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THE ROLE OF TRUE FINITENESS IN THE ADMISSIBLE RECURSIVELY ENUMERABLE DEGREES

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Abstract. When attempting to generalize recursion theory to admissible ordinals, it may seem as if all classical priority constructions can be lifted to any admissible ordinal satisfying a sufficiently strong fragment of the replacement scheme. We show, however, that this is not always the case. In fact, there are some constructions which make an essential use of the notion of finiteness which cannot be replaced by the generalized notion of α-finiteness. As examples we discuss both codings of models of arithmetic into the recursively enumerable degrees, and non-distributive lattice embeddings into these degrees. We show that if an admissible ordinal α is effectively close to ω (where this closeness can be measured by size or by cofinality) then such constructions may be performed in the α-r.e. degrees, but otherwise they fail. The results of these constructions can be expressed in the first-order language of partially ordered sets, and so these results also show that there are natural elementary differences between the structures of α-r.e. degrees for various classes of admissible ordinals α. Together with coding work which shows that for some α, the theory of the α-r.e. degrees is complicated, we get that for every admissible ordinal α, the α-r.e. degrees and the classical r.e. degrees are not elementarily equivalent.

§1. Introduction. The study of recursive ordinals and hyperarithmetic sets that began with the work of Church and Kleene [2], Church [1] and Kleene [9], suggested many analogies between the Π^1_1 and hyperarithmetic sets and the recursively enumerable and recursive ones, respectively. The analogy was not perfect, however. At the basic level, for example, the range of a hyperarithmetic function on a hyperarithmetic set is always hyperarithmetic rather than an arbitrary Π^1_1 set. At a deeper level, all nonhyperarithmetic Π^1_1 sets are of the same hyperarithmetic degree. Kreisel [10] studied this situation and came to the realization that while Π^1_1 is analogous to r.e., the correct analog for hyperarithmetic is not recursive but finite. This insight lead first to the development by Kreisel and Sacks [11, 12] of metarecursion theory as the study of recursion theory on the recursive ordinals (those less than ω_1^{CK}, the first nonrecursive ordinal) or, equivalently, on their notations

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in a Π^1_1 path through Kleene's $\mathcal{O}$. In this setting, the meta-r.e. subsets of ω are the Π^1_1 ones and the metafinite ones are hyperarithmetic.

Another approach to generalizing recursion theory to ordinals started with Takeuti's [42, 43] development of Gödel's [6] constructible universe L through a recursion theory on the class of all ordinals. These two approaches came together in the common generalization of recursion on *admissible* ordinals of Kripke [13] and Platek [27]. Here the domain of discourse is an ordinal α or the initial segment L_α of L up to α for admissible α. In general, we make the following definitions:

DEFINITION 1.1. Let β be a limit ordinal. A set $A \subset L_\beta$ is *β-recursively enumerable* if it is $\Sigma_1(L_\beta)$-definable. A partial function $f : L_\beta \to L_\beta$ is *β-recursive* if its graph is β-recursively enumerable. A set $A \subset L_\beta$ is *β-recursive* if it is $\Delta_1(L_\beta)$-definable, namely if it is both β-r.e. and β-co-r.e.

Now for all limit β, there is a β-recursive bijection between β and L_β[1], so β and L_β are used interchangeably as the domain of β-recursion theory. In particular, every element of L_β stands in β-recursive bijection with some $\gamma < \beta$ (of which we think as analogous to a finite number), which may at least partially justify the following definition:

DEFINITION 1.2. A *β-finite* set is an element of L_β.

A limit ordinal α is admissible if L_α satisfies Σ_1-replacement. In the defined terminology,

DEFINITION 1.3. A limit ordinal $\alpha > \omega$ is called *admissible* if the image of an α-finite set under an α-recursive function is bounded below α.

These notions coincide with those of metarecursion theory when $\alpha = \omega_1^{\mathrm{CK}}$, which is the least admissible ordinal.

