THE BULLETIN OF SYMBOLIC LOGIC
Volume 11, Number 4, Dec. 2005

ALGEBRAIC LOGIC, WHERE DOES IT STAND TODAY?

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Abstract. This is a survey article on algebraic logic. It gives a historical background leading up to a modern perspective. Central problems in algebraic logic (like the representation problem) are discussed in connection to other branches of logic, like modal logic, proof theory, model-theoretic forcing, finite combinatorics, and Gödel's incompleteness results. We focus on cylindric algebras. Relation algebras and polyadic algebras are mostly covered only insofar as they relate to cylindric algebras, and even there we have not told the whole story. We relate the algebraic notion of neat embeddings (a notion special to cylindric algebras) to the metalogical ones of provability, interpolation and omitting types in variants of first logic. Another novelty that occurs here is relating the algebraic notion of atom-canonicity for a class of boolean algebras with operators to the metalogical one of omitting types for the corresponding logic. A hitherto unpublished application of algebraic logic to omitting types of first order logic is given. Proofs are included when they serve to illustrate certain concepts. Several open problems are posed. We have tried as much as possible to avoid exploring territory already explored in the survey articles of Monk [93] and Németi [97] in the subject.

§1. Introduction. We start our article by the following two quotations of De Morgan and Russell indicating the importance of the work of Boole, the father of algebraic logic.

> *When the ideas thrown out by Boole shall have borne their full fruit, algebra though founded on the idea of number in the first instance will appear as a sectional model of the whole form of thought. Its forms considered apart from their matter will be seen to contain all the forms of thought in general. The anti-mathematical logician says that it makes thought a branch of algebra, instead of algebra a branch of thought. It* makes *nothing. It* finds *and it finds the laws of thought symbolized in the forms of algebra.* De Morgan in [26].

> *Pure mathematics was discovered by Boole, in a work which he called the Laws of Thought. His book was in fact concerned with formal logic and this is the same thing as mathematics.* Bertrand Russell as quoted in [16].

Received October 22, 2002; revised December 8, 2004.

§2. Algebraic logic. Algebraic logic arose as a subdiscipline of algebra mirroring constructions and theorems of mathematical logic. It is similar in this respect to such fields as algebraic geometry and algebraic topology, where the main constructions and theorems are algebraic in nature, but the main intuitions underlying them are respectively geometric and topological. The main intuitions underlying algebraic logic are, of course, those of formal logic. Investigations in algebraic logic can proceed in two conceptually different, but often (and unexpectedly) closely related ways. First one tries to investigate the algebraic essence of constructions and results in logic, in the hope of gaining more insight that could add to his understanding, thus his knowledge. Second, one can study certain "particular" algebraic structures (or simply algebras) that arise in the course of his first kind of investigations as objects of interest in their own right and go on to discuss questions which naturally arise independently of any connection with logic. But often such purely algebraic results have impact on the logic side.[1]

Throughout this article we will have occasion to deal with both types of investigations (algebraic and metamathematical) and their unexpected and indeed intriguing interplay and interaction. Also one of the aims of this article is see what the ideas of Boole have led to. Have they borne their full fruit, or not yet? But let us start with some history.

§3. History. In the middle of the nineteenth century George Boole initiated the investigation of a class of algebraic structures which were subsequently called boolean algebras. The theory of these algebras is directly related to the development of the most elementary part of mathematical logic, namely propositional logic. As is well known, however, the theory of boolean algebras can be developed in a purely algebraic fashion. It has at present numerous connections with several branches of mathematics–(independence results in) set theory, topology and analysis—and hence it can be understood and appreciated by mathematicians unfamiliar with the logical problems to which it owes its birth. The work of Boole was the starting point for a continuous flow of inquiries into the algebraization of *quantifier* logics (like first order logic and infinitary reducts of Keisler's logic) which through various intermediate stages led to (but did not end with) the foundation of the theory of cylindric and polyadic algebras.

[1]A new example of this phenomenon is the investigation of the algebraic notions of *complete representation* and that of *atom-canonicity*. The class of the so-called completely representable relation and cylindric algebras is not an elementary class (i.e., it is not axiomatizable) as proved by Hirsch and Hodkinson [44]. The class of representable cylindric algebras of finite dimension is not atom-canonical in the sense of [36]. These purely algebraic results were used by Andréka and the present author in [13] and [114] to show that the Henkin-Orey Omitting Types Theorem fails for the finite variable fragment of first order logic. This connection will be further elaborated upon below.

The history of the subject is interesting and it is worth briefly reviewing. The story begins in the nineteenth century when the pre-modern tradition of algebraic logic began. We find De Morgan in a restive frame of mind trying to extend Aristotle's syllogistic tradition that held sway for 2000 years to encompass more complex situations when he came across the slightly earlier work of Boole who had come forward with a highly successful algebra of propositions. In 1860 De Morgan published [26] thereby launching an investigation into the algebra of relations. This developed into the subject known now as algebraic logic, though in the nineteenth century it was simply thought of as mathematical logic. This work, along with Frege's quantifier logic, became the foundation of modern logic and model theory.

So in the nineteenth century there were two approaches to the formalization of *quantification* in logic. The first[2] (algebraic) approach, due to Boole and De Morgan and taken up by Peirce and Schröder led to what we now call (following Tarski) relation algebra. The other approach, due to Frege, and boosted by Russell and Whitehead in their momentous work *Principia Mathematica*, became the standard formalism of first order logic with its explicit universal and existential quantifiers. Both can express quantification, albeit in different ways.[3]

Historians of mathematical logic frequently tell us that there are two traditions, the algebraic tradition of Boole, Schröder, De Morgan, and Peirce, arising from the algebraization of analysis as opposed to the quantification-theoretical (or logistic) tradition of Peano, Frege, and Russell arising from the development of the theory of functions. It is said that these two traditions, together with the independent set-theoretic tradition of Cantor, Dedekind, and Zermelo, arising out of the search for a foundation for real analysis in the work of Cauchy, Weierstrass and others, were united by Whitehead and Russell in their Principia Mathematica to create mathematical logic. However, the dual *algebraic* and *quantification-theoretic* traditions, as a matter of of historical fact, simply did not exist for logicians in the turn of the century. It is a false retrospective duality which derives from the Principia and is post Principia phenomena. There was no such dichotomy in the nineteenth century, algebraic logic was simply *the* mathematical logic of its time.[4]

[2]historically, too, it was prior.

[3]For example, in first order logic we can say that there exists a person who is my parent and your sibling, which could be expressed in relation algebra as you are either my uncle or my aunt. The fundamental operations on binary relations considered since De Morgan's time and highlighted by Tarski are the boolean operations of intersection and complementation, the unary converse operation and the binary operation of composition. It can be seen that composition gives a qualified (relativized) form of existential quantification, and indeed [16] argues that both can be considered foundations of modern logic.

[4]It is worthy of note that a unification of the two approaches occurred in the twentieth century when Leon Henkin was concerned with algebraizing Russell's theory of types. Henkin's

§**4. Cylindric algebras.** In the twentieth century first order logic was given an algebraic setting by Tarski in the framework of *cylindric algebra*. This was a natural outcome of Tarski's formalization of the notion of truth in set theory, for indeed the prime examples of cylindric algebras are those algebras whose elements are sets of sequences (i.e., relations) satisfying first order formulas.

While relation algebras constitute an algebraization of binary relations, n-dimensional cylindric algebras are an algebraization of n-ary relations. (Here n is any (not necessarily finite) ordinal.) Ever since relation and cylindric algebras were defined, researchers have been investigating the connections between them. Tarski and Henkin investigated such connections, cf. [41, Thm. 5.3.8]. More recent references include the work of Maddux [70, 73] and the work of Hirsch and Hodkinson [46, 48]. Indeed, every cylindric algebra of dimension > 3 has a natural relation algebra reduct obtained by taking the 2-*neat reduct* i.e., the essentially 2-dimensional elements abstracting binary relations, and defining converse and composition using one spare dimension. Conversely, one can construct cylindric algebras of arbitrary dimension n, $3 < n \leq \omega$ from relation algebras that possess what Maddux calls an n-dimensional cylindric basis [73].[5] Making another leap into a modern perspective, the motivation for studying the theory of relation algebras and cylindric algebras, or hereinafter algebras of relations for short, today comes from at least three areas:

(i) Logic: predicate logics, propositional multi-modal logics, dynamic logics.
(ii) Algebra: the algebraic theory of these algebras is of interest in its own right. Indeed boolean and cylindric algebras served as a starting point for the (by now well-developed) theory of discriminator varieties.
(iii) The theory of relations: relations are of interest in themselves.

We could further motivate algebraic logic by surveying its numerous applications in temporal reasoning, planning data bases, modal logic and elsewhere, cf. the survey article [97] for an exposition of such applications. Instead, we prefer to recommend the part of algebraic logic most relevant to our investigations here as a very well established mathematical procedure, *algebraization*, applied to a fundamental entity, viz. a *relation*.

For this purpose, let us focus for a while on cylindric algebras, which is an abstraction from algebras of (not necessarily binary) relations. The concept of cylindric algebras permits the use of algebraic methods in treating two related parts of mathematics. One of these is a very general kind of geometry associated with basic set-theoretic notions. Indeed, Andréka, Madárasz

work is not well known. I came to know about it from his student Mohamed Amer, who happened to be my supervisor.

[5]We shall deal with such relation algebras to construct cylindric algebras, showing that the Omitting Types Theorem fails for the finite variable fragments of first order logic.

and Németi have used their geometric intuition originating from their investigations in the theory of cylindric algebra to formulate Einstein's general theory of relativity in first order logic, an indeed exotic and exciting novel application of algebraic logic [10]. The other—as previously mentioned—is the theory of deductive systems in mathematical logic. The two parts are indeed interconnected, because models of deductive systems (like first order theories) give rise in a natural way to structures within the set-theoretic "spaces" (set algebras based on these models.)

§5. Cylindric algebras of formulas. To illustrate further this connection, let L be any first order relational language, i.e., L has no function symbols nor individual constants. Then L has an infinite countable sequence of individual variables $x_0, x_1, \dots, x_n$ $(n < \omega)$, logical constants $\neg$, $\Longrightarrow$, $\forall$ $\exists$ $=$, and non-logical constants—say a system $\langle R_i : i \in I \rangle$ of relation symbols, where R_i is of rank $\rho_i < \omega$ for each $i \in I$.

We assume as known the usual syntactical primitive notions defined in terms of L, e.g., the notions of a formula, a sentence (formula without free occurrences of variables), the conjunction $\phi \wedge \psi$ of formulas of L, the notion of a formal proof of a formula from a set of sentences, etc. Now given a set Γ of sentences, sometimes referred to as a theory, we may call two formulas ϕ and ψ equivalent under (or modulo) Γ, in symbols

$$\phi \equiv_\Gamma \psi$$

provided that the biconditional $\phi \longleftrightarrow \psi$ is provable from Γ, in symbols $\Gamma \vdash \phi \longleftrightarrow \psi$. Here $\vdash$ denotes a standard (complete) proof system.[6] It is easy to check that the relation $\equiv_\Gamma$ is actually an equivalence relation on

[6] We defined the congruence relation via syntactical means, i.e., by fixing a standard complete proof system and defining the relation in terms of provability. The congruence relation we defined is denoted by $\equiv^p_\Gamma$ in [41, §4.3] with the superscript p indicating that the relation is defined via some fixed proof system. This relation could have just as well been defined by semantical means, i.e., by resorting to the notion of satisfiability in models. In more detail, define $\equiv^s_\Gamma$—with the superscript s now indicating that we are adopting *semantical* means—as follows: $\phi \equiv^s_\Gamma \psi$ if for every model M of Γ and every assignment $s \in {}^\omega M$, s satisfies ϕ if and only if s satisfies ψ. Because of Gödel's celebrated Completeness Theorem it follows these two congruences coincide (on "ordinary" first order languages) i.e., $\equiv^s_\Gamma = \equiv^p_\Gamma$, for any Γ. However in other contexts, such as the finite-variable fragments of first order logic and the finitary logics of infinitary relations studied [41, §4.3] the semantical congruence defined via satisfiability in models, is not easily captured by "finitary" means such as finite Hilbert- style axiomatizations. A lot of research in algebraic logic is devoted to investigating completeness for these logics.

Such investigations are referred to in the literature as the *Representation Problem*; or rather the logical counterpart thereof. The Budapest group has called this problem the *Finitization Problem*. The adjective "finitization" refers to the attempt to capture the infinity of validities by thoroughly finitary means. See [41, Prob. 4.16], for one incarnation of this problem, and Németi [97] for several other incarnations. We will have occasion to discuss in some depth the (algebraic aspect of the) Finitization Problem.

the set of formulas. In fact, it is a congruence relation compatible with the boolean operations of conjunction and negation on formulas. Even more, if we let A_Γ denote the set of equivalences classes under $\equiv_\Gamma$, we find that certain algebraic operations can be introduced on A_Γ which reflect the syntactical operations of building formulas. Besides the boolean operations

$$[\phi]_\Gamma + [\psi]_\Gamma = [\phi \vee \psi]_\Gamma,$$
$$[\phi]_\Gamma . [\psi]_\Gamma = [\phi \wedge \psi]_\Gamma,$$
$$-[\phi]_\Gamma = [\neg\phi]_\Gamma,$$

we have

$$c_i[\phi]_\Gamma = [\exists v_i \phi]_\Gamma,$$
$$d_{ij} = [v_i = v_j]_\Gamma.$$

Here, of course, $[\phi]_\Gamma$ is the equivalence class of ϕ under Γ. Note that the congruence relation defined via provability is now compatible with the newly added operations c_i and d_{ij} for each $i, j < \omega$. The resulting quotient algebra $\mathcal{A}_\Gamma$ thus associated with L and Γ is one of the fundamental algebras studied in algebraic logic, and it is in fact the prime source of inspiration for cylindric algebras. It turns out that many constructions and theorems in logic can be algebraically reflected using these algebras. For example Γ is complete and consistent if $\mathcal{A}_\Gamma$ is a simple algebra, i.e., has no proper congruences. The theorem that any consistent theory can be extended to a complete and consistent theory is mirrored by the theorem that any algebra $\mathcal{A}_\Gamma$ with $|A_\Gamma| > 1$ has a simple homomorphic image. For more on such connections between metalogical notions and algebraic ones, the reader is referred to [41, §4.3]. The algebra $\mathcal{A}_\Gamma$ is referred to as the *Tarski-Lindenbaum cylindric algebra of formulas corresponding to* Γ, or simply an algebra of formulas.

§6. Abstract cylindric algebras. The notion of a cylindric algebra is obtained from algebras of formulas by a process of abstraction. Let α be an ordinal. A *cylindric algebra of dimension* α, for brevity a CA_α, is an algebra of the form

$$\mathcal{A} = \langle A, +, \cdot, -, c_i, d_{ij} \rangle_{i,j<\alpha}$$

where $\langle A, +, \cdot, - \rangle$ is a boolean algebra with $+$ denoting the *boolean join*, $\cdot$ denoting the *boolean meet* and $-$ denoting *boolean complementation.* The c_i's, $i \in \alpha$, are unary operations of *cylindrification* on A (the domain of $\mathcal{A}$) and the d_{ij}'s, $i, j \in \alpha$, the *diagonal elements*, are distinguished elements of A. The operation c_i is an abstract version of the unary operation on first order formulas of existential quantification with respect to the i-th variable x_i. The diagonal element d_{ij} is an abstract version of the atomic identity formula $x_i = x_j$ in first order logic. The class CA_α of all cylindric algebras of dimension α is axiomatized by finitely many equational schemata that aim at capturing the essential algebraic properties of existential quantification and atomic identity formulas (cf. [40, Def. 1.1.1].) Intuitively, the axioms say the following for all $i, j, k \in \alpha$.

(i) c_i is an additive complemented closure operator.
(ii) $c_i c_j x = c_j c_i x$, i.e., the cylindrifications commute.
(iii) The diagonal elements satisfy the usual axioms one would expect of equality, such as $d_{ij} \cdot d_{jk} \leq d_{ik}$ (transitivity).
(iv) $d_{ij} \cdot c_i x \leq x$ for all $x \leq d_{ij}$.

