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NOTES ON QUASIMINIMALITY AND EXCELLENCE

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Abstract. This paper ties together much of the model theory of the last 50 years. Shelah's attempts to generalize the Morley theorem beyond first order logic led to the notion of excellence, which is a key to the structure theory of uncountable models. The notion of Abstract Elementary Class arose naturally in attempting to prove the categoricity theorem for $L_{\omega_1,\omega}(Q)$. More recently, Zilber has attempted to identify canonical mathematical structures as those whose theory (in an appropriate logic) is categorical in all powers. Zilber's trichotomy conjecture for first order categorical structures was refuted by Hrushovski, by the introducion of a special kind of Abstract Elementary Class. Zilber uses a powerful and essentailly infinitary variant on these techniques to investigate complex exponentiation. This not only demonstrates the relevance of Shelah's model theoretic investigations to mainstream mathematics but produces new results and conjectures in algebraic geometry.

Zilber proposes [63] to prove 'canonicity results for pseudo-analytic' structures. Informally, 'canonical' means 'the theory of the structure in a suitable possibly infinitary language (see Section 2) has one model in each uncountable power' while 'pseudoanalytic' means 'the model of power $2^{\aleph_0}$ can be taken as a reduct of an expansion of the complex numbers by analytic functions'. This program interacts with two other lines of research. First is the general study of categoricity theorems in infinitary languages. After initial results by Keisler, reported in [31], this line was taken up in a long series of works by Shelah. We place Zilber's work in this context. The second direction stems from Hrushovski's construction of a counterexample to Zilber's conjecture that every strongly minimal set is 'trivial', 'vector space-like', or 'field-like'. This construction turns out to be a very concrete example of an Abstract Elementary Class, a concept that arose in Shelah's analysis. And the construction is a crucial tool for Zilber's investigations. This paper examines the intertwining of these three themes. For simplicity, we work in a countable vocabulary.

The study of $(\mathbb{C}, +, \cdot, \exp)$ leads one immediately to some extension of first order logic; the integers with all their arithmetic are first order definable in $(\mathbb{C}, +, \cdot, \exp)$. Thus, the first order theory of complex exponentiation is

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horribly complicated; it is certainly unstable and so its first order theory cannot be categorical in power. That is, the first order theory of complex exponentiation cannot have exactly one model in each uncountable cardinal. One solution is to use infinitary logic to pin down the pathology. Insist that the kernel of the exponential map is fixed as a single copy of the integers while allowing the rest of the structure to grow. We describe in Section 5 Zilber's theorem that, modulo certain (*very serious*) algebraic hypotheses, $(\mathbb{C}, +, \cdot, \exp)$ can be axiomatized by a categorical $L_{\omega_1,\omega}(Q)$-sentence.

The notion of amalgamation is fundamental to model theory. Even in the first order case, the notion is subtle because there are several variants depending on the choice of a class of models $\boldsymbol{K}$ and a notion $\prec$ of substructure. The pair $(\boldsymbol{K}, \prec)$ has the amalgamation property if whenever $M \in \boldsymbol{K}$ is embedded by f_0, f_1 into N_0, N_1 so that the image of the embeddings $f_0M, f_1M \prec N_0, N_1$ respectively, there is an N^* and embeddings g_0, g_1 of N_0, N_1 into N^* with $g_0 f_0$ and $g_1 f_1$ agreeing on M. If $\boldsymbol{K}$ is the class of models of a complete first order theory then the amalgamation property holds with $\prec$ as elementary embeddings of models. If $\boldsymbol{K}$ is the class of substructures of models of a complete *quantifier eliminable* first order theory then the amalgamation property holds for $\prec$ as *arbitrary* embeddings. Morley [39] observed that by adding names for each definable relation, we can assume, for studying the spectrum problm, that any first order theory has elimination of quantifiers. Shelah [47], noted that this amalgamation hypothesis allows us to assume the existence of a 'monster model' which serves as a universal domain. In this domain the notion of type of an element a over a set A can be thought of either semantically as the orbit of a under automorphisms that fix A or syntactially as the collection of formulas with parameters from A that are satisfied by a.

Of course, the extension from first order logic causes the failure of the compactness theorem. For example, it is easy to write a sentence in $L_{\omega_1,\omega}$ whose only model is the natural numbers with successor. But there are some more subtle losses. The duality between the syntactic and semantic concept of type depends on the amalgamation property. Here is a simple example showing that amalgamation fails in models of a sentence of $L_{\omega_1,\omega}$. Consider the theory T of a dense linear order without endpoints, a unary predicate $P(x)$ which is dense and codense, and an infinite set of constants arranged in order type $\omega + \omega^*$. Let $\boldsymbol{K}$ be class of all models of T which omit the type of a pair of points, which are both in the cut determined by the constants. Now consider the types p and q which are satisfied by a point in the cut, which is in P or in $\neg P$ respectively. Now p and q are each satisfiable in a member of $\boldsymbol{K}$ but they are not simultaneously satisfiable. So the amalgamation property has failed for $\boldsymbol{K}$ and elementary embeddings. This shows that a more subtle notion than consistency is needed to describe types in this wider context.

We took 'canonical' above as meaning 'categorical in uncountable cardinalities'. That is, the class has exactly one model in each uncountable cardinality. The analysis of first order theories categorical in power is based on first studying strongly minimal sets. A set is strongly minimal if every definable subset of it is finite or cofinite. A natural generalization of this, particularly since it holds of simply defined subsets of $(\mathbb{C}, +, \cdot, \exp)$, is to consider structures where every definable set is countable or cocountable. As we will see, the useful formulation of this notion requires some auxiliary homogeneity conditions. The role of homogeneity in studying categoricity in infinitary languages has been known for a long time. There is a rough translation between 'homogeneity' hypotheses on a model and and corresponding 'amalgamation' hypotheses on the class of substructures of the model (Section 2). A structure is $\aleph_1$-homogeneous if for any two countable sequences $\boldsymbol{a}, \boldsymbol{b}$, which realize the same type, and any c, there is a d such that $\boldsymbol{a}c$ and $\boldsymbol{b}d$ realize the same type. Thus, $\aleph_1$-homogeneity corresponds to amalgamation over arbitrary countable subsets. Keisler [31] proved the natural generalization of Morley's theorem for a sentence ψ in $L_{\omega_1,\omega}$ modulo two assumptions:

1. Every model of ψ has arbitrarily large elementary extensions.
2. Every model of ψ is $\aleph_1$-homogeneous.

Keisler asked whether every $\aleph_1$-categorical sentence in $L_{\omega_1,\omega}$ satisfies assumption 2. The answer is no. Marcus [37] gave an example of a minimal prime model with infinitely many indiscernibles and a modification by Shelah provides an example of a totally categorical (categorical in each uncountable cardinality) sentence in $L_{\omega_1,\omega}$ which has no $\aleph_1$-homogeneous models. Shelah's notion of an excellent class (extremely roughly: 'amalgamation over (independent) n-dimensional cubes for all n' and '$\aleph_0$-stability') provides a middle ground. An excellent class (See paragraph 2.0.9.) is a strengthening of Keisler's first assumption (provides not only arbitrarily large models but a certain control over their construction) while weakening the second to assert amalgamation only over certain configurations.

Recall that the logic $L(Q)$ adds to first order logic the expression $(Qx)\phi(x)$ which holds if there are uncountably many solutions of ϕ. I had asked whether a sentence in $L(Q)$ could have have exactly one model and that model have cardinality $\aleph_1$. Shelah proved in [45] using $\diamond$ that an $\aleph_1$-categorical sentence in $L_{\omega_1,\omega}(Q)$ must have a model of power $\aleph_2$. There is a beautiful proof of this result in ZFC in [51]. Shelah has moved this kind of argument from $(\aleph_1, \aleph_2)$ to (λ, λ^+) in a number of contexts. But, getting arbitrarily large models just from categoricity in a single cardinal has remained intractable, although Shelah reported substantial but not yet written progress in the summer of 2003.

Shelah proved an analogue to Morley's theorem in [48], [49] for 'excellent' classes defined in $L_{\omega_1,\omega}$. Assuming $2^{\aleph_n} < 2^{\aleph_{n+1}}$, for all $n < \omega$, he also proved the following kind of converse: every sentence in $L_{\omega_1,\omega}$ that is categorical in $\aleph_n$ for all $n < \omega$ is excellent and categorical in all cardinals. The assumption of categoricity all the way up to $\aleph_\omega$ is shown to be essential in [18] by constructing for each n a sentence ψ_n of $L_{\omega_1,\omega}$ which is categorical up to $\aleph_n$ but has the maximal number of models in all sufficiently large cardinalities. He also asserted that these results 'should be reproved' for $L_{\omega_1,\omega}(Q)$. This 'reproving' has continued for 20 years and the finale is supposed to appear in the forthcoming Shelah [53], [54].