We should also note that care has to be taken in the definition of "α-recursive in", the analog of Turing reducibility. Here too, the crucial issue is that of finiteness. It no longer suffices to require that one be able to answer single membership question about A in a computation from B to say that A is reducible to B (this relation is called "weak reducibility" and is in general not transitive.) Instead one defines α-reducibility, $\leqslant_\alpha$, by requiring that all α-finite sets of such questions about A can be computed on the basis of α-finitely much information about B:

DEFINITION 1.4. $A \in 2^\alpha$ is *α-recursive* in $B \in 2^\alpha$ if there is some α-r.e. set Φ consisting of pairs of partial strings (i.e., partial functions from α to 2) such that for all α-finite partial strings σ, $\sigma \subset A$ iff there is some $\tau \subset B$ such that $(\tau, \sigma) \in \Phi$.

[1]In fact it is preferable to work with Jensen's hierarchy J_β; we make no distinction in this announcement since for all admissible α we have $L_\alpha = J_\alpha$.

The motivation and goals for generalizing recursion theory in this way included the hopes of elucidating the underlying nature of the notions fundamental to recursion theory and the essences of the constructions that are used to prove its most important theorems. In accordance with Kreisel's insight, a prominent role should be played by the analysis of finiteness along with recursive and recursively enumerable. Such an analysis might lead to a good axiomatic treatment or reveal approaches that would be less dependent on the specific combinatorial properties of ω exploited in these notions and constructions. In this way the study might also produce applications to both classical recursion theory and other domains (set theory, model theory, proof theory and, in hindsight, computer science) where the notions of effectiveness play many roles.

It was relatively easy to formalize the basic notions of recursion theory in these settings but also in much more general ones. Kreisel's test of a generalization worthy of investigation was the Freidberg-Muchnik theorem solving Post's problem by showing that there are incomparable r.e. degrees. As Sacks [31, p. ix] puts it, this brings us from the static or syntactic realm into the dynamic one. It is in this domain that priority arguments and the deeper investigations into the notion of enumerability and relative computability were developed in classical recursion theory. First metarecursion theory (Sacks [30]) and then α-recursion theory (Sacks and Simpson [28]) passed this test.

The route to the solution to Post's problem in α-recursion theory was the ability to make Σ_1-replacement suffice for arguments that in classical recursion theory seemed to naturally rely on Σ_2-replacement (or induction). Further investigations in α-recursion theory indicated that many of the more complicated priority arguments of the classical subject used yet higher levels of replacement and did not generalize so readily to all admissible α. The density theorem was successfully generalized to all admissible α (Shore [34]) but to this day the theorems epitomizing the basic construction of classical recursion theory have not been settled for all admissible ordinals. Almost always more admissibility suffices and at times other conditions as well. Early examples include the existence of an incomplete high α-r.e. degree (Shore [33]) and minimal pairs (Lerman and Sacks [20]) for which Σ_2 admissibility suffices and at times something less. Eventually, an elementary difference between the r.e. degrees and the α-r.e. degrees for some α was established by finding certain admissible ordinals for which, contrary to Lachlan's [15] nonsplitting theorem, one can combine splitting and density for all pairs of α-r.e. degrees (Shore [35]). (That is, for certain α it is always possible to find, for every pair $\boldsymbol{a} < \boldsymbol{b}$ of α-r.e. degrees, two incomparable α-r.e. degrees $\boldsymbol{b}_0$ and $\boldsymbol{b}_1$ between $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{b}_0 \vee \boldsymbol{b}_1 = \boldsymbol{b}$.) This work did indeed elucidate the role of various replacement or induction-like principles in recursion theoretic arguments and much later played a role in analyzing such arguments in reverse mathematics (e.g., Slaman and Woodin [40] and

Mytilinaios [24]). Other aspects of generalized recursion theory found applications in complexity theory (e.g., Shinoda and Slaman [32]). They did not however have much to say directly about the role of finiteness. Moreover, once the basic techniques are understood, all these constructions can be fairly easily carried out in metarecursion theory.

The crucial fact about ω_1^{CK} needed to carry out all these arguments is that there is a metarecursive projection of ω_1^{CK} into ω. This allows one to arrange priority requirements in an ω list and so carry out constructions in such a way that one only ever really needs to worry about there being truly finitely many predecessors of any requirement. For example, density was proved by Driscoll [4] and minimal pairs constructed by Sukonick [41]. It seemed as if everything one could do in classical recursion theory could be done in metarecursion theory as well. It was in this setting that Sacks [29] posed as his final question whether $\mathcal{R}_{\omega_1^{\mathrm{CK}}}$, the meta-r.e. degrees with ω_1^{CK}-reducibility, and $\mathcal{R}$, the r.e. ones with Turing reducibility, are elementarily equivalent. This seemed possible at the time. Indeed, at that time people still thought that there should be some nice characterization of the structure $\mathcal{R}$ that would indicate that it was simple in some way. Shoenfield's conjecture that it was ω-saturated and so categorical had been disproven with the construction of a minimal pair of r.e. degrees but, nonetheless, Sacks still conjectured in [29] that the theory was decidable and that the structure was isomorphic to the degrees r.e. in and above $\boldsymbol{d}$ for every degree $\boldsymbol{d}$.

