some computable function $f : \mathcal{B} \to \mathcal{A}$ which is an isomorphism. (For more about computable categoricity, see [1].)

If all computable members of a class K are computably categorical, then $E(K)$ must be Σ^0_3, at least within K. Any torsion-free Abelian group of finite rank is computably categorical. Using this, it is possible to prove the following [8].

PROPOSITION 3.9. *Let K be either the class of all torsion-free Abelian groups of finite rank, or the class of torsion-free Abelian groups of rank* 1. *Then $E(K)$ is m-complete Σ^0_3 within K.*

We mention some further exciting, and very deep, results of Thomas and Hjorth on torsion-free Abelian groups of finite rank. Let K_n be the class of torsion-free Abelian groups of rank n. Hjorth showed that $TFA_1 <_B TFA_2$. Thomas showed that for all $n \geq 1$, $TFA_n <_B TFA_{n+1}$. (See [57] for an account of these results.) A group $\mathcal{G}$ is said to be *p-local* if for all primes $q \neq p$, all elements of $\mathcal{G}$ are divisible by q. Let TFA^p_2 be the class of p-local torsion-free Abelian groups of rank 2. Hjorth and Thomas [26] have shown that for distinct primes p and q, TFA^p_2 and TFA^q_2 are Borel incomparable.

3.1. Optimal Scott sentences and index sets. Classification involves describing the structures in a class, up to isomorphism. One way to do this is by giving Scott sentences. If we consider only computable structures, it is enough to give pseudo-Scott sentences. How do we know when we have an optimal Scott sentence, or pseudo-Scott sentence?

For inspiration, we go to the setting of descriptive set theory, with arbitrary countable structures and Scott sentences in $L_{\omega_1\omega}$. As we said in Section 1, a class K closed under isomorphism is Borel iff it is the class of models of some $L_{\omega_1\omega}$ sentence. More is true. Vaught [59] showed that the class K is $\Sigma^{\boldsymbol{0}}_\alpha$, or $\Pi^{\boldsymbol{0}}_\alpha$, iff it is the class of models of some Σ_α, or Π_α sentence. If $K(\mathcal{A})$ is the class of copies of a given $\mathcal{A}$, then the form of the optimal Scott sentence matches the complexity of K. D. Miller [40] showed that if $K(\mathcal{A})$ is $\Delta^{\boldsymbol{0}}_\alpha$, then it is $d\text{-}\Sigma^{\boldsymbol{0}}_\alpha$; i.e., it can be expressed as a difference of $\Sigma^{\boldsymbol{0}}_\alpha$ sets. A. Miller [39] gave examples of structures $\mathcal{A}$ such that $K(\mathcal{A})$ is $\Sigma^{\boldsymbol{0}}_\alpha$, $\Pi^{\boldsymbol{0}}_\alpha$, or $d\text{-}\Sigma^{\boldsymbol{0}}_\alpha$ for various countable ordinals. He showed that $K(\mathcal{A})$ cannot be properly $\boldsymbol{\Sigma^0_2}$.

Now, we return to the computable setting, with computable structures and computable infinitary sentences. Let $\mathcal{A}$ be a computable structure, and let K be the class of structures for the language of $\mathcal{A}$. If $\mathcal{A}$ has a Scott sentence or pseudo-Scott sentence that is computable Π_α, then it is easy to see that $I(\mathcal{A})$ is Π^0_α within $I(K)$. Similarly, if $\mathcal{A}$ has a Scott sentence, or pseudo-Scott sentence, that is "computable $d\text{-}\Sigma_\alpha$" (i.e., it has the form $\varphi \,\&\, \neg\psi$, where φ and ψ are computable Σ_α), then $I(\mathcal{A})$ is $d\text{-}\Sigma^0_\alpha$ within $I(K)$.

In [12], there are preliminary results supporting the thesis that for computable structures $\mathcal{A}$ in various familiar classes K, when we have found an

optimal Scott sentence or pseudo-Scott sentence, then the complexity of $I(\mathcal{A})$, within K, matches that of the sentence, and $I(\mathcal{A})$ is m-complete in the complexity class. We begin with vector spaces.

PROPOSITION 3.10. *Let K be the class of vector spaces over $\mathbb{Q}$, let K_f consist of the finite dimensional members of K, and let $\mathcal{A}$ be a computable member of K.*

1. *If $\dim \mathcal{A} = 0$ then $I(\mathcal{A})$ is m-complete Π^0_1 within K_f.*
2. *If $\dim \mathcal{A} = 1$ then $I(\mathcal{A})$ is m-complete Π^0_2 within K_f.*
3. *If $1 < \dim \mathcal{A} < \aleph_0$ then $I(\mathcal{A})$ is m-complete d-Σ^0_2 within K_f.*
4. *If $\dim \mathcal{A} = \aleph_0$ then $I(\mathcal{A})$ is m-complete Π^0_3 within K.*

PROOF. For 3, the Scott sentence for $\mathcal{A}$, within K_f, says that there are at least n linearly independent elements and there are not at least $n+1$ such elements. For 2, we might first write a Scott sentence of the same kind, but this is not optimal. The optimal sentence says that any two elements are linearly dependent. ⊣

The situation for Archimedean ordered real closed fields is similar.

