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GEOMETRIC CARDINAL INVARIANTS, MAXIMAL FUNCTIONS AND A MEASURE THEORETIC PIGEONHOLE PRINCIPLE

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Abstract. It is shown to be consistent with set theory that every set of reals of size $\aleph_1$ is null yet there are $\aleph_1$ planes in Euclidean 3-space whose union is not null. Similar results will be obtained for other geometric objects. The proof relies on results from harmonic analysis about the boundedness of certain harmonic functions and a measure theoretic pigeonhole principle.

§1. **Introduction.** Cantor's realization that there is a bijection between the plane and the line shifted the direction of mathematical thought by putting an end to any speculation that spaces of larger dimension had more points than spaces of smaller dimension. An unfortunate consequence of the revelation that cardinality can not distinguish so fundamental a geometric invariant as dimension is the impression that set theory and geometry are disjoint domains having little to say about each other. Certainly the vigorous and longstanding interaction between combinatorics and set theory overshadows the small number of isolated results at the interface of set theory and geometry. But this is not say that there are has been no progress in this direction. A noteworthy recent example is the solution by Jackson and Mauldin of a problem of Steinhaus about transversals of isometric copies of the lattice of integers in the plane [8, 7, 6]. A solution to the still open problem of the existence of a Borel set meeting each line on exactly 2 points may some day provide another example. However, if one asks whether geometry can directly influence the calculation of a cardinal number then results are even harder to find. If one is willing to accept a sufficiently broad definition of geometry then one can look to Cichon and Morayne's characterization of the value of the continuum in terms of differentiability properties of Peano maps onto cubes of specified dimension [2]; Osofsky had earlier obtained similar results on homological dimension [15, 14, 13]. The program of studying the cardinal invariants related to convexity in Euclidean and other spaces initiated by Kojman and his co-authors [4, 10, 9, 5, 11, 17] provides examples which are more clearly rooted in a geometric tradition.

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This paper will examine another family of geometric cardinal invariants by drawing upon results in classical harmonic analysis. At the summer meeting of the ASL in Veszprém Hungary in the August of 1992 Peter Komjáth asked whether the assertion that every set of reals of size $\aleph_1$ is Lebesgue null implies that the union of any $\aleph_1$ lines in the plane is also Lebesgue null. That this is truly a geometric question was established in [18]. To be precise, it is shown there that a positive answer cannot be found using only the methods of topology and measure theory. More generally though, it is possible to use the framework of small cardinals to define for any family $\mathcal{B}$ of Borel subsets of a Polish measure space (X, μ) the cardinal $\mathrm{add}(\mathcal{B})$ to be the least cardinal of a subfamily $\mathcal{F} \subseteq \mathcal{B}$ such that $\cup\mathcal{F}$ is not μ-null. So, letting $\mathcal{S}$ be the points in $\mathbb{R}^2$ and $\mathcal{L}$ be the lines in $\mathbb{R}^2$ Komjáth's question becomes whether or not $\mathrm{add}(\mathcal{S}) = \mathrm{add}(\mathcal{L})$. (Note that $\mathrm{add}(\mathcal{S})$ is a well known cardinal usually denoted by $\mathrm{non}(\mathcal{N}\sqcap\updownarrow\updownarrow)$.) Letting $\mathcal{P}$ be the planes in $\mathbb{R}^3$ and $\mathcal{C}$ be the circles in $\mathbb{R}^2$ one ask about the relationships between $\mathrm{add}(\mathcal{C})$ and $\mathrm{add}(\mathcal{P})$ as well as other similarly defined invariants.

One might imagine an equality such as $\mathrm{add}(\mathcal{L}) = \mathrm{add}(\mathcal{S})$ being proved by defining an ingenious mapping from lines to points is some way that preserves measure. Such a mapping would have to encode geometric information in a manner difficult to imagine but not yet ruled out. Indeed, this remains an intriguing possibility about which something will be said in §5. The main goal of this note, however, is to discuss strict inequalities and, in particular, the construction of models of set theory where $\mathrm{add}(\mathcal{B}) < \mathrm{add}(\mathcal{S})$ for certain Borel families $\mathcal{B}$ of geometric figures.