Axiom (iv) for example is an algebraic version of Leibniz's law of equality saying that equal objects are indistinguishable by formulas, i.e.,

$$\phi \Longrightarrow v_i = v_j \vdash (v_i = v_j \wedge \exists v_i \phi) \Longrightarrow \phi.$$

Now what are the advantages of working with the abstract algebras rather than the concrete structures? There are a number of possible answers to this rather deep question. Firstly, the technique of focusing our whole attention on a small number of properties to the exclusion of all others is quite common in scientific analysis. Secondly and precisely because we are ignoring many concrete features of our structures, we may find that our approach is more general, and that the complexity of certain problems may be lower. Finally, algebra is a very thoroughly studied discipline and there is a range of methods and results that we can apply to the study of relations. Universal Algebra and model theory have techniques and theorems that can be fruitfully exploited here.

That the abstraction from the algebras of formulas to abstract algebras is indeed sound, is established by the following logical representation theorem, which is not very difficult to prove [40]. Loosely speaking, the proof of Fact 1 consists of simply checking that the postulates characterizing cylindric algebras form an adequate algebraic transcription of the axioms of any standard complete system for first order logic, cf. [41, Thm. 4.2.28(iii)].

FACT 1. [40, 2.6.49] For any algebra $\mathcal{A}$ having the same signature as CA_ω the following two conditions are equivalent:

(i) $\mathcal{A} \cong \mathcal{A}_\Gamma$ for some set of first order formulas Γ.
(ii) $\mathcal{A}$ is a CA_ω such that the set $\Delta x = \{i \in \omega : c_i x \neq x\}$ is finite for every $x \in A$.

Fact 1 justifies the choice of equations axiomatizing CA_α. In the infinite dimensional case, these equations force locally finite algebras satisfying them to be isomorphic to algebras of formulas. Historically that was the reason Tarski stipulated the (by now) official axiomatization of cylindric algebras adopted in [40]. Δx is referred to as the *dimension set* of x, and sometimes the *support* of x.[7] An algebra for which Δx is finite for all x is called a locally finite-dimensional algebra. The class of all such algebras of dimension α is

[7] The support of x comes from terminology of polyadic algebras, due to Halmos.

denoted by Lf_α. Of course all finite dimensional algebras are locally finite-dimensional, but for the ω-dimensional algebras, it is precisely the property of local finite-dimensionality that singles out algebras of formulas from arbitrary cylindric algebras (up to isomorphism.) In algebras of formulas, Δx is finite because every first order formula is a finite string of symbols and hence contains only finitely many free variables.

§7. Models and set algebras. The other more concrete source of cylindric algebras is that of *cylindric set algebra*, which Tarski introduced as an algebraic counterpart of the semantics of first order logic.[8] Such algebras, the cylindric set algebras, arise naturally from models of first order theories, and therefore they are closely related to the algebras of formulas. To explain further the connection of cylindric set algebras to algebras of formulas, we now turn to some basic semantical notions.

Fix a first order relational language L. Let M be an L-structure. We take it as well-known what it means for a sequence $s \in {}^{\omega}M$ to *satisfy* a formula ϕ in M. We write $M \models \phi[s]$ if s satisfies ϕ in M. For $\phi \in L$, let ϕ^M be the set of all ω-ary sequences, or assignments, that satisfy ϕ in M, that is

$$\phi^M = \{s \in {}^{\omega}M : M \models \phi[s]\}.$$

Then ϕ^M is an ω-ary relation on M; it is a point-set in the ω-dimensional space ${}^{\omega}M$, and the set of all these, i.e., the set

$$A_M = \{\phi^M : \phi \in L\}$$

is the universe of a cylindric algebra $\mathcal{A}_M$ of dimension ω. It is easy to see that A_M is a boolean field of sets; it is closed under intersection and complementation (hence under unions), for indeed

$$\phi^M \cap \psi^M = (\phi \wedge \psi)^M,$$

and

$${}^{\omega}M \smallsetminus \phi^M = (\neg\phi)^M.$$

Certain set-theoretic operations similar to the classical operations of descriptive set theory can now be introduced corresponding to the basic non-boolean operations. In harmony with the notation of Henkin, Monk and Tarski [40], we use capital letters for the interpretation of the cylindric operations in set algebras: The i^{th} cylindrification, algebraizing existential

[8] Cylindric set algebras can be viewed as multi-dimensional boolean set algebras, (their dimension can be finite or transfinite). They play a role in the theory of cylindric algebras analogous to that played by boolean set algebras in boolean algebras and rings of matrices in ring theory. Such algebras are building blocks, in the sense that the standard models of CA-theory, the so-called *representable* algebras, are defined to be *subdirect products* (in the universal algebraic sense) of set algebras.

quantification with respect to the i^{th} variable, is defined as follows:

$$C_i(\phi^M) = \{t \in {}^{\omega}M : \text{there exists } s \in \phi^M \text{ such that } t(j) = s(j) \text{ for all } j \neq i\} = (\exists v_i \phi)^M.$$

This is simply the cylinder obtained by moving ϕ^M parallel to the i-axis, hence the terminology of i^{th} cylindrification. On the other hand, the diagonal elements, denoted by D_{ij} for $i, j \in \omega$, are defined to be the following hyperplanes:

$$D_{ij} = \{s \in {}^{\omega}M : s_i = s_j\} = (x_i = x_j)^M.$$

Now let Γ be an L-theory. Suppose further that M is a model of Γ, i.e., that every sentence in Γ is true in M. Then this is reflected algebraically by the fact that $\mathcal{A}_M$ is a homomorphic image of $\mathcal{A}_\Gamma$ via the natural map

$$[\phi]/\Gamma \mapsto \phi^M.$$

If ψ is satisfiable in M, then this can be expressed algebraically by the fact that the image of ψ under the above map is non-zero, for indeed ψ^M contains a sequence that satisfies ψ in Γ, i.e., is non-empty.

Having at hand the concrete notion of a cylindric set algebra based on models, we now make another abstraction from this notion to obtain a more general (but still concrete) set-theoretic object. There is no reason why we should restrict our attention to ω-dimensional set algebras. Accordingly, relaxing the restriction of dimension, a cylindric set algebra of dimension α, α an *arbitrary* ordinal, and base U, for short a Cs_α^U or simply a Cs_α, when U is clear from context, is an algebra

$$\mathcal{A} = \langle A, \cup, \cap, \smallsetminus, C_i, D_{ij}\rangle_{i,j\in\alpha}$$

such that A is a boolean field of subsets of ${}^{\alpha}U$ closed under each C_i and containing the D_{ij}'s for all $i, j \in \alpha$ defined as above by replacing ${}^{\omega}M$ by ${}^{\alpha}U$ and making the obvious changes. Note that set algebras can have arbitrary dimensions. It turns out that in the ω-dimensional case, the cylindric set algebras corresponding to models are precisely the locally finite *regular* ones.

The property of local finiteness reflects the fact that such algebras consist of relations that are essentially “finitary”, though of course the rank of such relations can be unbounded. On the other hand, $\mathcal{A} \in Cs_\omega$ is regular if for all $X \in A$ and $s, t \in {}^{\omega}U$, whenever s and t agree on ΔX, then $s \in X$ implies that $t \in X$. This, in turn, reflects the metalogical property that if two assignments agree on the indices of the free variables occurring in a formula, then they both satisfy the formula or neither does. The class of regular cylindric set algebras of dimension ω is denoted by Cs_ω^{reg} in [41]. In fact, there is an isomorphism, $h : Models \to Lf_\omega \cap Cs_\omega^{reg}$, definable in *ZFC* (Zermelo-Fraenkel set theory with choice) by an absolute formula without parameters, cf. [94, p. 406]. Because of its connection with first order models, the class $Cs_\omega^{reg} \cap Lf_\omega$ has been quite extensively studied in the literature. Therefore it is no coincidence that a lot of problems in algebraic logic frequently boil down to constructing certain models satisfying certain properties.[9] For a detailed and extensive exposition of cylindric set algebra (and related structures like relativized cylindric set algebras) the reader is referred to Monk's survey article [93].

§8. The completeness and incompleteness theorems of Gödel, algebraically. In terms of the concept of a cylindric set algebra, a purely algebraic form of Gödel's Completeness Theorem can now be stated. For that we need to recall that a subdirect product of a family of algebras is a subalgebra of the product of this family such that the (natural) projections are onto.

FACT 2 (Tarski). If $\mathcal{A} \in CA_\omega$ is locally finite, then $\mathcal{A}$ is isomorphic to a subdirect product of Cs_ω's. Furthermore these Cs_ω's can be chosen to be locally finite and regular.

The proof of Fact 2 is somewhat deeper than that of Fact 1. In fact, Fact 2 is equivalent in ZF (Zermelo-Fraenkel set theory without choice) to Gödel's classical Completeness Theorem. To illustrate this connection we give a sketch of the proof of one implication, namely that Gödel's Completeness Theorem implies Fact 2. We use the (equivalent) form that every consistent set of first order formulas has a model.

SKETCH OF PROOF OF FACT 2. Let $\mathcal{A} \in CA_\omega$ be locally finite. It suffices to show that for every non-zero element $a \in A$, there is a regular locally finite $\mathcal{C} \in Cs_\omega$ and a homomorphism $h : \mathcal{A} \to \mathcal{C}$ such that $h(a) \neq 0$. We can assume without loss of generality that $|A| > 1$, for else there is nothing more to prove. By Fact 1, $\mathcal{A} \cong \mathcal{A}_\Gamma$, for some consistent theory Γ. Now let a be a non-zero element in A_Γ. Then a is the equivalence class of some ψ that is consistent with Γ, i.e., $a = [\psi]/\Gamma$, and no contradiction is derivable from $\Gamma \cup \psi$. By Gödel's Completeness Theorem there is a model M of Γ in which ψ is satisfiable. Then it is not hard to check that the natural map $h : \mathcal{A}_\Gamma \to \mathcal{A}_M$ defined by

$$[\phi]/\Gamma \mapsto \phi^M$$

is an onto homomorphism for which $h(a) = \psi^M$ is non-empty. We note that if Γ is a complete theory, then this map is further an isomorphism. $\dashv$

For the opposite implication, the reader is referred to [41, §4.3] where this and other deeper connections are further elaborated. As regular locally finite set algebras of dimension ω are simple, Fact 2 shows that every locally finite algebra is actually *semisimple* i.e., isomorphic to a subdirect product of simple algebras. Since, on universal algebraic grounds, the kernel of a homomorphism is a maximal ideal if and only if its range is simple, this in

[9] In this connection, we mention for example [18, 115, 59], and [130]. Lately, Hodkinson and Hirsch [50] applied rather sophisticated machinery from model theory to solve long-standing open problems in algebraic logic.

turn is equivalent to the fact that the intersection of maximal ideals in any locally finite algebra is the trivial algebra. We note that Halmos' proof that semisimplicity of (locally finite) polyadic algebras is equivalent to Gödel's Completeness Theorem was to him a revelation. In fact, Halmos [37] was concerned with the algebraization of two of the deepest theorems in logic, Gödel's Completeness and Incompleteness Theorems. The original formulations and proofs of these justly celebrated theorems are due to Gödel. The algebraic counterpart of Gödel's Completeness Theorem is the semisimplicity of locally finite polyadic algebras, as illustrated in Fact 2, bearing in mind that locally finite polyadic algebras are essentially the same as locally finite cylindric algebras. Now algebraizing Gödel's Incompleteness results is more intricate, and is still (to the best of our knowledge) an unfinished task. However, Halmos [38] has work in this direction, which we now briefly review. There is no difficulty in constructing locally finite polyadic algebras, and for that matter locally finite cylindric algebras, with sufficiently rich structure to mirror the axioms of complex systems such as Peano arithmetic or even the stronger Zermelo-Fraenkel set theory in detail. The resulting algebra is what Halmos calls a Peano algebra. A Peano algebra is adequate to mirror elementary arithmetic; so for example we can "talk about" recursive functions in a Peano algebra. Now Gödel's Incompleteness Theorem says (very roughly) that there are propositions that are neither refutable nor provable in any formal system strong enough to encode recursive functions. We recall that a simple algebra is one that has no proper congruences. The algebraic version or formulation of Gödel's Incompleteness Theorem turns out surprisingly rather simple. It states:

FACT 3 (Halmos). Not every Peano algebra is simple.

Halmos identifies a Peano algebra with Mathematics, and rephrases his result in the following pun: "Mathematics is not simple". We refer the interested reader to Halmos [38] for a very interesting survey of the history of polyadic algebras.

In passing, we note that Tarski and Givant [133] formalized set theory in the calculus of relations, i.e., in relation algebras. A similar task was done by Németi [96], when he formalized an essentially undecidable fragment of Peano Arithmetic due to Tarski and Robinson in the so-called class of semi-associative relation algebras of Maddux. This interpretation of essentially undecidable fragments of set theory into the calculus of relations was used to prove undecidability of the equational theories of several classes of relation algebras [133]. Recent results on undecidability of various classes of relation and cylindric algebras involve the more sophisticated methods of the interpretation of the (undecidable) tiling problem and the (undecidable) word problem of finitely presented semigroups [41, 4.2] and [50, Ch.18]. The relevant papers are [49] for the tiling problem and [69] and [78] for word problems in finitely presented semigroups. A recent reference for undecidability in various classes of relation algebras is [6]. In [6] an easily applicable deep method, based on Von Neumann's coordination theorem, is developed to prove undecidability results answering outstanding open problems posed by Tarski.

Conversely, Gödel's Incompleteness Theorem was used to solve problems in algebraic logic. In cylindric algebra, Gödel's Incompleteness was used by Németi [96] to show that the (boolean reduct of the) free cylindric algebras of dimension at least 3 are not atomic. Recently, in [112] the author proved that the free quasi-polyadic algebras are not atomic either, by showing that their equational theory is still strong enough to encode Peano arithmetic. Quasi-polyadic algebras are diagonal-free reducts of cylindric algebras expanded with the so-called unary operations of finite substitutions [97]. A long-standing open question here posed by Maddux and independently by Németi is whether an analogous result holds for diagonal-free reducts of cylindric algebras (without substitutions), more precisely:

Open Question 1 (Maddux-Németi). Is the equational theory of the diagonal-free reducts of cylindric algebras of dimension 3, or Df_3 for short, strong enough to encode Peano arithmetic?

In fact, it is conjectured by Németi that set theory (and not just Peano arithmetic) might be interpreted in Df_3, in the manner of Tarski-Givant [133]. Nemeti suceeded to interpret set theory in CA_3. Omitting diagonals keeps the problem on the very tough side. The latter is also related to the "Finitization Problem" in algebraic logic to be discussed in a while.[10] Another interesting application of Gödel's incompleteness results is the Németi-Sagi result in [99], which shows that the equational theory of Halmos' polyadic equality algebras is strong enough to "encode" second order Peano arithmetic; thus the representable polyadic *equality* algebras cannot be axiomatized by any reasonable schema, let alone a finite one, which is an interesting contrast to the classical Daigneualt-Monk representation theorem for polyadic algebras *without* equality [25].

§9. The algebraization of first order logic, the representation problem. Coming back from this fascinating short detour into the algebraization of Gödel's Incompleteness results, we turn our attention for a while to the class Lf_ω. Soon in the development of the subject, it transpired that the class Lf_ω, the algebraic counterpart of first order logic, had some serious defects when treated as the sole subject of research in an autonomous algebraic theory. In universal algebra one prefers to deal with *equational classes* of algebras i.e., classes of algebras characterized by a postulate systems in which

[10]Here the problem can be paraphrased as how far can we "finitize metamathematics

every postulate has the form of an equation (an identity); such classes are also referred to as *varieties*.