PROPOSITION 3.11. *Let K be the class of computable Archimedean real closed ordered fields, and let $\mathcal{A}$ be a computable member of K.*

1. *If the transcendence degree of $\mathcal{A}$ is 0, then $I(\mathcal{A})$ is m-complete Π^0_2 within K.*
2. *If the transcendence degree of $\mathcal{A}$ is finite but greater than 0, then $I(\mathcal{A})$ is m-complete d-Σ^0_2 within K.*
3. *If the transcendence degree of $\mathcal{A}$ is infinite then $I(\mathcal{A})$ is m-complete Π^0_3 within K.*

PROOF. There is a computable Π_2 sentence φ characterizing the Archimedean real closed ordered fields. For 1, there is a computable Π_2 Scott sentence, the conjunction of φ and a sentence saying that all elements are algebraic. For 2, fix a transcendence base of any size n. There is a computable Σ_2 Scott sentence—the conjunction of φ and a sentence saying that the cuts of the transcendence base are filled, and there do not exist $n+1$ algebraically independent elements. For 3, there is a computable Π_3 Scott sentence—the conjunction of φ and a sentence saying the appropriate cuts are filled, and every element lies in one of the cuts that should be filled. ⊣

Now, we turn to Abelian p-groups. The proof of Theorem 3.7 gives some index set calculations. In each case, the completeness was witnessed by a single reduced Abelian p-group. We get the following.

PROPOSITION 3.12. *Let $\mathcal{A}$ be the reduced Abelian p-group of length $\omega \cdot \alpha$, with $u_\gamma(\mathcal{A}) = \infty$ for all $\gamma < \alpha$. Let $\hat{\alpha} = sup_{\omega\gamma<\alpha}(2\gamma+3)$. Then $I(\mathcal{A})$ is m-complete $\Pi^0_{\hat{\alpha}}$ within the class of reduced Abelian p-groups of length $\leq \omega \cdot \alpha$.*

A theorem of Khisamiev [32], [33] gives a description of the Ulm invariants of computable Abelian p-groups of length less than ω^2. For each of these groups, we can determine the complexity of the index set.

THEOREM 3.13 (Calvert-Harizanov-Knight-S. Miller). *Let K be the class of reduced Abelian p-groups of length $\omega M + N$ for some $M, N \in \omega$. Let $\mathcal{A} \in K$.*

1. *If $\mathcal{A}_{\omega M}$ is minimal for the given length (i.e., it has the form $\mathbb{Z}_{p^N}$), then $I(\mathcal{A})$ is m-complete Π^0_{2M+1} within K.*
2. *If $\mathcal{A}_{\omega M}$ is finite but not minimal for the given length, then $I(\mathcal{A})$ is m-complete d-Σ^0_{2M+1} within K.*
3. *If there is a unique $k < N$ such that $u_{\omega M+k}(\mathcal{A}) = \infty$, and for all $m < k$, $u_{\omega M+m}(\mathcal{A}) = 0$, then $I(\mathcal{A})$ is m-complete Π^0_{2M+2} within K.*
4. *If there is a unique $k < N$ such that $u_{\omega M+k}(\mathcal{A}) = \infty$ and for some $m < k$ we have $0 < u_{\omega M+m}(\mathcal{A}) < \infty$, then $I(\mathcal{A})$ is m-complete d-Σ^0_{2M+2} within K.*
5. *If there exist $m < k < N$ such that $u_{\omega M+m}(\mathcal{A}) = u_{\omega M+k}(\mathcal{A}) = \infty$, then $I(\mathcal{A})$ is m-complete Π^0_{2M+3} within K.*

PROOF. In most cases here, the first Scott sentence that we write down, describing the Ulm sequence in a straightforward way, is not optimal. To specify that $u_{\omega M+k}(\mathcal{A}) = \infty$, it suffices, via a Ramsey's Theorem argument, to say that for all r, the group $\mathcal{A}_{\omega M}$ has a subgroup of type $\mathbb{Z}^r_{p^{k+1}}$. This gives a simpler Scott sentence. ⊣

In [12], the models of the original Ehrenfeucht theory T illustrate a further pattern. Recall that T is the theory of a dense linear ordering without endpoints, with an infinite increasing sequence of constants. The three non-isomorphic models are as follows:

1. the prime model, in which the sequence has no upper bound,
2. the "middle" model, in which the sequence has a least upper bound,
3. the saturated model, in which the sequence has an upper bound but no least upper bound.