Zilber's approach to categoricity theorems is more analogous to the Baldwin-Lachlan approach than to Morley's. Baldwin-Lachlan [8] provide a structural analysis; they show each model of an $\aleph_1$-categorical theory is prime over a strongly minimal set. This allows one to transfer the 'geometric' proof of categoricity in power for a strongly minimal theory to show categoricity in $\aleph_1$ implies categoricity in all cardinalities. In fact, Zilber considers only the quasiminimal case. But a 'Baldwin-Lachlan' style proof was obtained by Lessmann for homogeneous model theory in [34] and for excellent classes in [35]. That is, he proves every model is prime and minimal over a quasiminimal set.

We begin in Section 1 by recalling the basic notions of the Fraïssé construction and the notion of homogeneity. In Section 2, we sketch some results on the general theory of categoricity in non-elementary logics. In particular, we discuss both reductions to the 'first order logic with omitting types' and the 'syntax-free' approach of Abstract Elementary Classes. We turn to the development of the special case of quasiminimal theories in Section 3. This culminates in Zilber's first approximation of a quasiminimal axiomatization of complex exponentiation. In Section 4 we formulate the generalized Fraïssé construction and place it in the setting of Abstract Elementary Classes. We analyze this method for constructing first order categorical theories; we then see a variant to get examples in homogeneous model theory. Then we discuss the results and limitations of the program to obtain analytic representations of models obtained by this construction. Finally in Section 5 we return to Zilber's use of these techniques to study complex exponentiation. We describe the major algebraic innovations of his approach and the innovations to the Hrushovski construction which result in structures that are excellent but definitely not first order axiomatizable.

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§1. The Fraïssé construction. In the early 1950's Fraïssé [13] generalized Hausdorff's back and forth argument for the uniqueness of the rationals as a countable dense linear order (without end points). He showed that any countable class $\boldsymbol{K}$ of finite relational structures closed under substructure and satisfying the joint embedding and amalgamation properties (see Definition 4.1.6) has a unique countable (ultra)-homogeneous member (denoted $\mathbb{G}$): any isomorphism between finite subsets of $\mathbb{G}$ extends to an automorphism. There are easy variants of this notion for locally finite classes in a language with function symbols. The existence of such structures is proved by iterating the amalgamation property and taking unions of chains. (See [21] for a full account.) Jónsson [28] extended the notion to arbitrary cardinals and Morley-Vaught [40] created an analogous notion for the class of models of first order theories with elementary embeddings as the morphisms. They characterized the homogeneous universal models in this situation as the saturated models. In general the existence of saturated models in power κ requires that $\kappa = \kappa^{<\kappa}$ and $\kappa > 2^{|L|}$; alternatively, one may assume the theory is stable. In particular, κ-saturated models are κ-homogeneous. Morley proved every uncountable model of a theory categorical in an uncountable power is saturated. Abstract versions of the Fraïssé construction undergird the next section; concrete versions dominate the last two sections of the paper.

§2. Syntax, stability, amalgamation. This section is devoted to investigations of categoricity for non-elementary classes. We barely touch the immense literature in this area; see [15]. Rather we just describe some of the basic concepts and show how they arise from concrete questions of categoricity in $L_{\omega_1,\omega}$ and $L_{\omega_1,\omega}(Q)$. In particular, we show how different frameworks for studying nonelementary classes arise and some relations among them. Any serious study of this topic begins with [30], [31].

In its strongest form Morley's theorem asserts: Let T be a first order theory having only infinite models. If T is categorical in some uncountable cardinal then T is complete and categorical in every uncountable cardinal. This strong form does not generalize to $L_{\omega_1,\omega}$; take the disjunction of a sentence which is categorical in all cardinalities with one that has models only up to, say, $\beth_2$. Using both the upward and downward Löwenheim-Skolem theorem, Łos [36] proved that a first order theory that is categorical in some cardinality is complete. Since the upwards Löwenheim-Skolem theorem fails for $L_{\omega_1,\omega}$, the completeness cannot be deduced for this logic. However, if the $L_{\omega_1,\omega}$-sentence ψ is categorical in κ, then, applying the downwards Löwenheim-Skolem theorem, for every sentence ϕ either $\psi \to \phi$ or all models of ϕ have cardinality less than κ. So if ϕ and ψ are κ-categorical sentences with a common model of power κ they are equivalent. We say a sentence of $L_{\omega_1,\omega}$ is complete if it either implies or contradicts every other $L_{\omega_1,\omega}$-sentence. Such a sentence is necessarily $\aleph_0$-categorical (using downward Löwenheim-Skolem). Moreover, every countable structure is characterized by a complete sentence, which is called its *Scott sentence*. So if a model satisfies a complete sentence, it is $L_{\infty,\omega}$-equivalent to a countable model. In particular, any model M of $\psi \in L_{\omega_1,\omega}$ is *small*. That is, for every n it realizes only countably many $L_{\omega_1,\omega}$-n-types (over the empty set). Moreover, if ϕ has a small model then ϕ is implied by a complete sentence satisfied in that model.

In the first order case it is trivial to reduce the study of categoricity to complete (for $L_{\omega,\omega}$) theories. Moreover, first order theories share the fundamental properties of sentences—in particular, Löwenheim-Skolem down to $\aleph_0$. But an $L_{\omega_1,\omega}$-theory need not have a countable model. The difficulty is that an $L_{\omega_1,\omega}$-theory need not be equivalent to a countable conjunction of sentences, even in a countable language. So while we want to reduce the categoricity problem to that for complete $L_{\omega_1,\omega}$-sentences, we cannot make the reduction trivially. We first show that if $\psi \in L_{\omega_1,\omega}$ has arbitrarily large models and is uncountably categorical then ψ extends to a complete sentence. A key observation is that if ψ has arbitrarily large models then ψ has models that realize few types.

LEMMA 2.0.1. *Suppose $\psi \in L_{\omega_1,\omega}$ has arbitrarily large models.*

1. *In every infinite cardinality ψ has a model that realizes only countably many $L_{\omega_1,\omega}$-types over the empty set.*
2. *Thus, if N is the unique model of ψ in some cardinal, ψ is implied by a consistent complete sentence ψ' which holds of N.*

PROOF. Since ψ has arbitrarily large models we can construct a model with indiscernibles (Chapters 13–15 of [31]). Now take an Ehrenfeucht-Mostowski model M for ψ over a set of indiscernibles ordered by a k-transitive dense linear order. (A ordering is k-transitive if any two properly ordered k-tuples are in the same orbit under the automorphism group. These orders exist in every cardinal; take the order type of an ordered field.) Then for every n, M has only countably many orbits of n-tuples and so realizes only countably many types in any logic where truth is preserved by automorphism—in particular in $L_{\omega_1,\omega}$. If ψ is κ-categorical, let ψ' be the Scott sentence of this Ehrenfeucht-Mostowski model with cardinality κ. $\dashv$

If we do not assume ψ has arbitrarily large models the reduction to complete sentences, sketched below, is more convoluted and uses hypotheses (slightly) beyond ZFC. In particular, the complete sentence ψ' does not hold, *a priori* of the categoricity model. The natural examples of $L_{\omega_1,\omega}$-sentences which have models of bounded cardinality (e.g., a linear order with a countable dense subset, or coding up an initial segment of the V_α hierarchy of all sets) have the maximal number of models in the largest cardinality where they have a model. Shelah discovers a dichotomy (Theorem 2.0.2) between such sentences and 'excellent' sentences. We expand on the notion of excellence at 2.0.9 and later in the paper. For the moment just think of the assertion that a complete $L_{\omega_1,\omega}$-sentence (equivalently, its class of models) is excellent as a step into paradise.

For any class $\boldsymbol{K}$ of models, $I(\lambda, \boldsymbol{K})$ denotes the number of isomorphism types of members of $\boldsymbol{K}$, with cardinality λ. We may write ψ instead of $\boldsymbol{K}$ if $\boldsymbol{K}$ is the class of models of ψ. We say that a class $\boldsymbol{K}$ has *many* models of cardinality $\aleph_n$ if $I(\aleph_n, \boldsymbol{K}) \geq \mu(n)$ (and *few* if not; there may not be any). We use as a black box the function $\mu(n)$ (defined precisely in [49]). Either GCH or $\neg O^{\#}$ imply $\mu(n) = 2^{\aleph_n}$ but it is open whether it might be (consistently) smaller. The difficult heart of the argument is the following theorem of Shelah [48], [49]; we don't discuss the proof of this result but just show how this solution for complete sentences gives the result for arbitrary sentences of $L_{\omega_1,\omega}$.