Classes of algebras which are not varieties are often introduced in discussions as specialized subclasses of varieties. One often treats fields as a special case of rings. Probably the reason for this preference is the fact that every variety is closed under certain general operations frequently used to construct new algebras from given ones; we mean here the operations of forming subalgebras, homomorphic images and direct products. By a well known theorem of Garrett Birkhoff, varieties are precisely those classes of algebras that have all three of these closure properties. Local finiteness does not have the the form of an identity, nor can it be equivalently replaced by any identity or system of identities, nor indeed any set of first order axioms. This follows from the simple observation that the ultraproduct of infinitely many algebras in Lf_ω is not, in general, locally finite, and a first order axiomatizable class is necessarily closed under ultraproducts.

When Alfred Tarski introduced cylindric algebras, he introduced them in the form satisfying (ii) in Fact 1, and proved the representation theorem expressed in Fact 2. In [40] it was mentioned that Tarski's proof of Fact 2 was not published. This means that Tarski's proof was not published as of 1971. We do not know whether it was published at all, but it seems to us that this is unlikely. In any case a simple purely algebraic proof due to Andréka and Németi can be found in [9]. But some modifications in the definition of Tarski's cylindric algebras seemed desirable. The definition contains certain assumptions which considerably restrict the applicability of the definition and thus can be dispensed with. One such assumption is the fixed dimension ω. The other is local finiteness. The restrictive character of these two notions becomes obvious when we turn our attention to cylindric set algebras. We find there are algebras of all dimensions, and set algebras that are not locally finite are easily constructed. For these reasons the original conception of a cylindric algebra was extended. The restriction to dimension ω and local finiteness were removed, and the class CA_α was introduced.

A central and indeed still (*very*) *active part of research in algebraic logic is the vaguely posed frequently discussed problem concerning improvements of* Fact 2.

This problem is referred to as *The Finitization Problem* by the Budapest group specifically by Andréka and Németi [97] while it is referred to as the *Representation Problem* by the London group specifically by Hodkinson and Hirsch [50]. Two points we need to clarify here.

First, strictly speaking, the Finitization Problem and the Representation Problem are not one and the same thing. The Finitization Problem is more restrictive than the Representation Problem. The Finitization Problem is the Representation Problem restricted to the case when in the search for axioms that enforce representability, one requires that such axiomatizations are not only "simply, elegant, transparent, etc" but also *finite*. In our subsequent discussion, when we refer to the above problem we shall use both words relying on context.

Second, the attribution of names to schools could be a little misleading. For example, the early Jónsson-Tarski paper [62] was called "*Representation problems* for relation algebras". Their representation problem is, given the equationally defined class of relation algebras and the set-theoretically defined class of special relation algebras arising from binary relations, is every one of the former isomorphic to one of the latter, i.e., is every relation algebra representable. The famous answer is "no". So one can say that the Representation Problem can be traced back to the work of Tarski and his students on relation algebras back in the forties of the 20th century. This, however, does not change the fact that Hirsch and Hodkinson attribute the name "Representation Problem" to the problem above. In the course of our discussion we will give more tangible and concrete forms of the Representation Problem.

To start with, the Finitization Problem asks for an equationally defined class of algebras, each with only finitely many finitary operations, such that the class defined by finitely many equations and every algebra in the class is representable (assuming that that notion has been suitably defined) with the additional (rather vague) property that the equational theory of that class can serve as an algebraic version of first-order logic (with or without equality—Sain [106] did this for logic without equality; whether it can be done for logic with equality is unlikely). For attempts to explain what this means, see the original formulations of Henkin, Monk, and Tarski, for example, in paragraph 1, page 20, of [88], Problem 1 in [39] and Section 3.5 of [133]. For recent extensive discussions of the Finitization Problem, one can of course look at Németi [97] or the papers of Sain [106], Simon [131]. In what follows we give our reading of the Finitization and for that matter the Representation problem. This problem has invoked extensive amount of research and still is. To get a grasp of how substantial the problem is let us start "from the beginning".

Let RCA_α be the class of CA_α's isomorphic to subdirect products of Cs_α's. We first explain why the class RCA_α is indeed a plausible "natural" candidate to substitute for boolean set algebras in the quest of a Representation Theorem for cylindric algebras analogous to that of Stone. It is easily seen that every cylindric set algebra of given dimension $\alpha < \omega$ is simple (has no proper congruences) and therefore subdirectly (and directly) indecomposable in the sense of the general theory of algebras [40]. Hence when discussing the problem as to which CA_α's are isomorphic to cylindric set algebras, it is natural to restrict ourselves to subdirectly indecomposable algebras. On the other hand, as a consequence of a classical theorem of Birkhoff, every CA_α is isomorphic to a subdirect product of subdirectly indecomposable CA_α's. Therefore we are naturally led to the problem of characterizing those CA_α's which are isomorphic to subdirect products of set algebras. Henkin, Monk and Tarski declare that these are the *representable* algebras, thus the notation RCA_α.

RCA_α plays a role in cylindric algebra analogous to the role played by boolean set algebras in boolean algebra and rings of matrices in ring theory. The Representation Problem in boolean algebras is completely resolved by Stone's Theorem. In ring theory we see for example the Wederburn-Artin Theorem and Goldie's Theorem, which giving nice intrinsic conditions for an abstract ring to be isomorphic to a subdirect product of rings of matrices. As opposed to boolean algebras, the Representation Problem for the CA case proves to be much harder and richer.

The definition of representability, without any change in its formulation, is extended to algebras of infinite dimension. In this case, however, an intuitive justification is less clear since cylindric set algebras of infinite dimension are not in general subdirectly indecomposable. In fact, for $\alpha \geq \omega$ no intrinsic property is known which singles out the algebras isomorphic to set algebras among all representable CA_α's, as opposed to the finite dimensional case where such algebras can be intrinsically characterized by the property of being simple. But in any case, members of RCA_α can be still represented as algebras consisting of genuine α-ary relations over a disjoint union of Cartesian squares, the class consisting of all such algebras is denoted by Gs_α, with Gs standing for *generalized set algebras.*

Generalized set algebras thus differ from the ordinary cylindric set algebras in one respect only: the unit of the algebra, i.e., the α-dimensional Cartesian space ${}^\alpha U$, is replaced everywhere in their construction by any set that is a disjoint union of arbitrarily many pairwise disjoint Cartesian spaces of the same dimension. The class of generalized cylindric set algebras, just as that of ordinary cylindric set algebras, has many features that make it well qualified to represent CA_α. The construction of the algebras in this (bigger) class retains its concrete character, all the fundamental operations and distinguished elements are unambiguously defined in set-theoretic terms, and the definitions are uniform over the whole class; geometric intuition underlying the construction gives us good insight into the structures of the algebras. Thus there is (geometric) justification that RCA_α consists of the standard models of CA-theory. Its members consist of genuine α-ary relations. Now, in elementary terms the definition of RCA_α runs as follows: $\mathcal{A}$ is representable iff for every non-zero $x \in A$ there is a homomorphism h from $\mathcal{A}$ onto a Cs_α such that $h(x) \neq 0$.

Thus Fact 2 is equivalent to the statement that every locally finite CA_ω is representable. But it soon transpired that the CA axioms (originating from the (complete) axiomatization of locally finite algebras, cf. Fact 2) do not exhaustively generate all valid principles governing α-ary relations, when

$\alpha > 1$. More precisely, for $\alpha > 1$, RCA_α is properly contained in CA_α. CA_α, for $\alpha > 1$, is only an *approximation* of RCA_α.

Tarski [40] proved that RCA_α is a variety. Henkin [41] proved that RCA_2 is finitely axiomatizable. However for $\alpha > 2$, the class RCA_α cannot be axiomatized by a finite schema of equations analogous to that axiomatizing CA_α, a classical result of Monk [87], cf. Fact 6 below. Furthermore, there is an unavoidable and inevitable degree of complexity to any (potential) axiomatization of RCA_α, as shown by Andréka [3] for any $\alpha > 2$.

The Finitization Problem (and for that matter a form of the Representation Problem) is thus the attempt to circumvent or sidestep such complexity. If we look at RCA_α as the standard models to which the CA_α's aspire, the Finitization Problem can thus be rephrased as the attempt to capture the essence of the standard models by thorough "finitary" means, or else find other broader comprehensible classes of "standard models" that are sufficiently concrete and tangible and most important of all would exhaust the class CA_α, or at worst possibly a slightly smaller class, i.e., a variety that is finitely axiomatizable (by equations) over CA_α.

There has been work in this direction by representing cylindric algebras using quasigroups, cf. [90], [41, p. 91–93], [23] or sheaves [22]. Groups were used to represent CA_3's and relation algebras in [31]. A classical result of Resek [41, p. 101] that is relevant in this connection shows that algebras satisfying the CA axioms plus the so-called *merry-go-round identities*, or MGR for short, can be represented as *relativized* set algebras, a primary advance in development of the theory of CA's, as indicated in the introduction of [41]. Resek's result was polished and "finitized" by Thompson, cf Andreka Thompson [14]. The replacement of MGR by a finite scheme is entirely due to Thompson and appears in his dissertation; see [41, 3.2.8]. The proof in [14] is due to Andreka. It might be not be appropriate to use the word "finitized" here, since, if $\alpha < \omega$, Resek's result already produces a *finite* axiom set, and if $\alpha \geq \omega$, then Thompson's simplification is still an *infinite* axiom set. What is meant in this context, is that, in case $\alpha \geq \omega$, there are originally infinitely many MGR schemata, and that Thompson reduced this infinity to 2. Thompson actually proved much more than is stated here—among other things, he weakened the commutivity of cylindrifications, showed that atoms of the algebras are represented as orbits of single sequences under groups of permutations of the underlying set. A complete statement (with proofs) of Resek's and Thompson's theorems can be found in [71].

Recently, Simon [132] proved that any abstract 3-dimensional cylindric algebra satisfying the MGR can be obtained from a Cs_3 by the so-called methods of *twisting and dilation*, studied in [41, p. 86–91] which adds to our understanding of the distance between the abstract notion of cylindric algebra and its concrete one, at least in the case of dimension 3.

However, Simon had to broaden Henkin's notion of dilation to exhaust the class CA_3. Also he showed that Henkin's more restrictive notion of dilation does not fit the bill; there are abstract CA_3's satisfying the MGR that cannot be obtained by the methods of relativization, twisting and dilation, the latter understood in the sense of Henkin. The analogous problem for higher dimension is an intriguing open problem. This problem, originally due to Leon Henkin and slightly modified by Simon [132] is:

Open Question 2 (Henkin-Simon). Can every CA_n, $3 < n < \omega$, be obtained from a Cs_n by a finite sequence of the operations of relativization, twisting (as defined in [41]), and dilation (as defined by Simon in [132])—not necessarily in this order?

A different slant on the Representation Problem is: How can we abstract away from the subject matter of specific concrete α-ary relations to arrive at their essential forms? For unary relations the answer is given by Stone's Theorem: Every boolean algebra is isomorphic to a set algebra, i.e., is representable. The significance of this is that the (finitely many) axioms of boolean algebras exactly capture the true properties of unary relations: all and only those properties that hold in every domain of individuals endowed with unary relations are derivable from the axioms. This is a great for the algebraic viewpoint. The same situation holds for locally finite cylindric algebras. The axiomatization provided by Tarski and recalled herein is sufficient for representability as indicated in Fact 2 above. However, if we want to *algebraize first order logic*, then we have to remove the (non-first order) restriction of local finiteness, and keep only equations or at worst quasi-equations.

According to Blok and Pigozzi [19] *algebraizable* logics are those logics whose algebraic counterparts are quasi-varieties, i.e., classes of algebras axiomatized by quasi-equations. But then infinity seems unavoidable, it backlashes in a strong sense: Any first order axiomatization of the class RCA_α for $\alpha > 2$ is essentially infinite.

The metalogical aspect of the Finitization Problem asks for an algebraizable extension of first order logic that admits a finite complete and sound Hilbert style axiomatization of the valid formula schemata involving only valid formula schemata. This is formulated as problem 4.16 in [41]. This form of the Finitization Problem, it seems, is not unrelated to Hilbert's program of finitizing metamathematics, and indeed it seems to add to our knowledge of reasoning about reasoning.

On the other hand, the Finitization Problem (in its algebraic form) is indeed non-trivial as the following quote from Henkin, Monk and Tarski in [40] might suggest: "An outstanding open problem in cylindric algebra theory is that of exhibiting a class of cylindric algebras which contains an isomorphic image of every cylindric algebra and hence serves to represent the class of all these algebras, and which at the same time is sufficiently concrete and simply constructed to qualify for this purpose from an intuitive point of view. It is by no means certain or even highly plausible that a satisfactory solution of this problem will ever be found!" (our exclamation mark).

Fortunately today the situation seems to be not as drastic! However, it still involves some open questions.

§10. Possible solutions, the feedback between algebraic logic, modal logic, finite combinatorics and games. Several different stratagems to get round the obstacle of the non-finite axiomatizability of the class of representable algebras were evolved. One promulgated by Tarski especially was to find elegant intrinsic conditions for representability. For example, certain comprehensible subclasses of abstractly defined CA_α's turn out to be representable. In this connection, examples include locally finite, dimension complemented, semisimple, and diagonal algebras of infinite dimension, cf. [41, Thm. 3.2.11]

Another sample of such results in this direction is the classical result of Henkin and Tarski, formulated as Theorem 3.2.14 in [41], that states that any atomic CA_α whose atoms are rectangular is representable. This was strengthened by Andréka et al. [5] by looking at dense subsets consisting of rectangular elements that are not necessarily atoms. Venema [137] extended this result to the diagonal free case.

This approach of finding simple intrinsic sufficient conditions for representability has continued to the present, and now forms an extensive field, cf. [97].

Another strategy of attacking the Finitization Problem is to define variants of RCA_α, $\alpha > 2$ that are finitely axiomatizable and are still adequate to algebraize first order logic. Such an approach originates with Craig [24] and is further pursued by Sain [106, 107], Simon [131] and the present author [122] and [121].

The reasoning here is that maybe the negative results we already mentioned are merely a historical accident resulting from the particular (far from unique) choice of extra non-boolean operations, namely the cylindrifications and diagonal elements. This approach typically involves changing the signature of CA_α by either taking reducts or expansions or perhaps even changing the signature altogether but bearing in mind that cylindrifications and diagonal elements are term definable in the new signature, cf. [24] and [106], and/or broadening the notion of representability, allowing representation on arbitrary subsets of α-ary relations, rather than just (disjoint unions of) Cartesian squares [82]. The approach of broadening the permissible units is referred to in the literature as *relativization* or the *non-square* approach. The term relativization is a term that is already old in logic, and in cylindric algebras it has been used from the very beginning, see [40, §2.2].

The second term comes from the Amsterdam group, in fact it is due to Venema [135]. Relativization might involve adding new operations that become no longer term definable after relativization, uch as the difference operator [135]. This approach is related to *dynamic logic*, cf. [15]. Modalizing set algebras yields variants of the n-variable fragment of first order logic differing from the classical Tarskian view because the unit W of the set algebra in question may not be of the form of a "square" ${}^nU$, but merely a subset thereof. These mutant logics are under intensive study at the present time and we cite [15] and [83] as sources.

We should mention, in this connection, that [106] provides a solution for first order logic without equality. Indeed in [106] a stricty finite set of axioms are given for a class of representable algebras that is an extension of the class of representable quasi-polyadic algebras. This provides an algebraizable extension of first order logic *without equality* that admits a finite complete and sound Hilbert style axiomatization of the valid formula schemata involving only valid formula schemata answering the equality free version of Problem 4.16 in [41]. Adding equality is problematic so far.

Another more adventurous approach is to "stay inside", so to speak, the "CA-RCA infinite discrepancy" and to try to capture the essence of (the equations holding in) RCA_α in as simple a manner as possible. It is not hard to show that the set of equations holding in RCA_α for any countable α is recursively enumerable, and indeed, using a well-known trick of Craig, several (recursive) axiomatizations of RCA_α exist in the literature, the first such axiomatization originating with Monk, building on work of McKenzie [84], [41, p. 112] and [87]. Robinson's finite forcing in model theory proves extremely useful here as shown by Hirsch and Hodkinson. The very powerful recent approach of synthesizing axioms by games due to Hirsch and Hodkinson [45], building on work of Lyndon [64], is a typical instance of giving an intrinsic characterization of the class of representable algebras by providing an *explicit* axiomatization of this class in a step-by-step fashion. This approach is of a very wide scope; using Robinson's finite forcing in the form of games, Hirsch and Hodkinson [50] axiomatize, not only the variety of representable algebras, but almost all pseudo-elementary classes existing in the literature, an indeed remarkable achievement.