THEOREM 3.14 (S. Miller). *Let K be the class of models of the original Ehrenfeucht theory, and let $\mathcal{A}^1$, $\mathcal{A}^2$, and $\mathcal{A}^3$ be the prime model, the middle model, and the saturated model, respectively. Then*

1. *$I(\mathcal{A}^1)$ is m-complete Π^0_2 within K.*
2. *$I(\mathcal{A}^2)$ is m-complete Σ^0_3 within K.*
3. *$I(\mathcal{A}^3)$ is m-complete Π^0_3 within K.*

In related work, Csima, Montalban, and Shore [14] calculated the index sets for Boolean algebras up to elementary equivalence. The calculation was based on elementary invariants isolated by Tarski (see Chapter 7 of [36], or [18]).

§4. **Computable embeddings.** In [10], there is a definition of *computable* embedding, inspired by that of *Borel* embedding. The goal was a notion that would allow meaningful comparisons even of classes of finite structures. There are different possible definitions. The one chosen in [10] is essentially uniform enumeration reducibility. Recall that B is *enumeration reducible* to A, or $B \leq_e A$, if there is a computably enumerable set Φ of pairs (α, b), where α is a finite set and b is a number such that $B = \{b : (\exists \alpha \subseteq A)\, (\alpha, b) \in \Phi\}$.

NOTE. Given Φ, for each set A, there is a unique set B such that $B \leq_e A$ via Φ. Thus, Φ yields a function from $P(\omega)$ to $P(\omega)$.

As in the setting of descriptive set theory, we consider structures that need not be computable. We suppose that the universe of each structure is a subset of ω. For simplicity, we suppose that the language consists of finitely many relation symbols—we could allow more general languages. Each class consists of structures for a single language, and, modulo the restriction on the universes, each class is closed under isomorphism.

DEFINITION 4.1. Let K, K' be classes of structures. A *computable transformation* from K to K' is a computably enumerable set Φ of pairs (α, φ), where

1. α is a finite set of sentences appropriate to be in the atomic diagram of a member of K,
2. φ is a sentence appropriate to be in the atomic diagram of a member of K', and
3. for each $\mathcal{A} \in K$, there exists $\mathcal{B} \in K'$ such that $\Phi(D(\mathcal{A})) = D(\mathcal{B})$; i.e., $D(\mathcal{B}) \leq_e D(\mathcal{A})$ via Φ.

Identifying the structures with their atomic diagrams, we write $\Phi(\mathcal{A}) = \mathcal{B}$.

DEFINITION 4.2. Let K, K' be classes of structures. A *computable embedding of K in K'* is a computable transformation Φ such that for all $\mathcal{A}, \mathcal{A}' \in K$, $\mathcal{A} \cong \mathcal{A}'$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{A}')$

We write $K \leq_c K'$ if there is a computable embedding of K into K'.

EXAMPLE 1. Let FLO be the class of finite linear orders, and let FVS be the class of finite-dimensional vector spaces over $\mathbb{Q}$. Then $FLO \leq_c FVS$.

PROOF. Let $\mathcal{V}$ be a computable vector space, with a computable sequence $(b_k)_{k \in \omega}$ of linearly independent elements. For each finite set S, let $\mathcal{V}(S)$ be the linear span of $\{b_k : k \in S\}$. Let Φ be the set of pairs (α, φ) such that for some finite linear order $\mathcal{L}$, with universe S, $\alpha = D(\mathcal{L})$ and $\varphi \in D(\mathcal{V}(S))$. $\dashv$

EXAMPLE 2. Let UG be the class of undirected graphs, and let F_χ be the class of fields of any characteristic χ (0, or p, for some prime p). Then $UG \leq_c F_\chi$.

PROOF. We adapt a Borel embedding given by Friedman and Stanley [16]. Let $\mathcal{F}$ be a computable algebraically closed field of characteristic χ, with a computable sequence $(b_k)_{k\in\omega}$ of algebraically independent elements. For $\chi \neq 2$, for each graph $\mathcal{G}$, let $\mathcal{F}(\mathcal{G})$ be the subfield of $\mathcal{F}$ generated by the elements

$$\Big(\bigcup_{k\in\mathcal{G}} acl(b_k)\Big) \cup \big\{\sqrt{d_i + d_j} : (d_i, d_j) \in R\big\}$$

where R is the set of all pairs (d_i, d_j) such that i, j are elements of $\mathcal{G}$ joined by an edge, and we have $acl(d_i) = acl(b_i)$ and $acl(d_j) = acl(b_j)$.[1] For $\chi = 2$, we use cube roots instead of square roots to indicate edges. Let Φ be the set of pairs (α, φ), where for some finite undirected graph $\mathcal{G}$, $\alpha = D(\mathcal{G})$ and $\varphi \in D(\mathcal{F}(\mathcal{G}))$. It is not at all easy to see that if $\Phi(\mathcal{A}) \cong \Phi(\mathcal{A}')$, then $\mathcal{A} \cong \mathcal{A}'$. The difficulty is showing that square roots of the kind representing edges do not appear in the field by accident. One of several proofs in the literature is given in [52]. ⊣

EXAMPLE 3. Let LO be the class of linear orderings. Then $UG \leq_c LO$.

