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FUTURE TASKS FOR GÖDEL SCHOLARS

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Abstract. As initially envisioned, Gödel's *Collected Works* were to include transcriptions of material from his mathematical workbooks. In the end that material, as well as some other manuscript items from Gödel's *Nachlass*, had to be left out. This note describes some of the unpublished items in the *Nachlass* that are likely to attract the notice of scholars and surveys the extent of shorthand transcription efforts undertaken hitherto. Some examples of sources outside Gödel's *Nachlass* that may be of interest to Gödel scholars are also indicated.

At the time the Gödel Editorial Project began in the summer of 1982 the cataloguing of Gödel's *Nachlass* had only just begun. Nevertheless, despite limited knowledge of the extent of that collection, enthusiasm and expectations were high: Volume I of Gödel's *Collected Works*, which appeared four years later, confidently declared that "Succeeding volumes are to contain Gödel's ... lecture notes, as well as extracts from his scientific notebooks."

In the end, volume III of those *Works* [5], devoted to previously unpublished essays and lectures, contained the texts of a number of individual lectures that Gödel gave on various occasions, as well as those of several manuscripts that he left in relatively finished form, including some items transcribed from his Gabelsberger shorthand. But none of the five volumes ultimately published contain any of the notes that Gödel prepared for his lecture courses at the University of Vienna or at Notre Dame, nor any extracts from the three series of notebooks he compiled on mathematics and philosophy. In particular, to the great disappointment of the editors and other scholars, details of Gödel's reputed proof of the independence of the axiom of choice in type theory are conspicuously absent.

There were several reasons for the editors' inability fully to realize their intentions: gross underestimation of the time required to do the editorial work; difficulties in reconstructing some of the texts; changes in personnel (in particular, the tragic loss of Jean van Heijenoort on the eve of the appearance of volume I); and eventually, after twenty years of effort, exhaustion of sources of funding. The present note discusses various items that were considered for inclusion in the *Collected Works* but remain unpublished. Its

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aim is to explain why, in the end, they were left out and to encourage younger scholars to take up where we left off. Some of the items, it appears to us, are too fragmentary to deserve further attention. Others, however, are likely to reward future study.

§1. Unpublished items in Gödel's *Nachlass* of potential scholarly interest. Volume V of Gödel's *Collected Works* contains a detailed outline of the contents of his *Nachlass*, housed at the Firestone Library of Princeton University. That inventory includes cross-references to a set of preservation microfilms of the *Nachlass* that was prepared under a grant to the Gödel Editorial Project from the Alfred P. Sloan Foundation.[1] The items described below are cited by archival item, box and folder numbers, together with the number(s) of the commercially available microfilm reel(s) on which the items appear.[2] They fall into several categories:

I. Topical notebooks. The three series of shorthand notebooks mentioned above are likely to be of the greatest interest to mathematicians and philosophers. They are the *Arbeitshefte* (workbooks), comprising 16 handwritten volumes together with a partial index thereof (items 030016–030034, box 5c, folders 12–28; reels 2 and 3); *Resultate Grundlagen* (results on foundations), four handwritten volumes plus subject index (items 030115–030119, box 6c, folders 82–86; reel 7); and a series designated *Max/Phil* (philosophical notebooks), of whose 15 volumes one (volume XIII) is missing (items 030086–030100, box 6b, folders 63–72; reels 5 and 6). In addition, a volume entitled *Protokoll* ("minutes"), also in shorthand, but with German and English headings, contains some interesting notes on conversations with and lectures by others (item 030114, box 6c, folder 81; reel 7).

The *Arbeitshefte* are much like a physicist's laboratory notebooks. They record false starts, the working out of ideas, and approaches that might be pursued, and so have the potential to shed considerable light on Gödel's ways of thinking and working. Altogether the *Arbeitshefte* comprise 1212 pages, of which 750 have been transcribed in some form. Their topics are in no particular order, but Gödel himself prepared a partial index to the volumes (translated below as Appendix A). In addition, the editors compiled a list of the section headings found within those volumes (reproduced here as Appendix B). Both were used as guides for judging which sections appeared to be of greatest importance for transcription. Transcripts of those sections were then prepared and circulated among the editors and some outside experts in an effort to determine what material might be suitable for publication. The resulting transcriptions are presently in various stages: some have been

[1]A project carried out in the face of substantial initial opposition from library officials then in charge of the collection.

[2]Copies of individual microfilm reels, excluding Gödel's correspondence, are available for purchase from IDC Publishers, Leiden. For online ordering information see `http://www.ide.nl/catalog/index.php?c=375`.

typed and vetted, some are pencil drafts that will probably require much revision, and some are accompanied by draft translations.

The *Resultate Grundlagen*, as the title suggests, are statements of foundational results, written up more formally than material in the *Arbeitshefte*. The four notebooks contain a total of 391 pages, all of which have been roughly transcribed in some form. They appear to consist of Gödel's own results, in contrast to another set of six notebooks, entitled *Logik und Grundlagen* ("Logic and Foundations"), in which Gödel primarily recorded results obtained by others. A list of the headings found in the *Resultate* notebooks is given in Appendix C, and an index to the topics in the *Logik und Grundlagen* series in Appendix D.