THEOREM 2.0.2. 1. (*For* $n < \omega$, $2^{\aleph_n} < 2^{\aleph_{n+1}}$) *A complete* $L_{\omega_1,\omega}$*-sentence which has few models in* $\aleph_n$ *for each* $n < \omega$ *is excellent* (*see* 2.0.9).
2. (ZFC) *An excellent class has models in every cardinality.*
3. (ZFC) *Suppose that* ϕ *is an excellent* $L_{\omega_1,\omega}$*-sentence. If* ϕ *is categorical in one uncountable cardinal* κ *then it is categorical in all uncountable cardinals.*

So a nonexcellent class defined by a complete $L_{\omega_1,\omega}$-sentence ψ may not have arbitrarily large models but, if not, it must have many models in some cardinal less than $\aleph_\omega$. Combining several results of Keisler, Shelah [48] shows:

LEMMA 2.0.3. *Assume* $2^{\aleph_0} < 2^{\aleph_1}$. *Let* ψ *be a sentence of* $L_{\omega_1,\omega}$ *that has at least one but less than* $2^{\aleph_1}$ *models of cardinality* $\aleph_1$. *Then* ψ *has a small model of cardinality* $\aleph_1$.

PROOF. By Theorem 45 of [31], for any countable fragment L^* containing ψ and any $N \models \psi$ of cardinality $\aleph_1$, N realizes only countably many L^* types over the empty set. Theorem 2.2 of [45] says that if ψ has a model M of cardinality $\aleph_1$ which realizes only countably many types in each fragment then ψ has a small model of cardinality $\aleph_1$. We sketch a proof of that theorem. Add to the language a linear order $<$, interpreted as a linear order of M with order type ω_1. Using that M realizes only countably many types in any fragment, write $L_{\omega_1,\omega}$ as a continuous increasing chain of fragments L_α such that each type in L_α realized in M is a formula in $L_{\alpha+1}$. Add new $2n+1$-ary predicates and $n+1$-ary functions f_n. Let M satisfy $E_n(\alpha, \boldsymbol{a}, \boldsymbol{b})$ if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ realize the same L_α-type and let f_n map M^{n+1} into the initial ω elements of the order, so that $E_n(\alpha, \boldsymbol{a}, \boldsymbol{b})$ implies $f_n(\alpha, \boldsymbol{a}) = f_n(\alpha, \boldsymbol{b})$. Note: (i) $E_n(\beta, \boldsymbol{y}, \boldsymbol{z})$ refines $E_n(\alpha, \boldsymbol{y}, \boldsymbol{z})$ if $\beta > \alpha$; (ii) $E_n(0, \boldsymbol{a}, \boldsymbol{b})$ implies $\boldsymbol{a}$ and $\boldsymbol{b}$ satisfy the same quantifier free formulas; (iii) if $\beta > \alpha$, $E_n(\beta, \boldsymbol{a}, \boldsymbol{b})$ implies $(\forall x)(\exists y)E_{n+1}(\alpha, x\boldsymbol{a}, y\boldsymbol{b})$. Thus, (iv) for any $a \in M$ each equivalence relation $E_n(a, \boldsymbol{y}, \boldsymbol{z})$ has only countably many classes. All these assertions can be expressed by an $L_{\omega_1,\omega}$ sentence ϕ. Now add a unary predicate symbol P and a sentence χ which asserts that M is an *end* extension of $P(M)$. For every $\alpha < \omega_1$ there is a model M_α of

$\phi \wedge \psi \wedge \chi$ with order type of $(P(M), <)$ greater than α. (Start with P as α and alternately take an elementary submodel for the smallest fragment L^* containing $\phi \wedge \psi \wedge \chi$ and close down under $<$. After ω steps we have the P for M_α.) Now by Theorem 12 of [31] there is countable structure $(N_0, P(N_0))$ such that $P(N_0)$ contains a copy of $(Q, <)$ and N_0 is an end extension of $P(N_0)$. By Theorem 28 of [31], N_0 has an L^* elementary extension of cardinality $\aleph_1$. Fix an infinite decreasing sequence $d_0 > d_1 > \dots$ in N_0. For each n, define $E_n^+(\boldsymbol{x}, \boldsymbol{y})$ if for some i, $E_n(d_i, \boldsymbol{x}, \boldsymbol{y})$. Now using (i), (ii) and (iii) prove by induction on the quantifier rank of ϕ that $N_1 \models E_n^+(\boldsymbol{a}, \boldsymbol{b})$ implies $N_1 \models \phi(\boldsymbol{a})$ if and only if $N_1 \models \phi(\boldsymbol{b})$ for every $L_{\omega_1,\omega}$-formula ϕ. For each n, $E_n(d_0, \boldsymbol{x}, \boldsymbol{y})$ refines $E_n^+(\boldsymbol{x}, \boldsymbol{y})$ and by (iv) $E_n(d_0, \boldsymbol{x}, \boldsymbol{y})$ has only countably many classes; so N is small. ⊣

Using these two results, we easily derive a version of Morley's theorem for an $L_{\omega_1,\omega}$-sentence.

Theorem 2.0.4. *Assume* $2^{\aleph_n} < 2^{\aleph_{n+1}}$ *for* $n < \omega$. *If an* $L_{\omega_1,\omega}$*-sentence* ψ *has an uncountable model, then either*

1. ψ *has many models in* $\aleph_n$ *for some* $n < \omega$ *or*
2. ψ *has arbitrarily large models and if* ψ *is categorical in one uncountable cardinal* κ *then it is categorical all uncountable cardinals.*

Proof. Suppose ψ has few models in $\aleph_n$ for each $n < \omega$. By Lemma 2.0.3, choose a small model of ψ, say with Scott sentence ψ'. Assuming $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for each n, Theorem 2.0.2 (1) implies ψ' is excellent. By Theorem 2.0.2 (2) ψ' and thus ψ have arbitrarily large models. Now suppose ψ is categorical in $\kappa > \aleph_0$. Then so is ψ' whence, by Theorem 2.0.2 (3), ψ' is categorical in all uncountable powers.

To show ψ is categorical above κ note that by downward Löwenheim-Skolem all models of ψ with cardinality at least κ satisfy ψ'; the result follows by the categoricity of ψ'. If ψ is not categorical in some cardinality $\mu < \kappa$, there must be a sentence θ which is inconsistent with ψ' but consistent with ψ. Applying the entire analysis to $\psi \wedge \theta$, we find a complete sentence ψ'' which has arbitrarily large models, is consistent with ψ and contradicts ψ'. But this is forbidden by categoricity in κ. ⊣

One corollary of this result is

Corollary 2.0.5. *Assume* $2^{\aleph_0} < 2^{\aleph_1}$. *If an* $L_{\omega_1,\omega}$*-sentence is categorical in* $\aleph_n$ *for* $n < \omega$, *then it is categorical in all cardinalities.*

Hart and Shelah [18] have shown the necessity of the hypothesis of categoricity up to $\aleph_\omega$.

A key tool in the study of *complete* $L_{\omega_1,\omega}$-sentences is the reduction of the class of models of such sentences to classes which are 'closer' to being first order. We now give a full account of this easy reduction. Chang proved in [12] that the class of models of any sentence in $L_{\kappa^+,\omega}$ could be viewed as the class of reducts to L of models of a first order theory in an expansion L'

of L which omitted a family of types. Chang (Lopez-Escobar [12]) used this observation to prove that the Hanf number for $L_{\kappa^+,\omega}$ is same as the Hanf number for omitting a family of κ types. Shelah [45] took this reduction a step further and showed that the class of models of a *complete* sentence in $L_{\omega_1,\omega}$ are in 1-1 correspondence (mapping $L_{\infty,\omega}$-submodel to elementary submodel) with the class of *atomic* models of an appropriate first order theory in an expanded language. That is, to study the generalization of Morley's theorem to complete $L_{\omega_1,\omega}$-sentences it suffices to study classes of structures defined by a special type of *finite diagram*. By a finite diagram we mean an $EC(T, \Gamma)$ class: those models of first order theory that omit all types from a specified collection Γ of types in finitely many variables over the empty set. Abusing the $EC(T, \Gamma)$ notation, $EC(T, Atomic)$ denotes the class of *atomic* models of T (i.e., to conform to the notation we should write nonatomic). Most detailed study of the spectrum of $L_{\omega_1,\omega}$-sentences [45], [48], [49], [35], [16], [27] just work with finite diagrams or more restrictively atomic models (and usually under stronger homogeneity conditions). In general, an atomic class might be defined by omitting uncountably many types; in the case of interest only countably many types have to be omitted.