PROOF. Again we adapt a Borel embedding given by Friedman and Stanley [16]. Let $\mathcal{L}$ be $\mathbb{Q}^{<\omega}$, under the lexicographic ordering. Let $(Q_a)_{a\in\omega}$ be a uniformly computable disjoint family of dense subsets of $\mathbb{Q}$. Let $(t_n)_{n\in\omega}$ be a computable list of the atomic types of tuples in the language of graphs. For a graph $\mathcal{G}$, we let $\mathcal{L}(\mathcal{G})$ be the restriction of the ordering $\mathcal{L}$ to the set of sequences that are associated with tuples $a_1, \ldots, a_n$ in $\mathcal{G}$ in the way described below. Let t_m be the atomic type realized by $a_1, \ldots, a_n$ in $\mathcal{G}$. The sequences $r_0 q_1 r_1 q_2 r_2 \ldots q_n r_n k$ associated with $a_1, \ldots, a_n$ are the ones that satisfy the following conditions:

1. for $1 \leq i \leq n$, $q_i \in Q_{a_i}$,
2. for all $i < n$, $r_i \in Q_0$, and $r_n \in Q_n$,
3. k is a natural number such that $k < m + 1$.

We let Φ be the set of pairs (α, φ), where for some finite undirected graph $\mathcal{G}$, $\alpha = D(\mathcal{G})$, and $\varphi \in D(\mathcal{L}(\mathcal{G}))$. ⊣

4.1. The c-degrees. We have given some examples of computable embeddings. The following is a basic fact about the relation $\leq_c$.

PROPOSITION 4.3. *The relation $\leq_c$ is a partial order on the set $\mathcal{C}$ of all classes of structures.*

As usual, from the partial ordering $\leq_c$ on $\mathcal{C}$, we get an equivalence relation $\equiv_c$, also on $\mathcal{C}$, where $K \equiv_c K'$ iff $K \leq_c K'$ and $K' \leq_c K$. Moreover, $\leq_c$

[1] Friedman and Stanley put in only $\sqrt{b_i + b_j}$. This is enough for a computable embedding. The added elements $\sqrt{d_i + d_j}$ give the embedding a further property of "rank preservation," which will be helpful in Section 5.

induces a partial order on the $\equiv_c$-classes, which we call *c-degrees*. Let $\boldsymbol{C}$ be the resulting degree structure $(\mathcal{C}/_{\equiv_c}, \leq_c)$.

If K and K' are classes of structures, each consisting of just $\aleph_0$ isomorphism types, then $K \equiv_B K'$. We shall see that K and K' may differ in c-degree. The following is a useful tool for showing non-embeddability, with no analogue in the setting of Borel embeddings.

PROPOSITION 4.4. *Suppose $K \leq_c K'$ via Φ. If $\mathcal{A}, \mathcal{B} \in K$, where $\mathcal{A} \subseteq \mathcal{B}$, then $\Phi(\mathcal{A}) \subseteq \Phi(\mathcal{B})$.*

It follows that if K contains a properly increasing chain of structures of length α, and K' has no such chain, then $K \not\leq_c K'$. Using this proposition, we can see that there is no computable embedding of the class LO of linear orderings in the class of vector spaces.

Familiar classes of *finite* structures seem to lie in one of two c-degrees. Recall that FLO is the class of finite linear orders. Let PF be the class of finite prime fields.

PROPOSITION 4.5.

1. $PF <_c FLO$,
2. *the c-degree of PF also contains the class of finite cyclic graphs,*
3. *the c-degree of FLO, maximum for classes of finite structures, also contains the following classes: finite undirected graphs, finite cyclic groups, and finite simple groups.*

Considering infinite structures, we come to further landmark c-degrees. Recall that FVS is the class of finite dimensional vector spaces over $\mathbb{Q}$. Let LO be the class of linear orders, infinite as well as finite.

PROPOSITION 4.6.

1. $FLO <_c FVS <_c LO$,
2. *the c-degree of LO, maximum over-all, also contains the class of undirected graphs, and the class of fields of characteristic 0 (or p).*

The class of vector spaces over $\mathbb{Q}$, the class of algebraically closed fields of fixed characteristic, and the class of models of $Th(\mathbb{Z}, S)$ (the integers with successor) share some important features. We do not know whether they represent the same c-degree.

We have located some familiar classes of structures in the c-degrees. There are natural questions about the degree structure itself. Is it a linear order? Is it a lattice? The first of these questions is answered in [10], and the second is answered in [34].