The *Max/Phil* series of philosophical notebooks proved more difficult to survey. Of their 1576 pages, 715 have been transcribed in some form, and a number of short excerpts relating to Gödel's "Ontological Proof" were published in volume III of the *Collected Works*. Some idea of the contents of those notebooks may be gleaned from the many names that appear in longhand. (It is a principle of the Gabelsberger system that proper names should be rendered in longhand. Gödel, however, did not always observe that rule.) A list of those names and where they occur was prepared, which may prove useful to future scholars.

II. Notes for seminars and lecture courses. During the years 1931–32, Gödel, then a *Privatdozent*, was responsible for choosing the topics and preparing the students for Hans Hahn's logic seminar at the University of Vienna. Between 26 October 1931 and 4 July 1932 the seminar met in twenty-two weekly sessions, seven of them featuring talks by Gödel himself. Typescript notes were prepared, copies of which are preserved both in Gödel's *Nachlass* (items 004030 and 004031, box 7b, folder 16; reel 10) and among Rudolf Carnap's papers at the University of Pittsburgh. The notes are legible and coherent. Most of the topics, however, such as Gödel's completeness theorem, are familiar and available in other presentations. For that reason it was decided not to publish those notes, or even just the parts constituting Gödel's own contributions. Among the latter, however, are discussions of Herbrand's thesis (session 19, 13 June 1932) and Herbrand's consistency proof for a fragment of arithmetic (sessions 21 and 22, 27 June 1932 and 4 July 1932). They do not duplicate Gödel's other writings and may well be of interest to students of Herbrand's work.

During his years as *Privatdozent* Gödel also gave three lecture courses at the University of Vienna, entitled Foundations of Arithmetic (spring 1933), Topics in Mathematical Logic (summer 1935), and Axiomatic Set Theory (May–June 1937). Notes for the first of those occupy part of a spiral-bound notebook (item 40104, box 7b, folder 25; reel 10), written partly in German longhand and partly in Gabelsberger shorthand, that consists mostly of mathematical symbols. There is no connective text, and the content appears to consist merely of derivations of some elementary theorems of arithmetic.

The notes for the 1935 lectures (item 040126, box 7b, folder 31; reel 10) are in Gabelsberger shorthand. Topics include truth tables, predicate logic, Skolem normal form, the Skolem-Löwenheim theorem, the decision problem, and set theory. They are more extensive than the notes from the 1933 lectures, but still appear to be sketchy, intended for Gödel's own use. For that reason, as well as the familiar nature of the material, the editors decided against transcribing them.

The lectures on axiomatic set theory, also in shorthand, occupy a 58-page notebook (item 040140, box 7b, folder 36; reel 10), the first 17 pages of which have been transcribed. It was during June of 1937 that Gödel carried out the crucial step in his proof of the consistency of the generalized continuum hypothesis relative to Zermelo-Fraenkel set theory, so it was thought that those notes might be of special interest. They, too, however, are very sketchy, with little in the way of connective commentary. There appears to be no mention in them of the consistency results, but further study may be warranted.

Gödel lectured on the consistency results both at the Institute for Advanced Study in the fall of 1938 and at Notre Dame the following semester. Notes of the former occupy seven notebooks (items 040150–040154, box 7c, folders 39–43; reel 11), and those of the latter another five (items 040194–040198, boxes 7c and 8a, folders 52–56; reels 11 and 12). Written in English, they are detailed and coherent. They were not included in volume III of the *Collected Works* because of their overlap with other published treatments by Gödel of the same material.

At Notre Dame Gödel also lectured on logic in a joint seminar with Karl Menger. Notes on those lectures are preserved in eight notebooks (items 040209–040218, box 8a, folders 58–65; reel 12), written in English except for a few passages (notably, the examination questions!) in Gabelsberger. The lectures constitute a basic introduction to logic, presented in Gödel's usual clear style. Volume I, actually the second of the notebooks, appears to be a rewritten, somewhat condensed version of the first (numbered 0). There are some gaps in the page numbering and some breaks in the continuity of the text, but no loose pages. It and the next two volumes are devoted to the development of the propositional calculus, including the completeness proof and a demonstration of the independence of the axioms employed. Volume IV begins with some remarks about Gentzen's sequent calculus (several pages of which are missing) and goes on to introduce the predicate calculus. Volume V develops that calculus further; completeness is stated but not proved, as is the negative solution for the decision problem for validity. Volume VI introduces the calculus of classes and relations and mentions type distinctions and Russell's paradox. The final volume presents Gödel's analysis of that antinomy and briefly mentions the liar paradox. Although the material is standard, the choice and ordering of topics, as well as some of the examples that are discussed, may well be of pedagogical interest.

Gödel also delivered two series of lectures at the Institute for Advanced Study following his emigration in 1940. In April of that year, not long after his arrival there, he lectured on the constructible sets, notes for which fill two notebooks in his *Nachlass* (items 040239 and 040242, box 8a, folders 70 and 71; reel 12); their contents are too similar to his other presentations on that topic to warrant publication. The following year, also in April, he lectured at the I.A.S. on intuitionism. Notes on that topic take up 117 notebook pages (items 040407–040409, box 8c, folders 121–123; reel 15), handwritten in English except for a few shorthand annotations (now transcribed). The lectures form a self-contained introduction to intuitionism that, while not polished, is generally coherent and readable. It breaks off, however, at the end. Gödel begins by saying that in contrast to the intuitionists themselves, who take the meaning of the intuitionistic logical notions as primitive, he believes they "can be defined in terms of simpler and much clearer ones", and declares that "the chief purpose of these lectures [is] to give such a definition and consequent proof of the intuitionistic axioms". Some references to these lectures are to be found in the introductory note in volume III of Gödel's *Collected Works* to his lecture at Yale on intuitionism, which was given at nearly the same time. The I.A.S. lectures were not published because of their incomplete nature and their overlap with that Yale talk. Scholars of intuitionism, however, will likely wish to examine them in more detail.