THEOREM 2.0.6. *Let ψ be a* complete *sentence in $L_{\omega_1,\omega}$. Then there is a countable language L' extending L and a first order L'-theory T such that the reduct map is* 1-1 *from the* atomic *models of T onto the models of ψ.*

PROOF. Let L^* be a countable fragment of $L_{\omega_1,\omega}$ which contains all subformulas of ψ and the conjunction of each $L_{\omega_1,\omega}$-type that is realized in a model of ψ. (This set is countable since complete sentences are small.) Expand L to L' by inductively adding a predicate $P_\phi(\boldsymbol{x})$ for each L^*-formula ϕ. Fix a model of ψ and expand it to an L'-structure by interpreting the new predicates so that the new predicates represent each finite Boolean connective and quantification faithfully: E.g.,

$$P_{\neg\phi(\boldsymbol{x})} \leftrightarrow \neg P_\phi(\boldsymbol{x}),$$

and

$$P_{(\forall \boldsymbol{x})\phi(\boldsymbol{x})} \leftrightarrow (\forall \boldsymbol{x}) P_\phi(\boldsymbol{x}),$$

and that, as far as first order logic can, the P_ϕ preserve the infinitary operations: for each i,

$$P_{\bigwedge_i \phi_i(\boldsymbol{x})} \rightarrow P_{\phi_i(\boldsymbol{x})}.$$

Let T be the first order theory of any such model and consider the countable set Γ of types

$$p_{\bigwedge_i \phi_i(\boldsymbol{x})} = \{\neg P_{\bigwedge_i \phi_i(\boldsymbol{x})}\} \cup \{P_{\phi_i(\boldsymbol{x})} : i < \omega\}.$$

Note that if q is an $L_{\omega_1,\omega}$-type realized in a model of T, $P_{\bigwedge q}$ generates a principal type in T. Now if M is a model of T which omits all the types in Γ (in particular, if M is an atomic model of T), $M \mid L \models \psi$ and each model of

ψ has a *unique* expansion to a model of T which omits the types in Γ (since this is an expansion by definitions in $L_{\omega_1,\omega}$). ⊣

So in particular, *any complete sentence of $L_{\omega_1,\omega}$ can be replaced (for spectrum purposes) by considering the atomic models of a first order theory*. Since all the new predicates are $L_{\omega_1,\omega}$-definable this is the natural extension of Morley's procedure of replacing each first order formula ϕ by a predicate symbol P_ϕ. Morley's procedure resulted in a theory with elimination of quantifiers thus guaranteeing amalgamation over sets for first order categorical T. A similar amalgamation result does *not* follow in this case. Nor, In general, do finite diagrams satisfy the upwards Löwenheim-Skolem theorem.

Remark 2.0.7 ($L_{\omega_1,\omega}(Q)$). The situation for $L_{\omega_1,\omega}(Q)$ is more complicated. The example [18] of a sentence of $L_{\omega_1,\omega}$ that is $\aleph_1$-categorical and not categorial in all uncountable powers is quite complicated. But the $L(Q)$ theory of two disjoint infinite sets illustrates this phenomena trivially. Some of the analysis of [48], [49] goes over directly. But many problems intervene and Shelah has devoted several articles (notably [52], [53], [54] to completing the analysis; a definitive version has not appeared. The difficulty in extending from $L_{\omega_1,\omega}$ to $L_{\omega_1,\omega}(Q)$ is in constructing models with the proper interpretation of the Q-quantifier. Following Keisler's analysis of this problem in [30] the technique is to consider various notions of strong submodel. Two notions are relevant: in the first, the relation of $M \prec_{\boldsymbol{K}} N$ holds when definable sets which are intended to be countable ($M \models \neg(Qx)\phi(x)$) do not increase from M to N. The seconds adds that definable sets intended to be uncountable ($M \models (Qx)\phi(x)$) increase from M to N. The first notion gives an AEC (Definition 2.0.8); the second does not. The reduction [51], [53] is actually to an AEC along with the second relation as an auxiliary that guarantees the existence of standard models.

When Jónsson generalized the Fraïsse construction to uncountable cardinalities [28], [29], he did so by describing a collection of axioms, which might be satisfied by a class of models, that guaranteed the existence of a homogeneous-universal model; the substructure relation was an integral part of this description. Morley and Vaught [40] replaced substructure by elementary submodel and developed the notion of saturated model. Shelah [51], [50] generalized this approach in two ways. He moved the amalgamation property from a basic axiom to a constraint to be considered. (But this was a common practice in universal algebra as well.) He made the *substructure* notion a 'free variable' and introduced the notion of an *Abstract Elementary Class*: a class of structures and a 'strong' substructure relation which satisfied variants on Jonsson's axioms. To be precise

Definition 2.0.8. *A class of L-structures, $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$, is said to be an* abstract elementary class: AEC *if both $\boldsymbol{K}$ and the binary relation $\prec_{\boldsymbol{K}}$ are closed under isomorphism and satisfy the following conditions.*

- **A1**. *If* $M \prec_{\boldsymbol{K}} N$ *then* $M \subseteq N$.
- **A2**. $\prec_{\boldsymbol{K}}$ *is a partial order on* $\boldsymbol{K}$.
- **A3**. *If* $\langle A_i : i < \delta \rangle$ *is* $\prec_{\boldsymbol{K}}$*-increasing chain*:
  1. $\bigcup_{i<\delta} A_i \in \boldsymbol{K}$;
  2. *for each* $j < \delta$, $A_j \prec_{\boldsymbol{K}} \bigcup_{i<\delta} A_i$
  3. *if each* $A_i \prec_{\boldsymbol{K}} M \in \boldsymbol{K}$ *then* $\bigcup_{i<\delta} A_i \prec_{\boldsymbol{K}} M$.
- **A4**. *If* $A, B, C \in \boldsymbol{K}$, $A \prec_{\boldsymbol{K}} C$, $B \prec_{\boldsymbol{K}} C$ *and* $A \subseteq B$ *then* $A \prec_{\boldsymbol{K}} B$.
- **A5**. *There is a Löwenheim number* $\kappa(\boldsymbol{K})$ *such that if* $A \subseteq B \in K$ *there is a* $A' \in \boldsymbol{K}$ *with* $A \subseteq A' \prec_{\boldsymbol{K}} B$ *and* $|A'| < \kappa(\boldsymbol{K}) + |A|$.

With $\prec_{\boldsymbol{K}}$ as the notion of elementary submodel for such logics as first order logic, $L_{\omega_1,\omega}$, finite variable logic, classes defined in those logics become examples of AEC. Note that $L_{\omega_1,\omega}(Q)$ with the standard notion of elementary submodel is *not* an AEC (an uncountable union of countable sets can become uncountable). By interpreting $\prec_{\boldsymbol{K}}$ in the first way described in Remark 2.0.7, sentences of $L_{\omega_1,\omega}(Q)$ define AEC's with Löwenheim number $\aleph_1$. The generalization to AEC is motivated by the fact that many arguments for the model theory of $L_{\omega_1,\omega}(Q)$ work as well in the abstract setting. We discuss a particularly relevant AEC for the Zilber program in Section 4.

By a very straightforward and short argument, Shelah shows in [51] that for every AEC $\boldsymbol{K}$ with vocabulary L, there is a vocabulary L' such that K is the class of reducts to L of L'-structures which omit a certain set of types $(PC(T_1, L, \Gamma))$. In particular, by the same argument as in [12] any AEC in a countable vocabulary with countable Lowenheim number which has a model of cardinality $\beth_{(2^\omega)^+}$ has arbitrarily large models. Moreover, the same procedure allows the construction of Ehrenfreucht-Mostowski models and the deduction (modulo some amalgamation hypotheses) of stability from categoricity [53].

Note the following hierarchy of ease of definition where $A > B$ (read $>$ as 'is more general than') means every B class is an A class:

$$PC(T_1, L, \Gamma) > \text{AEC} > L_{2^{\kappa^+},\omega} > EC(T, \Gamma) = \textit{finite diagrams} > EC(T, \textit{Atomic}) > \textit{first order},$$

for an arbitrary cardinal κ.

The distinction given by the first inequality is very sharp. Silver (Chapter 18 of [31]) gives a simple example of a pseudoelementary class where the categoricity spectrum and its complement are both cofinal in the class of cardinals. Morley's theorem for AEC and even the weaker conjecture, that the aberrant behaviour of the last sentence cannot hold in an AEC, remain open.

