THE BULLETIN OF SYMBOLIC LOGIC
Volume 11, Number 2, June 2005

GÖDEL'S CONCEPTUAL REALISM

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Kurt Gödel is almost as famous—one might say "notorious"—for his extreme platonist views as he is famous for his mathematical theorems. Moreover his platonism is not a myth; it is well-documented in his writings. Here are two platonist declarations about set theory, the first from his paper about Bertrand Russell and the second from the revised version of his paper on the Continuum Hypotheses.

> Classes and concepts may, however, also be conceived as real objects, namely classes as "pluralities of things" or as structures consisting of a plurality of things and concepts as the properties and relations of things existing independently of our definitions and constructions.
>
> It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence.[1]

> But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception.[2]

The first statement is a platonist declaration of a fairly standard sort concerning set theory. What is unusual in it is the inclusion of concepts among the objects of mathematics. This I will explain below. The second statement expresses what looks like a rather wild thesis.

From such statements, one might get a picture of set theory as the study of a realm of abstract objects our epistemic access to which is direct, through

Submitted December 6, 2004; accepted December 20, 2004.

*I would like to thank John W. Dawson, Jr., Solomon Feferman, and Charles Parsons for corrections to, suggestions for, and comments on an earlier version of this paper. I would also like to thank Joseph Almog for comments on a talk with the same subject as the paper.

[1]Gödel [4], p. 128.

[2]Gödel [6], p. 268.

1079-8986/05/1102-0008/$2.80

perception. Gödel does subscribe to such a picture. Nevertheless, there is another aspect of Gödel's view about mathematics, an aspect that makes the picture look a good deal different. It is this other aspect that I want to discuss.

Gödel calls his brand of platonism "conceptual realism," and what is distinctive about the other aspect of his view is the role of concepts as opposed to objects, e.g., of the concept of set as opposed to sets themselves. But matters are complicated by Gödel's use of the word "concept," so I will begin by trying to say what he means by this word.

In the unpublished text of his Josiah Willard Gibbs Lecture, Gödel talks of "two separate worlds (the world of things and the world of concepts)."[3] He rejects one-world "Aristotelian realism (according to which concepts are parts or aspects of things)."[4]

Evidently classes are examples of *things*. The first passage quoted on page 207 above indicates that both classes and concepts are examples of *objects*. A contrast Gödel makes while discussing the paradoxes in Gödel [5] helps to explain how classes and concepts differ.

> This concept of set, however, according to which a set is anything obtainable from the integers (or some other well-defined objects) by the iterated application of the operation "set of", and not something obtained by dividing the totality of existing things into two categories, has never led to any antinomy whatsoever; that is, the "naive" and uncritical working of this concept has so far proved completely self-consistent.[5]

One might think of "classes" in the Russell paper passage as meaning sets in this iterative sense. The operation "set of" Gödel also calls "set of x's." It is the operation that gives all the sets whose members are particular given objects, the x's. Sets in the iterative sense are gotten by starting with, e.g., the integers, applying the "set of" operation to the integers, then applying the operation to the integers and sets of integers, and so on, repeating the process an "absolute infinity" of times, to use an expression of Cantor. Gödel stresses that a set of x's need not be definable (for example, "random sets are not excluded"[6]).

Concepts are for Gödel things obtained by dividing the universe into two categories, or rather they are properties that yield such divisions. Gödel's concepts differ from Frege's concepts in two ways:

(1) They are intensional, whereas the consensus seems to be that Fregean concepts are extensional.
(2) They are objects.

[3]Gödel [7], p. 321.
[4]Gödel [7], p. 321.
[5]Gödel [5], p. 180 and Gödel [6], pp. 258–259.
[6]Gödel [6], p. 259.

The second difference might, though, be merely a difference in what "object" means for Frege and for Gödel. For Frege objects are distinguished from concepts by being "saturated." There is no special reason for thinking that Gödel counts concepts as being saturated. I suspect that Gödel talks of concepts as "objects" mainly to to stress their independent existence.

The classes of Gödel-Bernays or Morse-Kelly set theory are, I believe, examples of extensions of Gödelian concepts. These concepts are ones that apply to things (sets). There are also concepts that apply to concepts. In Gödel [4], Gödel speculates about the possibility of a mathematical theory of concepts, a theory that would have universal concepts and somehow avoid the paradoxes.

A concept that I will be very much concerned with in the sequel is the concept of set. One might think that this is the property of being a set: the intensional property whose possession is what makes an object a set. I will call this sense of "concept of set" the *straightforward* sense. I will argue that, although Gödel's sense of the term may be the straightforward sense, this is not so in a straightforward way. I will describe another sense of "concept of set" and argue that this other sense fits better than the straightforward one with much of what Gödel says about set theory. I will suggest that the only "perception ... of the objects of set theory" that plays a role in Gödel's account of actual and hoped-for set-theoretic knowledge is "perception" of the concept of set in this other sense.

