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PARTITION THEOREMS AND COMPUTABILITY THEORY

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§1. Introduction. The connections between mathematical logic and combinatorics have a rich history. This paper focuses on one aspect of this relationship: understanding the strength, measured using the tools of computability theory and reverse mathematics, of various partition theorems. To set the stage, recall two of the most fundamental combinatorial principles, König's Lemma and Ramsey's Theorem. We denote the set of natural numbers by ω and the set of finite sequences of natural numbers by $\omega^{<\omega}$. We also identify each $n \in \omega$ with its set of predecessors, so $n = \{0, 1, 2, \dots, n-1\}$.

DEFINITION 1.1.

1. A *tree* is a subset T of $\omega^{<\omega}$ such that for all $\sigma \in T$, if $\tau \in \omega^{<\omega}$ and $\tau \subseteq \sigma$, then $\tau \in T$.
2. If T is a tree and $S \subseteq T$ is also a tree, we say that S is a *subtree* of T.
3. A tree T is *bounded* if there exists $h\colon \omega \to \omega$ such that for all $\sigma \in T$ and $k \in \omega$ with $|\sigma| > k$, we have $\sigma(k) \leq h(k)$.
4. A *branch* of a tree T is a function $f : \omega \to \omega$ such that $f \upharpoonright n \in T$ for all $n \in \omega$.

THEOREM 1.2 (König's Lemma). *Every infinite bounded tree has a branch.*

DEFINITION 1.3.

1. Given a set $Z \subseteq \omega$ and $n \in \omega$, we let $[Z]^n = \{x \subseteq Z : |x| = n\}$.
2. Suppose that $n, p \geq 1$ and $f : [\omega]^n \to p$. Such an f is called a p-*coloring* of $[\omega]^n$ and n is called the *exponent*. We say that a set $H \subseteq \omega$ is *homogeneous* for f if H is infinite and $f(x) = f(y)$ for all $x, y \in [H]^n$.

THEOREM 1.4 (Ramsey's Theorem [20]). *Suppose that* $n, p \geq 1$ *and* $f : [\omega]^n \to p$. *There exists a set* H *homogeneous for* f.

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König's Lemma and Ramsey's Theorem are intimately related, as several proofs of partition theorems in set theory (such as Ramsey's Theorem) utilize paths through trees, and vice-versa. In the realm of large cardinals, those cardinals for which the appropriate analogue of Ramsey's Theorem holds are exactly those for which the appropriate analogue of König's Lemma holds (see [13, Theorem 7.8]).

Another interesting thread in the investigation of the logical strength of partition theorems is the use of infinitary methods to prove finite combinatorial results. One can use Ramsey's Theorem together with König's Lemma to derive the following finite version of Ramsey's Theorem.

THEOREM 1.5 (Finite Version of Ramsey's Theorem [20]). *Suppose that* n, $p, k \geq 1$. *There exists* $\ell \in \omega$ *such that for every* $f : [\ell]^n \to p$, *there exists a set* $Z \subseteq \ell$ *such that* $|Z| \geq k$ *and* $f(x) = f(y)$ *for all* $x, y \in [Z]^n$.

These infinitary methods give no indication of how large ℓ must be as a function of n, p, and k. However, the finite version of Ramsey's Theorem can also be proved using nothing but basic finite combinatorics, and reasonable bounds can be read off from such a proof (although improving these bounds is a major problem in combinatorics). On the other hand, Paris and Harrington [18] used these infinitary methods to derive the following natural finite partition theorem stronger than the finite version of Ramsey's Theorem.

THEOREM 1.6 (Paris-Harrington Principle [18]). *Suppose that* $n, p, k \geq 1$. *There exists* $\ell \in \omega$ *such that for every* $f : [\ell]^n \to p$, *there exists a set* $Z \subseteq \ell$ *such that* $|Z| \geq k$, $|Z| \geq \min(Z)$, *and* $f(x) = f(y)$ *for all* $x, y \in [Z]^n$.

The interest of this innocuous looking strengthening of the finite version of Ramsey's Theorem lies in the fact that Paris and Harrington showed that it is not provable in Peano Arithmetic. Hence, any proof of this finite combinatorial result needs, in a precise sense, to use infinite sets. A few years later, Kanamori and McAloon [14] found another, perhaps more natural, finite partition theorem about regressive functions which is true but not provable in Peano Arithmetic. We will discuss their partition theorem (both its finite and infinite versions) below.

Our interest is in the effective content of partition theorems such as Ramsey's Theorem. For example, we may ask whether every *computable* $f : [\omega]^2 \to 2$ must have a *computable* homogeneous set. If the answer is negative, we may wonder about the complexity of homogeneous sets for computable $f : [\omega]^2 \to 2$ as measured using the tools of computability theory, such as the Turing degrees and the arithmetic hierarchy. From a related perspective, we may seek to understand the strength of the set existence axioms inherent in Ramsey's Theorem, as measured by tools of reverse mathematics. We might expect that the above mentioned relationship between partition theorems and König's Lemma manifests itself in their corresponding computability-theoretic or reverse mathematical strengths.

Aside from Ramsey's Theorem, our focus in this discussion is two partition theorems which allow infinitely many colors: the Canonical Ramsey Theorem of Erdös and Rado and the Regressive Function Theorem of Kanamori and McAloon. We first set up some notation that will be useful when discussing partition theorems.

DEFINITION 1.7.

1. If $x \subseteq \omega$ is finite and $a \in \omega$, we write $x < a$ if a is greater than every element of x.
2. Suppose that $n \geq 1$, $f : [\omega]^{n+1} \to \omega$, $x \in [\omega]^n$, and $a \in \omega$. When we write $f(x, a)$, we implicitly assume that $x < a$, and we let $f(x, a) = f(x \cup \{a\})$. Also, if $n = 1$ and $a, b \in \omega$, when we write $f(a, b)$, we implicitly assume that $a < b$, and we let $f(a, b) = f(\{a, b\})$.