III. Other shorthand items. Almost all of the shorthand material found in Gödel's Nachlass was written for his own personal use, rather than for publication or for sharing with others. In addition to the notebooks and notes for lecture courses described above, there are several other categories of such material, to wit:

(1) Incidental jottings, such as short comments written on letters, envelopes or manuscripts; notes written in books; lists of various items; and so on. Where they were found to be relevant to items published in the *Collected Works*, such items were transcribed. Otherwise, they are likely to be of interest mainly to scholars studying the particular papers on which they are found.

(2) Drafts of various papers and individual lectures, such as the *Vortrag bei Zilsel*; "The modern development of the foundations of mathematics in the light of philosophy"; and Gödel's lecture at Göttingen on Cantor's continuum hypothesis, all of which were included in volume III of the *Collected Works*. Other items are less coherent or less suitable for publication. A prime example of the latter is a notebook (item 040019, box 71, folder 10; reel 9) containing Gödel's draft for a proposed joint publication with

Arend Heyting. (For further discussion of that endeavor see *Dawson 1997*, pp. 83–85, or Charles Parsons' introductory note to the Heyting correspondence, pp. 27–33 in volume V of the *Collected Works*.) The fragmentary nature of that notebook reveals how little progress Gödel made toward an organized presentation of his part of the projected volume, and after some initial attempts at reconstruction the editorial committee decided that the chances of making sense of it were too small to warrant the effort.

(3) Sheaves of loose bibliographic notes on the writings of particular individuals, such as Leibniz, Husserl, or Wronski. Evidently these will be of interest primarily to those studying the particular person to whom the notes refer.

(4) Other loose manuscript notes. Of particular interest here are two sets of pages, one (item 060001, box 11b, folder 1; reel 27) entitled "American constitution and government" and the other (item 060103, box 11b, folder 11; reel 28) "Finitismus; Cohen'sche Methode 1942" (evidently so labeled after 1963). The first has been of interest to legal scholars and others intrigued by the well-known anecdote about Gödel's attempt at his citizenship hearing to explain to the judge how the U.S. Constitution might permit the establishment of a dictatorship. In response to several queries, we have transcribed portions of those notes, but so far we have found nothing relevant to the citizenship incident. The material appears rather to be just the sort of notes one would expect a candidate for citizenship to compile in preparation for an examination on the Constitution and the structure of the U.S. government.

The second was of immediate interest to set theorists, since, together with some passages from *Arbeitshefte* 14–15, it appears to contain Gödel's notes on his reputed proofs of the independence of the axioms of constructibility and choice in the framework of type theory. Early on it was transcribed in full, and as noted in Feferman's article "The Gödel Editorial Project: A synopsis", the transcript was sent for examination to several eminent set theorists, including Robert Solovay and A. D. (Tony) Martin. But none of those who have studied the notes have been able to develop a clear picture of what Gödel was thinking. Again, the notes are quite sketchy, having been made for Gödel's own use; and in later correspondence with Wolfgang Rautenberg (reproduced on pp. 182–183 of volume V of the *Collected Works*) Gödel admitted that on the basis of his "very incomplete memoranda from that time" he himself could "without difficulty" reconstruct only one of the "partial results" he had obtained during the summer of 1942. (See Charles Parsons' introductory note to the *Rautenberg* correspondence for further reflections on Gödel's claims.) Given that admission, and in light of the failure of the experts we consulted to make much headway in interpreting Gödel's notes, the editors regretfully concluded that that material was not suitable for publication. We will be delighted if future scholars are able to penetrate further.

Other archival sources of potential interest to Gödel scholars. Already, scholars have unearthed some interesting documents relating to Gödel in collections outside his own *Nachlass*. Paolo Mancosu, for example, in his papers [8] and [9], mined the Carnap archives at the University of Pittsburgh, the Wissenschafthistorisches Archiv at the E.T.H. in Zürich, the Felix Kauffmann and Hans Reichenbach archives at Konstanz, and the Heinrich Behmann archive at Erlangen. And Maria-Elena Schimanovich-Galidescu, in her [10], has drawn upon holdings in the archives of the Vienna Rathaus and the University of Vienna to fill in details of Gödel's Vienna years. In particular, she has compiled a nearly complete list of the courses in which Gödel enrolled at the university.

The diaries of Oskar Morgenstern, held at the Perkins Library of Duke University, were a valuable resource in the writing of Gödel's biography (*Dawson 1997*), and recently the papers of Karl Menger, containing at least one item of correspondence with Gödel not known to the editors of volume IV of the *Collected Works*, have been opened to scholars at that same repository. More recently still, the Firestone Library at Princeton, where Gödel's *Nachlass* is housed, has made the papers of Alonzo Church available to scholars. An on-line finding aid is now available at `http://libweb.princeton.edu/libraries/firestone/rbsc/aids/church`.

To date scholars have not been given access to the papers of Hao Wang, held by Rockefeller University. If and when they become available, they may well be of considerable interest for Gödel studies. Further exploration may perhaps also yield greater insight into Gödel's physical and mental health problems. Whether any medical records of his survive outside the *Nachlass* remains to be determined.