It turns out, as pointed out by Hirsch and Hodkinson, that representations of an algebra can be described in a first order 2-sorted language. The first sort in a model of this *defining theory* is the algebra itself, while the second sort is a representation of it. The defining theory specifies the relation between the two, and its axioms depend on what kind of representation is considered. Thus the representable algebras are those models of the first sort of the defining theory with the second sort providing the representation. This method has its roots for relation algebras in McKenzie's dissertation [84].

The class of all structures that arise as the first sort of a model of a two-sorted first order theory is an old venerable notion in model theory introduced by Maltsev in the forties of the 20th century. Ever since it was studied by Makkai and others. It is known as a pseudo-elementary class. What is meant here is a PC_Δ class in the sense of [57] but expressed in a two sorted language. The term pseudo-elementary class strictly means PC_Δ when the second sort is empty, but the two notions were proved to be equivalent by Makkai [79].

Any elementary class is pseudo-elementary, but the converse is not true; the class of α-dimensional neat reducts of β-dimensional cylindric algebras for $1 < \alpha < \beta$ is an example; see [110] and [115]. Another is the class of strongly representable atom structures and the completely representable ones, as proved by Hirsch and Hodkinson in [51]. Many classes in algebraic logic can be seen as pseudo-elementary classes. The defining theory is usually finite, simple and essentially recursively enumerable,

According to Hirsch and Hodkinson a fairly but not completely general *definition* of the notion of representation is just the second sort of a model of a two-sorted (more often than not recursively enumerable) first order theory, where the first sort of the theory is the algebra.

Put in this form, Hirsch and Hodkinson apply model-theoretic finite forcing to the representation problem. Model-theoretic forcing, as described in Hodges [57], and indeed in the proof of the classical Completeness Theorem by Henkin, and in his Neat Embedding Theorem to be recalled below, typically involves constructing a model of a first order theory by a *game.*

The game builds the model step-by-step, elements of the model being produced by the second player called $\exists$, in his response to criticism by the first player, called $\forall$. The approach of Hirsch and Hodkinson is basically to combine the forcing games with the pseudo-elementary approach mentioned above to representations to build the second sort of a model of the defining theory whose first sort is already the algebra whose representability is at issue. Taking the defining theory of the pseudo-elementary class to be given, this defines the notion of representation to be axiomatized.

We refer the reader to [44] and [50] for applications of this technique to axiomatize the classes of representable relation and cylindric algebras. The step-by-step technique in op.cit of building representations, especially when viewed as a game, can be extremely potent and inspiring. Not only does it allow the construction of axiomatizations of relation algebras and cylindric algebras and other kinds of related algebras, but close examination of the way that games can be played on given algebras provides very fine and detailed information about their structure. In [118], [121], and [114] we use a disguised form of Robinson's finite forcing in model theory to prove interpolation and omitting for certain infinitary expansions of first order logic. This involves building models in an essentially step-by-step manner, although we do not resort to games. But in any case, the step-by-step approach, whether by games or otherwise is widely accepted and has been used by many authors.

Another approach initiated by van Benthem and Venema, consists of viewing cylindric set algebras as subalgebras of complex algebras of Kripke frames that have the same signature as atom structures of cylindric algebras [135], thus opening an avenue to techniques and methods coming from modal logic. This typically involves introducing Gabbay-style rules on the logic side. These extremely liberal Gabbay-style inference systems correspond to classes that are inductive, i.e., axiomatized by $\forall\exists$-formulas. An example of such a class is the class of rectangularly dense cylindric algebras, [5] and [137]. We should mention that this approach is an instance, or rather, an application of the triple duality, in the sense of Goldblatt [35], existing between abstract modal logic, Kripke frames or relational structures, and boolean algebras with operators. In this connection, we refer to the article by Venema in [83] an explanation and application of this duality to relation algebras and to [135], [81], and [6] for further elaboration on this duality.

Actually there is a categorial approach to the representation problem. Instead of viewing the representation problem as a two-sorted first order theory, one can present it in the context of a *functor* going from from one sort to the other, or rather from one category to the other. We have two categories, the category of abstract algebras and the category of concrete algebras and adjoint functors between the two categories. Indeed the representation problem can be seen as a typical duality, be it between models and theories, or boolean algebras with operators and modal logic [33], or quasi-varieties and algebraizable logics [19]. Indeed it is argued in [66] that duality theories, representation theorems and adjoint functors are different words for the same thing.

However, it is debatable whether any of the intrinsic characterizations of the class of the representable algebras are "good enough." In particular, axiomatizations of this class existing in the literature see, e.g., [41, 3, 135, 50], it seems, are not considered satisfactory from the algebraic point of view. Maybe a better description of the present situation would be that none of these axiomatizations are considered final.

This also applies to the modal approach initiated by Venema [135, 137]. A similar situation occurs in modal logic when people say that Gabbay's irreflexitivity rule [30] introduces variables by the back door and is *inimical* to the true nature of modal logic.

Quoting Hirsch and Hodkinson, cf. p. 9 in [50] "The precise objections are hard to pin down, but broadly it seems that these axiomatizations are regarded as unsatisfactory in some way: they are too complicated—or perhaps too trivial, just paraphrasing the original problem without providing any significant new 'algebraic insight'."

There are also some reservations to the approach initiated by Hirsch and Hodkinson adopted in [44], [47], [48], and [50]. It is sometimes argued that the use of games is just a presentational matter. In any case the problem to find simpler axiomatizations of representable algebras is still open.

§11. Back to classical algebraic logic; neat reducts. A form of the Representation Problem is to describe properties of the class RCA_α and try to give a useful characterization of it in *abstract* terms. An old result of Henkin which gives an abstract sufficient and necessary condition for representability fits here. An algebra $\mathcal{A}$ is in RCA_α if and only if for every $\beta > \alpha$, it can be embedded as a *neat subreduct* in some cylindric algebra of dimension β, or, equivalently, using ultraproducts, into an algebra with ω extra dimensions. This brings us to the central venerable notion in the representation theory of cylindric algebras, namely the notion of *neat reducts.* The notion of neat reducts, which we now recall, is also due to Henkin [40, p. 401]. A *neat reduct* of a cylindric algebra $\mathcal{A}$ is basically a new algebra of lesser dimension obtained from $\mathcal{A}$ by overlooking some of its operations and discarding some of its elements. More precisely:

DEFINITION 4 (Henkin). Let $\alpha < \beta$ be ordinals, and let $\mathcal{A} \in CA_\beta$. Then the *neat-α reduct* of $\mathcal{A}$, in symbols $\mathcal{N}r_\alpha\mathcal{A}$, is the CA_α whose domain is the set of all *α-dimensional* elements of $\mathcal{A}$ defined by

$$Nr_\alpha A = \{a \in A : c_i^{\mathcal{A}} a = a \text{ for all } i \in \beta \smallsetminus \alpha\}.$$

The operations of $\mathcal{N}r_\alpha\mathcal{A}$ are those of the α-dimensional reduct

$$\mathcal{R}d_\alpha\mathcal{A} = \langle A, -^{\mathcal{A}}, \cdot^{\mathcal{A}}, c_i^{\mathcal{A}}, d_{ij}^{\mathcal{A}}\rangle_{i,j\in\alpha}$$

of $\mathcal{A}$, restricted to $Nr_\alpha A$.

When no confusion is likely, we omit the superscript $\mathcal{A}$. $Nr_\alpha A$ as easily checked, is closed under the indicated operations and indeed is a CA_α. $\mathcal{N}r_\alpha\mathcal{A}$ is thus a special subreduct of $\mathcal{A}$, i.e., a special subalgebra of a reduct in the universal algebraic sense.

For a class K, let SK denote the class of all algebras embeddable into members of K. Neat reducts play a key role in the representation theory of cylindric algebras as indeed illustrated in the next celebrated theorem of Henkin, which is basically a completeness result:

FACT 5 (Henkin). For any ordinal α and any $\mathcal{A} \in CA_\alpha$, the following two conditions are equivalent:

(i) $\mathcal{A} \in RCA_\alpha$
(ii) $\mathcal{A} \in SNr_\alpha CA_{\alpha+\omega}$.

Infinity manifests itself in (ii) above, and it does so essentially in the case when $\alpha > 2$, in the sense that if $\mathcal{A}$ neatly embeds into an algebra in finitely many extra dimensions, then it might not be representable, as shown by Monk (cf. Fact 6 below.) All ω extra dimensions are needed for representability One cannot truncate ω to any finite ordinal. The ω

extra dimensions play the role of added constants or witnesses in Henkin's classical proof of the completenes theorem. Therefore it is no coincidence that variations on Fact 5 lead to metalogical results concerning interpolation and omitting types for the corresponding logic. Such results can be proved by Henkin's method of constructing models out of constants. In this connection we refer to [114], [129], [68] and [118].

An interesting result of Ferenczi [29] shows that we can dispense with some of the CA axioms when we get to ω extra dimensions. More precisely: For every $\alpha > 2$ there exists a variety $K_{\alpha+\omega}$ having the same signature as $CA_{\alpha+\omega}$ such that $CA_{\alpha+\omega} \subset K_{\alpha+\omega}$ (here the inclusion is proper) and $RCA_\alpha = SNr_\alpha K_{\alpha+\omega}$. Here the class $Nr_\alpha K_{\alpha+\omega}$ is defined as in the CA case. Coming back to Henkin's Neat Embedding Theorem, we have the following. On the one hand, the characterization established by Henkin is an abstract characterization of the class RCA_α, and it does help occasionally to prove that certain subclasses consist exclusively of representable algebras. This occurs when it is easier to prove that an algebra embeds neatly into another algebra in ω extra dimensions, and applies for example in the cases of the classes of the so-called diagonal cylindric algebras and the semisimple ones, [40, 2.6.50] and [41, 3.2.11]. On the other hand, this characterization is not satisfactory from the set-theoretic representation point of view, since it is not *intrinsic* and refers to algebras outside the algebra considered, namely, algebras with ω extra dimensions.[11]

In [40] algebras in the class $SNr_\alpha CA_{\alpha+\omega}$ are said to have the *neat embedding property*. Therefore an algebra has the neat embedding property if it neatly embeds into an algebra with ω extra dimensions, and according to Henkin's Neat Embedding Theorem the class RCA_α coincides with the class of cylindric algebras that have the neat embedding property. Since locally finite algebras have the neat embedding property, Fact 5 can be seen as an indeed substantial generalization of Fact 2, pushing it to the limit. Quoting Henkin-Monk and Tarski [40, p. 400] "The notion of the neat embedding property appears to be more suitable for an abstract algebraic treatment than that of representability. This is the main reason why our discussion of neat reducts and their subalgebras in the present section will be comprehensive and detailed."

It is known (cf. [118] or [129]) that the Neat Embedding Theorem, or the NET for short, proved by Henkin in the fifties is an algebraization of Henkin's celebrated proof of the completeness of first order logic, or rather an

[11]By an intrinsic property of an algebra $\mathcal{A}$ we understand, loosely speaking, a property which can be expressed entirely in terms of symbols denoting the operations of $\mathcal{A}$, and variables ranging exclusively over elements of the universe of $\mathcal{A}$, subsets of $\mathcal{A}$, relations between elements of A, sets of such subsets and relations, etc., and not, e.g., in case A is a family of sets, in terms of variables ranging over $\cup A$. (Note that the notion of an intrinsic property is of mathematical nature; to be made precise it must be relativized to a well defined formal language).

extension thereof. Indeed it can be viewed as a typical instance of Robinson's finite forcing in model theory [44]. Referred to by Hirsch and Hodkinson [43] as one of the earliest examples of step-by-step building representations in algebraic logic, variants of the NET have been successfully applied to (algebraically) prove the completeness of several versions of quantifier logics, that are extensions, variants, or reducts of first order logic. Examples include Keisler's logics investigated algebraically in [25], and various reducts thereof like the logics studied in [7] and much later in [106], under the name of *typeless finitary logics of infinitary relations*, see also [41, §4.3] for a systematic treatment of such logics. Other contexts to which the NET applies to prove completeness are the higher order logics investigated by Sagi [103] and Sagi, Sayed [104]. Conversely the NET conjoined with some form of Ramsey's theorem has been applied to show the essential incompleteness of L_n, the first order logic restricted to the first n variables when $\omega > n > 2$, see, e.g., [89, 53]. This follows from the following classical algebraic result of Monk that established the "infinite distance" between CA's and RCA's. Monk's result marked a turning point in the development of the subject, and is considered one of the most, if not *the* most, important model-theoretic result concerning cylindric algebras.

FACT 6 (Monk). Let $\omega > n > 2$ and $m \in \omega$. Then RCA_n is properly contained in Nr_nCA_{n+m}. In particular, RCA_n is properly contained in SNr_nCA_{n+m}. Thus RCA_n is not finitely axiomatizable.

SKETCH OF PROOF. Monk used Ramsey's Theorem to construct for each $m \in \omega$ and $2 < n < \omega$, an algebra $A_m \in Nr_nCA_{n+m}$ that is not representable. The ultraproduct of the A_m's constructed by Monk (relative to any non-principal ultrafilter on ω) is in $SNr_nCA_{n+\omega}$, hence by the NET, is representable. Using elementary model theory, it follows thus that the class RCA_n for $\omega > n > 2$, is not finitely axiomatizable. $\dashv$

The A_m's are referred to in the literature as Monk or Maddux algebras. Both authors used them. The key idea of the construction of a Monk algebra is not so hard. Such algebras are finite, hence atomic, more precisely their boolean reduct is atomic. The atoms are given colors, and cylindrifications and diagonals are defined by stipulating that monochromatic triangles are inconsistent. If a Monk algebra has many more atoms than colors, it follows from Ramsey's Theorem that any representation of the algebra must contain a monochromatic triangle, so the algebra is not representable. Andréka's splitting as seen in [3] is a variation on the same theme, and indeed leads to a refinement of Monk's result. Here splitting refers to splitting an atom into more atoms that enforces non-representability, in which case the original atom before the process of splitting is no longer an atom after the process of splitting. However, Andréka's splitting does not appeal to any form of Ramsey's theorem. In [110] Andréka's splitting is used to show that the class of neat n reducts of β-dimensional cylindric algebras is not elementary for $2 < n < \beta \cap \omega$. We note that Monk established a very interesting connection between finite combinatorics and algebraic logic [90], a recurrent theme in algebraic logic. A recent use—establishing this link—of Monk algebras with a powerful combinatorial result of Erdös has shown that the class of the so called strongly representable atom structures of relation algebras and 3-dimensional representable cylindric algebras is not elementary [51, 127].

Such counterexamples were used in [114] and [4] to show that the omitting types theorem fails for the finite variable fragments of first order logic, as long as the number of variables available is at least 3. We will further elaborate on this below.

In this connection, we should mention that Lyndon's three papers [63, 64], and [65] on relation algebras were very influential. [64] is the basis for Hirsch-Hodkinson's (step-by-step) approach to the representation problem. [65], on the other hand contains results of constructing non-representable relation algebras from projective geometries. This led to Monk's famous result that the class of representable relation algebras is not finitely axiomatizable [85]. The second key paper in this context is [86] where Monk extended his result (of non-finite axiomatizability) to 3 dimensional representable cylindric algebras. In both of these papers, Monk uses projective geometries. In his 1964 paper [85] the relation algebras used arise from Lyndon's construction applied to projective lines, and in the 1965 paper [86] the algebras dealt with are 3-dimensional cylindric algebras constructed from Lyndon's relation algebras. Monk extended his results to cylindric algebras of dimension larger than 3 in his 1969 paper [87] using the algebras based on Ramsey's Theorem. So it took Monk quite some time to replace projective geometries by graphs.