THEOREM 4.7 (Calvert-Cummins-Knight-S. Miller). *There is a family of $2^{\aleph_0}$ pairwise incomparable c-degrees. Therefore, $\boldsymbol{C}$ is not a linear order.*

The proof of Theorem 4.7 uses the following notion.

DEFINITION 4.8. We say that sets A, B are *mutually immune*[2] if for any partial computable function f,

1. if $ran(f \upharpoonright A)$ is infinite, then $ran(f \upharpoonright A) \not\subseteq B$,
2. if $ran(f \upharpoonright B)$ is infinite, then $ran(f \upharpoonright B) \not\subseteq A$.

For each natural number n, we have a finite prime field of size p_n, where p_n is the n^{th} prime. Below, we write $PF(X)$ for the class of finite prime fields having size p_n for $n \in X$.

LEMMA 4.9. *If A, B are mutually immune, then $PF(A) \not\leq_c PF(B)$ and $PF(B) \not\leq_c PF(A)$.*

Lemma 4.9 reduces the proof of Theorem 4.7 to the following.

LEMMA 4.10. *There is a family of $2^{\aleph_0}$ pairwise mutually immune sets.*

By relativizing the notion, and the lemma, we obtain further families of incomparable c-degrees in various intervals.

The result below says that $\boldsymbol{C}$ is not a lattice.

PROPOSITION 4.11. *There is a pair of c-degrees with no join and no meet.*

PROOF. Let A, B be mutually generic. The forcing conditions are the pairs (p_1, p_2) with $p_1, p_2 \in 2^{<\omega}$ (representing finite partial characteristic functions for A, B, respectively). Take the c-degrees of $PF(A)$ and $PF(B)$. With some effort, it can be shown that this pair of degrees has neither meet nor join. ⊣

For computable embeddings, the connection between the classes is much tighter than for Borel embeddings. One piece of evidence for this is Proposition 4.4, on preservation of the substructure relation. As further evidence, we consider transfer of invariants. Suppose $K \leq_c K'$ via Φ. If there are nice invariants for elements of K', we may use them to describe elements of K. There are two methods for doing this. The first is trivial. We describe $\mathcal{A} \in K$ as the structure whose Φ-image satisfies the description of $\Phi(\mathcal{A})$. The second method is more satisfying, as it allows us to describe $\mathcal{A}$ in its own language. The following result is from [34].

THEOREM 4.12 (Knight). *Suppose $K \leq_c K'$ via Φ. There is a partial computable function taking computable infinitary sentences φ in the language of K' to computable infinitary sentences φ^* in the language of K such that for all $\mathcal{A}$ in K, $\Phi(\mathcal{A}) \models \varphi$ iff $\mathcal{A} \models \varphi^*$. Moreover, if φ is computable Σ_α, then so is φ^*.*

PROOF. For all $\mathcal{A} \in K$, we can build generic copies. The forcing conditions are members of the set $\mathcal{F}_\mathcal{A}$ of finite $1-1$ partial functions from ω to $\mathcal{A}$. From a generic copy $\mathcal{A}^*$ of $\mathcal{A}$, we get $\Phi(\mathcal{A}^*) \in K'$. The forcing language, which describes $\Phi(\mathcal{A}^*)$, is quite different from the language of K. However, forcing is definable in the language of K. For each sentence σ in the forcing

[2]In [10], we said "bi-immune". However, this term was used earlier in a quite different way in, for instance, [28].

language and each tuple $\overline{b}$ in ω, we can find a formula $Force(\overline{x})$ such that $\mathcal{A} \models Force(\overline{a})$ iff we have $p \in \mathcal{F}_{\mathcal{A}}$ taking $\overline{b}$ to $\overline{a}$ and p forces σ. The formulas defining forcing are in the language of K. Moreover, they do not depend on $\mathcal{A}$. The sentence φ^* says, in all $\mathcal{A} \in K$, $(\exists p \in \mathcal{F}_{\mathcal{A}})$ [$p \Vdash$ "$\Phi(\mathcal{A}^*) \models \varphi$"]. ⊣

§5. Computable structures of high Scott rank. We return to computable structures. By Theorem 2.8, these structures all have Scott rank at most $\omega_1^{CK} + 1$. We consider examples of computable structures having different Scott ranks. We begin with structures having computable ranks. The exact value is unimportant—it depends on our choice of definition.