The first partition theorem is the Canonical Ramsey Ramsey Theorem due to Erdös and Rado which considers arbitrary functions $f : [\omega]^n \to \omega$. Of course, we can not expect to always have homogeneous sets, as witnessed by the following simple functions $f : [\omega]^2 \to \omega$ (where $\langle\cdot\rangle$ is a fixed effective bijection from ω^2 to ω):

1. $f(a, b) = a$,
2. $f(a, b) = b$,
3. $f(a, b) = \langle a, b\rangle$.

However, the Canonical Ramsey Theorem for exponent 2 says that given any $f : [\omega]^2 \to \omega$, there exists an infinite set $C \subseteq \omega$ which either is homogeneous, or on which f behaves like one of the above functions. Precisely, given any $f : [\omega]^2 \to \omega$, there exists an infinite C such that either

1. For all $a_1, b_1, a_2, b_2 \in C$, we have $f(a_1, b_1) = f(a_2, b_2)$.
2. For all $a_1, b_1, a_2, b_2 \in C$, we have $f(a_1, b_1) = f(a_2, b_2) \leftrightarrow a_1 = a_2$.
3. For all $a_1, b_1, a_2, b_2 \in C$, we have $f(a_1, b_1) = f(a_2, b_2) \leftrightarrow b_1 = b_2$.
4. For all $a_1, b_1, a_2, b_2 \in C$, we have $f(a_1, b_1) = f(a_2, b_2) \leftrightarrow (a_1 = a_2$ and $b_1 = b_2)$.

In the general case of an $f : [\omega]^n \to \omega$, we get 2^n different possibilities. In what follows, we identify each $x \in [\omega]^n$ with the n-tuple listing x in increasing order which we think of as function from n to ω.

DEFINITION 1.8. Suppose that $n \geq 1$ and $f : [\omega]^n \to \omega$. Given $u \subseteq n$, we say that a set $C \subseteq \omega$ is *u-canonical* for f if C is infinite and for all $x_1, x_2 \in [C]^n$, we have $f(x_1) = f(x_2) \leftrightarrow x_1 \restriction u = x_2 \restriction u$. We say that a set C is *canonical* for f if there exists $u \subseteq n$ such that C is u-canonical for f.

THEOREM 1.9 (Canonical Ramsey Theorem [5]). *Suppose that $n \geq 1$ and $f : [\omega]^n \to \omega$. There exists a set C canonical for f.*

Our other main interest is the Regressive Function Theorem of Kanamori and McAloon. As discussed above, this result is the infinite version of the finite combinatorial statement that Kanamori and McAloon showed was true but not provable in Peano Arithmetic. The Regressive Function Theorem (for exponent n) is a straightforward consequence of the Canonical Ramsey Theorem (for exponent n), but it has its own intrinsic interest. Like the Canonical Ramsey Theorem, it deals with colorings which allow infinitely many colors, but it places a restriction on which such colorings it considers.

DEFINITION 1.10.

1. Suppose that $n \geq 1$ and $f : [\omega]^n \to \omega$. We say that f is *regressive* if for all $x \in [\omega]^n$, we have $f(x) < \min(x)$ whenever $\min(x) > 0$, and $f(x) = 0$ whenever $\min(x) = 0$.
2. Suppose that $n \geq 1$ and $f : [\omega]^n \to \omega$ is regressive. We say that a set $M \subseteq \omega$ is *minhomogeneous* for f if M is infinite and $f(x) = f(y)$ whenever $x, y \in [M]^n$ satisfy $\min(x) = \min(y)$.

THEOREM 1.11 (Regressive Function Theorem [14]). *Suppose that $n \geq 1$ and $f : [\omega]^n \to \omega$ is regressive. There exists a set M minhomogeneous for f.*

It is a simple exercise to derive Ramsey's Theorem (for exponent n) from the Regressive Function Theorem (for exponent n), so we now have three interesting partition theorems which form a natural hierarchy in terms of their strength. Also, as mentioned above, Kanamori and McAloon showed that the following finite version of their theorem is not provable in Peano Arithmetic.

THEOREM 1.12 (Finite Version of the Regressive Function Theorem [14]). *Suppose that $n, k \geq 1$. There exists $\ell \in \omega$ such that for every $f : [\ell]^n \to \omega$ which satisfies $f(x) < \min(x)$ for all $x \in [\ell]^n$ with $\min(x) > 0$, and $f(x) = 0$ for all $x \in [\ell]^n$ with $\min(x) = 0$, there exists a set $Z \subseteq \ell$ such that $|Z| \geq k$ and $f(x) = f(y)$ whenever $x, y \in [Z]^n$ satisfy $\min(x) = \min(y)$.*

In what follows, we will concentrate on the complexity of solutions of computable colorings in each of these partition theorems. These results are often closely related to results in reverse mathematics.

DEFINITION 1.13. The following definitions are made in second-order arithmetic.

1. RT^n_p is the statement that every $f : [\omega]^n \to p$ has a homogeneous set.
2. CRT^n is the statement that every $f : [\omega]^n \to \omega$ has a canonical set.
3. REG^n is the statement that every regressive $f : [\omega]^n \to \omega$ has a minhomogeneous set.

Our goal is to see how the relationship between these partition theorems and König's Lemma are reflected in the effective analysis.

§2. Effective analysis of König's Lemma and Ramsey's Theorem. There has been an extensive study of the strength of König's Lemma and Ramsey's Theorem using the tools of computability theory and reverse mathematics. From the viewpoint of computability theory (see [25] for the necessary background information about computability theory), one may ask where solutions to computable instances of these problems lie either in the Turing degrees or the arithmetical hierarchy. Also, one may seek to classify the strength of these statements with respect to the reverse mathematics hierarchy (see [24] for the necessary background information about reverse mathematics). Before embarking on our analysis of the above partition theorems, we will discuss some of the known results for König's Lemma and Ramsey's Theorem.