Both of these conjectures turned out to be false (Harrington-Shelah [8], Shore [36]). Indeed, these results and others showed that $\mathcal{R}$ was very complicated in various ways. Shore [36] showed that it is not recursively presentable and later Harrington and Slaman and Slaman and Woodin (see Slaman [39]) showed that its theory is recursively isomorphic to true arithmetic. These sorts of results changed the paradigm for understanding $\mathcal{R}$ from a hope for simplicity to an approach to its characterization by its complexity. (For more of the history and further discussion, see Shore [37] and [38]). Once one had this view of $\mathcal{R}$, it became natural to believe that the answer to Sacks' question was "no" just because it seemed that one could prove all the results of classical recursion theory in metarecursion theory. If the meta-r.e. degrees, like the r.e. ones, are as complicated as possible then $\mathcal{R}_{\omega_1^{\mathrm{CK}}}$ is more complicated than $\mathcal{R}$. In this way, Odell [26] established an analog of Shore [36] for the meta-r.e. degrees to show that $\mathcal{R}_{\omega_1^{\mathrm{CK}}}$ is not arithmetically presentable and so not isomorphic to $\mathcal{R}$. Once Harrington and Slaman and Slaman and Woodin had proven that the theory of $\mathcal{R}$ is recursively isomorphic to true arithmetic, it became "morally certain" that the two structures are not even elementarily equivalent.

Shore and Slaman, as announced in Shore [37], managed to carry out enough of the relevant constructions in metarecursion theory to prove this result. The proof was fairly elaborate and required lifting several major theorems of classical recursion theory to ω_1^{CK}. It also failed to give a full characterization of the degree of the theory of $\mathcal{R}_{\omega_1^{\mathrm{CK}}}$. The expected result was that it should be recursively isomorphic to the theory of $\langle L_{\omega_1^{\mathrm{CK}}}, \in\rangle$ or, equivalently, of degree $\mathcal{O}^{(\omega)}$. This result awaited further developments in classical recursion theory. Nies, Shore and Slaman [25] provided a definable standard model of arithmetic in $\mathcal{R}$ and so a more direct proof that the degree of its theory is $\boldsymbol{0}^{(\omega)}$. In [7], the same original intuition from the 60s about the similarity of $\mathcal{R}$ and $\mathcal{R}_{\omega_1^{\mathrm{CK}}}$ was followed, to lift enough of Nies, Shore and Slaman [25] to metarecursion theory to prove that a standard model of arithmetic with a predicate for $\mathcal{O}$ is definable in $\mathcal{R}_{\omega_1^{\mathrm{CK}}}$ and so its theory, as expected, is recursively isomorphic to both that of $L_{\omega_1^{\mathrm{CK}}}$ and to $\mathcal{O}^{(\omega)}$. These results thus answered Sacks's original question by providing an elementary difference between $\mathcal{R}$ and $\mathcal{R}_{\omega_1^{\mathrm{CK}}}$. However, they did so by continuing along the path following the intuition that one can lift all constructions of r.e. degrees to ω_1^{CK} by using projectability to convert requirements lists to ones of length ω; and to any admissible ordinal satisfying enough replacement to handle requirements in order type α.

These illusions are dispelled in the work described here, whose aim is to illuminate the role of true finiteness in various classical constructions in the setting of the r.e. *degrees* (Lerman and Simpson [21] and Lerman [16] gave such results in the context of the lattice of r.e. sets). We show that constructions given in [25] can be performed in the α-r.e. degrees (for Σ_2-admissible α) if and only if the cofinality of α, as measured by some relatively effective class of functions, is ω. Another line of investigation considers constructions which are used to embed some nondistributive lattices into the r.e. degrees. Lachlan [14] has shown that the 1-3-1 lattice, the one of the two basic nondistributive lattices which includes a *critical triple*, is embeddable in the r.e. degrees. We show that this construction uses finiteness in an essential way; it can only be performed in the α-r.e. degrees (here α is any admissible ordinal) if α is countable in some effective sense. All of the work taken together shows that no $\mathcal{R}_\alpha$ is elementarily equivalent to $\mathcal{R}$.