An indication that there are subtleties in Gödel's platonism is his assertion that mathematical truths are analytic. Near the end of the Russell paper, he considers whether the axioms of ***Principia*** are analytic. He distinguishes two senses of "analytic."

> First it may have the purely formal sense that the terms occurring can be defined (either explicitly or by rules for eliminating them from sentences containing them) in such a way that the axioms and theorems become special cases of the law of identity and disprovable propositions become negations of this law.[7]

Gödel says that even number theory is demonstrably not analytic in this sense.

> In a second sense a proposition is called analytic if it holds "owing to the meaning of the concepts occurring in it", where this meaning may perhaps be undefinable (i.e., irreducible to anything more fundamental).[8]

The subject of analyticity also comes up in Gödel's Gibbs Lecture. There Gödel uses "analytic" only in the second of the two senses. He explains his use as follows.

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[7]Gödel [4], pp. 138–139.

[8]Gödel [4], p. 139.

> I wish to repeat that "analytic" here does not mean "true owing to our definitions", but rather "true owing to the nature of the concepts occurring [therein]", in contradistinction to "true owing to the properties and the behavior of things". This concept of analytic is so far from meaning "devoid of content" that it is perfectly possible that an analytic proposition might be undecidable (or decidable only with [a certain] probability). For our knowledge of the world of concepts may be as limited and incomplete as that of [the] world of things."[9]

Gödel contends that all mathematical truths are analytic in his sense. In the Gibbs Lecture, he argues at length against the view that mathematical truths are tautologies (analytic in the Russell paper's first sense) or true in virtue of linguistic conventions. He then says:

> However, it seems to me that nevertheless one ingredient of this wrong theory of mathematical truth is perfectly correct and really discloses the true nature of mathematics. Namely, it is correct that a mathematical proposition says nothing about the physical or psychical reality existing in space and time, because it is true already owing to the meaning of the terms occurring in it, irrespectively of the world of real things. What is wrong, however, is that the meaning of the terms (that is, the concepts they denote) is asserted to be something man-made and consisting merely in semantical conventions. The truth, I believe, is that these concepts form an objective reality of their own, which we cannot create or change, but only perceive and describe.[10]

What does this passage imply for the truths of set theory? The most obvious interpretation is one that takes "concept of set" in the straightforward sense. On this interpretation, Gödel is saying that the truths of set theory are truths solely in virtue of the meanings of the words "set" and "$\in$," i.e., in virtue of what it means to be a set and what it means for a set (or other object) to be a member of a set. On the face of it, this is an astonishing assertion. Why should, for example, the existence of an infinite set follow from the meaning of "set" (the concept of set) any more than the existence of a rocking chair follows from the meaning of "chair" (the concept of chair)? Indeed why should the existence of any sets at all follow from the concept of set?

Fortunately, we are not left to our own devices in applying Gödel's analyticity thesis in the case of set theory. In both the Russell paper and the Gibbs Lecture, Gödel explicitly discusses the case of set theory, and in the latter he also discusses the case of number theory. In the Russell paper, the

[9] Gödel [7], p. 321. The insertions in square brackets are by the editors of [3].

[10] Gödel [7], p. 320.

passage describing the second sense of analyticity is immediately followed by the following.

> It would seem that all the axioms of ***Principia***, in the first edition, (except the axiom of infinity) are in this sense analytic for certain interpretations of the primitive terms, namely if the term "predicative function" is replaced either by "class" (in the extensional sense) or (leaving out the axiom of choice) by "concept", since nothing can better express the meaning of the term "class" than the axiom of classes (cf. page 140) and the axiom of choice, and since, on the other hand, the meaning of the term "concept" seems to imply that every propositional function defines a concept.[11]

At this point there is a footnote, which begins as follows.

> This view does not contradict the opinion defended above that mathematics is based on axioms with real content, because the very existence of the concept of e.g., "class" constitutes already such an axiom; since, if one defined e.g., "class" and "$\in$" to be "the concepts satisfying the axioms", one would be unable to prove their existence.[12]

In the Gibbs Lecture, the assertion that mathematical truths are truths in virtue of meaning is followed by the following paragraph.