Request. At this time, no decision has been reached regarding the ultimate disposition of the shorthand transcriptions prepared by the Gödel Editorial Project. How best to make them available to interested scholars is a matter on which the editors would welcome readers' suggestions.

Acknowledgments. We thank Aki Kanamori and Solomon Feferman for suggesting improvements to our original draft of this article.

Appendix A: Translated Entries from Gödel's Index to his *Arbeitshefte*.

1. Continuum [hypothesis using] recursive [functions]
   VI, 32; VII, 19–21, 28–42, 47–49.2, 52–54; VIII, 5–12

1a. Corrections to Herbrand
   IV, 29; V, 22, 35–53 (Thm 4: p. 47)

2. Consistency of the axiom of replacement
   VI, 59

3. Attempt [to prove] the consistency of analysis
   VII, 1–4, 10, 49.3

4. Ordinals in analysis
   VII, 5–12

4′. First ordinal number not constructible in an[alysis]
   VIII, 16–25

4″. Finite recursions for ordinals
   VIII, 13–15; X, 38

5. Consistency [of] Quine['s system]
   VII, 12, $-8--9$

6. Schönfinkel
   VII, 13–14

7. Intuitionistic many-valued logic and modal logic
   VII, 15–18, $-1--6$, -11

8. Completeness axiom for the calculus of relations (i.e., variable-free functional calculus)
   VII, 21–22, 24–28

9. Rosser's consistency proof [for] Church['s system]
   VII, 43–47, 49–52, -12, $-16--18.4$

10. Interpretation of intuitionistic logic
    VII, $-12--15$ (heur.), $-18--37$; IX, 5–15, 41–63, 74–81; X, 5–8; XII, 36–46
    Intuitionistic logic of premises
    IX, 64–72, 82–84
    Equality rule
    IX, 73–74

11. Consistency of analysis
    VII, -32.3; XI, 34–37; XIII, 56–60

12. Reduction of proofs [by induction] on ε_0 [to] definitions [by recursion] on ε_0
    VIII, 22 (impossible for usual); IX, 16–17

13. Connection between ε_0 and the system of finite tuples
    VIII, 26–32; IX, 25–41; X, 9–25, 30–32, 36–41

14. Definition and proof by induction for schemes of ramification (partial orderings)
    IX, 18

15. Consistency of $\neg(p)\ (p \vee \neg p)$
    IX, 85–88; X, 44–55; XI, 1–27

16. Proof of the axiom of extensionality for Σ
    X, 33–46, 42–43

17. Lemma for Gentzen's consistency proof with choice sequences
    XI, 28–33, 55–56

18. Uniqueness of the quaternion field
    XI, 37–41

19. Metrics in unbounded function spaces
    XI, 41, 46; XII, 26–28

20. Computable functions of higher type
    XI, 43–46; XII, 28–35, 49–59, 65–74: syst.; XIII, 61–96; XIV, 1–12
21. Souslin's problem
    XI, 47–54
22. Independence of the continuum hypothesis according to Brouwer's method
    XI, 56–57; XII, 22–26, 46–49, 60–62, 62–65 (Extensionality Axiom), 75–99; XIII, 1–6, 9–29, 31–38 (Functions of class K), 40–42
23. Germs of orderings
    XI, 58; XII, 13–19
24. Brouwer's [work in] the [Proceedings of?] the Academy of Sciences, Amsterdam
    XII, 3–12
25. [Crossed out]
26. Calculation with "pieces"
    XII, 51–58
27. Analytically measurable functions, with an application to the proof program for the independence of the continuum hypothesis
    XIII, 7–9
28. Extendability of elements in schemes of ramification
    XIII, 42–44
29. Regular sequences of ordinal numbers
    XIII, 44–46; XVI, 44–48
30. Extension of measures
    XIII, 46–50
31. Existence of undecidable arithmetic propositions for the system of all ordinal logics (for ordinal numbers of the second class that are constructible (in the strict sense))
    XIII, 51
32. Independence of the axiom of choice
    XIV, 14–34, 92–93
    For Zermelo set theory
    XIV, 34–43; Blue Hill: XV, 26–57
    Systems of representation
    XV, 57–96; XVI, 1–8
    Consistency of the existence of lawless sets
    XVI, 38–40; perhaps Max IV–VI?
33. Brouwerian non-Archimedean number systems
    XIV, 21–23
34. Refutation of the Fregean solution of the antinomies
    XIV, 44–48
    Refutation of other similar solutions
    XIV, 48–49

35. Attempt to resolve the antinomies by means of a notion of freedom from circularity
    XIV, 55–57
36. Proof that every primitive recursive function in the system of arithmetic (with $+$ and $\cdot$ as primitive notions) is representable and computable
    XIV, 50–54
37. Attempt at a consistency proof for the ordering axiom (unsuccessful)
    XIV, 58–76
    Another way (namely, with the help of constructible sets)
    XIV, 77–85
    General considerations about realizations of this kind
    XIV, 64–71, 89–90
    Only beginnings
    XIV, 86–91; XV, 1–2
38. The formula $\neg(p)\ (\neg p \vee p)$ can be adjoined consistently to arithmetic (proof by means of undecidable propositions)
    XIV, 91, 93
39. Functional calculus with inclusion of B and its realizations
    XV, 3–7 (something wrong here), 13–19
40. Consistency of the axiom of choice (initiation of the proof)
    XV, 8–11
41. Consistency of the axiom of extensionality
    XV, 11–13
42. Unprovability of consistency on the basis of few assumptions (wrong)
    XV, 19–25
43. Remarks on my work about Russell
    XVI, 9–10
44. Attempt at a consistency proof for ramified type theory (wrong?)
    XVI, 10–23
45. Aristotelean theory of modal inferences
    XVI, 23–26
46. Simpler derivation of Simpson's Rule, Euler's summation formula, and Peano's intermediate-value theorem
    XVI, 26–30
47. Coupled systems in quantum mechanics
    XVI, 31–32
48. Attempt [to formulate] the relativity theory of a classical system (e.g., hydrodynamics) in relation sentences
    XVI, 33–38
49. Metrization of the space of G_δ sets (respectively, of functions of the 2nd class) so that monotone (resp., functional) convergence implies metric convergence (or conversely)
    XVI, 42–44