Orthogonal to this syntactical hierarchy are various kinds of amalgamation hypotheses. There are a number of variants on homogeneity; here we mean the basic notion of sequential homogeneity. Perhaps the most important distinction is:

$$\textit{amalgamation over models} > \textit{homogeneity} = \textit{set amalgamation}.$$

The original Keisler hypothesis for the categoricity theorem: that the model of power $\aleph_1$ is homogeneous led to an important topic, finite diagrams with amalgamation [44], i.e., an $EC(T, \Gamma)$ class with set amalgamation. The term 'homogeneous model theory' is used for both this notion (cf. [17], [27]) and a more general context proposed by Buechler and Lessmann in [11]. Although formally different the techniques and results of the two areas are 'morally' the same. Many of the results of stability and simplicity theory have been developed in this context. In particular, one can prove a stablility spectrum theorem very similar to the first order case. The elaborate development during the last 30 years of the model theory of Banach spaces is an example of homogeneous model theory (12.5 of [19]). More general than any of the classes discussed here is the study of classes where structures, which are amalgamation bases for extensions of the same cardinality, are dense [55], [56]. An early overview of all these questions is in the hard to locate Lazy Model Theorists guide [46].

Remark 2.0.9 (Excellence). The notion of excellence as defined in [48] includes both an amalgamation component and a stability component. The idea arises from the attempt to construct arbitrarily large models. Vaught [57] proved that a theory with a countable atomic model M_0 that is not minimal has a model M of power $\aleph_1$: properly extend M_0 to M_1 (which is also atomic). Iterate, taking unions at limits. Shelah transferred the argument to convert a categorical model M of size $\aleph_1$ to a model of size $\aleph_2$. The key idea is to analyze how every model can be built up from countable submodels. Using categoricity, the problem reduces to finding a proper elementary extension of M. For this, write M as a union of a chain of countable models M_i and extend each M_i to an N_i. The simplest way to guarantee $\bigcup_{i<\aleph_1} M_i$ is properly contained in $\bigcup_{i<\aleph_1} N_i$ is to guarantee that $N_0 \cap M = M_0$, that is, that M and N_0 are disjointly amalgamated over M_0. For this, some stability is used. To construct a model of power $\aleph_3$, from an M of cardinality $\aleph_2$ the process is repeated. Now, after writing M as a union of models of cardinality at most $\aleph_1$, one tries to again extend model by model. Each model in the tower is decomposed into a chain of countable models. To reconstruct the tower one must amalgamate cubes of countable models and the system must be 'stable' to ensure that the towers expand. Excellence asserts that a free n-dimensional cube of models can be completed. We have just sketched the use of excellence to build arbitrarily large models. In Section 3, we will discuss how in a restricted setting it produces uniqueness. Excellent classes have been explored by a number of authors. The 'main gap' was carried over from first order logic to excellent classes by Grossberg and Hart [16]; Lessmann [35] expounds the categoricity situation, explaining the distinction between homogeneous and excellent categorical classes.

§3. Quasiminimality and excellence. In the first subsection we define (a slight variant of) Zilber's notion of a quasiminimal excellent class and sketch his proof that quasiminimal excellent classes are categorical in all uncountable powers. A quasiminimal excellent class is a class of structures such that each structure admits a combinatorial geometry and certain amalgamations over free configurations for this geometry are insured. In the second subsection, we describe a quasiminimal excellent approximation of complex exponentiation [60].

The notion of quasiminimality generalizes the first order notion of strong minimality. In first order logic $a \in \mathrm{acl}(B)$ if some first order formula $\phi(x)$ with parameters from B has only finitely many solutions, one of which in a. In a strongly minimal set acl determines a combinatorial geometry—that is, it satisfies the condition on an abstract cl relation we describe in Assumptions 3.1.1 and 3.1.5. This easily gives the categoricity in each uncountable power of a strongly minimal set.

3.1. Abstract quasiminimality. A class $\boldsymbol{K}$ is *quasiminimal excellent* [63] if it satisfies the following four conditions. We speak of *abstract* quasiminimality because the notion is defined here in terms of an unspecified combinatorial geometry. A specific application might, for example, define $a \in \mathrm{cl}(A)$ if a is in a countable set that is $L_{\omega_1,\omega}$-definable with parameters from A. This notion has a fundamental difficulty: in general, one may have elementarily equivalent sets X and Y whose closures are not isomorphic. In the ordinary notion of algebraic closure, a map can be extended to the algebraic closure by minimizing the size of the finite set witnessing $a \in \mathrm{acl}(X)$; in the quasiminimal case, the notion of excellence provides such an extension to the closure by a much more involved argument. Quasiminimal excellence is to Shelah's notion of excellence as strongly minimal sets are to the study of ω-stable first order theories.

Recall that an operator F on sets has *finite character* if $F(X)$ is the union of $F(X_0)$ for X_0 a finite subset of X. We adopt the convention here of writing concatenation for union, XY denotes $X \cup Y$.

Assumption 3.1.1 (Condition I). *Let $\boldsymbol{K}$ be a class of L-structures which admit a monotone idempotent closure operation* cl *which has finite character. Further, for every X, $\mathrm{cl}(X) \in \boldsymbol{K}$.*

Let G be a subset of H, H' and all three be in $\boldsymbol{K}$. A map from $X \subset H - G$ to $X' \subset H' - G$ is called a partial G-monomorphism if its union with the identity map on G preserves quantifier free formulas. Condition 3.1.2.3, implies, in particular that every member of $\boldsymbol{K}$ is ω-homogeneous.

Assumption 3.1.2 (Condition II). *Let $G \subseteq H, H' \in \boldsymbol{K}$ with G empty or in $\boldsymbol{K}$.*

1. *If f is a bijection between X and X' which are* separately cl-*independent* (*over G*) *subsets of H and H' then f is a partial G-monomorphism.*

2. *If f is a partial G-monomorphism from H to H' with finite domain X then for any $y \in H$ there is y' in an extension $H'' \in \boldsymbol{K}$ of H' such that $f \cup \{\langle y, y'\rangle\}$ extends f to a partial G-monomorphism.*
3. *If f is a partial G-monomorphism from H to H' taking $X \cup \{y\}$ to $X' \cup \{y'\}$ then $y \in \mathrm{cl}(XG)$ iff $y' \in \mathrm{cl}(X'G)$.*

Condition (3) has an *a priori* unlikely strength: quantifier free formulas determine the closure; in practice, the language is specifically expanded to guarantee this condition. Part 2 of Assumption 3.1.2 implies that each M with $G \subseteq M \in \boldsymbol{K}$ is finite sequence homogeneous over G.

In the following definition it is essential that $\subset$ be understood as *proper* subset.

Definition 3.1.3. 1. *For any Y, $\mathrm{cl}^-(Y) = \bigcup_{X \subset Y} \mathrm{cl}(X)$.*
2. *We call C (the union of) an n-dimensional* cl-independent system *if $C = \mathrm{cl}^-(Z)$ and Z is an independent set of cardinality n.*

To visualize a 3-dimensional independent system think of a cube with the empty set at one corner A and each of the independent elements z_0, z_1, z_2 at the corners connected to A. Then each of $\mathrm{cl}(z_i, z_j)$ for $i < j < 3$ determines a side of the cube: $\mathrm{cl}^-(Z)$ is the union of these three sides; $\mathrm{cl}(Z)$ is the entire cube.

Assumption 3.1.4 (Condition III). *Let $G \subseteq H, H' \in \boldsymbol{K}$ with G empty or in $\boldsymbol{K}$. Suppose $Z \subset H - G$ is an n-dimensional independent system, $C = \mathrm{cl}^-(Z)$, and X is a finite subset of $\mathrm{cl}(Z)$. Then there is a finite C_0 contained in C such that: for every G-partial monomorphism f mapping X into H', for every G-partial monomorphism f_1 mapping C into H', if $f \cup (f_1 \restriction C_0)$ is a G-partial monomorphism, $f \cup f_1$ is also a G-partial monomorphism.*

Thus Condition III, which is the central point of excellence, asserts (e.g., in dimension 3) that the type of any element in the cube over the union of the three given sides is determined by the type over a finite subset of the sides. The 'thumbtack lemma' of Subsection 3.2 verifies this condition in a specific algebraic context.