> Therefore a mathematical proposition, although it does not say anything about space-time reality, still may have a very sound objective content, insofar as it says somethng about relations of concepts. The existence of non-"tautological" relations between the concepts of mathematics appears above all in the circumstances that for the primitive terms of mathematics, axioms must be assumed, which are by no means tautologies (in the sense of being in any way reducible to $a = a$), but still do follow from the meaning of the primitive terms under consideration. For example, the basic axiom, or rather, axiom schema, for the concept of set of integers says that, given a well-defined property of integers (that is, a propositional expression $\varphi(n)$ with an integer variable n), there exists a set M of those integers which have the property φ. Now, considering the circumstance that φ may itself contain the term "set of integers", we have here a series of rather involved axioms about the concept of set. Nevertheless, these axioms (as the aforementioned results show) cannot be reduced to anything substantially simpler, let alone to explicit tautologies. It is true that these axioms are valid owing to the meaning of the term "set"—one might even say they express the very meaning of the term "set"—and therefore they might fittingly be called analytic; however,

[11] Gödel [4], p. 139.

[12] Gödel [4], p. 139, footnote 47.

> the term "tautological", that is, devoid of content, for them is entirely out of place, because even the assertion of the existence of a concept of set satisfying these axioms (or of the consistency of these axioms) is so far from being empty that it cannot be proved without again using the concept of set itself, or some other abstract concept of [a] similar nature.[13]

In discussing these two passages, I will at first ignore the footnote to the Russell paper passage and the final clause of the last sentence of the Gibbs Lecture passage (the clause beginning "however, ... ").

What the two passages say about the concept of set is, I think, essentially the same. The Russell passage is complicated by matters not directly relevant to my present interests: the context of ***Principia***; Gödel's concern, prominent in the Russell paper but absent in other writings, with the theory of concepts; the question of whether the Axiom of Choice is valid for concepts. Both passages are concerned with the concept of "set of x's," where the x's are integers in the Gibbs Lecture case. In both cases, comprehension axioms are said not merely to be analytic but also to express the very meaning of the term "set" (of, e.g., integers) or the synonym, "class." I will talk mainly about the Gibbs Lecture passage, since it is freer of irrelevancies.

The first thing to note is that what the axioms express is not really the concept of set of integers in what I've called the straightforward sense. The axioms say nothing about what it is that distinguishes sets from non-sets. They say nothing relevant to whether any particular object is a set. More importantly, they say things about sets that one would not think part of the concept in the straightforward sense: they say that many sets of integers exist. The concept that the axioms may well express is the concept of a set-of-integers structure or the concept of a set-of-integers concept. What the axioms do is put a constraint on any concept for counting as a concept of set of integers in the straightforward sense and put a constraint on any structure for counting as the structure of sets of integers.

I will call the indirect sense of "concept of set" just indicated *my sense*, even though I have no special claim to it. It would be natural just to call this the "indirect sense," but unfortunately the term "indirect sense" already has a different, a Fregean, meaning. My sense differs from the straightforward sense in that the instances of a concept of set in the straightforward sense—the objects that fall under the concept—are sets (or, at least, what the concept counts as sets). The instances of a concept of set in my sense are not sets. There are two versions of my sense. In one version, the instances are concepts: straightforward-sense concepts of set. In the other version, the instances might be described as set structures or universes of sets. The difference is that the instances in the first version are intensional. This

[13] Gödel [7], pp. 321–322.

difference will not matter for the issues I will discuss, so I will treat the two versions as if they were one.

A concept of set expressed by axioms such as comprehension axioms cannot put any constraint on which objects count as sets and which do not. Such axioms put constraints on the isomorphism type of a set-theoretic structure, but they permit any objects whatsoever to be elements of the structure. I want to make it clear, though, that a concept of set could be a concept in my sense even if it determined completely what objects count as sets and what counts as the membership relation. A concept of this sort would have at most one instance: it would allow at most one structure to count as a set-theoretic universe. (I have doubts that there is such a concept, but that is another matter.)

Let me for the moment assume that, in the Gibbs Lecture passage I am discussing, Gödel means "concept of set" in my sense. We will later see that this assumption cannot be completely true, but first we will see how the assumption would make more plausible the main content of the passage.

If we understand "concept of set" in this way, it no longer seems so strange to say that set-theoretic truths are analytic. We can count as set-theoretic truths those assertions about sets that would have to be true for any instance of the concept of set: for any universe of sets or concept of set (in the straightforward sense) that conforms to the concept (in the general case, to the iterative concept). This way of understanding "concept of set" also explains Gödel's talk of certain statements as being "implied by the concept of set."

This concept of set seems perfectly objective, whether or not one wants to count it as an *object*. We did not create it, though we have singled it out as something to study. We did not define it either, nor can we define it—except perhaps in terms of other, closely related, notions.

The objectivity of the concept does not, however, imply that we lack epistemic access to it. And I don't see why our access needs to be described as *perceiving* the concept, as Gödel describes it. We *understand* the concept and we can *explain* it, as Gödel does in the Continuum Hypothesis papers. In those papers, Gödel himself talks of explaining the concept of set and of understanding it. When we work in the mathematical subject of set theory, we can think of what we are doing as finding out what is implied by the concept, what has to be true of any instance of the concept.