§2. The prototypical partial order. In order to illustrate the general construction, a model of set theory will be constructed for the specific task of showing the consistency of $\mathrm{add}(\mathcal{S}) > \aleph_1 = \mathrm{add}(\mathcal{C})$. In this case it necessary to find a partial order which makes the reals from the ground model into a null set in a way which does not make the union of circles from the ground model into a null set. If it can be established that this partial order is proper, then it could be iterated ω_2 times to obtain the desired model. It turns out though, that taking products is a better strategy in this case. Nevertheless, the idea is the same and the key is to find the correct partial order to use as a building block for the final result.

This partial order $\mathbb{P}$ will consist of trees of approximations to a cover of the ground model by a null G_δ. This means that $\mathbb{P}$ will force the existence of a sequence of open sets $\{U_n\}_{n=0}^{\infty}$ such that the Lebesgue measure of U_n is $1/2^n$ and every real in the ground model belongs to infinitely many of the U_n. Moreover, the trees in $\mathbb{P}$ will consist of finite sequences t, ordered by end extension, which approximate $\{U_n\}_{n=0}^{\infty}$ in the sense that for each j in its domain $t(j)$ is an open set of measure $1/2^n$ which is a candidate to be U_j. The strategy for constructing $\mathbb{P}$ will require choosing a very quickly growing sequence of integers $\{M_n\}_{n=0}^{\infty}$ and letting $\mathcal{X}_n$ consist of all sets of the form $\bigcup_{i\in X}\left[i/M,(i+1)/M\right]$ where $X \subseteq M$ has size less than $M/2^n$. For each n an $\mathbb{N}$-valued norm ν_n will then be defined on $\mathcal{P}(\mathcal{X}_n)$ which will measure the size of subsets of $\mathcal{X}_n$. The trees belonging to $\mathbb{P}$ will then be subtrees of the tree of all sequences t such that $t(n) \in \mathcal{X}_n$ for each n in the domain of t. Furthermore, it will be required that for each $t \in T$ and each integer k there is some $s \supseteq t$ such that $\nu_{m+1}\left(\Sigma_T(s)\right) \geq k$ where $\Sigma_T(s)$ is the set of successors of s or, in other words, $\Sigma_T(s) = \{X \in \mathcal{X}_{m+1} \mid s^\frown X \in T\}$. As long as it is the case that $\cup\mathcal{Y} = [0,1]$ whenever $\nu_n(\mathcal{Y}) > 0$ then genericity will guarantee that every real is contained in infinitely many U_n. It therefore follows that $\mathbb{P}$ forces the ground model reals to become a null set. But why is it that the union of the ground model circles is not Lebesgue null after forcing with such a partial order? The answer lies in the construction of the norms ν_n.

It will turn out that the partial order $\mathbb{P}$ is ω^ω-bounding so that if T forces that the ground model circles are covered by an open set of measure less than 1 then it can be assumed that there is a continuous function Φ, defined on $\overline{T}$, the branches of T, and taking as values open sets in the plane[1] such that $\Phi(G)$ contains each circle from the ground model where G is the generic branch through T. Given a node $t \in T$, let $\Phi^*(t)$ be the interior of $\bigcap_{b\in\overline{T},t\subseteq b}\Phi(b)$; in other words, $\Phi(t)$ is that part of the open set about which all branches passing through t agree. Using the continuity of Φ it is possible to choose for each $X \in \Sigma_T(t)$ a node $t_X \supseteq t^\frown X$ such that $\Phi^*(t_X)$ is a very good approximation to $\Phi(G)$. Letting $F(X) = \Phi^*(t_X) \setminus \Phi^*(t)$ yields a function F defined on $\Sigma_T(t)$ which sends elements of $\Sigma_T(t)$ to open sets of small measure, the exact meaning of “small measure” being determined by t. Moreover, the condition below T determined by $t^\frown X$ forces $F(X)$ to be contained in $\Phi(G)$, the new set containing the ground model circles. The goal at this stage is to find many circles C such that

$$\nu_{|t|}\left(\left\{X \in \Sigma_T(t) \mid \lambda(C \cap F(X)) < \varepsilon_t\right\}\right) \geq \nu_{|t|}(\Sigma_T(t)) - 1 \tag{2.1}$$

where λ is the normalized linear measure on the circle. The constants ε_t depend on some calculations not yet discussed.