Assumption 3.1.5 (Condition IV). cl *satisfies the exchange axiom: $y \in \mathrm{cl}(Xx) - \mathrm{cl}(X)$ implies $x \in \mathrm{cl}(Xy)$.*

Zilber omits exchange in the fundamental definition but it arises in the natural contexts he considers so we make it part of quasiminimal excellence. Note however that in Section 4, the examples of first order theories with finite Morley rank greater than 1 (the parameter α of the construction is greater than 1) fail to satisfy exchange. We say a closure operation satisfies the *countable closure condition* if the closure of a countable set is countable. We easily see:

Lemma 3.1.6. *Suppose* cl *is defined on an uncountable structure M and satisfies Conditions I and II and the countable closure condition.*

1. *For any finite set $X \subseteq H \in \boldsymbol{K}$, if $a, b \in H - \mathrm{cl}(X)$, a, b realize the same $L_{\omega_1,\omega}$ type over X.*
2. *For every uncountable $M \in \boldsymbol{K}$, every $L_{\omega_1,\omega}$ definable set is countable or cocountable. This implies that $a \in \mathrm{cl}(X)$ iff it satisfies some ϕ over X, which has only countably many solutions.*

Proof. Condition (1) follows directly from (1) and (2) of Assumption 2 by constructing a back and forth. To see condition (2), suppose both ϕ and $\neg\phi$ had uncountably many solutions with ϕ defined over X. Then there are a and b satisfying ϕ and $\neg\phi$ respectively and neither is in $\mathrm{cl}(X)$; this contradicts (1).

The ω-homogenity yields by an easy induction:

Lemma 3.1.7. *Suppose Conditions I and II hold. If $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$ are countable and X is independent then any isomorphism between X and Y extends to an isomorphism of $\mathrm{cl}(X)$ and $\mathrm{cl}(Y)$*

For *algebraic* closure the countability restriction is unnecessary. We now use Assumption 3.1.4 to remove the restriction in excellent classes.

Theorem 3.1.8. *Let $\boldsymbol{K}$ be a quasiminimal excellent class and suppose $H, H' \in \boldsymbol{K}$ satisfy the countable closure condition. Let $\mathcal{A}, \mathcal{A}'$ be cl-independent subsets of H, H' with $\mathrm{cl}(\mathcal{A}) = H$, $\mathrm{cl}(\mathcal{A}') = H'$, respectively, and ψ a bijection between $\mathcal{A}$ and $\mathcal{A}'$. Then ψ extends to an isomorphism of H and H'.*

Thus, in each uncountable cardinality $\boldsymbol{K}$ has a unique model on which cl satisfies the countable closure condition.

We sketch the proof of Theorem 3.1.8. Fix a countable subset $\mathcal{A}_0$ of $\mathcal{A}$; without loss of generality, we can assume ψ is the identity on $\mathcal{A}_0$ and work over $G = \mathrm{cl}(\mathcal{A}_0)$. So from now on monomorphism means monomorphism over G and $\mathrm{cl}(X)$ means $\mathrm{cl}(\mathcal{A}_0X)$.

Note that ψ is a monomorphism and so is $\psi_0 = \psi \mid \mathcal{A}_0$. By Lemma 3.1.7 and induction, for any independent X with $|X| \leq \aleph_0$, $\psi \mid X$ extends to a isomorphism from $\mathrm{cl}(X)$ to $\mathrm{cl}(\psi(X))$. Taking unions of an increasing chain, we can even assume $|X| = \aleph_1$.

Note also that $H = \lim_{X \subset \mathcal{A}; |X| < \aleph_0} \mathrm{cl}(X)$. We have the obvious directed system on $\{\mathrm{cl}(X) \colon X \subset \mathcal{A}; |X| < \aleph_0\}$. So the theorem follows immediately if for each finite X we can choose $\psi_X \colon \mathrm{cl}(X) \to H'$ so that $X \subset Y$ implies $\psi_X \subset \psi_Y$. We prove this by induction on $|X|$. Suppose $|Y| = n + 1$ and we have appropriate ψ_X for $|X| < n + 1$. We will prove two statements by induction.

1. $\psi_Y^- \colon \mathrm{cl}^-(Y) \to H'$ defined by $\psi_Y^- = \bigcup_{X \subset Y} \psi_X$ is a monomorphism.
2. ψ_Y^- extends to ψ_Y defined on $\mathrm{cl}(Y)$.

The first step is done by induction and ω-homogeneity using Lemma 3.1.7. The exchange axiom is used to guarantee that the maps ψ'_Y for $Y' \supset Y$ agree where more than one is defined. The second step follows by Assumption 3.1.4 and induction using Lemma 3.1.7 and the fact that $\mathrm{cl}(Y)$ is countable. We have shown that the isomorphism type of a structure in $\boldsymbol{K}$ is determined by the cardinality of a basis for the geometry. If $\boldsymbol{K}$ satisfies the countable closure condition the size of an uncountable model is the same as its dimension, so we get categoricity. ⊣

A natural way to require countable closure condition of all members of $\boldsymbol{K}$ is to axiomatize the class in $L_{\omega_1,\omega}(Q)$; for the next example $L_{\omega_1,\omega}$ suffices because of a clever choice of the closure relation.

3.2. Covers of the multiplicative group of $\mathbb{C}$. The first approximation to a quasiminimal axiomatization of complex exponentiation considers short exact sequences of the following form.

$$0 \to Z \to H \to F^* \to 0. \tag{1}$$

H is a torsion-free divisible abelian group (written additively), F is an algebraically closed field, and exp is a homomorphism from $(H, +)$ to $(F^*, \cdot)$, the multiplicative group of F. We can code this sequence as a structure:

$$(H, +, E, S),$$

where $E(h_1, h_2)$ iff $\exp(h_1) = \exp(h_2)$. We pull the additive structure of the field back to H by the defining $H \models S(h_1, h_2, h_3)$ iff $F \models \exp(h_1) + \exp(h_2) = \exp(h_3)$. Thus H now represents both the multiplicative and additive structure of F.

Lemma 3.2.1. *There is an $L_{\omega_1,\omega}$-sentence Σ such that there is a* 1-1 *correspondence between models of Σ and sequences* (1).

The sentence asserts first that the quotient of H by E with $+$ corresponding to $\times$ and S to $+$ is an algebraically closed field. We use $L_{\omega_1,\omega}$ to guarantee that the kernel is isomorphic to Z. This same proviso insures that the relevant closure condition has countable closures. Now the key result asserts.

Theorem 3.2.2. *For an appropriate definition of closure, Σ is quasiminimal excellent with the countable closure condition and categorical in all uncountable powers.*

In this context the appropriate cl on the domain H of a model of Σ is defined by

$$\mathrm{cl}(X) = \exp^{-1}(\mathrm{acl}(\exp(X))$$

where acl is the field algebraic closure in F. It is easy to check that cl gives a combinatorial geometry such that the countable closure of countable sets is countable. (Strictly speaking, the language will have to be expanded to guarantee Assumption 3.1.2.3.) The main algebraic ingredient in this argument arises from the treatment of the divisible closure (in the multiplicative group of the field).

Definition 3.2.3. *By a* divisibly closed multiplicative subgroup *associated with $a \in \mathbb{C}^*$, $a^{\mathbb{Q}}$, we mean a* **choice** *of a multiplicative subgroup containing a and isomorphic to the group* $(\mathbb{Q}, +)$.

Definition 3.2.4. *We say* $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \dots, b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$, determine the isomorphism type *of* $b_1^{\mathbb{Q}}, \dots, b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$ *over the subfield* k *of* $\mathbb{C}$ *if given subgroups of the form* $c_1^{\mathbb{Q}}, \dots, c_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$ *and* ϕ_m *such that*

$$\phi_m\colon k\big(b_1^{\frac{1}{m}} \dots, b_\ell^{\frac{1}{m}}\big) \to k\big(c_1^{\frac{1}{m}} \dots, c_\ell^{\frac{1}{m}}\big)$$

is a field isomorphism, ϕ_m *extends to*

$$\phi_\infty\colon k\big(b_1^{\mathbb{Q}}, \dots, b_\ell^{\mathbb{Q}}\big) \to k\big(c_1^{\mathbb{Q}}, \dots, c_\ell^{\mathbb{Q}}\big).$$

In the following, $\sqrt{1}$ denotes the subgroup of roots of unity. We call this result the thumbtack lemma based on the following visualization of Kitty Holland. The various *nth* roots of $b_1, \dots, b_m$ hang on threads from the b_i. These threads can get tangled; but the theorem asserts that by sticking in a finite number of thumbtacks one can ensure that the rest of strings fall freely. The proof involves the theory of fractional ideals of number fields, Weil divisors, and the normalization theorem. For $a_1, \dots, a_r$ in $\mathbb{C}$, we write $\mathrm{gp}(a_1, \dots, a_r)$ for the multiplicative subgroup generated by $a_1, \dots, a_r$. The following general version of the theorem is applied for various sets of parameters to prove quasiminimal excellence.