§2. The results.

NOTATION. Throughout, α denotes an admissible ordinal. We let $\mathcal{R}_\alpha$ denote the structure of the α-r.e. degrees together with the partial ordering induced by α-reducibility.

We first introduce two ways to "effectively" measure an admissible ordinal α.

DEFINITION 2.1. Let $n < \omega$. The *Σ_n-projectum of α*, ϱ_α^n, is the least ordinal $\gamma \leqslant \alpha$ such that there is a set $A \subset \gamma$ which is $\Sigma_n(L_\alpha)$-definable but is not α-finite.

Jensen showed that for all n, there is a partial Σ_n-definable function from ϱ^n_α onto α. Thus in some sense, ϱ^n_α measures the "effective size" of α. If $\varrho^n_\alpha = \omega$ then α is effectively countable. For example, we have $\varrho^1_{\omega_1^{\mathrm{CK}}} = \omega$.

Next, we effectively measure the cofinality of α.

DEFINITION 2.2. Let Γ be a class of functions. The *Γ-cofinality of α*, $\mathrm{cf}_\Gamma(\alpha)$, is the least ordinal β such that there is some function $f \in \Gamma$ with domain β and range cofinal in α.

Admissibility is equivalent to the statement that $\mathrm{cf}_\Gamma(\alpha) = \alpha$, where Γ is the class of α-recursive functions. We often examine $\mathrm{cf}_\Gamma(\alpha)$ for Γ the class of $\Sigma_n(L_\alpha)$-definable functions. Also, we can take the class of functions which a particular degree can compute:

DEFINITION 2.3. Let $\boldsymbol{a}$ be an α-degree. The *recursive cofinality of $\boldsymbol{a}$*, $\mathrm{rcf}(\boldsymbol{a})$, is $\mathrm{cf}_\Gamma(\alpha)$, where Γ is the class of functions which are weakly α-recursive in $\boldsymbol{a}$.

Since the analog of the limit lemma holds in α-recursion theory, we have that the $\Delta_2(L_\alpha)$-definable sets and functions are exactly those which are weakly α-recursive in $\boldsymbol{0}'$, and so $\mathrm{rcf}(\boldsymbol{0}') = \mathrm{cf}_{\Sigma_2(L_\alpha)}(\alpha)$.

2.1. Lattice embeddings. Embeddings of lattices into the classical r.e. degrees have been studied extensively; see [3] for details. The question of which finite non-distributive lattices can be embedded into the r.e. degrees remains open. These lattices may or may not contain a *critical triple*.

DEFINITION 2.4. A *critical triple* in a lattice L is a triple $a_0, a_1, b \in L$ such that $a_0 \vee b = a_1 \vee b$ and $a_0 \wedge a_1 \leqslant b$.

All lattices with critical triples contain the basic such lattice, the 1-3-1 (also known as M_5 or M_3). A necessary and sufficient criterion for embedding lattices *without* critical triples (also known as *principally decimposable* or *join semidistributive*) has been found by Lerman ([17, 19, 18]); no such criterion is yet known for lattices with critical triples.

In this section we study embeddings of the 1-3-1 into the α-r.e. degrees. We consider also a weak version of the notion of a critical triple, which we use in the context of upper semi-lattices.

DEFINITION 2.5. Suppose L is an upper semi-lattice. A *weak critical triple* in L is a triple a_0, a_1, b such that $a_0 \vee b = a_1 \vee b$, $a_0 \nleqslant b$ and there is no $e \leqslant a_0, a_1$ such that $a_0 \leqslant e \vee b$.

THEOREM 2.6. *Let α be admissible. The following are equivalent for an incomplete degree $\boldsymbol{a} \in \mathcal{R}_\alpha$:*

1. *There is a weak critical triple in $\mathcal{R}_\alpha(\leqslant \boldsymbol{a})$.*
2. *There is an embedding of the* 1-3-1 *lattice into $\mathcal{R}_\alpha(\leqslant \boldsymbol{a})$.*
3. $\mathrm{rcf}(\boldsymbol{a}) = \omega$.