Of course, there is no reason we should be able to discover everything that is implied by the concept. Our understanding of the concept may be incomplete. Gödel says, in a continuation of a passage I quoted earlier, that our knowledge of concepts may be incomplete or indistinct. He gives the example of the paradoxes as evidence of this. But even where there are no danger signals like the paradoxes, as in the case of the concept of natural number, we need not have a way to discover all that is implied by the concept.

Gödel himself discusses the concept of natural number (he uses the term "integer") right after the Gibbs passage about the concept of set.

> Of course, this particular argument is addressed only to mathematicians who admit the general concept of set in mathematics proper. For finitists, however, literally the same argument could be alleged for the concept of integer and the axiom of complete induction. For, if the general concept of set is *not* admitted in mathematics proper, the axiom of complete induction must be assumed as an axiom.[14]

Finitists or not, we might think of the induction axiom, plus axioms about order or about the successor operation, as expressing the meaning of "natural number." These axioms say nothing at all about what it is to *be* a natural number. Furthermore they imply statements, e.g., that there are infinitely many numbers, that could not be implied by a straightforward-sense concept of number. Hence they do not seem to characterize the concept of being a number any more than the axioms of set theory seem to characterize the concept of being a set. What they do perhaps characterize is the concept of a number structure or the concept of a straightforward-sense concept of number. Though nothing like the paradoxes makes us think our understanding of this concept is defective, there is also no reason to think that all number-theoretic truths are discoverable by us. Moreover the fact that the truths of first-order number theory do not form a recursive set shows that number-theoretic truth is not "tautologous" in the most obvious way it might have been.

Before turning to what I have been ignoring—the footnote to the Russell passage and the corresponding portion of the Gibbs passage—I will discuss whether we should indeed take Gödel as saying, e.g., that the comprehension axioms fully express the concept of set of natural number. One worry is that, partly because he talks of the axiom "schema," he may have first-order axioms, not second-order ones, in mind. He definitely does not believe that the first-order axioms fully express the concept. My opinion, which I won't argue for, is that it is second-order comprehension, or something akin to it, that he has in mind, at least when he says that the axioms express the concept. I think he means by "well-defined property" something like what Zermelo means by "definite property." A second worry (about whether Gödel thinks that comprehension axioms fully express the concept) is his inclusion, in the Russell paper, of the Axiom of Choice among the axioms said to express the concept of class. Since he does not include Choice in the case where concepts replace classes, it would seem that he doesn't rule out there being classes (sets) which are not the extensions of any concepts. If his "well-defined property" means "concept" in this narrow sense, then it is doubtful that

[14] Gödel [7], p. 321.

the comprehension axioms fully characterize the concept of set of numbers. I'm not sure what to think about this, but I will take him as construing "property" in a broad enough sense that for any particular set of integers there is a property of belonging to the set. Another worry comes from the Continuum Hypothesis papers. There Gödel says that the concept of set of x's cannot be characterized (at present) except in terms of concepts such as "multitude of x's." But surely the possibility of expressing the concept of set of x's via axioms involving the concept of well-defined property doesn't give a counterexample. Still another worry is that the full expression of the concept of set of natural numbers would seem to require adding to the comprehension axioms other axioms, in particular, the Axiom of Extensionality. When I talk of "the axioms" for sets of natural numbers, I will assume that such axioms are included along with second-order comprehension. A final worry is that Gödel might think that full expression of the concept of set might require specifying *what* objects particular sets are, e.g., what objects {Gödel, Russell} and the set of all even integers are. I doubt that this is Gödel's view, though it might be. I strongly doubt that there *is* any way of specifying what objects such particular sets are. But if I am wrong—and my opinion might be thought eccentric or worse—nothing important to the present paper will be affected.

An aside: It is often said that the Axiom of Extensionality, unlike other axioms of set theory, is an axiom about the concept of set proper (i.e., about what it is to be a set). This seems to me false. Extensionality says that a set is determined by its members only in saying that no other set has the same members. It says something about what the set is only within a set-structure. E.g., it says nothing about *what* object {Gödel, Russell} the objects Gödel and Russell determine.

It is time to stop ignoring the two passages I have been deliberately ignoring. Here they are again.

> This view does not contradict the opinion defended above that mathematics is based on axioms with real content, because the very existence of the concept of e.g., "class" constitutes already such an axiom; since, if one defined e.g., "class" and "$\in$" to be "the concepts satisfying the axioms", one would be unable to prove their existence.[15]

> ... however, the term "tautological", that is, devoid of content, for them is entirely out of place, because even the assertion of the existence of a concept of set satisfying these axioms (or of the consistency of these axioms) is so far from being empty that it

[15]Gödel [4], p. 139, footnote 47.