50. Absolute definability
    XVI, 51–53
51. Principle for constructing axioms of infinity (Mahlo)
    XVI, 54
52. Theta functions
    XVI, 55–58
53. Error theory for large angles
    XVI, 60–71

53′. Bessel's differential equation (solution by quadrature)
    XVI, 72–79

54. Impossibility of an absolute (that is reflexive) proof concept within a finite language, and similar [things]
    XVI, 80–89
55. Comparisons, also memoranda slips among the excerpts

$\Rightarrow$ This index begins only with volume VI

$\rightarrow$ volume IV, p. 26 is about consistency up to the kth step

Back of index page 1: Workbook I—Entire program of my research (approximately 100 questions)

Back of index page 2: [various page references, then] separation between mathematics and philosophy

Back of index page 5: addendum to 1a—Discussion of Herbrand and applicability goes further in volume V, page 79 (in particular Theorem 11 (p. 63), the theorem about: if P and $P \rightarrow Q$ are satisfiable in a certain finite field, then Q is satisfiable in an estimable field)

Appendix B: Translated Headings from Gödel's *Arbeitshefte*. Some of Gödel's workbooks (such as number 3) bear no headings at all; others are divided into many topically labelled sections. The latter are translated here. Negative page numbers refer to material written forward from the back of a volume.

Vol.	Page(s)	Translated Heading
I	inside cover	Continuum (work) [This volume appears to consist almost entirely of questions and ideas concerning the continuum hypothesis. There are no headings *per se*, but a few conspicuous phrases are listed below.]
	11	General basis for W_1 and W_2
	24	Resumé about the consistency of the axiom of choice and the generalized continuum hypothesis
	32	The general existence theorem for recursive definitions (whether ordinal numbers be introduced in von Neumann's or Russell's sense) rests on the axiom of replacement

Vol.	Page(s)	Translated Heading
	38	Results of reflections
	40	Program [of his research]
	41	Skolem and Bernays
	62	Consistency of the generalized continuum hypothesis
	63	Fundamental mapping of pairs of ordinal numbers to ordinal numbers

[In addition, workbook #1 contains 28 pages of algebraic and calculational entries, as well as 10 pages of notes labelled "Conversation with von Neumann, July 17, 1937" dealing mainly with rings and matrix algebra.]

Vol.	Page(s)	Translated Heading
II	Last 15	[Except for the last 15 pages, this volume contains no headings or page numbers.] The measure problem (1–12) Totally imperfect co-analytic sets (12–15) [The rest of the volume continues the work on the continuum hypothesis begun in volume 1.]
III		[Unlabelled. This volume contains only 14 pages.]
IV	1–3	Decision procedure for positive logic
	3–4	Attempt [to prove] the independence of the axiom of choice
	4–5	Directed graphs
	5–13	Representation of recursive functions by means of Diophantine equations with one alternation of quantifier
	14–17	Existence of automorphisms in models
	18	Unprovability of consistency
	19–20	Considerations preliminary to p. 23
	21–22	Different definitions of the concept of "formal system"
	23–28	Proof of consistency up to the Nth step
	29–30	False lemmas in Herbrand['s Thesis]
	30–33	Mostowski (well-ordering [is] independent of [the] ordering [principle])
	34–39	Gentzen
	39–40	Gentzen with choice sequences
	41	Formulation of the general theorem
	42–45	Second special case
	45	Third special case of the theorem
	46–51	Proof of the general theorem

Vol.	Page(s)	Translated Heading
	50.1–.7	Gentzen's consistency [proof]
	53–57	Consistency of the axiom of foundation without the axiom of replacement
V	1–6	Completeness of the system with transfinite rules for the natural numbers
	6–12	Transfinite disjunctive rules of inference
	13–14	Finitely many truth values in Heyting['s system]
	14–18	Herbrand: more on the false lemmas
	18–33	Herbrand's Thesis
	33–35	Attempt to find a counterexample to Herbrand's lemma (i.e., Theorem II)
	35–71	(Proofreading) correction to Herbrand: heuristic consideration
	72–73	Church's [definition of] pairs
	73–78	Herbrand's axioms equivalent with the preceding
	83–93	Infinite ramification schemes
VI	−9 – −5	Continuum hypothesis with recursive functions
	−4	Turing computability equivalent to that of Herbrand
	−3 – −2	Continuum hypothesis with recursive functions
	32–34, 42–45	Continuum hypothesis with recursive functions
	35–41	Insertion on absolute undecidability (draft and final copy)
	46–58	Absoluteness in models (continuum hypothesis)
	59–65	Higher transfinite types and unsuccessful attempt to prove the consistency of the ordering axiom
VII	1–6	Łukasiewicz's many-valued logic
	7–8	Russell's antinomy without negation, with general application of the positive propositional calculus
	8	An intuitionistically absolutely undecidable number-theoretic proposition
	8–10	Quine's theorems 181 and 180
	11	Three-valued logic
	12	Rosser's consistency proof
	12–16	Gentzen
	16–18	Rosser
	18–24	Gentzen
	24–25	Heyting's functional calculus
	25–32.3	Gentzen
	32.3–32.4	Consistency proof for analysis
	32.4–32.5	$\neg\sim A$ is provable if A is classically provable