Of course, the meaning of “many circles” will also have to be made more precise but, for the moment, it will be enough to know that it refers to a measure on the family of circles and that “many” means all but a set of small measure and the exact meaning of “small” depends on the node. The important point is that if a small set of circles is discarded for each node of the tree then at least one circle C will be left over. Now if T^* is defined to be the tree obtained by taking for each node t only those $Y \in \Sigma_T(t)$ such

[1] As usual, open sets can be coded as subsets of $\mathbb{N}$ by enumerating all the rational rectangles in the plane and identifying an open set with the integers enumerating rectangles contained in the open set. So Φ is continuous as a function to codes for open sets.

that $\lambda(C \cap F(Y)) < \varepsilon_t$ then it the follows that $T^* \in \mathbb{P}$ by Inequality 2.1 and, moreover, if the ε_t have been chosen such that the sum of all ε_t as t ranges over the nodes of the tree T is less than 1, then T^* forces that the linear measure of the intersection of C with $\Phi(G) = \bigcup_n (\cup G)(n)$ is less than 1, a clear contradiction. Note that this is not the same as finding a point on the circle which is not covered; indeed this can not be done if we are to succeed with the first goal of making the ground model reals into a null set.

The preceding informal discussion was intended to serve as motivation for the following definition of the norms v_n.

DEFINITION 2.1. For $n \in \mathbb{N}$ define a function $v\colon \mathcal{P}(\mathcal{X}_n) \to \mathbb{N}$ by using induction to provide increasingly better lower bounds for $v(\mathcal{Y})$.

- Define $v(\mathcal{Y}) \geq 0$ if $\mathcal{Y} \neq \emptyset$.
- Define $v(\mathcal{Y}) \geq 1$ if $\cup\mathcal{Y} = M$.
- If $j \geq 1$ define $v(\mathcal{Y}) \geq j+1$ if for every function F from $\mathcal{Y}$ to open set of measure less than $J(j)$ (where $\lim_{j\to\infty} J(j) = 0$ and this convergence is very rapid)

$$\lambda \times \sigma\Big(\Big\{(x,z) \in \mathbb{R}^2 \times [1,2] \mid v\big(\big\{Y \mid \lambda^s(C_x^z \cap F(Y)) < 1/n\big\}\big) < j\Big\}\Big) < 2/j \tag{2.2}$$

where C_x^z denotes the circle with centre x and radius z. Define $v(\mathcal{Y}) = j$ if $v(\mathcal{Y}) \geq j$ and $v(\mathcal{Y}) \not\geq j+1$.

The second item in the definition guarantees that $\bigcup_{n=j}^{\infty} U_n$ covers the ground model reals for every j. The measure $\lambda \times \sigma$ makes precise the notion of "many" circles. The crucial question, however, is:

QUESTION 2.1. Why should there be any $\mathcal{Y}$ such that $v_n(\mathcal{Y}) > 1$?

The answer to this is provided by a result of Bourgain. For families of sets other than circles there are analogous results from harmonic analysis that play a similar role.

§3. Maximal functions and a measure theoretic pigeonhole principle. The key to answering Question 2.1 is the following consequence of a theorem [1] of Bourgain on the boundedness of a certain maximal operator: There is a constant K such that for every measurable set $A \subseteq \mathbb{R}^2$

$$\lambda^2\Big(\Big\{x \in \mathbb{R}^2 \mid (\exists z \in [0,1])\ \lambda(C_x^z \cap A) \geq \varepsilon\Big\}\Big) \leq \frac{K\lambda(A)}{\varepsilon^2} \tag{3.1}$$