> cannot be proved without again using the concept of set itself, or some other abstract concept of [a] similar nature.[16]

In these passages, "concept of set" is not being used in my sense. It is seemingly being used in what I have been calling the "straightforward sense." The objects that instantiate such a concept are sets, not concepts or universes of sets. To put it another way, we could say that Gödel is talking about concepts that are instances of the concept of set in my sense.

The two passages are a bit different. In the Gibbs passage, the concept of set of integers is being considered in the context of the general concept of set, whereas in the Russell passage there is no reference to a wider concept of class than the type-theory classes of ***Principia***. A second difference is that the Russell passage implies uniqueness of the concept of class, because of the word "the" in "the concepts satisfying the axioms."

Let us first consider uniqueness, postponing all discussion of existence. We might describe this aspect of Gödel's view as that there is at most one instance of the concept of set in my sense. The passage above suggests this in the special case of the classes of ***Principia***. That he holds the view for the general iterative concept of set seems to be implied by the following passage from the second version of the Continuum Hypothesis paper.

> For if the meanings of the primitive terms of set theory as explained on page 262 and in footnote 14 are accepted as sound, it follows that the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false.[17]

The explanation to which he is referring is the one whose central part I quoted on page 208. The footnote 14 referred to contains the discussion, which I mentioned on page 215, of the concept of "set of x's."

Obviously uniqueness of instantiation cannot be literally true for a concept fully expressed by axioms. Given a concept satisfying the axioms, we can easily get one that is different, and not just intensionally different. Given one concept of set (or set of integers or whatever) in the straightforward sense, we can easily get another concept that has an isomorphic, but different, extension.

Perhaps Gödel does, after all, think that there is some way to distinguish the *genuine* set with such-and-such members from all pretenders. More likely, I believe, is that Gödel is only concerned with identity up to isomorphism. If so then the important question is: Can one rule out non-isomorphic instances of the concept (in my sense) of set of x's? Can one rule out non-isomorphic instances of the general iterative concept of set?

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[16] Gödel [7], pp. 321–322.
[17] Gödel [6], p. 260.

As I have said elsewhere (Martin [8]), I think that the answer to these questions is yes. Zermelo proved that any two models of second-order ZFC with the same number of urelements are isomorphic unless one is isomorphic to an initial part of the cumulative hierarchy of the other. The basic step in his proof shows that different concepts of set of x's have isomorphic structures as extensions, provided that the concepts have to obey second order comprehension—i.e., provided that the concept in my sense of set of x's requires that comprehension be satisfied. I tried to show in [8] that any two instances of the iterative concept of set have to have isomorphic extensions. To see how isomorphism is proved in the "set of x's" case, let us consider the case of "set of integers." Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are two models of Extensionality and the second-order comprehension axiom for sets of integers, and suppose that the two models have the same integers. Suppose, to derive a contradiction, that there is a set X of integers in the sense of $\mathfrak{M}$ such that no set of integers in the sense of $\mathfrak{N}$ has the same members in $\mathfrak{N}$ as X does in $\mathfrak{M}$. Then we get a contradiction by applying the comprehension axiom in $\mathfrak{N}$ to the property of being a member of X in $\mathfrak{M}$.

In summary, I think that Gödel is entitled to hold that the axioms for sets of integers implicitly define—at least, up to isomorphism—a concept of set of integers in the straightforward sense, provided that there exists such a concept. I also think that he is entitled to hold that the general iterative concept of set has, up to isomorphism, at most one instance. What remains at issue is whether there are any instances and whether it matters that there are.

Uniqueness up to isomorphism of instantiation differs in an important way from true uniqueness of instantiation. If the meaning of "set" is supposed to be a straightforward-sense concept of set, then uniqueness up to isomorphism does not supply us with a meaning for "set." Indeed it does not supply us even with an extension for "set." Hence if it is through such uniqueness that Gödel gets the "well-determined reality" he speaks of, then the meaning of "set" is not a straightforward-sense concept of set. True uniqueness, provided it is not just uniqueness of extension, does pick out a straightforward-sense concept.

Next let us turn to the main content of the Russell footnote. I will talk about it in a set-of-integers context instead of the ***Principia*** context, for simplicity and to have a more direct comparison with the Gibbs passage. To make the context analoguous to the ***Principia*** one, we should ignore the fact that there is a wider concept of set than that of set of integers. Gödel then would think that we need an axiom asserting the existence of a (straightforward-sense) concept of set of integers, i.e., the existence of a concept satisfying our axioms about sets of integers. Is this extra axiom supposed to be part of the concept of set of integers, i.e., not involving additional concepts? It is not part of the straightforward-sense concept.