Vol.	Page(s)	Translated Heading
	32.6–32.11	Attempt at a direct proof of $\neg(p)\ (p \vee -p)$
	33	Extension of the consistency proof to the case that quantifiers appear in the recursive definitions (and thereby to ramified type theory)
	34–37	Gentzen's consistency proof
	$-54--52$	Continuum hypothesis with recursive functions
	$-51--49.3$	Attempt [to derive an] antinomy from the provability of the continuum hypothesis
	$-49.3--49.2$	Axiomatic characterization of the [unreadable character] relation
	$-49.1--49$	Rosser's consistency proof
	$-49--47$	Continuum hypothesis with recursive functions
	$-47--43$	Rosser's consistency proof
	$-42--36.1$	von Neumann's axioms with 2-place functions—proof
	$-35--34$	Proof of von Neumann's axioms
	$-34--28$	Continuum with recursive functions
	$-28--24$	Relation calculus
	$-24--23$	Continuation of continuum [with] recursive [functions]
	$-22--21$	Insertion on the relation calculus
	$-21--19$	Continuum hypothesis with recursive functions
	-19	Rosser's consistency proof
	-18	4-valued logic and Lewis' calculus
	loose page	Interpretation of 3-valued logic in Lewis' caluclus
	$-17--15$	3-valued logic interpreted in Lewis' calculus
	$-15--13$	Schönfinkel's Ann[als?] work
	-12	Consistency [of] Quine['s system]
	-10	Insertion on the consistency of set theory
	$-12--5$	Which ordinal numbers are definable in the system of analysis
	$-4--1$	Failed attempt to find a contradiction in analysis
VIII	1–3	[Missing]
	4	Program [of his work in progress or planned]
	5–12	[Unlabelled]
	13–15	Definability of ordinal numbers and ordinal-number functions
	16–22	The first ordinal not constructible in P_1 (P_2)
	22–23	Reduction of number-theoretic proofs by induction up to ε_0 to inductive definitions up to ε_0
	23–25	The first ordinal undefinable in P_1, according to Veblen's method of continuous functions

Vol.	Page(s)	Translated Heading
	26–30	Connection between the finitary system of finite types and ε_0
	30–32	Program goals
IX		(Intuitionistic mathematics)
	1	[Unlabelled list]
	2–3	Lecture, summer 1941
	4	[Blank]
	5–17	Gentzen's proof (interpretation of intuitionistic logic)
	18–25	Complete induction for schemes of ramification (definition and proof)
	25–41	Connection between ε_0 and the system of finite types
	42–63	Intuitionistic logic
	64–72	Intuitionistic logic of assumptions
	73–74	Equality rule
	74–81	Intuitionistic logic (without the assumption rule [deduction rule?])
	82–85	Intuitionistic logic of assumptions
	85–89	Proof that $A = \neg(f)\,[(x)\,f(x) = 0 \vee (\exists x)\,f(x) \neq 0]$ is consistent
X	1–4	[Missing]
	5–9	Intuitionistic logic
	10–21	[Missing]
	22–25	Connection between ε_0 and the system of finite types
	26–29	Reduction of schemas of inductive definition in Σ to exponentiation, and elimination of impredicative definitions
	30–33	Connection between ε_0 and the system of finite types
	33–36	Proof of the axiom of extensionality for Σ
	36–38	Connection between ε_0 and the system of finite types
	38	Finitary recursion for functions on ordinal numbers
	39–41	Connection between ε_0 etc. by means of finitary functions (schemes of ramification)
	42–43	Proof of the axiom of extensionality in Σ
	44–58	Proof of the consistency of $\neg(p)\,(p \vee \neg p)$

Vol.	Page(s)	Translated Heading
XI	1–24	Consistency of $\neg(p)\ (p \vee \neg p)$
	25–27	[Blank]
	28–33	Principal lemma of Gentzen's consistency proof with choice sequences
	34–37	Consistency of analysis
	37–41	Uniqueness of quaternion fields
	41–42	Metrics in spaces not restricted [to] functions on the whole [real] line
	43–46	Lowest-type computable functions
	46–47	Metrics in spaces not restricted [to] functions on non-comp[act?] sets
	47–54	Souslin's problem
	55–56	Principal lemma, Herbrand-Gentzen consistency proof
	56–57	Constraints on the independence of the continuum hyp[othesis]
	58	Germs of orderings
XII	1	[Page references]
	2	[Blank]
	3–12	Brouwer, Academy of Sciences, Amsterdam [reference to one of Brouwer's articles?]
	13–16	Germs of orderings
	16–19	Proof of lemma on p. 58 of workbook 11
	19–21	Herbrand's axioms for the functional calculus
	22–26	Abstract content of the Herbrand/Gentzen consistency proof
	26–28	Metric [on] continuous functions in [Baire's] null space
	28–35	Computable choice-sequence functions of first type
	36–45	Heyting's and Lewis' propositional calculus
	46	Mathematics and philosophy
	46–49	Brouwer's interpretation of the quantifiers
	49–59	Computable functions
	60–62	Brouwer's interpretation of analysis
	62–65	The axiom of extensionality in Brouwer's interpretation
	65–74	Computable functions
	75–100	Brouwer's interpretation of analysis
XIII	i	[Page references]
	1–6	Brouwer's interpretation of analysis