where λ^2 is Lebesgue measure in the plane. To see the connection between Inequality 2.2 and Inequality 3.1 observe that Inequality 2.2 says that for most circles the norm of those Y for which $F(Y)$ intersects it on a set of small linear measure is large. One way that this could fail is if there were a set W of small Lebesgue measure which intersected many circles on a set of large linear measure and such that $F(Y) = W$ held for each Y. Bourgain's Inequality 3.1 does not quite say that this can not happen, but it does say that there are many centres x for which there is some radius z such that $C_x^z \cap W$ must have large linear measure. In order to exploit this, a new norm ρ based on a measure theoretic pigeonhole principle has to be introduced. The norm is designed to ensure that if the Inequality 3.1 holds for each of some collection of sets of positive rank then the set has sufficiently large measure to guarantee that it can not serve as W. Indeed, it will be shown that there is a satisfactory answer to the variant of Question 2.1 with ρ in place of ν. It will then be shown that $\rho \leq \nu$ and this will answer Question 2.1.

The measure theoretic pigeonhole principle upon which this argument hinges says, in the 1-dimensional case, that a measurable mapping of a product of probability spaces either maps most of a vertical section to small set, or, it maps most of the graph of a function to small set; the graph of a function case is the one which appears in the conclusion of Bourgain's result. The following two definitions will help to make this precise.

Definition 3.1. Let (Q_1, μ_1) and (Q_2, μ_2) be probability spaces and $\gamma > 0$. For $Z \subseteq X \times Y$ and $x \in X$ let $Z\langle x\rangle = \{y \in Y \mid (x, y) \in Z\}$. For A and B subsets of $\prod_k (Q_1 \times Q_2)$ define $A \Subset_\gamma B$ by induction on k. If $k = 1$ then $A \Subset_\gamma B$ if and only if $A \subseteq B$ and either

$$(\exists p \in Q_1)\, B\langle p\rangle \neq \emptyset \text{ and } \mu_2\,(B\langle p\rangle \setminus A\langle p\rangle) < \gamma \tag{3.2}$$

or

$$\mu_1\,(\{p \in Q_1 \mid A\langle p\rangle = \emptyset \neq B\langle p\rangle\}) < \gamma. \tag{3.3}$$

If $k > 1$ then $A \Subset_\gamma B$ if and only if $A \subseteq B$ and

$$\left\{x \in Q_1 \times Q_2 \mid A\langle x\rangle \Subset_\gamma B\langle x\rangle\right\} \Subset_\gamma \pi(B). \tag{3.4}$$

where $\pi(B) = \{x \in Q_1 \times Q_2 \mid B\langle x\rangle \neq \emptyset\}$.

In the next definition the probability spaces (Q_1, μ_1) and (Q_2, μ_2) will be taken to be $[0, 1]^2$ with Lebesgue measure and $[0, 1]$ with Lebesgue measure. The first can be thought of as the space of centres of circles in the plane and the second can be thought of as the space of radii.

Definition 3.2. Given $M \in \mathbb{N}$ and $\xi > 0$ and $\gamma > 0$ define a function $\rho\colon \mathcal{P}(\mathcal{P}_\xi(M)) \to \mathbb{N}$ by induction similarly to the definition of ν:

- Define $\rho(\mathcal{Y}) \geq 0$ if $\mathcal{Y} \neq \emptyset$.
- Define $\rho(\mathcal{Y}) \geq 1$ if $\cup\mathcal{Y} = M$.
- If $j \geq 1$ define $\rho(\mathcal{Y}) \geq j+1$ if for every measurable $A \subseteq (Q_1 \times Q_2)^j$ and every measurable $\Psi\colon A \to M$ there is $Y \in \mathcal{Y}$ such that $\Psi^{-1} Y \Subset_\gamma A$.

Define $\rho(\mathcal{Y}) = j$ if $\rho(\mathcal{Y}) \geq j$ and $\rho(\mathcal{Y}) \not\geq j + 1$.