If it is not part of the concept of set of integers, then it must be a truth involving an additional concept. (Remember, we are in the Russell context, so we are not appealing to a wider concept of set than that of set of integers.) Since all mathematical truths are analytic, it must be analytic, true in virtue of the meaning of the terms it contains. This seems very odd. For it to be analytic—or even just to have a meaning—we seem to need a further axiom asserting the existence of the extra concepts that this axiom involves. Thus the mere existence of a concept of set seems to involve an infinite sequence of axioms and concepts.

If the existential axiom Gödel speaks of involves only the concept of set, then it seemingly must be true in virtue of a concept whose very existence it states.

Whether or not there is a coherent account of the role of the existential axiom, it is very strange to say that it is analytic. The alleged necessity of adding the axiom seems to do more to undermine Gödel's assertion that set theory is analytic than it does to support his view that set theory is not tautological. Indeed the need for the axiom has little to do with the real reasons why set theory is not tautological in his sense. These real reasons have to do with the fact that the concept of set is not definable in any simple sense and the fact that set-theoretic truth is not decidable in any natural way. These reasons would obtain if set theory were entirely concerned with the concept of set in my sense and it were irrelevant to set-theoretic truth whether that concept was instantiated.

An example may make this point clearer. Suppose that our only mathematical interest were in things and the identity relation. Suppose we came up with the concept, in my sense, of infinitely many things. An instance of this concept would be a concept in the straightforward sense that applied to infinitely many things. Gödel would seemingly say that we need an "axiom" saying that there is such a concept. Nevertheless, truth (first-order truth) in the theory of our concept would be recursive, so there would be a mechanical method for determining truth or falsity. Hence truth for our concept would seemingly qualify as tautological. (This example can be challenged on several points, but I hope it at least shows that there is no immediate connection between the non-provability of the existence of a concept and the non-tautologicality of its theory.)

Like the Russell passage, the Gibbs passage assumes that we need to know that there is a concept satisfying the axioms expressing the concept of set of integers—that the concept in my sense of set of integers is instantiated. But the latter passage talks of proving the existence of such a concept, not of adding an axiom asserting it. Gödel says that such a proof is possible only by using the concept of set itself. The positive part of what I think he has in mind is something like the following. Using the theory of the general iterative concept of set, or just using the second-order theory of sets of sets of integers, one can prove that there is a model of the second-order theory of sets of integers. But this amounts to the fact that we can show that there is an instance of one fragment of the general concept of set (in my sense) by assuming that there is an instance of a larger fragment. This would not seem much help if we want to know that there is an instance of the general iterative concept of set. When Gödel does want to assert that there is such an instance, as he does in the papers on the Continuum Hypothesis, either he gives no argument or he appeals to the "soundness" of the concept rather than to proof or to an axiom.

The Gibbs passage fares no better than the Russell passage in showing that the truths of set theory are not tautologies. Like the other passage, it does more to make one doubt analyticity than to make one merely doubt tautologicality.

Whether or not the two passages make the point Gödel thinks they do, they certainly provide evidence as to what Gödel takes the concept of set to be, though the evidence is not easy to interpret.

One try at an interpretation is the following. The concept of set in my sense implicitly defines a concept of set in the straightforward sense, because—up to isomorphism, at least—there is exactly one instance of the concept of set in my sense. Why Gödel might believe the "at least one" half of "exactly one" I will discuss below. When he says "concept of set," he means the unique instance of the concept of set in my sense. The word "set" means something like "element of the structure that is the unique instance of the concept of set in my sense." Thus set-theoretic truths are truths in virtue of the meaning of "set" (and the meaning of "$\in$"). If this interpretation of Gödel is right, then there are two problems with his view. (1) As I noted earlier, if uniqueness of instantiation holds only up to isomorphism or even if it holds only extensionally, then we haven't really gotten hold of any particular straightforward-sense concept. This problem disappears if there is a way of determining exactly what object the set with any given members is. (2) The claim of truth in virtue of meaning seems fraudulent.

Whatever the right interpretation of Gödel's notions of concept of set and analyticity, it seems right to say that for Gödel the concept of set is built out of a combination of the concept of set in my sense and the instantiation of that concept. It might not be too inaccurate to say that Gödel's concept of set is my concept plus instantiation.