Vol.	Page(s)	Translated Heading
	7–9	Insertion: measurability of the function which actually guarantees a well-ordering
	9–29	Brouwer's interpretation of analysis
	30–31	Program [of work planned or in progress]
	31–39	Functions of the classes F_σ, G_δ
	40–42	Brouwer's interpretation of analysis
	42–44	Extendability of elements in schemes of ramification
	44–46	Regular sequences of ordinal numbers
	46–51	Extension of measures
	51	Undecidable propositions in systems of ordinals
	52–54	Program [of work planned or in progress]
	54–56	Problems and questions
	56–58	Construct[ive] model of analysis according to my first method
	59–end	Theory of computable functions of arbitrary type
XIV	1–13	[Continuation of workbook 13]
	14–20	Independence of the axiom of choice
	21–23	Insertion: Brouwer
	24–34	Independence of the axiom of choice
	34–43	Zermelo set theory
	44–49	Fregean solution of the antinomies
	50–54	Computable functions of integers
	55–57	Solution of the antinomies without type theory
	58–93	Consistency of the axiom of replacement
XV	1	[Index]
	1–2	[Unlabelled]
	3–7	Existence of models for consistent B-systems (wrong)
	8–11	Consistency of the axiom of choice according to the new method
	11–13	Relative consistency of the axiom of extensionality
	13–16	New axiom system for all B-tautologies
	16–19	Omission of $B(A) \supset A$ and replacement by iteration of the B-symbol (wrong)
	19–25	Generalization of the unprovability of consistency, with few assumptions (wrong)
	26–28	Independence of the axiom of choice
	28–38	General theory of B theories about objects that form a topological space
	38–41	Application to the independence of the axiom of choice

Vol.	Page(s)	Translated Heading
	41–42	Attempt to extend the above to functions (instead of relations)
	42–44	Theory of the B symbol under the assumption that the truth of the propositions considered depends on the wording of the definition, the objects occurring therein, and a "world" given by a point of a topological space
	44–48	The axiom of complete induction and the descriptive operator in this theory
	48–49	The axiom of choice
	49–51	Modification of the earlier definition of $W_t(f(g))$ and of the type-hierarchy
	51	Possible reasonable extension of the B-axioms (insertion)
	52	Which notions of truth (not falsity) are suitable for consistency proofs?
	53–96	Conjecture about a property of the notion of well ordering in the B-theory
XVI	i	[Index]
	1–8	[Unlabelled]
	9–10	On the work [his own essay] about Russell's logic
	10–23	Ramified type theory (attempt at a consistency proof)
	23–26	Aristotelean theory of modal inferences
	26–27	Simpson's Rule
	27–30	Euler's summation formula
	31–32	Quantum mechanics (one-body system)
	33–38	General relativity theory [is] perhaps classically interpretable
	38–41	Independence of the axiom of choice (consistency of the existence of lawless sets)
	42–44	Metrization of the space of G_δ-[sets] (respectively, function of the second class) so that monotone (respectively, functional) convergence implies metric convergence (or conversely)
	44–48	Non-existence of certain "regular" assignments of $\alpha_i \to \alpha$ for each $\alpha < \Omega$
	49	Difference sequences from $n!$, and similar [things]
	50	Problems of abstract set theory
	50–53	Absolute definability
	54	Principle for constructing axioms of infinity (Mahlo)

Vol.	Page(s)	Translated Heading
	55–59	Theta functions
	60–61	Error theory for large angles
	62–71	Correction due to subdivision into only 6 intervals
	72–81	Bessel's differential equation (solution by quadrature)
	80–85	Impossibility of an absolute (i.e., reflexive) proof concept within a finite language, and similar [things]
	86–89	Idea of the proof

Appendix C: Translated Headings in Gödel's "Results on Foundations" Notebooks.

Vol.	Pages	Topic
I	1–35	[Unlabelled lemmas and remarks]
	36–37	Intensional antinomies
	37–38	The following axioms are contradictory
	39	Other axioms from which a contradiction likewise follows
	40–42	Complete axiom system for $\sim$ [and] $\cdot$
	42–45	Another complete axiom system for $\sim$ [and] $\cdot$
	45–end	Some theorems about definability of sets
	insert	Class calculus
II	53–66	[Unlabelled]
	66–153	Positive propositional calculus
III	154–187	Continuation of volume II
	188–191	Every function of the strictly intuitionistic system is computable (January 1, 1941)
	191–193	Theorem about the lengths of proofs
	193–196	Remarks about quaternions
	196–204	[Unlabelled (logic)]
	204–215	Principal theorem of Gentzen's consistency proof
	216–222	Souslin's problem and questions dependent on it
	223–225	Undecidable propositions according to Rosser
	225–241	Equivalence of Heyting's and Lewis' calculi
	242–248	Topology and Lewis' calculus
	249–254	Continuation of p. 241
	254–285	Existence of infinitely many truth values in Heyting's and Lewis' calculi
	258–262	Monotone sequences of F_σ [sets] of type $\geq \omega_1$
	263–267	Decomposition of complete graphs of power $\aleph_0$