An effective analysis of König's Lemma depends on both the complexity of f and the complexity of the bound. We will mostly be concerned with subtrees of $2^{<\omega}$ (that is, trees which are bounded by $h(k) = 1$). It is straightforward to effectively code computable trees bounded by a computable function using computable subtrees of $2^{<\omega}$, so for our purposes there is no loss in restricting attention to the following case.

Corollary 2.1 (Weak König's Lemma). *Every infinite subtree of $2^{<\omega}$ has a branch.*

Definition 2.2. Let $\mathbf{a}$ and $\mathbf{b}$ be Turing degrees. We write $\mathbf{a} \gg \mathbf{b}$ to mean that every infinite $\mathbf{b}$-computable subtree of $2^{<\omega}$ has an $\mathbf{a}$-computable branch.

The notation $\mathbf{a} \gg \mathbf{b}$ was introduced in Simpson [23], and many of the basic properties of this ordering can be found there. It is well known that $\mathbf{a} \geq \mathbf{b}' \rightarrow \mathbf{a} \gg \mathbf{b} \rightarrow \mathbf{a} > \mathbf{b}$. The following proposition gives some equivalent characterizations of this ordering.

Proposition 2.3 (Scott [21], Solovay). *Let $\mathbf{a}$ and $\mathbf{b}$ be Turing degrees. The following are equivalent*:

1. $\mathbf{a} \gg \mathbf{b}$.
2. *Every partial $\{0, 1\}$-valued $\mathbf{b}$-computable function has a total $\mathbf{a}$-computable extension.*
3. $\mathbf{a}$ *is the degree of a complete extension of the theory of Peano Arithmetic with an additional unary predicate symbol P, axioms $P(n)$ for all $n \in B$ and $\neg P(n)$ for all $n \notin B$ (where B is a fixed set in $\mathbf{b}$), and induction axioms for formulas involving P.*

Using the existence of a computable tree in which the branches code complete extensions of Peano Arithmetic, it follows that there is a "universal" computable subtree of $2^{<\omega}$.

Corollary 2.4. *There exists an infinite computable subtree T of $2^{<\omega}$ such that given any branch B_T of T, and any infinite computable subtree S of $2^{<\omega}$, there exists a branch B_S of S such that $B_S \leq_T B_T$.*

In [12], Jockusch and Soare established the following fundamental result.

THEOREM 2.5 (Low Basis Theorem [12, Theorem 2.1]). *There exists* $\mathbf{a} \gg \mathbf{0}$ *with* $\mathbf{a}' = \mathbf{0}'$.

Adding the formal statement of Weak König's Lemma in second order arithmetic to the base axiom system RCA_0 of reverse mathematics, we get the important system WKL_0.

We turn now to Ramsey's Theorem. Specker [26] was the first to analyze the effective content of Ramsey's Theorem, and he showed that there exists a computable $f : [\omega]^2 \to 2$ with no computable homogeneous set. It follows that RT^2_2 is not provable in RCA_0. Before discussing further bounds on the complexity of homogeneous sets, we first discuss a few proofs of Ramsey's Theorem.

DEFINITION 2.6. Suppose that $n, p \geq 1$ and $f : [\omega]^{n+1} \to p$. We say that a pair (A, g), where $A \subseteq \omega$ is infinite and $g : [A]^n \to p$, is a *prehomogeneous pair* for f if $f(x, a) = g(x)$ for all $x \in [A]^n$ and all $a \in A$.

Most proofs of Ramsey's Theorem break down into the following three steps, and differ only in their treatment of (1):

1. Given $f : [\omega]^{n+1} \to p$, construct a prehomogeneous pair (A, g) for f.
2. Apply induction to $g : [A]^n \to p$.
3. Show that any set homogeneous for g is homogeneous for f.

The standard way to construct a prehomogeneous pair proceeds by repeatedly thinning down a set of candidates to add to the prehomogeneous pair, while ensuring that this set of candidates is always infinite. For simplicity, consider a function $f : [\omega]^2 \to 2$. We will enumerate A in increasing order as $a_0, a_1, \dots$. We begin by letting $a_0 = 0$. If there are infinitely many $b \in \omega$ with $f(a_0, b) = 0$, then we can define $g(a_0) = 0$ and restrict attention to the set $I_0 = \{b \in \omega : f(a_0, b) = 0\}$. Otherwise, there are infinitely many $b \in \omega$ with $f(a_0, b) = 1$, so we can define $g(a_0) = 1$ and restrict attention to the set $I_0 = \{b \in \omega : f(a_0, b) = 1\}$. We then let $a_1 = \min I_0$ and continue in this fashion. If we succeed infinitely many times in this manner with color 0, then the corresponding elements form a homogeneous set colored 0, while if we succeed with infinitely many times with color 1, then the corresponding elements form a homogeneous set colored 1. Notice that this decision (infinitely many colored 0 or infinitely many colored 1) amounts to finding a homogeneous set for $g : [A]^1 \to p$.

This general idea can be extended to higher exponents n and to all $p \geq 1$. Suppose that $f : [\omega]^n \to p$ is computable. A simple analysis of this proof shows that there exists a prehomogeneous pair (A, g) for f with $\deg(A \oplus g) \leq \mathbf{0}''$ because the questions that need to be answered are whether or not certain effectively given sets are infinite. Following this outline, one arrives at the following result.

THEOREM 2.7. *Suppose that $n, p \geq 2$ and $f : [\omega]^n \to p$ is computable. There exists a set H homogeneous for f such that $deg(H) \leq \mathbf{0}^{(2n-2)}$.*

Another approach is to build a prehomogeneous pair by coding such pairs into the branches of a $\mathbf{0}'$-computable subtree of $2^{<\omega}$. Using an argument along these lines it follows that for every $\mathbf{a} \gg \mathbf{0}'$, there exists a prehomogeneous pair (A, g) for f with $\deg(A \oplus g) \leq \mathbf{a}$. Iterating this and making use of the Low Basis Theorem, we conclude the following.