Theorem 3.2.5 (thumbtack lemma). [60] *Let* $P \subset \mathbb{C}$ *be a finitely generated extension of* $\mathbb{Q}$ *and* $L_1, \dots, L_n$ *algebraically closed subfields of the algebraic closure of* P. *Fix multiplicatively divisible subgroups* $a_1^{\mathbb{Q}}, \dots, a_r^{\mathbb{Q}}$ *with* $a_1, \dots, a_r \in P^*$ *and* $b_1^{\mathbb{Q}}, \dots, b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$. *If* $b_1 \dots, b_\ell$ *are multiplicatively independent over* $\mathrm{gp}(a_1, \dots, a_r) \cdot \sqrt{1} \cdot L_1^* \cdot \dots \cdot L_n^*$ *then for some* m, *the elements* $b_1^{\frac{1}{m}} \in b_1^{\mathbb{Q}}, \dots, b_\ell^{\frac{1}{m}} \in b_\ell^{\mathbb{Q}} \subset \mathbb{C}^*$, *determine the isomorphism type of* $b_1^{\mathbb{Q}}, \dots, b_\ell^{\mathbb{Q}}$ *over* $P(L_1, \dots, L_n, \sqrt{1}, a_1^{\mathbb{Q}}, \dots, a_r^{\mathbb{Q}})$.

We describe these notions in terms of formulas rather than elements.

Definition 3.2.6. 1. *Let* V *be an irreducible variety over* $C \subsetneq F$. *The* sequence associated with V *over* C *is a sequence*

$$\big\{V^{\frac{1}{m}} : m \in \omega\big\}$$

such that $V^1 = V$ *and for any* $m, n \in \omega$, *raising to the* m*th power maps* $V^{\frac{1}{nm}}$ *to* $V^{\frac{1}{n}}$.

2. *If* $V' \subseteq V$ *are varieties in* n*-variables over* C, *the pair*

$$\tau = \big(V - V', \big\{V^{\frac{1}{m}} : m \in \omega\big\}\big)$$

is called an almost finite n-type *over* C.

3. *Zilber calls a principal type given by a difference of varieties* $V - V'$ *a* finite n-type *over* C.

One of the key ideas discovered by Shelah in the investigation of non-elementary classes is that in order for types to be well-behaved one may have to make restrictions on the domain. (E.g., we may be able to amalgamate types over models but not arbitrary types.) This principle is illustrated by the following definition and result of Zilber.

DEFINITION 3.2.7. *$C \subseteq F$ is* finitary *if C is the union of the divisible closure (in $\mathbb{C}^*$) of a finite set and finitely many algebraically closed fields.*

To prove the following result, apply the thumbtack lemma with the L_i as the fields and the $\boldsymbol{a}_i$ as the finite set.

COROLLARY 3.2.8. *Any almost finite n-type over a finitary set is a finite n-type.*

SKETCH OF PROOF OF THEOREM 3.2.2. Application of the thumbtack lemma gives directly the homogeneity conditions of Assumption 3.1.2. Exchange, Assumption 3.1.5, is immediate from the definition of closure (3.2). Finitary sets are more general than the n-dimensional independent systems in the definition of quasiminimal excellence, since the subsets do not have to be independent. So if X is a sequence associated with a variety V over an n-dimensional independent system C, Corollary 3.2.8 allows us to reduce X to a formula over a finite set yielding Assumption 3.1.4. So the class is quasiminimal and categoricity follows by Theorem 3.1.8. $\dashv$

We have shown the expansion of the complex numbers by naming the congruence (on the additive group) induced by exponentiation is quasiminimal excellent. This argument is rather *ad hoc*; one just checks the property of quasiminimal excellence with no specific model theoretic innovations in the argument. In the next section we see a family of constructions for quasiminimal excellent classes.

§4. The generalized Fraïssé construction.

In the 1950's Fraïssé generalized the Cantor-Hausdorff proof of the uniqueness of countable dense linear orders (without endpoints) by showing a class of finite relational structures that has the amalgamation property over arbitrary substructures gives rise to a countable homogeneous structure. This construction was generalized to uncountable cardinals by Jónnson and inspired the Morley-Vaught invention of saturated models. Shelah generalized the notion still further with various approximations to his notion of an abstract elementary class; key to this generalization is replacing the concrete notion of substructure by a 'strong submodel' notion which is described axiomatically. Although the Fraïssé models were $\aleph_0$-categorical, all but the most trivial were inherently unstable. Hrushovski [24], [20] constructs stable examples by defining a notion of strong submodel in terms of a function mapping finitely generated structures into discrete additive subgroups of the reals (or into the integers).

4.1. Weak ranks and strong submodels. We reprise the general construction in the form we described in [4]. Many explicit examples are discussed in that paper. Let $\langle K(N), \wedge, \vee\rangle$ be a lattice of substructures of a model N. For purposes of this paper a *rank* is a function δ from $K(N)$ to a discrete additive subgroup of the reals that is defined on each N in a class $\boldsymbol{K}$. This notion of rank is much weaker than any other rank notion used in stability theory. We write $\delta(A/B) = \delta(A \vee B) - \delta(B)$ to indicate the relativization of the rank. We demand *only* that δ is monotonic: if $B \subseteq A, C \subseteq N$ and $A \wedge C = B$,

$$\delta(A/B) \geq \delta(A/C).$$

This requirement can be rephrased as asserting that δ is *lower semimodular*: for any A, B,

$$\delta(A \vee C) - \delta(C) \leq \delta(A) - \delta(A \wedge C).$$

We say δ is *modular* if the inequality is an equality. Examples of δ include cardinality, relation size (number of instances of a relation), vector space dimension, and transcendence degree. All of these but the last are modular. The simplest example of 'relation size' is just the number of edges in a (symmetric) graph. As in [61] we say the rank is a *predimension* when the range of δ is the integers. There are many variants of this construction. Each depends on the choice of a class $\boldsymbol{K}$ and a rank function on members of $\boldsymbol{K}$. Many of the ranks are obtained by standard combinations of ones that are already known. If δ_1, δ_2 are ranks defined on a class $\boldsymbol{K}$, so are

$$\delta = \alpha\delta_1 + \beta\delta_2$$

for any *positive* reals α, β and

$$\delta = \alpha\delta_1 - \beta\delta_2$$

for any *positive* reals α, β if δ_2 is modular! With this observation, most of the examples of this construction can be seen as built up from the examples in the previous paragraph. Irrational α, β correspond to the construction of strictly stable structures [10], [9], [20] and will play no further role after the next example.

EXAMPLE 4.1.1. *Let* $\delta = \alpha\delta_1 - \beta\delta_2$.

1. *The class* $(\boldsymbol{K}, \delta)$ *is called* ab initio *if* $\boldsymbol{K}$ *is a universal class of relational structures,* δ_1 *is cardinality and* δ_2 *is the number of relations. This gives rise to the new strongly minimal set* $(\alpha = \beta = 1)$ [24], *a non-Desarguesian projective plane* $(\alpha = 2,\ \beta = 1)$ [3], *a strictly stable* $\aleph_0$*-categorical theory* (*Hrushovski* α *irrational,* $\beta = 1$), *and theories with infinite weight* (*sequence of irrational* α) [20] *and the almost sure theorey of graphs with edge probability* $n^{-\alpha}$ (α *irrational,* $\beta = 1$ *but different* $\boldsymbol{K}$) [9], [10].
2. *Bicolored fields* $(\alpha = k, \beta = 1)$ [42], [5], [7] *are expansions of a field by a unary predicate*; $\delta_1 = d_f$ *is transcendence degree*; δ_2 *counts the number of points in* P.

Let T_{-1} be a first order theory such that for any subset X of a model N of T_{-1}, there is a minimal submodel M of N that contains X; this implies there is a natural notion of a finitely generated model. We denote this submodel

$\langle X \rangle_N$, dropping the subscript N when the choice of N is evident. This condition is clearly satisfied if T_{-1} is universally axiomatized or strongly minimal and almost all of our examples fall into one of these two classes. Let $\overline{\boldsymbol{K}}_{-1} = \mod(T_{-1})$; $\boldsymbol{K}_{-1}$ is the *finitely generated* members of $\overline{\boldsymbol{K}}_{-1}$.

The construction of the homogeneous model is made with respect to a notion of strong substructure.