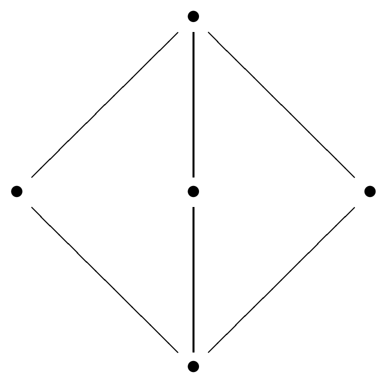

FIGURE 1. The 1-3-1 lattice.

In terms of definability, this gives us two first-order formulas φ_0 and φ_1, which are not in general equivalent in partial orderings or even in upper semi-lattices, such that for every admissible α,

$$\varphi_0(\mathcal{R}_\alpha) = \varphi_1(\mathcal{R}_\alpha) = \{\boldsymbol{a} \in \mathcal{R}_\alpha : \boldsymbol{a} < \boldsymbol{0}' \ \& \ \mathrm{rcf}(\boldsymbol{a}) = \omega\}.$$

In one direction, suppose that $\mathrm{rcf}(\boldsymbol{a}) > \omega$, and suppose that $C \leqslant_\alpha \boldsymbol{a}$. One can then find a closed, unbounded (in α) set of points which are closed under the use function for the computation. If A_0, A_1 and B below $\boldsymbol{a}$ are given then we can construct a set E witnessing the fact that A_0, A_1 and B do not form a weak critical triple as follows. We examine both reductions $A_i \leqslant_\alpha A_{1-i} \oplus B$. The set E will code points which are closed under both computations. The key observation is that if at some stage a point γ appears to be such a closure point but in fact is not one, due to a later change in one A_i, then another change, below γ, must occur in either A_{1-i} or in B. To construct E below both A_0 and A_1, we wait for such a γ and such change in A_i, and if the guaranteed further change occurs in A_{1-i} then this gives us permission from both A_0 and A_1 for γ to enter E. To recover both A_i from $E \oplus B$, we again look for suspected closure points γ with correct B-use; now E coded the information if further changes in the A_is occur at a later stage, and so can identify closure points correctly.

This argument reflects the fact that under the condition $\mathrm{rcf}(\boldsymbol{a}) > \omega$, Lachlan's strategy of continuous retracing, which he used in the embedding of the 1-3-1 into the r.e. degrees, fails miserably. The closure points can be viewed as bounding infinite increasing sequences of uses of the reductions $A_i \leqslant_\alpha A_{1-i} \oplus B$. These correspond to infinite sequences of traces which would result if an attempt to mimic Lachlan's construction were made in this context. In the truly finite case, these sequences of traces always have a last element, even though they are unbounded in size. This indicates that in a sense, Lachlan's strategy is the only way an embedding of a weak critical triple can be performed in the r.e. degrees.

In the other direction, if $\mathrm{rcf}(\boldsymbol{a}) = \omega$ and $\boldsymbol{a}$ is incomplete then $\boldsymbol{a}$ can compute a counting of α. This allows us to trim the relevant traces of balls so that they are indeed truly finite at every stage of the construction. Thus the Lachlan strategy can be pursued.

By Shore ([34]), $\boldsymbol{a}$ is high. Using the counting of α we can translate properties of $\boldsymbol{a}$ into the Turing degrees, using techniques of Maass ([22]) and S. Friedman ([5]). The highness results in domination properties of functions recursive in the so-called "collapse" of $\boldsymbol{a}$, which we can use back in the realm of α-degrees. This allows us to use techniques from [3] to get an embedding of the 1-3-1 below $\boldsymbol{a}$.

It follows from Shore's [34] that incomplete α-r.e. degrees $\boldsymbol{a}$ such that $\mathrm{rcf}(\boldsymbol{a}) = \omega$ exist iff both $\mathrm{cf}_{\Sigma_2(L_\alpha)}(\alpha) = \omega$ and $\varrho^2_\alpha = \omega$. These ordinals α are thus distinguished (among all admissible ordinals) by a first-order property of $\mathcal{R}_\alpha$.

THEOREM 2.7. *Let α be an admissible ordinal. Then there is an embedding of the* 1-3-1 *lattice with an incomplete top iff both $\varrho^2_\alpha = \omega$ and* $\mathrm{cf}_{\Sigma_2(L_\alpha)}(\alpha) = \omega$.