THEOREM 2.8 (Jockusch [11, Theorem 5.6]). *Suppose that n, $p \geq 2$, $f : [\omega]^n \to p$ is computable, and $\mathbf{a} \gg \mathbf{0}^{(n-1)}$. There exists a set H homogeneous for f such that $deg(H) \leq \mathbf{a}$.*

Jockusch also characterized the location of homogeneous sets for computable colorings in the arithmetical hierarchy for all exponents.

THEOREM 2.9 (Jockusch [11, Theorem 5.1, Theorem 5.5]). *Suppose that n, $p \geq 2$ and $f : [\omega]^n \to p$ is computable. There exists a Π^0_n set homogeneous for f. Furthermore, for each $n \geq 2$, there exists a computable $f : [\omega]^n \to 2$ with no Σ^0_n set homogeneous for f.*

Moreover, Jockusch showed how to code nontrivial information into the homogeneous sets of computable colorings for exponent $n \geq 3$.

THEOREM 2.10 (Jockusch [11, Lemma 5.9]). *Let $n \geq 3$. There exists a computable $f : [\omega]^n \to 2$ such that $deg(H) \geq \mathbf{0}^{(n-2)}$ for every set H homogeneous for f.*

Formalizing the proofs of Ramsey's Theorem and the previous theorem in second-order arithmetic, we arrive at the following result in reverse mathematics.

COROLLARY 2.11. *Let $n \geq 3$ and $p \geq 2$. Over RCA_0, RT^n_p is equivalent to* ACA_0.

At this point, we are still left with many questions about the degrees of homogeneous sets for computable colorings of exponent 2. For a computable $f : [\omega]^2 \to 2$, we know that we can find homogeneous sets below any $\mathbf{a} \gg \mathbf{0}'$, but we don't know if we code anything nontrivial. Furthermore, this gap for exponent 2 propagates up to higher exponents. A major step toward resolving this question was taken by Seetapun, who showed that it was not possible to code nontrivial information into the homogeneous sets of a computable coloring of exponent 2.

THEOREM 2.12 (D. Seetapun [22]). *Suppose that $p \geq 2$, $f : [\omega]^2 \to p$ is computable and $\{\mathbf{b}_k\}_{k \in \omega}$ is a family of nonzero degrees. There exists a set H homogeneous for f such that $\mathbf{b}_k \not\leq deg(H)$ for all $k \in \omega$.*

Seetapun iterated his result to arrive at the following important reverse mathematical fact.

COROLLARY 2.13. *For each $p \geq 2$, $\mathsf{RCA}_0 + \mathsf{RT}^2_p$ does not imply* ACA_0.

Using the Low Basis Theorem and Theorem 2.9, one can show that for any $p \geq 2$, RT^2_p is not provable from WKL_0, hence RT^2_p is not equivalent to any of the standard systems of reverse mathematics over RCA_0. To get more information about the complexity of Ramsey's Theorem for exponent 2, we look for guidance from yet another proof of Ramsey's Theorem.

There is another proof of Ramsey's Theorem which, although quite similar to the outline above, uses a nonprincipal ultrafilter on ω to guide the inductive construction. This changes the argument in following fundamental manner. In the above outline, the key question is how to define $g(a_n)$ so that the corresponding thinned out set remains infinite. We know that some choice will succeed, but there may be many possible choices which work. In contrast, the ultrafilter guides us because exactly one of the corresponding sets will remain in the ultrafilter. In our context of effectively analyzing these proofs, the nonprincipal ultrafilter can be replaced by a more basic object.

Definition 2.14. A set $V \subseteq \omega$ is *r-cohesive* if V is infinite and for every computable set Z, either $V \cap Z$ is finite or $V \cap \overline{Z}$ is finite.

Notice that if V is an r-cohesive set, then $\{Z \subseteq \omega : Z \text{ is computable and } V \subseteq^* Z\}$ is a nonprincipal ultrafilter in the boolean algebra of computable sets. Hence, if $f : [\omega]^n \to p$ is computable, we can use an r-cohesive set in place of a nonprincipal ultrafilter on ω in the above construction. Jockusch and Stephan [9] (see also [10] for a correction) characterized the Turing degrees of jumps of r-cohesive sets.

Theorem 2.15 (Jockusch and Stephan [9, Theorem 2.2(ii)]). *Suppose that* $\mathbf{a} \gg \mathbf{0}'$. *There exists an r-cohesive set V such that* $deg(V)' \leq \mathbf{a}$. *Furthermore, every r-cohesive set V satisfies* $deg(V)' \gg \mathbf{0}'$.

Using this result along with a suitable r-cohesive set, we arrive at another proof of the fact that for every computable $f : [\omega]^n \to p$ and every $\mathbf{a} \gg \mathbf{0}'$, there exists a prehomogeneous pair (A, g) for f with $\deg(A \oplus g) \leq \mathbf{a}$. Hence, we get another proof of Theorem 2.8. However, using this approach for exponent 2, Cholak, Jockusch, and Slaman showed that it is also possible to force the jump of a homogeneous set in the construction.

Theorem 2.16 (Cholak, Jockusch, Slaman [2, essentially Lemma 4.6]). *Suppose* $p \geq 2$, $f : [\omega]^2 \to p$ *is computable, and* $\mathbf{a} \gg \mathbf{0}'$. *There exists a set H homogeneous for f such that* $deg(H)' \leq \mathbf{a}$.

Furthermore, they showed that this characterization is sharp in the following sense.

Theorem 2.17 (Cholak, Jockusch, Slaman [2, Theorem 12.5]). *There exists a computable* $f : [\omega]^2 \to 2$ *such that* $deg(H)' \gg \mathbf{0}'$ *for all sets H homogeneous for f.*

Therefore, as remarked on pp. 50-51 of [2], we get a corollary about Ramsey's Theorem for exponent 2 similar to Corollary 2.4 about König's Lemma with "jump universal" in place of "universal".