The key fact about this relation is the following:

Lemma 3.1. *Suppose that ξ and γ are positive reals and that j is a non-zero integer. There is an integer M such that for any measurable $B \subseteq (Q_1 \times Q_2)^j$ and any measurable function $\Psi\colon B \to M$ there is $W \in \mathcal{P}_\xi(M)$ such that $\Psi^{-1}W \Subset_\gamma B$.*

The proof of Lemma 3.1 in the case $j = 1$ is not difficult. Choose M large enough that probabilistic arguments can be used. Given Ψ there are two cases to consider. The first is that there is some $q \in Q_1$ such that the image of $B\langle q\rangle$ under Ψ has cardinality less than ξM. In this case simply let W be this image. In the second case, the cardinality of the image of $B\langle q\rangle$ under Ψ is at least ξM for every $q \in Q_1$. If W is chosen by letting $m \in M$ belong to W if and only if a $\{0,1\}$-valued, independent, random variable with mean ξ is equal to 1 then with high probability W intersects any $B\langle q\rangle$. Applying Fubini's Theorem yields that W intersects most of the $B\langle q\rangle$. The argument for $j > 1$ is complicated by having to inductively combine these two arguments—one deterministic and the other probabilistic.

From Lemma 3.1 it follows that there are sets $\mathcal{X}$ such that $\rho_n(\mathcal{X})$ is as large as desired. Hence the following lemma answers the crucial Question 2.1 at the end of §2.

Lemma 3.2. *If $j \in \mathbb{N}$, $\xi < 2^{-n}$ and γ is sufficiently small then for any $\mathcal{Y} \subseteq \mathcal{X}_n$ if $\rho(\mathcal{Y}) \geq j$ then $\nu(\mathcal{Y}) \geq j$.*

§4. The general theorem.

Definition 4.1. A mapping Λ will be said to be *appropriate* is there are Polish probability space (P_0, σ_0) and (P_1, σ_1) such that:

- $\Lambda\colon P_0 \times P_1 \to \mathrm{Compact}(\mathbb{R}^n)$ is a continuous mapping to the space of compact subsets of $\mathbb{R}^n$ under the Hausdorff metric.
- There is an integer $s = s(\Lambda)$ such that $1 < \lambda^s(\Lambda(p_0, p_1)) < \infty$ for all $(p_0, p_1) \in P_0 \times P_1$.
- There is a continuous function $J\colon (0,1)^2 \to (0,1)$ such that for all $(\theta, \delta) \in (0,1)^2$ and for all measurable $A \subseteq \mathbb{R}^n$

$$(\forall p_1 \in P_1) \text{ if } \mu\left(\{p_0 \in P_0 \mid \lambda^s(\Lambda(p_0, p_1) \cap A) > \theta\}\right) > \delta \qquad \text{then } \lambda(A) > J(\theta, \delta). \tag{4.1}$$

In most cases the probability space P_1 can be ignored since it consists of a singleton with all the measure concentrated on it, but there is an important exception. Two examples would be useful at this point.

Example 4.1. Let $(P_0, \sigma_0) = ([1,2], \lambda)$ and let $P_1 = \{0\}$ and let σ_1 concentrate the probability measure on $\{0\}$. Let $\Lambda\colon P_0 \times P_1 \to \mathrm{Compact}(\mathbb{R}^k)$ be defined by letting $\Lambda(t, 0)$ be the surface of the n-sphere of radius t with centre at the origin. In this case $s(\Lambda) = k - 1$.

EXAMPLE 4.2. Let $(P_0, \sigma_0) = ([0,1], \lambda)$ and let $(P_1, \sigma_1) = (S^2, \lambda^2/4\pi)$ where S^2 denotes the surface of the unit sphere in $\mathbb{R}^3$. Let $\Lambda\colon P_0 \times P_1 \to \text{Compact}(\mathbb{R}^3)$ be defined by letting $\Lambda(t,x)$ be the disk of radius 1 with centre at tx and so that x is normal to the plane determined by the disk $\Lambda(x)$. In this case $s(\Lambda) = 2$.

It is easy to apply Fubini's Theorem to verify that each of the examples satisfy the Hypothesis 4.1.