Whether there are any embeddings of the 1-3-1 preserving either top or bottom remains open.

2.2. Effective successor models. Nies, Shore and Slaman ([25]) introduce a scheme interpreting models of arithmetic in the r.e. degrees, using *Slaman-Woodin sets*. Let $\chi_0(\bar{\boldsymbol{p}})$ be a (first-order) *correctness condition* which states that the parameters $\bar{\boldsymbol{p}}$ code a model $M_{\bar{\boldsymbol{p}}}$ which models Robinson arithmetic. In [25], the authors also construct *effective successor models*. We need a first-order definition.

DEFINITION 2.8. Let $\bar{\boldsymbol{p}}$ satisfy χ_0. We say that $M_{\bar{\boldsymbol{p}}}$ is an *effective successor model* if the following two conditions hold.

1. There is some quadruple $\bar{\boldsymbol{e}} = (\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{f}_0, \boldsymbol{f}_1)$ such that for all $\boldsymbol{x} \in M_{\bar{\boldsymbol{p}}}$, if $(\boldsymbol{x} > 0)^{M_{\bar{\boldsymbol{p}}}}$, $i < 2$ and $M_{\bar{\boldsymbol{p}}} \models$ "$\boldsymbol{x} = i \mod 2$", then
$$\boldsymbol{x} = (\boldsymbol{e}_i \vee (\boldsymbol{x} - 1)^{M_{\bar{\boldsymbol{p}}}}) \wedge \boldsymbol{f}_i.$$
2. For every $\boldsymbol{x} \in M_{\bar{\boldsymbol{p}}}$, the set
$$\{\boldsymbol{y} \in M_{\bar{\boldsymbol{p}}} : \boldsymbol{y} <^{M_{\bar{\boldsymbol{p}}}} \boldsymbol{x}\}$$
has a least upper bound (in $\mathcal{R}_\alpha$) which we denote by $\sum_{\bar{\boldsymbol{p}}} \boldsymbol{x}$. Further, for all $\boldsymbol{y} \in M_{\bar{\boldsymbol{p}}}$ such that $\boldsymbol{y} \geq^{M_{\bar{\boldsymbol{p}}}} \boldsymbol{x}$ we have $\boldsymbol{y} \nleq \sum_{\bar{\boldsymbol{p}}} \boldsymbol{x}$.

Let $\chi(\bar{\boldsymbol{p}}, \bar{\boldsymbol{e}})$ state that $M_{\bar{\boldsymbol{p}}}$ is an effective successor model, witnessed by $\bar{\boldsymbol{e}}$.

THEOREM 2.9. *Let α be Σ_2-admissible such that* $\mathrm{cf}_{\Sigma_3(L_\alpha)} = \omega$. *Let $\boldsymbol{u}$ be a promptly simple degree.*[2] *Then there are $\bar{\boldsymbol{p}}, \bar{\boldsymbol{e}} \leqslant \boldsymbol{u}$ such that $M_{\bar{\boldsymbol{p}}}$ is a standard, effective successor model of arithmetic, and furthermore, in $\mathcal{R}_\alpha(\leqslant \boldsymbol{u})$, the α-r.e. degrees below $\boldsymbol{u}$, the elements of $M_{\bar{\boldsymbol{p}}}$ do not have a least upper bound.*

The fact that $\mathrm{cf}_{\Sigma_3(L_\alpha)} = \omega$ allows us to approximate in some weak sense an ω-sequence cofinal in α. This allows us to imitate the construction of an SW set; in this construction, true finiteness is used in ensuring the minimality (M_i in the notation of [25]) requirements hold by passing "chits" from one functional to the next, bumping against some last one. The functionals are tied to elements of the SW set and so it is crucial that we construct a set of degrees of order-type ω.

To ensure that the elements of $M_{\bar{\boldsymbol{p}}}$ do not have a least upper bound below $\boldsymbol{u}$ we construct, below $\boldsymbol{u}$, an *exact pair* for the elements of $M_{\bar{\boldsymbol{p}}}$. Luckily, the tracing procedures used in the construction of the effective successor model and of the exact pair do not clash too violently and so can be combined into one construction.

For other ordinals we get the opposite results.