COROLLARY 2.18. *There exists a computable* $f: [\omega]^2 \to 2$ *such that that given any set* H_f *homogeneous for* f, *and any computable* $g: [\omega]^2 \to 2$, *there exists a set* H_g *homogeneous for* g *with* $H_g' \leq_T H_f'$.

With the base case of exponent 2 settled, we can handle higher exponents. As the exponent increases, the bounds that we obtain in the Turing degrees increases by one jump each time.

THEOREM 2.19. [15] *Suppose that* $n, p \geq 2$, $f: [\omega]^n \to p$ *is computable, and* $\mathbf{a} \gg \mathbf{0}^{(n-1)}$. *There exists a set* H *homogeneous for* f *such that* $deg(H)' \leq \mathbf{a}$. *Furthermore, for every* $n \geq 2$, *there exists a computable* $f: [\omega]^n \to 3$ *such that for all sets* H *homogeneous for* f, *we have* $deg(H) \geq \mathbf{0}^{(n-2)}$ *and* $deg(H)' \gg \mathbf{0}^{(n-1)}$.

The following question of whether we can replace the 3-coloring from the previous proposition by a 2-coloring is open.

QUESTION 2.20. *Let* $n \geq 3$. *Does there exist a computable* $f: [\omega]^n \to 2$ *such that for all sets* H *homogeneous for* f, *we have* $deg(H) \geq \mathbf{0}^{(n-2)}$ *and* $deg(H)' \gg \mathbf{0}^{(n-1)}$?

§3. The Canonical Ramsey Theorem and computability theory. One important lesson to glean from Section 2 is that we can often improve an effective analysis of a theorem by examining a genuinely different proof of the result. In the original inductive proof of the Canonical Ramsey Theorem (see [5]), in order to prove the result for exponent $n \geq 2$, Erdös and Rado used Ramsey's Theorem for exponent $2n$ together with the Canonical Ramsey Theorem for exponent $n-1$. Using Theorem 2.9, an effective analysis of their proof gives the result that every computable $f: [\omega]^2 \to \omega$ has a Π^0_4 canonical set. However, as n increases, the use of induction causes the arithmetical bounds to grow on the order of n^2. Rado [19] discovered a noninductive proof of the Canonical Ramsey Theorem which still used Ramsey's Theorem for exponent $2n$ to prove the result for exponent n. An effective analysis of his proof shows that given $n \geq 2$ and a computable $f: [\omega]^n \to \omega$, there exists a Δ^0_{2n+1} canonical set for f.

We give a new proof of the Canonical Ramsey Theorem which is inductive and similar in broad outline to the proofs of Ramsey's Theorem sketched above. The basic question is how to define a "precanonical pair" (A, g) so that we can carry out the same outline to prove the Canonical Ramsey Theorem. Consider a function $f: [\omega]^2 \to \omega$. We will enumerate A in increasing order as $a_0, a_1, \ldots$. We begin by letting $a_0 = 0$. If there exists $c \in \omega$ such that there are infinitely many $b \in \omega$ with $f(a_0, b) = c$, then we can define $g(a_0) = c$, restrict attention to the set $I_0 = \{b \in \omega : f(a_0, b) = c\}$, and after letting $a_1 = \min I_0$, continue in this fashion. In this case, we've made progress toward achieving a u-canonical set with $1 \notin u$, because if we fix a_0 and vary $b \in I_0$, we do not change the value of f. If we succeed infinitely many times in this manner with a fixed c, then the corresponding elements form a $\emptyset$-canonical set, while if we succeed with infinitely many different c in this manner, then the corresponding elements form a $\{0\}$-canonical set. Notice that this decision (one fixed c versus infinitely many distinct c) amounts to finding a canonical set for exponent 1 for g restricted to the set of successes.

The problem arises when for each $c \in \omega$, there are only finitely many $b \in \omega$ with $f(a_0, b) = c$. Now we must seek to make progress toward achieving a u-canonical set with $1 \in u$. We therefore let $I_0 = \{b \in \omega : f(a_0, b) \neq f(a_0, b') \text{ for all } b' < b\}$, so that if we fix a_0 and vary $b \in I_0$, we always change the value of f. We now want to let $g(a_0)$ be some new, infinitary color d distinct from each $c \in \omega$. Suppose that we then set $a_1 = \min I_0$, and again are faced with the situation that for each $c \in \omega$, there are only finitely many $b \in I_0$ with $f(a_1, b) = c$. We first want to thin out I_0 to an infinite set I_0' so that $f(a_i, b_0) = f(a_j, b_1) \rightarrow b_0 = b_1$ whenever $0 \leq i, j \leq 1$ and $b_0, b_1 \in I_0'$ (which is possible by the assumption about a_0, a_1). This allows both a_0 and a_1 to be in the same u-canonical set with $1 \in u$. Next, we need to assign an appropriate infinitary color to $g(a_1)$ so that a canonical set for g will be a u-canonical set for f. Thus, if the set $\{b \in I_0' : f(a_0, b) = f(a_1, b)\}$ is infinite, we let $g(a_1) = g(a_0)$ and we let I_1 be this set. Otherwise we will set $g(a_1)$ to a new infinitary color and let $I_1 = \{b \in I_0' : f(a_0, b) \neq f(a_1, b)\}$. If we succeed infinitely many times in this manner with a fixed infinitary color d, then the corresponding elements form a $\{1\}$-canonical set, while if we succeed with infinitely many different d in this manner, then the corresponding elements form a $\{0, 1\}$-canonical set. Notice again that this decision (one fixed d versus infinitely many distinct d) amounts to finding a canonical set for exponent 1 for g restricted to those elements assigned infinitary colors.

For higher exponents, we can still make use of the above outline. In general, given $f : [\omega]^{n+1} \rightarrow \omega$, we can pursue the above strategy to get an infinite set A and a function $g : [A]^n \rightarrow \omega \times 2$, where we interpret each $(c, 0) \in \omega \times 2$ as a finitary color and each $(d, 1) \in \omega \times 2$ as an infinitary color. This motivates the following definition.