PROPOSITION 5.1 (folklore). *For all computable structures $\mathcal{A}$ in the following classes, $SR(\mathcal{A}) < \omega_1^{CK}$.*

1. *well orderings,*
2. *superatomic Boolean algebras,*
3. *reduced Abelian p-groups.*

PROOF. For 2, note that a countable superatomic Boolean algebra can be expressed as a finite join of α-atoms, for some countable ordinal α. In a computable Boolean algebra $\mathcal{A}$, any element that bounds α-atoms for all computable ordinals α can be split into two disjoint elements with this feature. It follows that if $\mathcal{A}$ bounds α-atoms for all computable ordinals α, then it has an atomless subalgebra, so it cannot be superatomic. For 3, we show that for a computable Abelian p-group, if the length is not computable, then there is a non-zero divisible element. ⊣

Next, we consider computable structures of Scott rank $\omega_1^{CK} + 1$. We have already discussed one example, the Harrison ordering. This gives rise to some other structures. In particular, the *Harrison Boolean algebra* is the interval algebra of the Harrison ordering. The *Harrison Abelian p-group* is the Abelian p-group of length ω_1^{CK}, with all infinite Ulm invariants, and with a divisible part of infinite dimension. Clearly, the Harrison Boolean algebra has a computable copy. The Harrison Abelian p-group does as well.

THEOREM 5.2 (folklore). *The Harrison Boolean algebra, and the Harrison Abelian p-groups have Scott rank $\omega_1^{CK} + 1$.*

PROOF. In the Harrison Boolean algebra, any non-superatomic element has Scott rank ω_1^{CK}. In the Harrison Abelian p-group, any divisible element has Scott rank ω_1^{CK}. (See [20] for more about these structures.) ⊣

Finally, we turn to Scott rank ω_1^{CK}. The following is in [38].

THEOREM 5.3 (Makkai). *There is an arithmetical structure of Scott rank ω_1^{CK}.*

In [35], Makkai's example is made computable.

THEOREM 5.4 (Knight-Millar). *There is a computable structure of Scott rank ω_1^{CK}.*

There are two proofs of Theorem 5.4 in [35]. The first simply codes Makkai's example in a computable structure, without examining it. The second proof re-works Makkai's construction, incorporating suggestions of Shelah and Sacks. The structures are "group trees". To obtain a group tree, we start with a tree. Next, we define a family of groups, one for each level of the tree. Using the groups, we get a new, more homogeneous tree structure. We then throw away most of the structure. We retain a family of unary functions, one for each element.

Morozov [43] gives a nice description of the group tree construction. He was interested in the fact that if we start with a computable tree T having a path but no hyperarithmetical path, then the resulting group tree (Morozov calls it a "polygon") has non-trivial automorphisms but no hyperarithmetical ones. The Harrison ordering (as originally constructed) has no hyperarithmetical infinite decreasing sequence, so it also has no non-trivial hyperarithmetical automorphisms.

5.1. Trees. In [13], there are simpler examples of computable structures of rank ω_1^{CK}, which are just trees. We consider subtrees of $\omega^{<\omega}$ and their isomorphic copies. We write $\emptyset$ for the top node. To describe the examples, we need some terminology. We define *tree rank* for $a \in T$, and for T itself.

DEFINITION 5.5.

1. $rk(a) = 0$ if a is terminal,
2. for an ordinal $\alpha > 0$, $rk(a) = \alpha$ if all successors of a have ordinal rank, and α is the first ordinal greater than any of these ranks,
3. $rk(a) = \infty$ if a does not have ordinal rank.

We let $rk(T) = rk(\emptyset)$.

FACT 5.6. $rk(a) = \infty$ iff there is a path through a.

For a tree T, let T_n be the set of elements at level n.

DEFINITION 5.7. The tree T is *rank-homogeneous* if the following conditions hold:

1. for all $a \in T_n$ and all ordinals α, if there exists $b \in T_{n+1}$ with $rk(b) = \alpha < rk(a)$, then a has infinitely many successors a' with $rk(a') = \alpha$,
2. for all $a \in T_n$ with $rk(a) = \infty$, a has infinitely many successors a' with $rk(a') = \infty$.

The following is clear.

PROPOSITION 5.8. *If T and T' are rank-homogeneous trees, and for all n, T_n and T'_n represent the same ranks, then $T \cong T'$.*

DEFINITION 5.9. A tree T is *thin* if for all n, the set of ordinal ranks of elements of T_n has order type at most $\omega \cdot n$.

We use thinness in the following way.

PROPOSITION 5.10. *If T is a computable thin tree, then for each n, there is some computable α_n such that for all $a \in T_n$, if $rk(a) \geq \alpha_n$, then $rk(a) = \infty$.*

To show that there is a computable tree of Scott rank ω_1^{CK}, we show the following.

PROPOSITION 5.11.

1. *There is a computable, thin, rank-homogeneous tree T such that T has a path but no hyperarithmetical path.*
2. *If T is a computable, thin, rank-homogeneous tree such that T has a path but no hyperarithmetical path, then $SR(T) = \omega_1^{CK}$.*

PROOF. Part 1 is proved using Barwise-Kreisel Compactness. There is a Π_1^1 set of computable infinitary sentences describing the desired tree, and we show, in several steps, that every Δ_1^1 subset has a computable model. For Part 2, first we note that the orbit of each tuple in T is defined by a computable infinitary formula—this is where we use thinness. It follows that $SR(T) \leq \omega_1^{CK}$. To show that $SR(T) \geq \omega_1^{CK}$, we show that if a and b are elements of T at the same level, and are such that $rk(a) = \infty$ and b has computable tree rank at least $\geq \omega \cdot \alpha$, then $a \equiv^{\alpha} b$. In fact, two tuples are α-equivalent if they generate isomorphic subtrees and for each pair of corresponding elements in the two subtrees, either the tree ranks match, or else both are at least $\omega \cdot \alpha$. ⊣

In [11], it is shown that there are structures of Scott rank ω_1^{CK} in further familiar classes.