THEOREM 2.10. *Let α and $\boldsymbol{u} \in \mathcal{R}_\alpha$ fall under one of the following cases*:

1. $\varrho^2_\alpha = \alpha$, $\mathrm{cf}_{\Sigma_3(L_\alpha)}(\alpha) > \omega$ *and $\boldsymbol{u}$ is low.*
2. *α is Σ_2-admissible*, $\mathrm{cf}_{\Sigma_3(L_\alpha)}(\alpha) > \omega$ *and $\boldsymbol{u}$ is* low_2.
3. *α is Σ_3-admissible*, $\mathrm{cf}_{\Sigma_4(L_\alpha)}(\alpha) > \omega$ *and $\boldsymbol{u}$ is any α-r.e. degree.*
4. $\varrho^3_\alpha = \alpha$, $\mathrm{cf}_{\Sigma_4(L_\alpha)}(\alpha) > \omega$ *and $\boldsymbol{u}$ is any r.e. degree.*

Then whenever $M_{\bar{\boldsymbol{p}}}$ is an effective successor model with $\bar{\boldsymbol{p}}, \bar{\boldsymbol{e}} \leqslant \boldsymbol{u}$, the standard part of $M_{\bar{\boldsymbol{p}}}$ has a least upper bound $\boldsymbol{c}$ in the degrees below $\boldsymbol{u}$. Further, that standard part is definable as the collection of elements $x \in M_{\bar{\boldsymbol{p}}}$ such that $\Sigma_{\bar{\boldsymbol{p}}} x \leqslant \boldsymbol{c}$.

The reason for this is that since $M_{\bar{\boldsymbol{p}}}$ is an effective successor model, a sequence of both $\boldsymbol{u}$- and α-r.e.-indexes for sets in $n^{M_{\bar{p}}}$ (for $n < \omega$) can be obtained in a $\Sigma_3(\boldsymbol{u})$ fashion (we also use Jensen's uniformization theorem here). Under the cases mentioned, all such sequences is α-finite, and so can be used to define an effective join of the elements of the standard part of $M_{\bar{\boldsymbol{p}}}$ and thus obtain a least upper bound. Again, this hints that true finiteness is crucial in the construction of effective successor models and perhaps in the construction of SW sets in general.

These results allow us to formulate further first-order differences between various $\mathcal{R}_\alpha$s. Let $\theta(x, \boldsymbol{c}, \bar{\boldsymbol{p}})$ state that $x \in M_{\bar{\boldsymbol{p}}}$ and that $\Sigma_{\bar{\boldsymbol{p}}} x \leqslant \boldsymbol{c}$. Let $\phi_0(y, \boldsymbol{c}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{e}})$ state that $\boldsymbol{c}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{e}} < y$, that $\chi(\bar{\boldsymbol{p}}, \bar{\boldsymbol{e}})$ holds, that $\theta(\mathcal{R}_\alpha, \boldsymbol{c}, \bar{\boldsymbol{p}})$ is a nontrivial initial segment of $M_{\bar{\boldsymbol{p}}}$ closed under the successor operation, and that $\boldsymbol{c}$ is the least upper bound for $\theta(\mathcal{R}_\alpha, \boldsymbol{c}, \bar{\boldsymbol{p}})$ in the degrees below y. Let $\phi(y)$ state the existence of some $\bar{\boldsymbol{p}}, \bar{\boldsymbol{e}}$ below y such that $\chi(\bar{\boldsymbol{p}}, \bar{\boldsymbol{e}})$ holds but

[2] The definition and basic properties of promptly simple r.e. sets and their degrees are identical to the classical ones (see [23]) for all admissible ordinals.

for no $\boldsymbol{c}$ below y does $\phi_0(y, \boldsymbol{c}, \bar{\boldsymbol{p}}, \bar{\boldsymbol{e}})$ hold. Theorem 2.9 shows that if α is Σ_2-admissible, $\mathrm{cf}_{\Sigma_3(L_\alpha)}(\alpha) = \omega$ and $\boldsymbol{u}$ is promptly simple, then $\phi(\boldsymbol{u})$ holds in $\mathcal{R}_\alpha$. For all pairs of α and $\boldsymbol{u}$ mentioned in Theorem 2.10, $\phi(\boldsymbol{u})$ fails in $\mathcal{R}_\alpha$.