Definition 3.1. Suppose that $n \geq 1$, and $f : [\omega]^{n+1} \rightarrow \omega$. We say that a pair (A, g) where $A \subseteq \omega$ is infinite and $g : [A]^n \rightarrow \omega \times 2$ is a *precanonical pair* for f if:

1. For all $x \in [A]^n$ with $g(x) = (c, 0)$, we have $f(x, a) = c$ for all $a \in A$.
2. For all $x_1, x_2 \in [A]^n$ with $g(x_1) = (d_1, 1)$ and $g(x_2) = (d_2, 1)$, and all $a_1, a_2 \in A$,
   (a) If $a_1 \neq a_2$, then $f(x_1, a_1) \neq f(x_2, a_2)$
   (b) If $a_1 = a_2$, then $f(x_1, a_1) = f(x_2, a_2) \leftrightarrow d_1 = d_2$.

The above outline can be used to give a proof that for every $n \geq 2$ and every $f\colon [\omega]^n \to \omega$, there exists a precanonical pair (A, g) for f. Now, before we can apply induction, it is important to thin out our set A to a set D so that either g maps all elements of $[D]^n$ to finitary colors, or g maps all elements of $[D]^n$ to infinitary colors. Of course, we can do this with a simple application of Ramsey's Theorem for exponent n. Although this strategy will succeed in proving the Canonical Ramsey Theorem, the use of Ramsey's Theorem is costly to an effective analysis. It is therefore necessary to pursue a slightly different approach which will roll this use of Ramsey's Theorem into the induction. This can be done by extending the notions of canonical sets and precanonical pairs to functions $f\colon [\omega]^n \to \omega \times p$ for $p \in \omega$, but the details do not concern us here. For details and proofs of the results in this and the next section, see [15] and [16].

A naive effective analysis of the above approach allows one to conclude that for every computable $f\colon [\omega]^n \to \omega$, there is a precanonical pair (A, g) for f such that $\deg(A \oplus g) \leq \mathbf{0}'''$. However, by making use of an r-cohesive set as in the proof of Ramsey's Theorem above, we get the following.

LEMMA 3.2. *Suppose that $n \geq 2$ and $f\colon [\omega]^n \to \omega$ is computable. There exists a precanonical pair (A, g) for f such that $\deg(A \oplus g) \leq \mathbf{0}''$.*

As in the case of Ramsey's Theorem, we can do better in the special case when the exponent is 2.

THEOREM 3.3. *Suppose that $f\colon [\omega]^2 \to \omega$ is computable, and $\mathbf{a} \gg \mathbf{0}'$. There exists a set C canonical for f such that $\deg(C) \leq \mathbf{a}$.*

Using this as a starting point and making use of an extended version of Lemma 3.2, we get the following theorem.

THEOREM 3.4. *Suppose that $n \geq 2$, $f\colon [\omega]^n \to \omega$ is computable, and $\mathbf{a} \gg \mathbf{0}^{(2n-3)}$. There exists a set C canonical for f such that $\deg(C) \leq \mathbf{a}$.*

In terms of the arithmetical hierarchy, Theorem 3.3 immediately implies that every computable $f\colon [\omega]^2 \to \omega$ has a Δ^0_3 set C canonical for f. However, by incorporating the basic ideas used by Jockusch to show Theorem 2.9 for exponent 2 into our effective analysis, and using a fairly delicate argument, we can show the following.

THEOREM 3.5. *Suppose that $f\colon [\omega]^2 \to \omega$ is computable. There exists a Π^0_2 set C canonical for f.*

Again, by applying an extended version of Lemma 3.2 inductively for higher exponents, we get the following result.

THEOREM 3.6. *Suppose that $n \geq 2$ and $f\colon [\omega]^n \to \omega$ is computable. There exists a Π^0_{2n-2} set C canonical for f.*

With the above notion of precanonical pair, we can also show the following, implying that these bounds are the optimal ones which can be mined from an effective analysis of the above proof.

THEOREM 3.7. *There exists a computable $f\colon [\omega]^3 \to \omega$ such that deg$(A) \geq \mathbf{0}''$ whenever (A, g) is a precanonical pair for f.*

In the next section, we give an effective analysis of the Regressive Function Theorem, which will also provide lower bounds.

§4. The Regressive Function Theorem and computability theory. The following proposition is straightforward, and shows how the Regressive Function Theorem is a special case of the Canonical Ramsey Theorem.

PROPOSITION 4.1 (Kanamori and McAloon [14]). *Suppose that $n \geq 1$ and $f\colon [\omega]^n \to \omega$ is regressive. If C is canonical for f, then C is minhomogeneous for f.*

Hence, we may use each of the upper bound results in the Turing degrees and the arithmetical hierarchy proved in the previous section to get upper bounds for solutions to computable instances of the Regressive Function Theorem. However, it is possible to give a simpler inductive proof of the Regressive Function Theorem using the notion of a "preminhomogeneous pair" and following the standard outline. Carrying out this approach, we get the same bounds in the Turing degrees and arithmetical hierarchy for exponent 2, but now the upper bounds on the location of minhomogeneous sets for computable colorings increases by one jump each time the exponent increases by 1, as in Ramsey's Theorem.

THEOREM 4.2. *Suppose that $n \geq 2$, $f\colon [\omega]^n \to \omega$ is computable and regressive, and $\mathbf{a} \gg \mathbf{0}^{(n-1)}$. There exists a set M minhomogeneous for f such that deg$(M) \leq \mathbf{a}$.*

THEOREM 4.3. *Suppose that $n \geq 2$ and $f\colon [\omega]^n \to \omega$ is computable and regressive. There exists a Π^0_n set M minhomogeneous for f.*

However, in the case of the Regressive Function Theorem, we can do enough coding to prove that these bounds are sharp. This result builds on a theorem of Hirst (see [6, Theorem 6.14]) which shows that it is possible to code $0'$ into the minhomogeneous sets of a computable h-regressive $f\colon [\omega]^2 \to \omega$ (where f is h-regressive if for all $x \in [\omega]^n$, we have $f(x) < h(\min(x))$ whenever $h(\min(x)) > 0$, and $f(x) = 0$ whenever $h(\min(x)) = 0$).