Vol.	Pages	Topic
IV	268–270	"Distinguished" enumerations of the ordinal numbers of the second class
	270–276	Decomposition of complete graphs of power $\aleph_1$
	277–283	Maximal well-ordered subsets in ordered sets
	283–287	König's Theorem for arbitrary cardinalities
	287–289	Countable fields of sets without absolutely additive measures
	289–294	Extension of finitely additive measures
	295	Fixed points of monotone set functions
	296–297	Analytic and recursively enumerable sets
	297–303	Interpretation of Heyting['s calculus] with quantifiers in Lewis' [calculus]
	304–311	Sufficient properties of a model for the B-calculus so that intuitionistic logic holds (in my interpretation)
	311–314	Axiomatization of the B-calculus without the B-adjunction rule
	315–319	Type-free logic is contradictory even if a substitution operation is used instead of ε
	320–322	The notion "definable" in the absolute sense
	322–353	Results and problems surrounding the continuum hypothesis
	354–356	Remarks about perturbation theory
	356–359	Results on foundations
	360–363	Consistency of analysis
	364–365	Cosmological problems
	365–366	Continuum hypothesis
	366–end	Remarks about the consistency of analysis

Appendix D: Logic and Foundations Notebooks: Translated Index Entries. Entries that are merely citations of works of others are not listed.

Topic	Pages
My own theory of cofinality	61–70
Properties of the first inaccessible cardinal and unsolvability of the measure problem and questions dependent upon it	78–88
Analytic and Borel sets	93–109
Questions and problems, etc.	
Considerations about Heyting's propositional calculus	113–116
Formalization of finitary mathematics [using] higher types, and the ordinal numbers in it	117–134

Topic	Pages
[Unlabelled notes]	135–147
Model for the finitary system of finite types in classical number theory	147–148
Ordinals in classical number theory	149–150
Characterization of the numbers $< \varepsilon_0$	151–156
Two remarks about the 2nd number class	156
Formalization of ε_0 in finitary systems of finite types	157–162, 165–170, 173–177
Abortive attempt to reconstruct Gentzen's proof in a finitary system	135–147, 162–164
General[ized] functional calculus (Church)	171–173
Schröder-Bernstein [done] constructively	181
My own remarks about the subdivision of the foundations [of mathematics]	219–220
[Crossed-out material]	221
Propositional calculus from one axiom	267–271
Error in *Principia Mathematica*	273
Exclusive "or"	274
Positive propositional calculus	295–302, 333–334, 336–348, 361–363
Equivalence of different definitions of a topological space	323–325
Theorem about the equivalence of the two definitions of "bicompact"	326–328
Pairs in Church's systems	354–360
Baire's example of sets of the 3rd class	364–367
Lusin's theorem on monotone sequences of F_σ [sets]	368–373
Kuratowski's and Hausdorff's proofs for sequences of $F_\sigma G_\delta$ [sets]	374–381
Foundations of the theory of Baire functions (incomplete)	390–395
Foundations of the theory of lattices (lecture by von Neumann, 1942)	395–432
Symbolism [of] *Principia*	432–439

Note: The 6 volumes are consecutively numbered. Entries are not dated, but "Begin America" is written on p. 282 of vol. 5.

REFERENCES

[1] JOHN W. DAWSON, JR., ***Logical dilemmas: The life and work of Kurt Gödel***, AK Peters, Wellesley, Mass., 1997.

[2] SOLOMON FEFERMAN, *The Gödel editorial project: A synopsis*, this BULLETIN, vol. 11 (2005), no. 2, pp. 132–149, this issue.

[3] KURT GÖDEL, ***Collected works***, vol. I: Publications, 1929–1936 (Solomon Feferman et al., editors), Oxford University Press, New York and Oxford, 1986.

[4] ———, ***Collected works***, vol. II: Publications, 1938–1974 (Solomon Feferman et al., editors), Oxford University Press, New York and Oxford, 1990.

[5] ———, ***Collected works***, vol. III: Unpublished Essays and Lectures (Solomon Feferman et al., editors), Oxford University Press, New York and Oxford, 1995.

[6] ———, ***Collected works***, vol. IV: Correspondence A—G (Solomon Feferman et al., editors), Oxford University Press, Oxford, 2003.

[7] ———, ***Collected works***, vol. V: Correspondence H—Z (Solomon Feferman et al., editors), Oxford University Press, Oxford, 2003.

[8] PAOLO MANCOSU, *Between Vienna and Berlin: The immediate reception of Gödel's incompleteness theorems*, ***History and Philosophy of Logic***, vol. 20 (1999), no. 1, pp. 33–46.

[9] ———, *On the constructivity of proofs. A debate among Behmann, Bernays, Gödel and Kaufmann*, ***Reflections on the foundations of mathematics*** (Wilfried Sieg, Richard Sommer, and Carolyn Talcott, editors), AK Peters, Natick, Mass., 2002, pp. 349–371.

[10] MARIA-ELENA SCHIMANOVICH-GALIDESCU, *Archivmaterial zu Kurt Gödels Wiener Zeit 1924–1940*, ***Kurt Gödel: Wahrheit & Beweisbarkeit, Bd. I: Dokumente und historische Analysen*** (Eckehart Köhler et al., editors), Hölder-Pichler-Tempsky, Vienna, 2002, pp. 135–147.

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