DEFINITION 4.2. For any appropriate function Λ with $s = s(\Lambda)$ the maximal operator $M_\Lambda\colon L^1([0,1]^n) \to L^\infty([0,1]^n)$ is defined by

$$M_\Lambda(f)(x) = \int_{p_1 \in P_1} \left(\sup_{p_0 \in P_0} \int_{z \in \Lambda(p_0, p_1)} \frac{f(z+x)\,d\lambda^s}{\lambda^s(\Lambda(p_0, p_1))} \right) d\sigma_1 \tag{4.2}$$

and the operator M_Λ will be said to be bounded if there are real numbers $p \geq 1$ and $q \geq 1$ and a constant K such that

$$\|M_\Lambda(f)\|_p \leq K\|f\|_q \tag{4.3}$$

for all $f \in L^q([0,1]^n)$.

Applying the methods outlined in the construction of the model for the special case of circles it is possible to prove the following.

THEOREM 4.1. *If Λ is appropriate and M_Λ is bounded then it is consistent that* $\text{add}(\mathcal{S}) > \aleph_1 = \text{add}(\mathcal{B})$ *where $\mathcal{B}$ is the range of Λ.*

In order to apply Theorem 4.1 it is necessary to look for families satisfying the hypotheses of Definition 4.1 and Inequality 4.2. Several such exist in the literature.

THEOREM 4.2 (Stein [16]). *If Λ is as defined in Example* 4.1 *and if $k \geq 3$ then the maximal operator M_Λ is bounded.*

THEOREM 4.3 (Bourgain [1]). *If Λ is as defined in Example* 4.1 *and if $k = 2$ then the maximal operator M_Λ is bounded.*

THEOREM 4.4 (Falconer [3] and Marstrand [12]). *If Λ is as defined in Example* 4.2 *then the maximal operator M_Λ is bounded.*

COROLLARY 4.1. *It is consistent that every set of reals of size $\aleph_1$ is null yet there are $\aleph_1$ spheres of dimension $k-1$ whose union is not null in $\mathbb{R}^k$.*

PROOF. Use Theorem 4.1 and Theorem 4.2 for the case $k \geq 3$ and Theorem 4.3 for the case $k = 2$. ⊣

COROLLARY 4.2. *It is consistent that every set of reals of size $\aleph_1$ is null yet there are $\aleph_1$ planes in the Euclidean* 3*-space whose union is not null.*

PROOF. Use Theorem 4.1 and Theorem 4.4. ⊣

§5. Further questions. In spite of their generality, the results of §4 leave open the original question posed by Komjáth: Does $\operatorname{add}(\mathcal{S}) = \aleph_1$ imply that $\operatorname{add}(\mathcal{L}) = \aleph_1$. The only stumbling block is that the appropriate maximal operator M_Λ—where Λ is the 2-dimensional version of the Λ from Example 4.2—is not bounded as was shown by Besicovitch with his construction of a Kakeya set, a null set containing a line segment in every direction. Therefore a negative answer to Komjáth's question will require a radically different approach from the one outlined here. However, the possibility exists that there is a positive answer and the arguments presented here provide the hint that any proof of the positive direction must use a Kakeya set or some similar construction. Since Kakeya sets are null sets containing many line segments it is conceivable that some small number of translations of a suitably constructed Kakeya set could cover any family of lines whose cardinality does not exceed $\operatorname{add}(\mathcal{S})$.

In this context it is worth mentioning a result of Talagrand [19] which shows the existence of an analogue of the Kakeya set for circles. In particular, he shows that there exists a family of circles of the plane whose union is of measure zero, but such that the set of the centres is of non-zero linear measure. This provides the same sort of obstacle as the Kakeya set to proving the consistency of: Every set of reals of size $\aleph_1$ is null yet there are $\aleph_1$ circles in the plane *with centres on the x-axis* whose union is not null. This remains an open question with the possibility that Talagrand's construction can be used to provide a positive result.

Finally, there is the question of establishing consistent inequalities such as $\operatorname{add}(\mathcal{L}) \neq \operatorname{add}(\mathcal{C})$, in other words inequalities not involving $\operatorname{add}(\mathcal{S})$. Several difficulties would have to be overcome to make progress in this direction. First, substantial changes to the forcing partial order would be required since the objective now would be to cover, for example, ground model circles with a null set while keeping the measure of the ground model lines positive. Assuming this could be done, a more serious and potentially more interesting challenge to harmonic analysis would arise in finding the correct boundedness theorems for maximal operators.

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