THEOREM 4.4. *Let $n \geq 2$. There exists a computable regressive $f\colon [\omega]^n \to \omega$ such that deg$(M) \gg \mathbf{0}^{(n-1)}$ for every set M which is minhomogeneous for f.*

As a corollary, we get that the arithmetical bounds are sharp. This result can also be obtained from Theorem 2.9.

COROLLARY 4.5. *Let $n \geq 2$. There exists a computable regressive $f\colon [\omega]^n \to \omega$ such that no Σ^0_n set is minhomogeneous for f.*

The coding techniques above completely resolve the question of the reverse mathematical strengths of the Canonical Ramsey Theorem and the

Regressive Function Theorem for each exponent $n \geq 2$. At the end of [14], Kanamori and McAloon state that the implication $\mathsf{REG}^2 \to \mathsf{ACA}_0$ over RCA_0 is originally due to Clote.

COROLLARY 4.6. *Let $n \geq 2$.*

1. *Over* RCA_0, REG^n *is equivalent to* ACA_0.
2. *Over* RCA_0, CRT^n *is equivalent to* ACA_0.

Using Proposition 4.1, it follows that the exponent 2 bounds in the Turing degrees and the arithmetical hierarchy given above for the Canonical Ramsey Theorem are sharp as well. In light of Theorem 3.7, I conjecture that the bounds for exponent $n \geq 3$ are also sharp.

CONJECTURE 4.7. *For every $n \geq 3$, there exists a computable $f : [\omega]^n \to \omega$ such that $deg(C) \gg \mathbf{0}^{(2n-3)}$ for every set C canonical for f.*

CONJECTURE 4.8. *For every $n \geq 3$, there exists a computable $f : [\omega]^n \to \omega$ with no Σ^0_{2n-2} canonical set.*

Finally, notice that the above results give the existence of "universal" computable colorings for the Regressive Function Theorem for each exponent, and for the Canonical Ramsey Theorem for exponent 2.

COROLLARY 4.9. *Let $n \geq 2$. There exists a computable regressive $f : [\omega]^n \to \omega$ such that given any set M_f minhomogeneous for f, and any computable regressive $g : [\omega]^n \to \omega$, there exists a set M_g minhomogeneous for g such that $M_g \leq_T M_f$.*

COROLLARY 4.10. *There exists a computable $f : [\omega]^2 \to \omega$ such that given any set C_f canonical for f, and any computable $g : [\omega]^2 \to \omega$, there exists a set C_g canonical for g such that $C_g \leq_T C_f$.*

§5. Ramsey degrees.

In Section 2, we saw that by shifting to an analysis of the jumps of the degrees of homogeneous sets for computable $f : [\omega]^2 \to 2$, we were able to get tight bounds in the Turing degrees. We now seek to understand the Turing degrees which are powerful enough to find homogeneous sets for every computable $f : [\omega]^2 \to 2$. Proofs of the results in this section can be found in [15] and [17].

DEFINITION 5.1. A degree $\mathbf{a}$ is *Ramsey* if every computable $f : [\omega]^2 \to 2$ has an $\mathbf{a}$-computable homogeneous set.

By Theorem 2.8, every degree $\mathbf{a}$ which satisfies $\mathbf{a} \gg \mathbf{0}'$ is Ramsey, but we know of no other natural classes of examples. One indirect way to build Ramsey degrees is to iteratively apply Theorem 2.12 to avoid cones for each computable $f : [\omega]^2 \to 2$, and get an upper bound for the resulting sequence of degrees. Hummel and Jockusch (see [7, Theorem 3.17]) use this technique to build a Ramsey degree which avoids one cone, but it's not hard to extend their result.

PROPOSITION 5.2. *Suppose that* $\{\mathbf{b}_k\}_{k\in\omega}$ *is a family of nonzero degrees. There exists a Ramsey degree* $\mathbf{a}$ *such that* $\mathbf{b}_k \not\leq \mathbf{a}$ *for all* $k \in \omega$.

Another indirect way to build Ramsey degrees is to iteratively apply Theorem 2.16 in conjunction with the Low Basis Theorem to find low$_2$ homogeneous sets (that is, homogeneous sets H which satisfy $H'' \leq_T 0''$) for each computable $f : [\omega]^2 \to 2$, and get an upper bound for the resulting sequence of degrees. Following this procedure, we get the following.

THEOREM 5.3. *There is a nonhigh Ramsey degree* (*that is, a Ramsey degree* $\mathbf{a}$ *which satisfies* $\mathbf{a}' \not\geq \mathbf{0}''$).

Building on the success of using an r-cohesive set to simplify an effective analysis of Ramsey's Theorem, we make the following definitions.

DEFINITION 5.4.

1. Suppose that $n, p \geq 1$ and $f : [\omega]^{n+1} \to p$. We say that f is *stable* if $\lim_{a\in\omega} f(x, a)$ exists for every $x \in [\omega]^n$.
2. A degree $\mathbf{a}$ is s-Ramsey if every computable stable $f : [\omega]^2 \to 2$ has an $\mathbf{a}$-computable homogeneous set.

Stable colorings were introduced in [8]. Apart from having intrinsic interest, they arise when we restrict a computable $f : [\omega]^n \to p$ to an r-cohesive set. Thus, we may hope to get information about Ramsey degrees by combining information about r-cohesive sets and s-Ramsey degrees. Furthermore, since every Ramsey degree is an s-Ramsey degree, any lower bounds that we prove for s-Ramsey degrees will automatically give us lower bounds on Ramsey degrees.

It's not hard to show every every $\mathbf{a} \geq \mathbf{0}'$ is s-Ramsey, but again it seems difficult to find other natural examples and it's nontrivial to find interesting lower bounds. Our first lower bound result shows that s-Ramsey degrees are array noncomputable.