§**12. Refinements of Monk's result.** Had it been otherwise, i.e., if for $\omega > n > 2$, RCA_n had turned out to be axiomatizable by a finite set of equations Σ say, then this Σ would have been probably taken as the standard axiomatization of CA_n. Unfortunately this turned out not to be the case. Quoting Hirsch and Hodkinson [50, p. 8] "As it seemed, the hopes of workers over a hundred years starting with De Morgan and culminating in Tarski's work to produce a (simple, elegant, or at least finite) set of algebraic properties—or in modern terminology—equations that captured exactly the true properties of n-ary relations for $\omega > n > 2$ were shattered by Monk's result."

This impasse is still provoking extensive research until the present day, in essentially two conflicting (but complementary) forms.[12] One form, which we already discussed, is to try to circumvent this negative non-finite

[12]To understand the "essence" of representable algebras, one often deals with the non-representable ones, the "distorted images" so to speak. Simon's result in [132], of representing non-representable algebras, seems to point out that this distortion is, after all, not completely

axiomatizability result. The other form is to sharpen it. Indeed, Monk's negative result—as far as non-finite axiomatizability is concerned—stated in Fact 6, was refined and strengthened by many authors in many directions, to mention a few, Andréka [3], Biro [17], Maddux [73], Sagi [102], and Hirsch and Hodkinson [48].

Maddux [73] proves that Monk algebras can be generated by a single element. This is far from being trivial. Making an algebra one-generated involves increasing complexity on its automorphism group. The structure becomes rigid.

Biro [17] proves that RCA_n, $\omega > n > 2$, remains non-finitely axiomatizable if we add finitely many *first order definable* operations, a result that is already implicit in Monk's and Maddux's non-finite axiomatizability results in [87] and [73]. The novelty occurring in Biro's result is making the notion "first order definable" explicit.

Andréka [3], building on work of Jonsson [61] for relation algebras, proves the same result in case we add other "kinds" of operations like for example *modalities*, i.e., operations distributing over the boolean join, as long as the added operations are finitely many. While Biro's results excludes axiomatizations by a finite set of equations, Andréka's, on the other hand, exclude axiomatizations involving universal formulas in which only finitely many variables occur.

Sagi [102], building on work of Lyndon [64], addresses the most general formulation of the problem showing that the Finitization Problem cannot be solved by adding finitely many *permutation invariant* operations in the sense of Tarski-Givant [133], as long as one hopes for particular (universal) axiomatizations involving only finitely many variables. We recall from [97] that a permutation invariant operation on a set algebra with unit ${}^nU$ is one that is invariant under permutations of U.

Madárasz [67] addresses the case when the (finitely many) added operation are binary and $L^3_{\infty\omega}$ definable. One general form of the Finitization Problem for both cylindric algebras and relation algebras is to the best of our knowledge still open. This more or less concrete form for cylindric algebras is the following:

Open Problem 3 (Tarski-Givant-Henkin-Monk-Maddux-Németi). Can we expand the language of Cs_n, $\omega > n > 2$ by finitely many permutation invariant operations so that the interpretation of these newly added operations in the resulting class of algebras is still of a concrete set-theoretic nature, and the resulting class generates a finitely axiomatizable variety or quasi-variety?

For further elaboration on this problem we refer to [102, 105] and [131]. We refer to the above problem as the *permutation invariant* version of the

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chaotic. This is similar to studying non-standard models of arithmetic, that do shed light on the standard model.

Finitization Problem. The requirement of permutation invariance here is crucial for it corresponds to the (meta-logical) fact that isomorphic models satisfy the same formulas, a basic requirement in abstract model theory. Without this requirement there are rather easy solutions to the Finitization Problem due to Biro [17], Maddux [73, 72, 76] and Simon [132].

An important result in [102] is reducing the Finitization Problem for relation algebras addressing (the infinitely many) permutation invariant expansions of relation algebras to working inside the class of relation algebras. The analogous result for cylindric algebras is proved in [125]. In other words, it is enough to construct a specific countable sequence of non-representable cylindric algebras to refute the permutation invariant version of the Finitization Problem. This of course does not settle the problem completely, instead it transforms it to a hopefully simpler one. The current "belief" is that the answer to the more restrictive (permutation invariant) form of the problem is either negative, or perhaps even independent of ZF (Zermelo-Fraenkel set theory). Indeed, in [98] and [108] it is proved that several versions of the Finitization problem are independent from ZF minus the axiom of foundation. The results of Hirsch and Hodkinson in [47] and [48] seem to be relevant to the permutation invariant form of the Finitization Problem. In [50, 17.4, p. 625] the problem of axiomatizing the class of relation algebras with a set of first order sentences using finitely many variables is reduced to a problem about (colorings of) certain graphs. On the face of it, this seems to be bad news for graphs, rather than good news for providing "simple' axiomatizations (using only finitely many variables) for representable relation algebras.

Finally, we note that from Monk's result, it follows that RCA_ω is also not finite schema axiomatizable, i.e., cannot be axiomatized by finitely many schemata analogous to those axiomatizing CA_ω. To avoid trivialities, here finite axiomatizability is replaced by finite schema axiomatizability because of the presence of infinitely many operations. The logic corresponding to RCA_ω is L_ω the *structural algebraizable* formalism of first order logic, studied for example in [135], [40], [131] and [41].

§13. Neat reducts and proof theory. Historically, one motivation for defining the class SNr_nCA_m, of all subalgebras of n-dimensional neat reducts of m-dimensional cylindric algebras for $n < m < \omega$, was connected to proof theory of first order logic. Algebraic logic, via the notion of neat reducts, is a useful tool for analyzing the number of variables needed in proofs. The result of Monk, for example, implies that for any Hilbert style axiomatization for languages with countablt many relation symbols and any $3 \leq n < m < \omega$, there are formulas built out of n variables that need more than m variables to be proved. Maddux's result [73] implies that such incompleteness results hold in languages with finitely many variables and with only one binary relation symbol. Maddux's result was strengthened by Hirsch, Hodkinson and

Maddux in [53], by showing that for a first order language with just one binary relation symbol, and for every $3 \leq n < m < \omega$, there is a formula built of n variables that can be proved using $m + 1$ variables, but cannot be proved using m variables. Note that by the soundness of the proof system adopted, Monk and Maddux's results pointed to before become special cases.

The reader may already have wondered why the negative result of Monk was so terminal. After all we may choose to reason about n-ary relations using classical first order logic. There are Hilbert systems with finite schema axiomatizations in the textbooks and a Completeness Theorem. The properties expressible in the language of SNr_nCA_m for $n < m$ can be all expressed using m variables. This is correct but such an approach takes us outside the language of CA_n and so at least lacks elegance.

§14. Neat reducts, interpolation and omitting types, some recent results. In a paper written by J. D. Monk in 1991, but published in the logic Journal of IGPL in 2000 [93], Hajnal Andréka writes the final survey section on the subject, to update the article. It is clear from Andréka's section that among the important problems that were still open in algebraic logic are those in Pigozzi's landmark paper on amalgamation [101] and two problems, both on neat reducts. We believe that Maddux's conjecture—open at the time—formulated in [70] is important too. This conjecture was confirmed by Hirsch and Hodkinson [51]. This problem is on relation algebras. In [70], Maddux introduces the concept of relation algebra of every dimension. The class of relation algebras of dimension n is denoted by RA_n. When $n = 3$ RA_3 coincides with Maddux's semi-associative relation algebras. RA_4 is just the class of relation algebras and RA_ω is the class of representable relation algebras. In between $RA_n \subseteq RA_m$ for$n > m$ forms a (strict) hierarchy approaching the class of representable relation algebras. Maddux proves that for $3 \leq n < m < \omega$ RA_n and RA_m are distinct [75]. What remained unresolved was whether RA_{n+1} is finitely axiomatizable over RA_n, for $4 \leq n < \omega$. Maddux conjectures that the answer is no, RA_{n+1} is not finitely axiomatizable over RA_n. Hirsch and Hodkinson [47, 48] confirm this conjecture using the so-called *rainbow construction*, a term introduced by Venema.

The rainbow construction is only superficially similar to Monk algebras. It is a very poweful technique and has many applications. Using the rainbow construction one can prove (as a sample) [50]: The theorem of Monk that the representable relation algebras are not finitely axiomatizable. The completely representable relation algebras do not form an elementary class. The so-called class of weakly representable relation algebras is not finitely axiomatizable. The class of relation algebras is not closed under completions.

Precursors to this construction can be found in [2] and [74]. In passing, we note that the same applies to the classes $SRaCA_m$ consisting of subalgebras of relation algebra reducts of CA_m when $m > 2$. They form a strict heirarchy as m varies.

All of Pigozzi's question are solved in [68]. The two problems on neat reducts are the consecutive Problems 2.11 and 2.12 posed by Henkin, Monk, Tarski in [40].

Hirsch and Hodkinson solve Problem 2.12 [54]. They show that for $2 < n < \omega$ and $k \in \omega$, $SNr_n CA_{n+k+1}$ is a proper subclass of $SNr_n CA_{n+k}$. Thus the decreasing sequence $CA_n \supseteq SNr_n CA_{n+1} \supseteq SNr_n CA_{n+2} \supseteq \cdots$ converging to RCA_n is not only not eventually constant, as proved by Monk, but is in fact *strictly* decreasing.[13]

In [123] and [94] it is proved that the class $Nr_\alpha CA_\beta$ is closed under homomorphic images, products and ultraproducts for all $\alpha < \beta$. Problem 2.11 in [40] asks whether the class $Nr_\alpha CA_\beta$ for $1 < \alpha < \beta$ is closed under forming subalgebras, for if it is then it would be a variety. Proved not to be closed under subalgebras by Németi [94], the question as to whether it is perhaps closed under elementary subalgebras appears as Problem 4.4 in the monograph [41]. This is equivalent to asking whether this class is elementary, because it is closed under ultraproducts. In [94], Németi conjectures that for $1 < \alpha < \beta$, the class $Nr_\alpha CA_\beta$ is not closed under elementary subalgebras, hence is not elementary, i.e., cannot be axiomatized by any set of first order axioms. In [110] and [109] Németi's conjecture is confirmed.

The notion of neat reducts is an old venerable notion in algebraic logic. But it often happens that an unexpected viewpoint yields new insights. Indeed, the repercussions of the seemingly innocent fact that the class of neat reducts is not closed under forming subalgebras turn out to be enormous, as we proceed to show next.

§15. Neat reducts and amalgamation. In algebraic logic, the notion of amalgamation occurs in essentially two forms. One of them is to construct representations in a step-by-step fashion in the spirit of Fraïssé. See for example the mosaic method of Németi [96], the work of Hirsch and Hodkinson in [50] and the work of Maddux [70, 73], on relation algebras with n-dimensional bases and n-dimensional cylindric bases.

Roughly a basis consists of (finite) approximations to a representation which Maddux call matrices. These matrices can be glued together or amalgamated in a step-by-step manner to give relativized representations, cf. [47] and [70].

Relativized semantics for algebras possessing n-dimensional relational or n-dimensional cylindric bases in the sense of Maddux [73] were provided by Hodkinson and Hirsch. Furthermore, Hodkinson and Hirsch generalized Maddux's bases by introducing the notions of hypernetworks and hyperbasis,

[13] This shows, as mentioned above, that for every $3 \leq n < m$ there is a formula built up of n variables that can be proved using $m + 1$ variables, but cannot be proved using m variables.

cf. [46], and [48], which are suitable for dealing with n-ary relations where $\omega > n > 2$, and not just binary relations. A basis (hyperbasis) consists of matrices (tensors) whose entries are elements of the algebra. When they consist of atoms they are called atomic matrices (tensors). Every such matrix (tensor) can be viewed as a finite graph (hypergraph) giving finite information on the potential representation of an algebra. Under favorable circumstances, specifically when these matrices form a cylindric basis or hyperbasis or relational basis, they can be viewed as a saturated set of mosaics in the sense of Németi [96], and consequently can be amalgamated in a step-by-step fashion to give a limiting genuine relativized representation [50]. If an algebra has an n-dimensional relational basis, the resulting representation is said to be n-square, while if it has an n-dimensional cylindric basis, the representation is said to be n-flat. The reader is referred to [50] for more details.

A typical Fraïssé construction was used in [115] to build a model that is ω-categorical and has quantifier elimination. The set algebra based on this model is used to show that the class $Nr_\alpha CA_\beta$ for $1 < \alpha < \beta \cap \omega$ is not even closed under elementary subalgebras, so that it cannot be defined by any set of first order axioms. In [55] hyperbases are to show that the class $RaCA_n$ of relation algebra reducts of CA_n for $n > 2$ is not elementary either.

The other form of amalgamation occurring in algebraic logic is typically of the following form: Which classes frequently studied in algebraic logic—like for example the class of representable cylindric or for that matter the class of polyadic algebras—have the amalgamation property? Amalgamation is proved in algebraic logic to be the algebraic counterpart of the metalogical property of interpolation, see for example [129] or [68].

Pigozzi [101] is a milestone in this respect. It gives a comprehensive picture of amalgamation for several classes of cylindric algebras. However several questions concerning the strong amalgamation property were posed as open questions by Pigozzi in [101]. We recall that a class has the strong amalgamation property if whenever two algebras in this clas s have a common subalgebra then they can be embedded in a third algebra in this class in such a way that the intersection of the images of the two algebras is the image of the common subalgebra. It turns out in [116], rather surprisingly, that the closure of the class of neat reducts under forming subalgebras for certain classes of cylindric algebras is closely related to the property of strong amalgamation for these classes, cf. [129], [101], [68], [120] and [118]. To formulate the connection between neat embeddings and strong amalgamation, we need:

Definition 7. Let α be an ordinal. Let $L \subseteq RCA_\alpha$. We say that L has the *NS* property, short for *neat reducts commuting with forming subalgebras*, if the following condition holds:

$$(\forall \mathcal{A} \in L)(\exists \mathcal{B} \in CA_{\alpha+\omega})[A \subseteq Nr_\alpha B \wedge (\forall X)(X \subseteq A) \Longrightarrow Sg^{\mathcal{A}} X = \mathcal{N}r_\alpha Sg^{\mathcal{B}} X].$$

Definition 7 singles out certain classes of algebras (having the neat embedding property) whose members embed neatly into algebras in ω extra dimensions in a special way. Let us dwell a little bit more on the property expressed in the *NS* property. Let $\mathcal{A} \subseteq \mathcal{N}r_\alpha\mathcal{B}$, with $\mathcal{A} \in CA_\alpha$ and $\mathcal{B} \in CA_{\alpha+\omega}$. Let $X \subseteq A \subseteq Nr_\alpha\mathcal{B}$. Then we can form the subalgebra of $\mathcal{N}r_\alpha\mathcal{B}$ generated by X, which is the same as the subalgebra of $\mathcal{A}$ generated by X. Or alternatively, we can form the subalgebra of $\mathcal{B}$ generated by X (here we are using ω extra operations, so that in principle we can get new α-dimensional elements), and then form the neat α-reduct of the resulting algebra. Then the *NS* says that taking any of these two paths we will arrive at the same algebra (up to isomorphism), i.e., we *do not* get new α-dimensional elements. In short, *the operation of taking neat α-reducts commutes with that of forming subalgebras.*[14]

Examples of subclasses of RCA_α that have the *NS* abound. These classes include Mn_α the class of minimal cylindric algebras [40, p. 254] more generally, the class of monadic-generated *CA*'s of dimension α, [40, p. 257] and CA_α's of positive characteristic $k < \alpha \cap \omega$ [41, p. 68]. A minimal cylindric algebra is an algebra with no proper subalgebras, equivalently, one that is generated by the diagonal element, while a monadic cylindric algebra is one that is generated by a set X such that $|\Delta x| \leq 1$ for all $x \in X$. Proofs of the results quoted above can be found in [129].