Does Gödel have a reason for believing that the concept of set in my sense is instantiated, or even a reason for thinking that it is important for set theory that it be instantiated? He gave no grounds for believing the existential axiom discussed in the Russell passage. Existence proofs, as described in the Gibbs passage, are ultimately circular. Moreover they do not apply to the full iterative concept. Gödel says, in a passage I have quoted, that the iterative concept must be instantiated if it is sound. (I think it is fair to construe him as talking of the iterative concept in my sense here.) "Sound" might mean "coherent" or even just "consistent," and perhaps he thinks we have good evidence that the iterative concept is sound. Another possible reason for believing in instantiation—a reason Gödel definitely has—is his statement, in the Russell paper, that classes and concepts are "in the same sense necessary to obtain a satisfactory systems of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions . . . ."[18] If he is right, then this provides a reason for believing in instantiation and, even more so, a reason for thinking that instantiation is important.

For myself, I have doubts both about the fact of instantiation and about whether it is important for mathematics. I don't see why the consistency or coherence or "soundness" of the concept of set implies that it is instantiated. Of course, if we knew that it was instantiated then we would know that it is consistent. But I see no reason why it couldn't be consistent and coherent without being instantiated. Suppose the nominalists are right and there are no abstract objects, and suppose there are only finitely many concrete ones. Would this mean that the concept of natural number (in the sense of the concept of a number sequence) was inconsistent or incoherent? Wouldn't it still make sense to consider what a number sequence would be like if there were one? Wouldn't it make sense to ask what is implied by the concept of number? *Applied mathematics* is concerned with instantiations of mathematical concepts. But why are instantiations needed for pure mathematics? One alleged reason, to which I give some weight but not a lot, is that mathematical statements, e.g., "There is an infinite set," cannot be taken at face value if there is no instantiation.

Of course, instantiations might have *epistemic* importance if we had an epistemic connection to the objects that make up an instance of a concept like that of set or number. And Gödel seems to say that we do have such a connection, via a kind of perception of the objects. The place, if there is one, where Gödel would make use of such a connection would have to be the 1947 paper, "What is Cantor's Continuum Problem," and its revised and expanded 1964 version. In these papers, Gödel is concerned with showing that the Continuum Hypothesis has a truth-value—or, at least, may have one—despite the fact that, as he predicted, it is independent of the first-order ZFC axioms for set theory. Moreover he is concerned with how we could learn what the truth-value is, specifically, how we could discover new axioms that imply the truth or falsity of the Continuum Hypothesis.

I begin the discussion of these papers by suggesting that there will be no serious misinterpretation—and probably no misinterpretation at all—if one takes Gödel to be using "concept of set" in my sense throughout these papers. If one considered in isolation Gödel's explanation of the iterative concept (part of which I quoted on page 208), then one might think he was

[18]Gödel [4], p. 128.

describing a straightforward-sense concept of set. He says that a set "is anything obtainable from the integers (or some other well-defined objects) by the iterated application of the operation 'set of.'" It is not obvious from this description that the "set of" operation need be always defined or even that it need ever be defined. Gödel does say in a footnote that transfinite iteration is included, but one might suppose that this means only that it is allowed. Moreover I was not quoting Gödel when I said that the iteration has an "absolute infinity" of stages. Nevertheless Gödel makes it abundantly clear that he thinks that the concept yields not only the ZFC axioms but also certain large cardinal axioms. Hence the concept in these papers is at least as much a concept in my sense as in the Russell and Gibbs papers. In those papers, instantiation of the concept in my sense seemed to be part (or all) of the concept in Gödel's sense. Here this is less clear. The quotation on page 216 above seems to take instantiation of the concept not as part of the concept but as a consequence of the soundness of the concept.

In the 1947 version, Gödel does not really argue that the Continuum Hypothesis has a truth-value. He simply says that someone who believes the concepts and axioms of set theory to "describe some well-determined reality"[19] must believe that the Continuum Hypothesis has a truth-value. In the 1964 version, Gödel gives one or possibly two reasons for holding that the Continuum Hypothesis must have a truth-value.

The first reason is the one already discussed, his contention that the soundness of the iterative concept implies a unique (or unique enough) instantiation, in which the Continuum Hypothesis is either true or false. I have already explained that I accept uniqueness but am unconvinced about existence. Indeed, I am unsure about existence *because* I am sure about uniqueness. I am convinced that we do not now know whether or not the Continuum Hypothesis has a truth-value. If the concept of set is instantiated, then uniqueness implies that all instances give the same truth-value. Hence if we are sure of both instantiation and uniqueness, then we seem to be sure that there is a truth-value. For me, this is a *reductio*.

Here is what may be another reason why Gödel is confident that the Continuum Hypothesis has a truth-value.

> However, the question of the objective existence of the objects of mathematical intuition (which, incidentally, is an exact replica of the question of the objective existence of the outer world) is not decisive for the problem under discussion here. The mere psychological fact of the existence of an intuition which is sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the

[19] Gödel [5], p. 181.