DEFINITION 5.5 (Downey, Jockusch, Stob [4]). A degree $\mathbf{a}$ is *array noncomputable* if for each $h \leq_{wtt} K$, there is an $\mathbf{a}$-computable function g such that $g(k) \geq h(k)$ for infinitely many k.

THEOREM 5.6. *Every s-Ramsey degree is array noncomputable.*

Another indication that s-Ramsey degrees are rare comes from measure-theoretic and category-theoretic considerations.

THEOREM 5.7. *The set of s-Ramsey degrees is meager and has measure* 0.

With the aim of understanding the reverse mathematical relationship between the Ramsey's Theorem for exponent 2 and Ramsey's Theorem restricted to stable colorings of exponent 2, Downey, Hirschfeldt, Lempp, and Solomon proved the following result.

THEOREM 5.8 (Downey, Hirschfeldt, Lempp, Solomon [3]). *There exists a computable stable* $f : [\omega]^2 \to 2$ *such that no low set* H *is homogeneous for* f.

In particular, no low degree is s-Ramsey. It is possible to extend this corollary in two ways.

THEOREM 5.9. *The only Δ^0_2 s-Ramsey degree is* $\mathbf{0}'$.

THEOREM 5.10. *There is no low$_2$ s-Ramsey degree.*

Theorem 5.10 has an interesting consequence. As alluded to above, using Theorem 2.16 together with the Low Basis Theorem, we may conclude that for every computable $f : [\omega]^2 \to 2$, there is a low$_2$ set H homogeneous for f. It follows that there is no "universal" computable $f : [\omega]^2 \to 2$.

COROLLARY 5.11. *There does not exist a computable $f : [\omega]^2 \to 2$ such that given any set H_f homogeneous for f, and any computable $g : [\omega]^2 \to 2$, there exists a set H_g homogeneous for g such that $H_g \leq_T H_f$.*

This result, in conjunction with the results in Section 2, is some evidence that the approach of Cholak, Jockusch, and Slaman to examine the jumps of degrees of homogeneous sets of individual computable $f : [\omega]^2 \to 2$ (instead of the degrees themselves or the collection of all computable $f : [\omega]^2 \to 2$ simultaneously) may be the right approach to get clean characterizations. Nonetheless, we have not given up hope that a nice characterization exists for the interesting class of Ramsey degrees.

§6. Conclusion.

Putting together the characterizations of Turing degrees of solutions for computable instances of König's Lemma and the above partition theorems for exponent 2, we see a close connection.

SUMMARY 6.1. *Let* $\mathbf{a}$ *be a Turing degree. The following are equivalent*:

1. $\mathbf{a} \gg \mathbf{0}'$
2. *For every computable $f : [\omega]^2 \to 2$, there is a set H homogeneous for f such that $deg(H)' \leq \mathbf{a}$.*
3. *For every computable regressive $f : [\omega]^2 \to \omega$, there is a set M minhomogeneous for f such that $deg(M) \leq \mathbf{a}$.*
4. *For every computable $f : [\omega]^2 \to \omega$, there is a set C canonical for f such that $deg(C) \leq \mathbf{a}$.*

For exponents $n \geq 3$, the Turing degrees characterizing the location of solutions for Ramsey's Theorem and the Regressive Function Theorem increase by one jump for each successive value of n, while our upper bounds for solutions for the Canonical Ramsey Theorem increase by two jumps for each successive value of n.

In terms of the arithmetical hierarchy, each of the above partition theorems for exponent 2 have Π^0_2 solutions for computable instances, but not necessarily Σ^0_2 solutions. For exponents $n \geq 3$ the location of solutions for Ramsey's Theorem and the Regressive Function Theorem increase by one jump for each successive value of n, while our upper bounds for solutions for the Canonical Ramsey Theorem increase by two jumps for each successive value of n.

Many open questions remain. A resolution of Conjecture 4.7 and Conjecture 4.8 is perhaps the most relevant to fill out the above web of connections between König's Lemma, Ramsey's Theorem, the Regressive Function Theorem, and the Canonical Ramsey Theorem. Furthermore, the following fundamental questions about the relationship between Ramsey's Theorem and König's Lemma remain open.

QUESTION 6.2. *Does every Ramsey degree* $\mathbf{a}$ *satisfy* $\mathbf{a} \gg \mathbf{0}$?

QUESTION 6.3. *Does every Ramsey degree* $\mathbf{a}$ *satisfy* $\mathbf{a} \cup \mathbf{0}' \gg \mathbf{0}'$?

A corresponding question in reverse mathematics is the following.

QUESTION 6.4 (D. Seetapun). *Over* RCA_0, *does* RT^2_2 *imply* WKL_0?

More interesting open questions arise when we examine other partition theorems. One such theorem which seems closely related to the ones we've been discussing is the Thin Set Theorem.

DEFINITION 6.5 (H. Friedman). Suppose that $n \geq 1$ and $f : [\omega]^n \to \omega$. We say that a set $T \subseteq \omega$ is *thin* for f if T is infinite and there exists $c \in \omega$ such that $f(x) \neq c$ for all $x \in [T]^n$.

THEOREM 6.6 (Thin Set Theorem, H. Friedman). *Suppose that* $n \geq 1$ *and* $f : [\omega]^n \to \omega$. *There exists a set* T *thin for* f.

The Thin Set Theorem (for exponent n) is a simple consequence of Ramsey's Theorem (for exponent n). After Friedman's initial work, Cholak, Guisto, Hirst, and Jockusch [1] furthered the effective analysis of the Thin Set Theorem, and gave a tight characterization of the location of thin sets for computable $f : [\omega]^n \to \omega$ in the arithmetical hierarchy. However, little is known about the Turing degrees of such solutions or the reverse-mathematical strengths of the principles themselves. For example, if it not known if it is possible to code any nontrivial information into the thin sets of a computable $f : [\omega]^n \to \omega$ for any n.

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