THEOREM 5.12 (Calvert-Goncharov-Knight). *Each of the following classes contains computable structures of Scott rank ω_1^{CK}:*

1. *undirected graphs,*
2. *fields of any fixed characteristic,*
3. *linear orderings.*

To prove Theorem 5.12, we compute embeddings with a special property.

DEFINITION 5.13. Let Φ be an embedding of K in K'. We say that Φ has the *rank preservation property* if for all computable $\mathcal{A} \in K$, either $\mathcal{A}$ and $\Phi(\mathcal{A})$ both have computable Scott rank, or else they have the same non-computable Scott rank.

PROPOSITION 5.14. *For each of the following pairs of classes (K, K'), there is a computable embedding of K in K' with the rank preservation property:*

1. *K is the class of trees and K' is the class of undirected graphs,*
2. *K is the class of undirected graphs and K' is the class of fields, of any desired characteristic,*
3. *K is the class of undirected graphs and K' is the class of linear orderings.*

PROOF. For 1, if $\mathcal{G}$ is the graph corresponding to a tree T, then we have a copy $T_{\mathcal{G}}$ of T, which is definable in $\mathcal{G}$ using existential formulas. Each automorphism of $T_{\mathcal{G}}$ extends to an automorphism of $\mathcal{G}$. This implies that

if $\mathcal{G}$ has Scott rank less than ω_1^{CK}, or at most ω_1^{CK}, then so does T. In addition, for each tuple $\overline{b}$ in $\mathcal{G}$, there is an existential formula defining the orbit of $\overline{b}$ under automorphisms that fix the elements of $T_{\mathcal{G}}$. This implies that if T has Scott rank less than ω_1^{CK}, or at most ω_1^{CK}, then so does $\mathcal{G}$.

For 2, we use the embedding of graphs in fields described in Section 4 (a variant of the one due to Friedman and Stanley). There is a copy of the graph sitting in the corresponding field as a quotient of a definable structure by a definable congruence relation, and we obtain rank preservation by proving properties analogous to the ones above.

For 3, we use the embedding of graphs in linear orderings described in Section refsec4 (due to Friedman and Stanley). There is no copy of the graph definable, even as a quotient structure, in the corresponding linear ordering. Even so, the orbit of each tuple in the graph corresponds to an orbit of an element in the linear ordering, and each orbit in the linear ordering is determined by an existential formula together with a finite tuple of orbits from the graph. ⊣

DEFINITION 5.15. A computable structure $\mathcal{A}$ of non-computable rank is *computably approximable* if every computable infinitary sentence σ true in $\mathcal{A}$ is also true in some computable $\mathcal{B} \ncong \mathcal{A}$.

For all we know, every computable structure of high rank is computably approximable. Failure to be computably approximable is equivalent to existence of a computable infinitary pseudo-Scott sentence.

Sometimes we want a stronger computable approximability.

DEFINITION 5.16. Suppose $\mathcal{A}$ is computable, with non-computable Scott rank. We say that $\mathcal{A}$ is *strongly computably approximable* if for all Σ_1^1 sets S, there is a uniformly computable sequence $(\mathcal{C}_n)_{n\in\omega}$ such that if $n \in S$, then $\mathcal{C}_n \cong \mathcal{A}$, and if $n \notin S$, then $\mathcal{C}_n$ has computable Scott rank.

EXAMPLE. The Harrison ordering is strongly computably approximable.

THEOREM 5.17 (Calvert-Knight-Millar). *There is a computable tree T such that $SR(T) = \omega_1^{CK}$ and T is strongly computably approximable.*

The proof of Theorem 5.17 in [13] uses the technical proposition below. The statement refers to Kleene's system for assigning numbers to ordinals. The system $\mathcal{O}$, with a partial ordering $<_{\mathcal{O}}$, assigns many notations to a single ordinal. However, for a path P through $\mathcal{O}$, there is a unique notation a for each computable ordinal α. In the proof of Theorem 5.17, the path P comes from an initial segment of a Harrison ordering. We write $|a|$ for the ordinal with notation a.