While every $L \subseteq RCA_1$ has the strong NS, this is not the case for higher dimensions. Indeed, thealgebra $\mathcal{A} \in Ws_\alpha$ constructed in [129, Thm. 1] shows that if $Ws_\alpha \subseteq L \subseteq RCA_\alpha$ then L fails to have even the *NS* when $\alpha > 1$. Here Ws_α is the class of weak set algebras of dimension α. Also the algebra $\mathcal{A}$ referred to above was used to confirm a conjecture of Tarski formulated as item (v) in the introduction of [41]. Madárasz [68] proves that any such L, when α is infinite, fails to have the super amalgamation property, or $SUPAP$ for short, in the sense of the definition below. The link between *NS* and $SUPAP$ is given in Theorem 9 (iii). Before formulating our theorem we recall from [129] the notions of amalgamation, strong amalgamation and super amalgamation. The latter is due to Maksimova [80].

Definition 8. (1) Let V be a class of algebras (usually but not always assumed to be a variety) and $L_1 \subseteq L_2 \subseteq V$. L_2 is said to have the amalgamation property, or AP for short over L_1, with respect to V, if for all $\mathcal{A}_0 \in L_1$, all $\mathcal{A}_1$ and $\mathcal{A}_2 \in L_2$, and all monomorphisms i_1 and i_2 of $\mathcal{A}_0$ into $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, there exist $\mathcal{A} \in V$, a monomorphism m_1 from $\mathcal{A}_1$ into $\mathcal{A}$, and a monomorphism m_2 from $\mathcal{A}_2$ into $\mathcal{A}$ such that $m_1 \circ i_1 = m_2 \circ i_2$. In this case we say that $\mathcal{A}$ is an amalgam of $\mathcal{A}_1$ and $\mathcal{A}_2$ over $\mathcal{A}_0$ via m_1 and m_2 or even simply an amalgam.

[14] Below we shall show that the *NS* is, in essence, a definability condition.

(2) Let everything be as in (1). If in addition, $(m_1 \circ i_1)A_0 = m_1(A_1) \cap m_2(A_2)$, then we say that L_1 has the strong amalgamation property, or SAP for short with respect to V. We say that $\mathcal{A}$ is a strong amalgam of $\mathcal{A}_1$ and $\mathcal{A}_2$ over $\mathcal{A}_0$ via m_1 and m_2 or even simply a strong amalgam.

(3) Let $L_1 \subseteq L_2 \subseteq V$. We say that L_2 has $SUPAP$ over L_1 with respect to V, if for all $\mathcal{A}_0 \in L_1$, $\mathcal{A}_1$ and $\mathcal{A}_2 \in L_2$, and all monomorphisms i_0 and i_1 of $\mathcal{A}_0$ into $\mathcal{A}_1, \mathcal{A}_2$, respectively, there exists $\mathcal{A} \in V$, a monomorphism m_1 from $\mathcal{A}_1$ into $\mathcal{A}$ and a monomorphism m_2 from $\mathcal{A}_2$ into $\mathcal{A}$ such that $m_1 \circ i_1 = m_2 \circ i_2$, and

$$(\forall x \in A_j)(\forall y \in A_k)(m_j(x) \leq m_k(y) \Longrightarrow (\exists z \in A_0)(x \leq i_j(z) \wedge i_k(z) \leq y))$$

where $\{j,k\} = \{1,2\}$. In this case we say that $\mathcal{A}$ is a super amalgam of $\mathcal{A}_1$ and $\mathcal{A}_2$ over $\mathcal{A}_0$, via m_1 and m_2, or even simply a super amalgam.

(4) When $L_1 = L_2 = L$ in (1) (2) and (3) above we say that L has AP, SAP, and $SUPAP$, respectively, with respect to V. If furthermore $L = V$, we say that V simply has AP, SAP, and $SUPAP$, respectively.

It is easy to see that for cylindric algebras $SUPAP$ implies SAP. Now the following Theorem relating *NS* to $SUPAP$ is proved in [116]. It is a generalization of a result of Pigozzi in [101], cf 2.12 therein.

THEOREM 9. *Let α be an ordinal $\leq \omega$. Let $L_1 \subseteq L_2 \subseteq RCA_\alpha$. Assume there there exists a "lifting" function $\boldsymbol{F}$ from $L_1 \cup L_2$ to $CA_{\alpha+\omega}$ such that $\mathcal{A} \subseteq \mathcal{N}r_\alpha \boldsymbol{F}(\mathcal{A})$ (here the neat embedding theorem is used) and such that the following two conditions hold*:

(i) *Forming subalgebras of $\mathcal{A}$, is the same as forming subalgebras of $\mathcal{N}r_\alpha\boldsymbol{F}(\mathcal{A})$ formally,*

$$(\forall \mathcal{A} \in L_1 \cup L_2)(\forall X \subseteq A)(Sg^{\mathcal{A}}X = Sg^{\mathcal{N}r_\alpha \boldsymbol{F}(\mathcal{A})}X),$$

(ii) *lifting and forming subalgebras commute, formally*

$$(\forall \mathcal{A} \in L_1 \cup L_2)(\forall X \subseteq A)(\boldsymbol{F}(Sg^{\mathcal{A}}X) = Sg^{\boldsymbol{F}(\mathcal{A})}X),$$

then L_2 has AP over L_1 with respect to RCA_α. Assume in addition to (i) *and* (ii) *that the following condition is satisfied.*

(iii) *L_1 has NS (with $\boldsymbol{F}(\mathcal{A})$ playing the role of $\mathcal{B}$ in* Definition 7) *i.e.,*

$$(\forall \mathcal{A} \in L_1)(\forall X \subseteq A)(Sg^{\mathcal{A}}X = \mathcal{N}r_\alpha Sg^{\boldsymbol{F}(A)}X),$$

then L_2 has $SUPAP$ over L_1 with respect to RCA_α.

PROOF. A proof can be found in [68]. All the same, here we highlight the main steps of the proof. Let $L_1 \subseteq L_2 \subseteq RCA_\alpha$. Let

$$\boldsymbol{M} = \{\mathcal{A} \in CA_{\alpha+\omega} : Sg^{\mathcal{A}}Nr_\alpha A = \mathcal{A}\}.$$

Then $\boldsymbol{M}$ consists of algebras generated by the set of their α-dimensional elements. In particular, $\boldsymbol{M} \subseteq Dc_{\alpha+\omega}$, the class of dimension complemented cylindric algebras of dimension $\alpha+\omega$ in the sense of [40, 1.11.1], that is, if $\mathcal{A} \in \boldsymbol{M}$, then $|(\alpha + \omega) \smallsetminus \Delta x| \geq \omega$. Let $\mathcal{B}$ be the *dimension restricted* free algebra $Fr_{\omega}^{\rho} CA_{\alpha+\omega}$, in the sense of [40, 2.5.34]. Here $\rho : \omega \to \wp(\alpha + \omega)$ the *restricting function* is defined by $\rho(i) = \alpha$ for all $i \in \omega$. Roughly $Fr_{\omega}^{\rho} CA_{\alpha+\omega}$ corresponds to a language and ρ specifies the arity of the atomic formulas in this language. Then $\mathcal{B}$ is dimension complemented, hence it has the strong interpolation property in the sense of [101, 2.1.16]. This is proved by Pigozzi in [101, 2.2.6]. It follows by [68] Proposition 5 that $\boldsymbol{M}$ has $SUPAP$. Now let $\mathcal{A}_0 \in L_1$, and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be in L_2. Let i_1 and i_2 be monomorphisms of $\mathcal{A}_0$ into $\mathcal{A}_1$, and $\mathcal{A}_2$, respectively. We first using (i) and (ii) find an amalgam. We lift our diagram to ω extra dimensions. More precisely, we have $\boldsymbol{F}(\mathcal{A}_0)$, $\boldsymbol{F}(\mathcal{A}_1)$ and $\boldsymbol{F}(\mathcal{A}_2)$ are in $\boldsymbol{M}$. Let $\bar{i}_1$, $\bar{i}_2$ be monomorphisms of $\boldsymbol{F}\mathcal{A}_0$ into $\boldsymbol{F}\mathcal{A}_1$, and $\boldsymbol{F}\mathcal{A}_2$, respectively, such that the restriction of $\bar{i}_1$ and $\bar{i}_2$ to $\mathcal{A}_0$ are i_1 and i_2. (i) and (ii) guarantee the existence of these functions. Let $\mathcal{E} \in \boldsymbol{M}$ be a super amalgam of $\boldsymbol{F}(\mathcal{A}_1)$ and $\boldsymbol{F}(\mathcal{A}_2)$ over $\mathcal{A}_0$ via i_1 and i_2. Let $\mathcal{D} = \mathcal{N}r_\alpha\mathcal{E}$. Then it is absolutely straightforward to check that $\mathcal{D}$ is an amalgam of $\mathcal{A}_1$ and $\mathcal{A}_2$ over $\mathcal{A}_0$ via the restrictions of i_1 and i_2 to $\mathcal{A}_0$. If in addition, L_1 has NS then $\mathcal{D}$ is a *super amalgam*. Note that the idea is simple. Given three algebras, we embed them into three other algebras in ω extra dimensions, find a superamalgam there, and then return to our diagram via the neat reduct operator. $\dashv$

As an application to Theorem 9, it is proved in [68] that the classes of monadic generated CA's and CA's of positive characteristic have $SUPAP$. In [129] it is proved that the classes of semisimple cylindric algebras Sc_ω and the class of diagonal cylindric algebras Di_ω, in the sense of [40], do not have NS. Emphasizing the link between NS and $SUPAP$, Madárasz [68] proves that the classes Sc_ω and Di_ω do not have SAP. In fact, these classes do not have the strong amalgamation property even if the strong amalgam is sought in the bigger class of representable algebras [68]. In [116] and [129] it is proved that the so-called strong amalgam base of RCA_α coincides with the class of algebras having NS. This answers a question raised by Monk [8].

In [120] Theorem 9 is generalized to the very general context of systems of varieties definable by schemes in the sense of [41]. Also Theorem 9 is compared to Németi's techniques on amalgamation adopted in [95]. Presenting Theorem 9 and Nemeti's technique formulated as Lemma 3 in [95] in a functorial context as adjoint situations, it is shown in [120] that both can be seen as instances of the use of the Keisler-Shelah ultrapower Theorem in proving Robinson's Joint Consistency Theorem. Another aspect of this unification consists of presenting Theorem 9 herein and Németi's Lemma 3 as transforming a diagram of algebras to be strongly amalgamated into certain saturated representations of these algebras that can be strongly amalgamated and then returning to the original diagram using an inverse operator. The categorical formulation makes the notion of inverse operator precise. In the case of Theorem 9 it is the neat reduct functor[15] (an inverse to a neat embedding functor—denoted by $\boldsymbol{F}$ taking an algebra into ω extra dimensions i.e., to a classical representation), while in Németi's Lemma 3 it is basically the operation of forming atom structures that is an inverse of taking an algebra to its canonical extension (which can be seen as a modal representation). This is yet another application instance of the triple duality existing between boolean algebras with operators Kripke frames and modal logic.

A purely modal approach to proving Németi's results in [95] was found independently by Weaver and Welaish in [138]. In [138] the argument used is a back and forth construction to establish a modal analogue of Robinson's joint consistency theorem. Also it is shown in [120] that both techniques take the representation problem expressed by a two-sorted defining theory, as used by Hirsch and Hodkinson, a step further, asking that the second sort to be a saturated representation.

This is not surprising since (in extensions of first order logic and multimodal logics) interpolation is stronger than completeness, see e.g., Henkin's proof of Craig interpolation in [21], and its algebraization to wider contexts in [101] and [121]. Indeed this is illustrated in Theorem 9, because of the following reasoning. Completeness corresponds to representability. A class of algebras consists of representable algebras, if each of its members embed neatly into ω extra dimensions. On the other hand, for this class to have $SUPAP$, its members should embed in ω extra dimensions in a special way; this class should have the NS.[16] An intriguing open question—among the very few remaining—on amalgamation in cylindric algebras is the following[17]

OPEN QUESTION 4 (Madárasz-Németi-Pigozzi). Find a variety or quasivariety or even just a class of cylindric algebras that has the strong amalgamation property but does not have the super amalgamation property.

We mention in passing that it is easy to find classes of algebras that have AP but not SAP. Examples are Sc_ω and Di_ω, as proved in [101] and [68]. Henkin's methods of constructing models out of constants was used to prove the completeness of first order logic and the Craig interpolation Theorem.

[15]It is easy to see that the operator of forming neat α-reducts of CA_β's is a functor from CA_β to CA_α for any $\alpha < \beta$.

[16]From a model-theoretic point of view, this is not too surprising. Let us explain why. Henkin's neat embedding theorem is an algebraization of his celebrated proof of the completeness theorem, which is an instance of Robinson's finite forcing in model theory. As Hodges [57] says: if the forcing conditions are chosen cleverly, this yields interpolation results. Pigozzi did exactly that in his proof that the dimension-restricted free algebras that happen to be dimension complemented have the interpolation property [101]. In [118] a similar argument is used to show that the class of algebras studied in [106] has the strong amalgamation property.

[17]We learned from Németi that this problem is already solved by Madárasz and Shelah, but the proof is not published yet.

And indeed its algebraization (and generalization) is expressed in the Neat Embedding Theorem, and a strengthening thereof, expressed in the *NS* property, was used to prove the completeness and interpolation for certain extensions of first order logic. Henkin's methods were also used to prove the Henkin-Orey omitting types theorem. Therefore it is no surprise that another variation on the Neat Embedding Theorem leads to proving an Omitting Types Theorem for these extensions, as we proceed to show next.

§16. Neat embeddings, complete representations omitting types, and Martin's axiom. The following new characterization of countable algebras in the class of the so-called completely representable finite dimensional cylindric algebras is given in [114]. Such algebras are investigated in, e.g., [63], [44] and [43]. We first define the notion of complete representations. For this we need to fix some notation. For an algebra $\mathcal{A}$ with a boolean reduct and $X \subseteq A$, $\sum X$ and $\bigwedge X$ denote the supremum and infimum of X, respectively. For algebras $\mathcal{A}$ and $\mathcal{B}$, $Ho(\mathcal{A}, \mathcal{B})$ denotes the set of all homomorphisms from $\mathcal{A}$ to $\mathcal{B}$.

Definition 10. Let $n < \omega$. Let $\mathcal{A} \in CA_n$. $\mathcal{A}$ is *completely representable* if for all non-zero $a \in A$, there exists $\mathcal{B} \in Cs_n$ and $f \in Ho(\mathcal{A}, \mathcal{B})$ such that $f(a) \neq 0$ and such that

$$f(\sum X) = \bigcup_{x \in X} f(x)$$

for all $X \subseteq A$ whenever $\sum X$ exists.

We shall use the following lemma that often facilitates proving that a given algebra possess a complete representation [43]:

Lemma 11 (Hirsch-Hodkinson [43]). *Let $\mathcal{A} \in CA_n$. Then the following two conditions are equivalent.*

(1) *$\mathcal{A}$ is completely representable.*
(2) *$\mathcal{A}$ is atomic, and has an atomic representation, i.e., there exists a $\mathcal{C} \in Gs_n$ with unit V, say, and an isomorphism $f : \mathcal{A} \to \mathcal{C}$ such that $\cup\{f(x) :$ x is an atom $\} = V$.*

Before formulating the connection between neat embedding and complete representations, we also need:

Definition 12. For K a class with a boolean reduct we define

$$S_cK = \{\mathcal{A} : \exists \mathcal{B} \in K : A \subseteq B, \text{ and whenever } \sum X = 1 \text{ in } \mathcal{A}$$
$$\text{then } \sum X = 1 \text{ in } \mathcal{B} \text{ for all } X \subseteq A\}.$$

Roughly S_cK denotes the operation of forming *complete* subalgebras of algebras in K. Our next Theorem is proved in [114]. It says that for finite n and for a countable atomic n-dimensional algebra to be completely representable, it has to embed neatly and *completely* into another algebra in ω extra dimensions, i.e., it is an algebra in $S_cNr_nCA_\omega$. We refer to algebras in the latter class as algebras having the *strong neat embedding property*. In [114] it is shown that the strong neat embedding property is indeed stronger than the neat embedding property. An example of a representable algebra that does not have the strong neat embedding property is given below in Theorem 15. However, the two notions coincide in other contexts, as is the case with the algebras studied in [106].