> truth or falsity of propositions like Cantor's continuum hypothesis.[20]

This passage—from the 1964 version of the paper—is one of the reasons I prefer the 1947 version. The natural reading of the passage makes it say that the fact Gödel cites is sufficient to guarantee that the Continuum Hypotheis has a truth-value. It is very hard to see why it is sufficient. Of course, Gödel may not intend that the meaningfulness of "the question of the truth or falsity" of the Continuum Hypothesis to imply it has a truth-value. Parsons [9] interprets the passage so that there is no such implication.

It would seem that everything Gödel says the intuition yields could also be said to be yielded by an understanding of the concept of set. No doubt Gödel would reject anything like an identification of the intuition with understanding the concept, I do not see any epistemic work that the former does that the latter doesn't do just as well. I would say something similar about the passage from the 1964 version which I quoted on page 207. The fact that the ZFC axioms "force themselves upon us as being true" can be explained without an appeal to perception. It is explained by the fact that the concept of set requires the truth of the axioms and the fact that we understand the concept well enough to see this implication.[21]

If the Continuum Hypothesis does in fact have a truth value, then Gödel has a good deal to say about how we might discover that truth-value and how we have found out the truth-values of other statements that are not decidable on the basis of the ZFC axioms. The great majority of what he has to say concerns the iterative concept of set (in my sense), and none of it concerns in any obvious way anything like perception of sets.

Two kinds of ZFC-undecidable statements whose truth Gödel says we have discovered are (1) sentences given by the proof of the incompleteness theorem and (2) certain large cardinal axioms. Here is what he says about these large cardinal axioms.

> For first of all the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the very concept of set on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation of "set of".[22]

The large cardinal axioms in question, for example those asserting the existence of inaccessible and Mahlo cardinals, are alleged to be "implied by the concept of set." He knows of much stronger large cardinal axioms, though he is says that their acceptance isn't yet justified:

[20]Gödel [6], p. 268. For a more serious discussion of this passage and of Gödel's notion of intuition, see Parsons [9] (pp. 65–68 for discussion of the passage).

[21]As Parsons [9] asserts, the perception Gödel is appealing to in this passage is surely perception of the concept of set, not perception of sets.

[22]Gödel [5], p. 181.

> That these axioms [e.g., the axiom that measurable cardinals exist] are implied by the general concept of set in the same sense as Mahlo's has not been made clear yet.[23]

Elsewhere he speculates about the possibility of our discovering new axioms of a different kind.

> Probably there are other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts.[24]

None of this depends on there being an instance of the concept of set. It is all concerned with understanding the concept of set, and perhaps understanding other concepts, e.g., those of logic. Nor is there any suggestion of using our alleged perception of mathematical objects, unless understanding the concept of set counts as perceiving it.

Gödel gives one possible method for deciding the Continuum Hypothesis that does not involve understanding the concept of set. One might, he says, come to a "probable" decision about the Continuum Hypothesis by discovering a new axiom that decides it and which might have to be accepted as true because of its "success," i.e., because of its consequences, in the same way a well-established physical theory is accepted. It is not obvious that the validity of this method depends on whether or not the concept of set is instantiated. It surely doesn't depend for its validity or its feasibility on our being able to perceive sets.

I will finish by summarizing what I have tried to show. Gödel has an account of mathematics as being concerned with relations of concepts, and he has a view that mathematical truths are true in virtue of the concepts they are about. I have tried to construe the concepts in question not as concepts of sets, numbers, etc. in the straightforward sense, but as concepts of set structures, number structures, etc. (or of concepts that determine such structures). I have tried to construe the mathematical truths as those mathematical propositions that would have to be true of any instances of the concepts in my sense. This way of interpreting Gödel accords with—and makes comprehensible—much of what Gödel says about mathematical concepts. However, it does not fit with everything he says about the concept of set in particular. Though his concept of set involves a concept of set in my sense, it is essential to the concept in his sense that the concept of set in my sense be instantiated. I am unsure about just what the concept of set in his sense is. But I have suggested that—whatever his concept is—it does not fit well with his account of mathematical truth, especially with his contention that mathematical truths are analytic. In the last part of the paper I have

[23] Gödel [6], p. 260.
[24] Gödel [5], p. 182.

considered Gödel's papers about the Continuum Hypothesis, where he may fairly be taken as using "concept of set" in my sense. I have also tried to show that Gödel, despite his assertion that we can perceive set-theoretic objects, nevertheless seems to assign no explicit role to such perception in his discussion of how we discover set-theoretic axioms. The one exception might be perception of the concept of set, but it is hard to see what such perception is supposed to yield that is not yielded by garden-variety understanding of the concept.

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