PROPOSITION 5.18. *Let P be a path through $\mathcal{O}$. Let $\mathcal{A}$ be a computable structure. Let $(\mathcal{A}_a)_{a\in P}$ be a uniformly computable family of structures, and let $(\leq_a)_{a\in P}$ be a family of binary relations on tuples from the structures. Suppose that the conditions following are satisfied*:

1. $\mathcal{A}_a$ *satisfies the computable* $\Sigma_{|a|}$ *sentences true in* $\mathcal{A}$.
2. $(\leq_a)_{a\in P}$ *are reflexive and transitive,*
3. *the relations have the* back-and-forth property; *i.e.,* $\leq_0$ *preserves satisfaction of quantifier-free formulas, and if* $(\mathcal{A}_a, \overline{a}) \leq_b (\mathcal{A}_{a'}, \overline{a}')$, *and* $b' <_{\mathcal{O}} b$, *then for all* $\overline{c}'$, *there exists* $\overline{c}$ *such that* $(\mathcal{A}_{a'}, \overline{a}', \overline{c}') \leq_{b'} (\mathcal{A}_a, \overline{a}, \overline{c})$,
4. *the restriction of* $\leq_b$ *to pairs* $(\overline{a}, \overline{a}')$, *where* $\overline{a} \in \mathcal{A}_a$ *and* $\overline{a}' \in \mathcal{A}_{a'}$, *is c.e. uniformly in* b, a, a'.

Then for any Σ^1_1 *set* S, *there is a uniformly computable sequence* $(\mathcal{C}_n)_{n\in\omega}$ *such that if* $n \in S$, *then* $\mathcal{C}_n \cong \mathcal{A}$, *and if* $n \notin S$, *then* $\mathcal{C}_n \cong \mathcal{A}_a$, *for some* $a \in P$.

PROOF. The proof of Proposition 5.18 is a priority construction with actions described by a Π^1_1 set Γ of computable infinitary sentences. To show that every Δ^1_1 subset of Γ is satisfied, we use an infinitely nested priority construction. We then apply Barwise-Kreisel Compactness. ⊣

The conditions of Proposition 5.18 seem very strong. However, with effort, we can show that they are satisfied by a special tree of Scott rank ω_1^{CK}, and a family of approximating trees of computable rank.

In [11], Theorem 5.17 is combined with Proposition 5.14, to produce further structures of Scott rank ω_1^{CK} that are strongly computably approximable.

THEOREM 5.19 (Calvert-Goncharov-Knight). *There are strongly computably approximable structures of Scott rank* ω_1^{CK} *in each of the following classes*:

1. *undirected graphs,*
2. *fields of any characteristic,*
3. *linear orderings.*

§6. Open problems and future directions. In this paper, we have focused on a particular body of work that we know. It is quite small and primitive in comparison with the work in model theory and descriptive set theory that inspired it. We have not attempted to be comprehensive even in describing work in computability having to do with classification. There are results of "Reverse Mathematics", articulating the resources needed to prove various classification results, such as Ulm's Theorem [55].

We have mentioned some open problems in the body of the paper.

PROBLEM 6.1. Do the class of vector spaces over $\mathbb{Q}$ and the class of models of $Th(\mathbb{Z}, S)$ have the same c-degree?

PROBLEM 6.2. Is there a computable structure of noncomputable Scott rank with a computable infinitary pseudo-Scott sentence?

In descriptive set theory, there are results characterizing classes with invariants of "Ulm type" [24]. We hope that Theorem 4.12 may be helpful in giving further such characterizations.

PROBLEM 6.3. Can we describe in an abstract way the classes with invariants like those for vector spaces over the rationals?

The recent paper of Sacks [50] gives a wealth of results related to Vaught's Conjecture and Scott rank. The organizing theme of the paper is "bounding theorems", where these are results saying that for a sufficiently nice partial function F from ω^ω to ω_1, there is a countable ordinal bounding the range. For a countable complete theory T, there is a natural family $(L_\alpha)_{\alpha<\omega_1}$ of fragments of $L_{\omega_1\omega}$ such that each countable model $\mathcal{A}$ of T is, for some α, an atomic model of its theory in the fragment L_α. The least such α is another possible Scott rank. If the function F assigning these ranks to the models of T satisfies the conditions for some bounding theorem, then T satisfies Vaught's Conjecture.

Sacks identifies a model $\mathcal{A}$ with its L_α-theory for this α, and he gives conditions on a theory T under which there is a nice enumeration of the models/theories. He describes what would happen if Vaught's Conjecture fails for T. There would be an ω_1-chain of countable models, each atomic in its own fragment, with the next model preserving satisfaction in that fragment but realizing some non-principal type. In addition to presenting many results, Sacks states a number of open problems. We add one more.

PROBLEM 6.4. If a countable complete theory T has only countably many countable models, up to isomorphism, must there be a bound on the Scott ranks that is computable relative to T?

In general, we feel that it is worthwhile to study specific mathematically interesting classes of structures, and to try to understand the invariants, or else to say, in a concrete way that any invariants are beyond understanding. We believe that for some of these classification problems, it will be helpful to combine ideas from model theory and descriptive set theory with those from computability.

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