We give a proof of a statement abbreviated by $(*)$ below from which Theorem 13 readily follows. The reason for including a proof is that we want to highlight the connection of $(*)$ with notions other than complete representations, namely the Henkin-Orey Omitting Types Theorem, Martin's Axiom, and the Baire Category Theorem.

THEOREM 13. *Assume that $n < \omega$. Let $\mathcal{A} \in CA_n$ be countable. Then $\mathcal{A}$ is completely representable if and only if $\mathcal{A}$ is atomic and $\mathcal{A} \in S_cNr_nCA_\omega$.*

PROOF. We only prove the non-trivial implication the if part.

Towards this end, we prove something stronger namely:

$(*)$. *If $\mathcal{A} \in S_cNr_nCA_{n+\omega}$ is countable, $n \leq \omega$ (note that here n is allowed to be infinite) and $\{X_i : i < \omega\}$ is a family of subsets of A such that $\bigwedge X_i = 0$ for all $i < \omega$, then for every non-zero $a \in A$ there exists $\mathcal{C} \in Ws_n$, with countable base, and $f \in Ho(\mathcal{A}, \mathcal{C})$ such that $f(a) \neq 0$ and for all $i \in \omega$ we have $\cap_{x\in X_i} f(x) = \emptyset$.*

We recall that a weak set algebra has unit of the form ${}^nU^{(p)} = \{s \in {}^nU : |\{i \in n : s_i \neq p_i\}| < \omega\}$, for some $p \in {}^nU$. U is called its base. The statement in $(*)$ is an algebraic version of an Omitting Types Theorem. Indeed, to emphasize the connection with the Omitting Types Theorem, we refer to the X_i's as *non-principal types* and to $\mathcal{B}$ as a representation *omitting* these types. Note that for $n < \omega$ we have $Ws_n = Cs_n$, so that a unit of a Ws_n is simply of the form ${}^nU$. We show how the only if part of Theorem 13 follows from $(*)$. Assume that $(*)$ is proved and let $n < \omega$. Let $\mathcal{A} \in S_cNr_nCA_\omega$ be countable and atomic. We can assume, without loss of generality, that $\mathcal{A}$ is simple, for if not then we could replace it by its simple components. Then taking $X_i = Y = \{-b : b$ is an atom of $\mathcal{A}\}$, and applying $(*)$ for any non-zero a in A, upon noting that $\bigwedge Y = 0$ since $\mathcal{A}$ is atomic, we get an atomic representation, hence by Lemma 11 a complete representation of $\mathcal{A}$.

Now we prove $(*)$. In the proof we use the following form of the celebrated Rasiowa-Sikorski Lemma:

$(**)$. *Let $\mathcal{B}$ be a boolean algebra. Let J and K be countable sets. Let $\{S_j : j \in J\}$ and $\{T_k : k \in K\}$ be families of subsets of B such that $\sum S_j$ and $\bigwedge T_k$ exist for each $j \in J$ and $k \in K$. Then for every non-zero $a \in B$, there exists an ultrafilter F containing a such that for all $j \in J$ if $\sum S_j \in F$ then $S_j \cap F \neq \emptyset$ and for $k \in K$ if $T_k \subseteq F$ then $\bigwedge T_k \in F$.*

We note that such an ultrafilter F is referred to as a *generic* ultrafilter in forcing constructions. Let $\mathcal{A} \in CA_n$ and $\{X_i : i < \omega\}$ be as in the hypothesis of $(*)$ and let $a \in A$ be non-zero. For brevity, let $\beta = n + \omega$. Note that $\beta = \omega$ when n is finite. We want to find a $\mathcal{C} \in Ws_n$ and a homomorphism $h : \mathcal{A} \to \mathcal{C}$ such that $h(a) \neq 0$ and $\cap_{x \in X_i} h(x) = \emptyset$ for all $i \in \omega$. Using Hodge's terminology [57, p. 369], we *force* with $\mathcal{A}$. Strictly speaking, Hodges forces with theories, here we force with their algebraic counterpart. Toward this end, let $\mathcal{B} \in CA_\beta$ be such that $A \subseteq Nr_n B$, A is a generating set for $\mathcal{B}$, and $\bigwedge^{\mathcal{B}} X_i = 0$ for each $i < \omega$. This is possible since $\mathcal{A}$ has the strong neat embedding property. In particular, $\mathcal{B}$, having a countable signature and generated by a countable set, is also countable and by [40, Thm. 2.1.5(ii)] is *dimension complemented* in the sense of [40, 1.11.1(ii)], that is,

$$(\forall x \in B)(|\Delta x \smallsetminus n| < \omega).$$

Recall that Δx, the *dimension set* of x, is defined to be the set $\{i \in \beta : c_i x \neq x\}$. Note that when $n < \omega$ then $\mathcal{B}$ is locally finite. We have by [40, Thm. 1.11.6] that

$$(1)\ \ (\forall j < \beta)(\forall x \in B)(c_j x = \sum_{i \in \beta} s_i^j x.)$$

Now let V be the weak space ${}^\beta\beta^{(Id)}$, where Id is the identity function on β, i.e.,

$$V = {}^\beta\beta^{(Id)} = \{\tau \in {}^\beta\beta : |\{i \in \beta : \tau_i \neq i\}| < \omega\}.$$

For each $\tau \in V$ for each $i \in \omega$, let

$$X_{i,\tau} = \{s_\tau^B x : x \in X_i\}.$$

Here s_τ^B is the unary operation as defined in [40, 1.11.9][18], which is a complete boolean endomorphism on B by [40, Thm. 1.11.12(iii)]. It thus follows that

$$(2)\ \ (\forall \tau \in V)(\forall i \in \omega) \bigwedge{}^B X_{i,\tau} = 0.$$

The superscript B indicates that these meets are evaluated (and indeed preserved) in the big algebra $\mathcal{B}$. Now using $(**)$, let F be the (generic) ultrafilter preserving the set of infinite meets and joins expressed in (1) and (2) and has $a \in F$. Then it is not difficult to check by the very construction of F that

$$(3)\ \ (\forall k < \beta)(\forall x \in B)(c_k x \in F \Longrightarrow (\exists i \notin \Delta x)(s_k^i x \in F).$$

and that

$$(4)\ \ (\forall \tau \in V)(\forall i \in \omega)(\exists x \in X_i)(s_\tau x \notin F).$$

[18] s_τ is the algebraic counterpart of the metalogical operation of the simultaneous *substitution* of variables (indexed by the range of τ) for variables (indexed by its domain), cf. [40], the remark after Theorem 1.11.8.

Next we form the *canonical* representation corresponding to F in which satisfaction coincide with genericity. To handle equality we define

$$R = \{(i, j) \in {}^2\beta : d_{ij} \in F\}.$$

Then, as easily checked, R is an equivalence relation on β. Let $\sigma, \tau \in V$. Then we write $\sigma \equiv_R \tau$ iff

$$(\forall i \in \beta)(\sigma(i), \tau(i)) \in R,$$

and we set

$$R^+ = \{(\sigma, \tau) \in {}^2V : \sigma \equiv_R \tau\}.$$

Then the following conditions hold:

$$R^+ \text{ is an equivalence relation on } V,$$

the set

$$C = \{X \subseteq V : (R^+)X = X\}$$

is a boolean field of sets, i.e., it is closed under set-theoretical intersection $\cap$, and complementation with respect to V $(\smallsetminus_V)$, and it is also closed under the i th cylindrification C_i, for each $i < \beta$. Finally, the algebra

$$\mathcal{C} = \langle C, \cap, \smallsetminus_V,\ C_i, R(d_{ij} \cap V)\rangle_{i,j<\beta}$$

is a weak set algebra with unit V, and it is not difficult to check that the map $h : \mathcal{A} \to \mathcal{C}$ defined by

$$h(x) = \{\sigma \in V : s_\sigma^B x \in F\}$$

is the desired homomorphism. Indeed, that h is a homomorphism follows from (3) (cf. the appendix of [7, p. 39] for a similar proof) that $h(a) \neq 0$ follows from the fact that Id, the identity function on β, is in $h(a)$ since $s_{Id}^\beta a = a \in F$, and finally, that h omits the non-principal X_i's follows directly from (4). $\dashv$

§17. Omitting types, the Baire category theorem and Martin's axiom. The property expressed in $(*)$ was used to prove an Omitting Types Theorem for certain infinitary extensions of first order logic studied in [7], cf. [114]. A variation thereof is used to prove an Omitting Types Theorem for other extensions of first order logic (without equality) introduced as one possible solution to the Finitization Problem in [107] and [106], cf. [121]. The proof of Theorem 13 is a typical omitting types construction; it is a variation on a theme occurring in [20] and [100]. In fact a slight modification of this proof gives the results in the latter two papers as shown in [114]. An instance of Robinson's finite forcing in model theory, the proof of Theorem 13 is indeed a Baire category argument at heart. It is not hard to show that $(**)$ is equivalent in ZFC to the property that every compact Hausdorff space is Baire. Indeed, instead of invoking $(**)$, one can instead find the desired ultrafilter F, by noting that the set X of such ultrafilters is dense in the

Stone space of A. This follows from the fact that X—consisting of a set of ultrafilters *respecting* a countable set of infinite meets and joins in the sense of [32, p. 414]—can be written as the intersection of a countable family of dense open sets in the (compact Hausdorff) Stone space of A. Therefore, by the Baire Category Theorem for compact Hausdorff space, X is dense. We refer in this connection to the topological proof in [32, p. 414] of the Rasiowa-Sikorski Lemma. An interesting result of Goldblatt [32] is that the Rasiowa-Sikorski Lemma is equivalent in ZF (Zermelo-Fraenkel set theory *without* choice) to the conjunction of the Ultrafilter Theorem and the Baire Category Theorem for compact Hausdorff spaces. We note that Theorem 13, basically a representation theorem, can be proved using a step-by step argument via games, as introduced in algebraic logic by Hirsch and Hodkinson. Having said that, it seems that one can make the translation from the "Baire category" approach adopted herein to Hirsch-Hodkinson's games, both being instances of Robinson's finite forcing, by way of the Banach-Mazur Theorem (cf. [57, p. 358]).

When one asks for the omission of $< 2^{\omega}$ types one is led to an instance of the famous independent Martin's axiom, establishing an interesting connection with set theory [129].

Indeed the resulting statement turns out to be equivalent (in ZF) to Martin's axiom restricted to countable boolean algebras. Also countability of the algebras involved cannot be omitted from the above characterization [56]. However in the absence of the continuum hypothesis and for that matter Martin's axiom, ω_1 many types can be omitted. In ZF one can omit $covK$ many types, where $covK$ is the least cardinal κ such that the real line can be covered by κ many closed nowhere dense sets, cf. [114]. $covK$ is an uncountable cardinal $< 2^{\omega}$. Martin's axiom implies that $covK = 2^{\omega}$ but it is consistent that $covK = \omega_1 < 2^{\omega}$.

For more on these descriptive set-theoretic notions in connection to omitting types the reader is referred to [20], [100] and [114].

§18. Omitting types for the finite variable fragments of first order logic. We now give yet another novel application of algebraic logic to first order logic. We show how the construction of certain relation algebras can be used to show that the classical Henkin-Orey Omitting Types Theorem fails for the finite variable fragments of first order logic, in a rather strong sense (to be made precise shortly).

These algebras were originally constructed to serve an entirely different purpose, namely to show that the classes of representable relation algebras and representable cylindric algebras of finite dimension > 2 are neither *atom-canonical* nor *single-persistent* nor *Sahlqvist* nor closed under *minimal completions* . We refer the reader to [36] for the definition of these notions. We will deal with some of these algebraic notions in a while. The author used cylindric algebras constructed by Hirsch and Hodkinson in [43] and Hodkinson in [59] to prove a result weaker than Theorem 15 below concerning the failure of omitting types, cf. [129, Thm. 4]. In Theorem 15 below, the construction of the relation algebras is due to Andréka [4], while the connection with omitting types is due to the present author. However, we will not give the construction of Andréka, since it will be published elsewhere. To formulate our joint result with Andréka, we need to fix some notation and recall some terminology.

Let L_n denote first order logic restricted to the first n variables. choices. Let T be a countable consistent L_n theory. Let Γ be a countable set of L_n formulas that is consistent over T, i.e., no contradiction is derivable from $T \cup \Gamma$ in any complete axiomatization of first order logic. For a formula ϕ and a model M, we recall that ϕ^M denotes the set of all assignments that satisfy ϕ in M, i.e.,

$$\phi^M = \{s \in {}^{\omega}M : M \models \phi[s]\}.$$

DEFINITION 14. We say that Γ is *implicitly principal* over T, if for all $M \models T$, $\bigcap_{\phi\in\Gamma}\phi^M \neq \emptyset$. We say that Γ is *explicitly k-principal* over T, if there exists a formula ϕ built up of at most k variables that isolates Γ, i.e., $T \models \phi \Longrightarrow \psi$ for all $\psi \in \Gamma$.

The classical Henkin-Orey Omitting Types Theorem [21, Thm. 2.2.9], or rather the contrapositive thereof, implies that if Γ is implicitly principal (over T) then Γ is explicitly k principal (over T) for some $k \in \omega$. The question is, do we guarantee that $k \leq n$, i.e., the formula isolating Γ stays inside L_n, or do we have to "step outside" L_n, resorting to extra variables? The following result was announced in [4] and [117]. It contrasts positive results on omitting types proved in [114].

THEOREM 15. *For all $3 \leq n < k$ there exists a countable L_n theory T, a type Γ consistent over T such that Γ is implicitly principal but not k-explicitly principal over T.*

We need the following Lemma. For undefined terminology in the coming key Lemma the reader is referred to [73] or [59].

We note that algebras with most of the properties listed below have been published first by Hodkinson [59], then again by Hirsch and Hodkinson [51] and in their book [50]. An explicit suggestion for constructing these algebras, based on published work, has been added to the proof.

LEMMA 16 (Andréka). (i) *Let $n > 2$. There is a countable symmetric atomic integral representable relation algebra R_n such that its completion, i.e., the complex algebra of its atom structure is not representable.*

(ii) *The (countable) set $B_n R_n$ of all $n \times n$ basic matrices over R_n constitutes an n-dimensional cylindric basis in the sense of* Maddux [73, Def. 4]. *Thus $B_n R_n$ is a cylindric atom structure and the full complex algebra $Ca(B_n R_n)$*

with universe the power set of B_nR_n is an n-dimensional cylindric algebra that is not representable.

(iii) *The term algebra over the atom structure B_nR_n, which is the countable subalgebra of $Ca(B_nR_n)$ generated by the countable set of n by n basic matrices, $Tm(B_nR_n)$ for short, is a countable representable CA_n.*

(iv) *$Tm(B_nR_n)$ is simple so that it is in fact (isomorphic to) a Cs_n.*

(v) *$Tm(B_nR_n)$ does not have a complete representation in the sense of* [43], *and is binary-generated i.e., generated by its 2-dimensional elements.*

(vi) *For $2 < n < k < \omega$, $Tm(B_nR_n)$ is isomorphic to $Nr_n(Tm(B_kR_k))$.*

Proof. Andréka [4] constructed such an algebra. However, as pointed to us by the referee if $n = 3$ or $n = 4$, and R_n is the algebra described in Exercise 8 of [50, p. 458], then R_n has properties (i)–(v). A sequence satisfying (i)–(vi) for $n \geq 5$ arises from [50, Ex.8] by changing the number of colors in Definition 14.10 of [50] from three (namely, red, green, and blue), to $n - 1$. ⊣

We now show how the construction of the relation algebras R_n can be used to show that the Omitting Types Theorem fails for L_n. In other words, we prove Theorem 15.