Let X be an additional unary predicate. Let **PS** denote the collection of promptly simple degrees; let $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ denote the classes of low and low$_2$ degrees respectively. Now there is always a low promptly permitting degree. Together with the results mentioned, we get the following:

THEOREM 2.11. *Let α be a Σ_2-admissible ordinal.*

1. $(\mathcal{R}_\alpha, \boldsymbol{PS}) \models \forall y \in X\ \phi(y)$ *iff* $\mathrm{cf}_{\Sigma_3(L_\alpha)}(\alpha) = \omega$.
2. $(\mathcal{R}_\alpha, \boldsymbol{L}_1) \models \exists y \in X\ \phi(y)$ *iff* $(\mathcal{R}_\alpha, \boldsymbol{L}_2) \models \exists y \in X\ \phi(y)$ *iff* $\mathrm{cf}_{\Sigma_3(L_\alpha)}(\alpha) = \omega$.

One would like of course to improve this by eliminating the extra unary predicate; one would think that the most likely candidate is the class of promptly simple degrees, which is definable in $\mathcal{R}_\omega$. The classical proof can be carried out if, for example, $\varrho^2_\alpha = \omega$, but fails miserably in other cases, and we, in fact, suspect that in some cases there may be a noncapping degree (i.e., a degree which is not half of a minimal pair) which is not promptly simple. It follows, though (as no degree which permits promptly can be half of a minimal pair), that if we let X state that y is noncappable then for every Σ_2-admissible ordinal α, if $\mathrm{cf}_{\Sigma_3(L_\alpha)}(\alpha) > \omega$ then $\mathcal{R}_\alpha \models \exists y \in X\ \phi(y)$ and if $\varrho^2_\alpha = \omega$ then $\mathcal{R}_\alpha \models \neg\exists y \in X\ \phi(y)$.

If we are willing to go one level higher to the Σ_3 level, then we get the following:

THEOREM 2.12. *If* $\mathrm{cf}_{\Sigma_4(L_\alpha)}(\alpha) > \omega$ *and either α is Σ_3-admissible or $\varrho^3_\alpha = \alpha$, then $\mathcal{R}_\alpha \models \neg\exists y\ \phi(y)$.*

This is an elementary difference between such α's and Σ_2-admissible α's such that $\mathrm{cf}_{\Sigma_3(L_\alpha)}(\alpha) = \omega$.

2.3. $\mathcal{R}_\alpha$ is sometimes complicated. If α is close to ω in some effective way, then the machinery of models of arithmetic and comparison maps of [25] can be constructed in $\mathcal{R}_\alpha$. This gives us a copy of the standard model of arithmetic which is interpreted in $\mathcal{R}_\alpha$ without parameters. Further, by coding additional subsets of ω in the various copies of $\mathbb{N}$ which are involved in this interpretation, we in fact get a parameter-less interpretation of an ω-model of *second*-order arithmetic. The reals in this model include all subsets of ω which are $\Delta_2(L_\alpha)$-definable, and in particular include Kleene's $\mathcal{O}$ and all Π^1_1 sets. Each such set can be definably identified in such an ω-model. This implies the following.

THEOREM 2.13. *Let α be an admissible ordinal. If $\varrho^2_\alpha = \omega$ or if α is Σ_2-admissible and* $\mathrm{cf}_{\Sigma_3(L_\alpha)}(\alpha) = \omega$ *then* $\mathcal{O}^{(\omega)} \leqslant_1 \mathrm{Th}(\mathcal{R}_\alpha)$.

Let ψ be the sentence stating that the 1-3-1 lattice can be embedded into the α-r.e. degrees with an incomplete top. The statement ψ holds in the classical r.e. degrees. For $\mathcal{R}_\alpha$ we get a dichotomy: if $\varrho^2_\alpha > \omega$ then ψ fails in $\mathcal{R}_\alpha$; and if $\varrho^2_\alpha = \omega$ then $\mathrm{Th}(\mathcal{R}_\alpha)$ is complicated (in particular, it is not hyperarithmetic, whereas $\mathrm{Th}(\mathcal{R}_\omega)$ which has complexity $\boldsymbol{0}^{(\omega)}$ lies low in the hyperarithmetic hierarchy.) We conclude:

THEOREM 2.14. *For every admissible ordinal α, $\mathcal{R}_\alpha$ and $\mathcal{R}_\omega$ are not elementarily equivalent.*

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