Proof of Theorem 15 modulo Lemma 16. For brevity, let $\mathcal{A} = Tm(B_nR_n)$, and let $\mathcal{B} = Tm(B_kR_k)$. Without loss of generality, we assume that $\mathcal{A} = \mathcal{N}r_n\mathcal{B}$. By [40, Thm. 4.3.28(ii)] there is a countable first order language $\boldsymbol{L}$, such that $\mathcal{B} \cong (Fm^{\boldsymbol{L}_k}/T)$ for some (countable) L_k theory $T \subseteq Fm^{\boldsymbol{L}_k}$. Here $\boldsymbol{L}_k$ is the restriction of the language $\boldsymbol{L}$ to the first k variables and $Fm^{\boldsymbol{L}_k}/T$ is the Lindenbaum-Tarski representable k-dimensional representable cylindric algebra corresponding to T. We can assume that T consists of sentences only, i.e., that no free variables occur in formulas in T. It follows that $\mathcal{A} \cong Fm^{\boldsymbol{L}_n}/T$. Fix θ an isomorphism from $Fm^{\boldsymbol{L}_n}/T$ to $\mathcal{A}$ and let $At\mathcal{A}$ denote the set of atoms of $\mathcal{A}$. Put

$$\Gamma = \cup\{\neg\phi/T : \theta(\phi/T) \in At\mathcal{A}\}.$$

Here ϕ/T denotes the equivalence class of ϕ, consisting of all formulas equivalent to ϕ modulo T. Since $\mathcal{A}$ is atomic and $A = Nr_n\mathcal{B}$, it follows that the supremum of $At(\mathcal{A})$ evaluated in $\mathcal{B}$ is equal to the unit, in symbols $\sum^{\mathcal{B}} At\mathcal{A} = 1$. Indeed, we have $\sum^{\mathcal{A}} At\mathcal{A} = 1$ since $\mathcal{A}$ is atomic and the boolean reduct of $\mathcal{A}$ is a complete subalgebra of that of $\mathcal{B}$. Thus we have that the following infimum is equal to the least element, in symbols

$$(+) \qquad \bigwedge\{\theta(\psi/T) : \psi \in \Gamma\} = 0.$$

Now we check that Γ is implicitly principal but not k-explicitly principal. To see that Γ is not explicitly k-principal, assume to the contrary that there exists $\phi \in Fm^{\boldsymbol{L}_k}$ such that ϕ is consistent with T and ϕ isolates Γ. We can assume without loss of generality that the free variables occurring in ϕ are among the first n. But then we get that for all $\psi \in \Gamma$

$$0 < \theta(\phi/T) \leq \theta(\psi/T).$$

This contradicts $(+)$. To see that Γ is implicitly principal, assume to the contrary that Γ can be omitted i.e., there exists $M \models T$ such that $\cap_{\phi \in \Gamma} \phi^M = \emptyset$. Then $\cup_{\phi \in \Gamma}(\neg \phi^M) = {}^n M$. But then $\{\phi^M : \phi \in Fm^{\mathbf{L}_n}\}$ would be (the domain of) an atomic representation of $\mathcal{A}$ in the sense of [43], Definition 4, thus by Theorem 5 of [43], it is a complete representation of $\mathcal{A}$, contradiction. ⊣

Since the algebra constructed by Andréka is binary generated i.e., generated by its two dimensional elements, it follow from [41] that the diagonal free reduct of this algebra is not completely representable. This gives:

THEOREM 17. *Let $n > 2$ then the Omitting Types Theorem fails for the equality free version of L_n, and the multi-modal logic $S5^n$.*

The multi-modal logic $S5^n$ is studied in e.g., [137] and [52]. Theorem 15 and Theorem 17 says that, though there are always formulas isolating or witnessing realizable types, we cannot control the number of variables occurring in such formulas. They grow without bound. This adds to the list of deep (negative) results known for L_n (and its equality free reduct) concerning the complexity of its proof theory for $n > 2$ [89], [3], and [53]), (failure of) interpolation for $n > 1$ ([129]) and undecidability of the validities for $n > 2$ [41, Thm. 4.2.18].

Also this adds to our knowledge of omitting types for usual first order logic, since it says that there is no bound on the number of variables needed to isolate non-omissable types. What this entails in general terms is that in isolating types, all ω variables are needed. This is indeed analogous to the situation with provability and interpolation. We point out that from Theorems 15 and Theorem 17 one can easily prove that the Omitting Types Theorem fails for any first order definable expansion of L_n (and its equality-free reduct) as defined in [17].. Biro [17] used the algebras in [73], but those same algebras apparently allow for a considerable extension of Biro's theorem, since, for example, Maddux added transitive closure (which is not first-order definable) in [72]. In contrast, we have:

THEOREM 18. *The Omitting Types Theorem holds for L_1, and more generally for countable L_n theories, $n \leq \omega$, with only (countably) many unary relation symbols.*

PROOF. This follows from Németi's algebraic result [94] that *monadic generated* cylindric algebras are neat reducts, that is, if $1 < n \leq \omega$ and $\mathcal{A} \in CA_n$ is generated by a set X such that every $x \in X$, $|\Delta x| \leq 1$,[19] then $\mathcal{A} \in Nr_n CA_\omega$. The proof of this can be easily recovered from the proof of "the only if part" of (ii) of Theorem 1 in [94, p. 404–405], or from [68] by noting that the class

[19] Recall that Δx is the dimension set of x, which is defined to be $\{i \in n : c_i x \neq x\}$.

of monadic generated CA_n's has the NS. In particular $\mathcal{A}$ has the strong neat embedding property for it is not difficult to see that $Nr_n CA_\omega \subseteq S_c Nr_n CA_\omega$. By Theorem 13 countably many non-principal types can be omitted. $\dashv$

While every atomic RCA_2 is completely representable; this follows easily from [41, Thm. 3.2.65], do not know whether the omitting types Theorem holds for L_2. We pose this as an open question:

Open Question 5. Does the omitting types Theorem hold for first order logic restricted to the first two variables?

As we mentioned earlier, the algebra R_n used above was originally designed to show that RRA and RCA_n are not Sahlqvist varieties, i.e., cannot be characterized by Sahlqvist equations. Thus they are not atom-canonical, nor closed under completions [36]. That RCA_ω is not Sahlqvist is proved in [126]. We now turn to surveying such notions, in the wider context of boolean algebras with operators.

§19. Atom-canonicity, omitting types and completions. Let V be a variety of completely additive boolean algebras with operators. Let $SigV$ denote the signature of V without the boolean operations. A completely additive variety is one for which the the non-boolean operations in $SigV$ distribute over arbitrary joins. Examples include the so-called *conjugated varieties* in the sense of [50], and more concretely RCA_α, α finite, and the class of representable relation algebras, or RRA for short,

Now let $\mathcal{A} \in V$ be atomic. We recall that the *atom structure* of $\mathcal{A}$ is the structure

$$(AtA, R_f)_{f \in SigV}$$

where R_f is an $n+1$ ary relation defined for each n-ary $f \in SigV$ by: For all atoms $a_1, \ldots, a_n, b$,

$$R_f(a_1, \ldots, a_n, b) \longleftrightarrow f(a_1, \ldots, a_n) \geq b.$$

$At(V)$ is the class of all such structures.

More generally an *atom structure* or a *Kripke frame* of type $Sig(V)$ is a pair $\mathcal{F} = (F, R_f)$ where F is a set and R_f is an $n+1$ ary relation symbol for each n-ary function $f \in SigV$. Given an atom structure $\mathcal{F}$, the complex algebra $Cm\mathcal{F}$ is the algebra with universe $\wp(F)$, and for each n-ary $f \in SigV$ and $X_1, \ldots, X_n \in \wp(\mathcal{F})$,

$$f(X_1, \ldots, X_n) = \{a \in \mathcal{F} : \langle a_1, \ldots, a_n, b\rangle \in R_f \quad \text{for some } a_1 \in X_1, \ldots, a_n \in X_n\}.$$

$Str(V)$, sometimes denoted by $Kripke(V)$, is the class of atom structures of full complex algebras in V, that is, $\mathcal{F} \in Str(V)$ iff $Cm\mathcal{F}$ is in V. It is known that both $At(V)$ and $Str(V)$ are closed under ultraroots.

The examples of Hodkinson [59] and Andréka [4] (which are the algebras we used in Theorem 15 show that for $3 \leq n < \omega$ and $V \in \{RRA, RCA_n\}$, $At(V) \neq Str(V)$. While $At(V)$ is closed under ultraproducts as well—hence is an elementary class by a result of Venema—a recent result of Hirsch and Hodkinson [51] shows that $Str(RRA)$, the so-called *strongly representable* atom structures of relation algebras, is not closed under ultraproducts, hence is not elementary. In [51] it is proved that there exists a sequence of strongly representable atom structures whose ultraproduct is (only weakly representable but) not strongly representable. The analogous question for RCA_n with $4 \leq n < \omega$ makes sense, and to the best of our knowledge is still open.

Open Question 6 (Hirsch-Hodkinson-Sayed). Let $n \geq 4$. Is $Str(RCA_n)$ closed under ultraproducts, equivalently is it an elementary class?

An unpublished result of the author [127] shows that $Str(RCA_3)$ is not elementary. It can be shown that for $\omega > n > 3$, $Str(RCA_n)$ is not finitely axiomatizable in first order logic nor in $L^{\omega}_{\infty\omega}$. In [43] on complete representations, there is a set of first order axioms characterizing the elementary closure of the class of completely representable cylindric algebras. All quantifiers in these conditions are already relativized to atoms, so they can be easily re-written as conditions on cylindric algebra atom structures. Any cylindric atom structure satisfying these conditions is in $Str(RCA_n)$. The example of Hodkinson in [59] and for that matter $Tm(B_nR_n)$ constructed above, however show that the converse fails. For a variety of boolean algebras with operators, we generally have $StrV \subset AtV$. A natural generalization of the above open question is:

Open Problem 7 (Goldblatt-van Benthem). For which varieties of boolean algebras with operators V is $Str(V)$ an elementary class?

Partial results in this direction are known. One such result covers the case when V is *atom-canonical*. A variety is atom-canonical if for atomic $\mathcal{A} \in V$, the completion of $\mathcal{A}$, i.e., the algebra $Cm(AtV) \in V$. Another such result covers the case when V is Sahlqvist, i.e., V can be axiomatized by Sahlqvist equations. In these two cases we actually have $At(V) = Str(V)$. Since $At(V)$ is always elementary, a result of Venema [136], the desired result readily follows. A recent generalization due to Goldblatt [36] is that $Str(V)$ is still equal to $At(V)$, hence is elementary, when V is only *single-persistent* [36]. Single persistence and atom-canonicity coincide in the case of completely additive varieties of boolean algebras with operators, but are generally distinct. Yet another generalization is the long standing open problem in the literature of both modal logic and algebraic logic is the following:

Open Problem 8 (Goldblatt-van Benthem). Is every canonical variety V, i.e., a variety closed under perfect extensions, generated by the complex algebras of an elementary class of frames? That is, is it true that if V is canonical, then there exists an elementary class K of frames such that V is the variety generated by CmK?

Here CmK is the class of complex algebras of frames in K. This is an important question in modal logic. Equivalently it can be rephrased by asking whether any canonical modal logic (one validated by its canonical frames) is characterized by an elementary class of Kripke frames, cf. [35]. The converse—a classical result in the literature of modal logic—was proved by van Benthem and independently by Goldblatt [34]. In fact, Van Benthem proves that if K is closed under ultraproducts then the variety generated by CmK is canonical. We note that in a series of publications, Goldblatt gives a partial answer to this question. When $At(V) = Str(V)$, then the (canonical) variety in question is generated by the elementary class $Str(V)$, since it is always the case that $At(V)$ is elementary and $HSPCmStr(V) \subseteq V$, where H S and P are the operations of forming homomorphic images, subalgebras and products, respectively. $Str(V)$ is always a (natural) candidate providing the desired class of frames. But unfortunately this class is not always elementary as indicated above for RRA and RCA_3, [127]. However, for $V \in \{RRA, RCA_n\}$, the canonical varieties V *can be* generated by an elementary class of frames that lie between $At(V)$ and $Str(V)$. In fact, several such classes are exhibited for the class of representable cylindric and relation algebras in [50] and [126]. This means that the condition of single persistence, introduced by Goldblatt, is only sufficient, i.e., there are varieties of boolean algebras with operators that are not single-persistent but are generated by an elementary class of frames.

The following Theorem relating atom-canonicity to omitting types is proved in [116].

THEOREM 19. *Let V be a variety of boolean algebras with operators. If V is not atom-canonical, then the Omitting types Theorem fails for the corresponding modal logic.*

SKETCH OF PROOF. Let $L(V)$ be the modal logic corresponding to V. The models of $L(V)$ are the usual Kripke frames. Let $\mathcal{A} \in V$ be atomic such that its completion i.e., the algebra $Cm(AtA) \notin V$. Exists since V is not atom-canonical, hence is not closed under completions. Let $Y = \{-x : x \in AtV\}$. Then $\bigwedge X = 0$. Then X is a non-principal type that cannot be omitted in any model. ⊣

As indicated above this gives that the Omitting Types Theorem fails for L_n and its equality free reduct, because the corresponding variety is not atom-canonical as witnessed by the algebra $Tm(B_nR_n)$. Also in [126] it is proved that the Omitting Types Theorem fails for L_ω the structural extension of first order logic studied in [41]. This is so because RCA_ω is not atom canonical. A converse holds in more restrictive contexts. Indeed in [128] and [121] it is proved that several algebraizable extensions of first order logic without equality and of L_ω enjoy an Omitting Types Theorem. The algebraic counterparts of such logics are atom-canonical varieties whose similarity types are reducts of polyadic algebras.

§20. To sum up, cylindric algebras and far beyond. In the history of the development of algebraic logic, there is, we believe, three important turning points. One as mentioned before is in the nineteenth century when De Morgan set out to find the Laws governing binary relations launching thereby an investigation into algebras of relations. This approach was taken up by Peirce and Schröder and was put on a modern footing by Tarski. This brings us to the other turning point. This was in the forties when Tarski and his school in Berkeley laid down the axioms for relation algebras, and in the fifties when the modern approach to algebraization of first order logic via cylindric algebras was initiated. At the same time Halmos initiated work on polyadic algebras. By 1991 when Andréka, Monk, and Németi edited a volume on algebraic logic, it was thought that a lot of the central problems were fairly understood. Quoting Monk: "The relationships between relation algebras, cylindric algebras, and polyadic algebras are well understood. The representation theory has been carefully studied, and the relationships between various types of representable algebras such as cylindric set algebras, has been thoroughly investigated". Today it seems that we are on the verge of a *third* turning point. The subject is taking new perspectives that is very likely to revive it. Hirsch and Hodkinson introduced Robinson's finite forcing in the form of games in their recent book [50], showing the power and elegance of game-theoretic techniques in model theory. Indeed they illustrate in a profusion of ways, games lie very close to some of the most general and fundamental notions of model theory such as axiomatizability, step-by-step constructions and even satisfaction of formulas. Quoting Hodges [50, p. vii], "It often happens that an unexpected view point yield dividends, and today relation algebras take their proper place as an important tool of theoretical computer science among other things."

In 1999, Steven Givant [134] published "Tarski's System of Geometry" in the Bulletin of Symbolic Logic. [134] contains extended remarks about Tarski's system of foundations of Euclidean geometry. Following this line of research based on Tarski's attempts to axiomatize Euclidean and non-Euclidean geometry, Suppes [66] suggested that theories like the special theory of relativity whose mathematical model is Minkowski's geometry can be, and indeed, should be formalized in first order logic. This logistic viewpoint to the special theory of relativity, it seems, originates with Reichenbach [66]. Following Reichenbach, Tarski, and Suppes, Andréka, Madárasz, and Németi applied algebraic logic to axiomatizing Einstein's *general* theory of relativity in first order logic [66] and [10], an indeed fascinating application of algebraic logic to geometry and physics. Németi and Etassi reap the harvest in their recent paper [28]. As one byproduct (among many others) of this logistic approach, Németi and Etassi show that if one takes into account the laws of general relativity more precisely, the dilation of time in strong gravitational fields, then one can imagine "thought experiments" in which Church's thesis is no longer valid. In these thought experiments one can compute non-computable functions (in the old sense) and one can prove that ZFC is consistent! So was Hilbert right after all? Ironically, it was Kurt Gödel who started considering such thought experiments (that involve rotating black holes with strong gravitational fields) but he was obviously not concerned with violating his own incompleteness results. In any case, it therefore seems that the ideas of Boole have not borne their full fruit yet!

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