Definition 4.1.2. 1. *For $A, B \in \overline{\boldsymbol{K}}_{-1}$, we say A is a* strong substructure *of B and write $A \prec_{\boldsymbol{K}} B$ if for every $B' \in \boldsymbol{K}_{-1}$ with $B' \subseteq B$, $\delta(B'/B' \wedge A) \geq 0$.*

2. *We denote by $\overline{\boldsymbol{K}}_0$ the set of $A \in \overline{\boldsymbol{K}}_{-1}$ which have $\delta(A') \geq 0$ for each $A' \subseteq A$ and by $\boldsymbol{K}_0$ those in $\overline{\boldsymbol{K}}_0$ which are finitely generated. T_0 denotes the theory of $\overline{\boldsymbol{K}}_0$.*

Now it is easy to show

Theorem 4.1.3. *Any class $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ where $\prec_{\boldsymbol{K}}$ is defined from a δ-function and a class $\overline{\boldsymbol{K}}_{-1}$ as in Definition* 4.1.2 *and that is closed under unions of increasing chains is an Abstract Elementary Class. If it has countable similarity type then the Löwenheim number is $\aleph_0$ (Definition* 2.0.8*).*

Since $\prec_{\boldsymbol{K}}$ is imposed by δ, the following properties hold, which are more special than AEC's in general or even the analysis of generic models in [33].

- **A6**. *If $A, B, C \in K(N)$, $A \prec_{\boldsymbol{K}} C$, $B \subseteq C$, then $A \cap B \prec_{\boldsymbol{K}} B$.*
  We can restrict to $\overline{\boldsymbol{K}}_0$ to obtain:
- **A7**. *$\emptyset \in \overline{\boldsymbol{K}}_0$ and $\emptyset \prec_{\boldsymbol{K}} A$ for all $A \in \overline{\boldsymbol{K}}_0$.*

A predimension δ also allows us to construct a combinatorial geometry.

Definition 4.1.4. 1. *For $M \in \overline{\boldsymbol{K}}_0$, $A \subseteq M$, $A \in \boldsymbol{K}_0$, $d_M(A) = \inf\{\delta(B) \colon A \subset B \subseteq M, B \in \boldsymbol{K}_0\}$.*

2. *For A, b contained M, $b \in \mathrm{cl}(A)$ if $d_M(bA) = d_M(A)$.*

Naturally we can extend to closures of sets that are not finitely generated by imposing finite character.

Lemma 4.1.5. 1. *The closure system defined in Definition* 4.1.4 *is monotone and idempotent as in Assumption* 3.1.1.

2. *If, in addition δ is a predimension (integer range) and for any finite X, $d_M(X) \leq |X|$ then the closure system satisfies exchange, Assumption* 3.1.5.

Definition 4.1.6. 1. *The pair $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$ has the* amalgamation property *if for $N, M \in \boldsymbol{K}$ with $A \prec_{\boldsymbol{K}} M, N$, there exists $N_1 \in \boldsymbol{K}$ and embeddings of M, N as strong submodels of N_1, which agree on A. It has the* joint embedding property *if any N, M have a common strong extension.*

2. *The model M is κ-$(\boldsymbol{K}, \prec_{\boldsymbol{K}})$*-homogeneous *(or rich* [42]*) if $A \prec_{\boldsymbol{K}} M$, $A \prec_{\boldsymbol{K}} B \in \boldsymbol{K}$ and $|B| \leq \kappa$ implies there exists $B' \prec_{\boldsymbol{K}} M$ such that $B \cong B'$ over A.*

3. *The* generic *model* $\mathbb{G}$ *is the unique countable* $\aleph_0$-$(\boldsymbol{K}, \prec_{\boldsymbol{K}})$-homogeneous-*model that is a union of a chain of finitely generated models, each of which is a strong extension of its predecessor.*

Now standard arguments show:

THEOREM 4.1.7. 1. *If a class* $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$ *has the amalgamation property and the joint embedding property then there is a countable generic structure* $\mathbb{G}$.
2. *Moreover, for every* κ, *there is a structure* M_κ *which is* κ-$(\boldsymbol{K}, \prec_{\boldsymbol{K}})$-*homogeneous.*

Note that we have amalgamation over models, not over sets and the homogeneity is with respect to strong substructures, not sequential homogeneity. To determine such properties of the generic model as ω-saturation and stability class requires that we introduce a second notion of closure. The existence of a 'unique minimal N' in the next definition follows from **A6**.

DEFINITION 4.1.8. 1. *Let* $A \subseteq M \in \boldsymbol{K}$. *The intrinsic (or self-sufficient) closure of* A *in* M, *denoted* $\mathrm{icl}_M(A)$ *is the unique minimal* N *such that*: $A \subseteq N$, $N \in \boldsymbol{K}$, $N \prec_{\boldsymbol{K}} M$.
2. *We say* B *is a minimal intrinsic extension of* A *if* $\delta(B/A) < 0$ *but* $\delta(B'/A) \geq 0$ *for every* B' *with* $B \supset B' \supseteq A$.

It is easy to check that $\mathrm{icl}_M(A)$ can also be constructed by interatively taking minimal intrinsic extensions. It is crucial that this notion be definable (in roughly the same sense one says Morley rank is definable). For example, this is necessary to guarantee that $\overline{\boldsymbol{K}}_0$ is axiomatizable. We say $\boldsymbol{K}$ has δ-formulas for minimal intrinsic extensions if for each pair (B, A) with B minimal intrinsic over A, there is a formula $\phi_{AB}(\boldsymbol{x}, \boldsymbol{y})$, satisfied by an enumeration of BA, such that if $\phi_{AB}(\boldsymbol{b}', \boldsymbol{a}')$ and B', A' are the structures generated $\boldsymbol{b}', \boldsymbol{a}'$ then $\delta(B'/A') \leq \delta(B/A)$ (and some other conditions we won't spell out here). The existence of δ-formulas is trivial in the *ab initio* case [24], routine for bicolored fields [5] and impossible (in full generality) for fields with a distinguished multiplicative subgroup [23], [43].

The following key facts about intrinsic closure follow from the definition of $\overline{\boldsymbol{K}}_0$ as the class of structures with hereditarily non-negative δ. The key points for (2) are that any minimal intrinsic extension B of A can have only finitely many copies in M and this implies that ϕ_{AB} is algebraic and any point in the intrinsic closure arises through finitely many iterations of minimal intrinsic extensions. The proofs of these results are not difficult.

LEMMA 4.1.9. *Suppose* $\boldsymbol{K}$ *has* δ-*formulas for minimal intrinsic extensions. Let* $A \subseteq M \in \overline{\boldsymbol{K}}_0$.

1. *If* A *is finitely generated then* $\mathrm{icl}_M(A)$ *is finitely generated.*
2. *For any* $A \subseteq M$, $\mathrm{icl}_M(A)$ *is contained in* $\mathrm{acl}_M(A)$.

DEFINITION 4.1.10. *$\boldsymbol{K}$ is* set-determined *if for every $X \subset A \in \boldsymbol{K}$ there is* (a) *a minimal X' contained in A with $X' \in \boldsymbol{K}$ and* (b) $\operatorname{tp}(X/\emptyset)$ *determines* $\operatorname{tp}(X'/\emptyset)$.

In either the *ab initio* case (see Section 4.2) or if $\boldsymbol{K}$ arises by naming a subset of an algebraically closed field, the class $\boldsymbol{K}$ is set-determined.

From Lemma 4.1.9 we easily see:

LEMMA 4.1.11. *Suppose $\boldsymbol{K}$ is set-determined. The countable generic model $\mathbb{G}$ is $\aleph_0$-homogeneous. More generally, a κ-$(\boldsymbol{K}, \prec_{\boldsymbol{K}})$-homogeneous M is κ-set homogeneous.*

PROOF. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be sequences of length less than κ from M which realize the same first order type and let $c \in M$; we must find d so that $\boldsymbol{a}c$ and $\boldsymbol{b}d$ also realize the same type. Part 2 of Lemma 4.1.9 implies $\operatorname{icl}_{\mathbb{G}}(\boldsymbol{a}) \cong \operatorname{icl}_{\mathbb{G}}(\boldsymbol{b})$ and they are finitely generated in the countable case (have cardinality $< \kappa$ in the uncountable case.) Since $\boldsymbol{K}$ is set-determined, we can replace these intrinsic closures by the models they generate. So there is an automorphism α of G taking one to the other and $\alpha(c)$ is the required d. ⊣

In [63], Zilber remarks that the categoricity of a structure depends on its 'dimension' and 'homogeneity'. Our countable model $\mathbb{G}$ is a candidate; the dimension theory is given by the geometry and it is homogeneous if $\boldsymbol{K}$ is set-determined. We describe below a variant of this construction to construct a quasiminimal excellent class which is not homogeneous.

We will discuss two strategies for producing categoricity via the Hrushovski construction: the first order strategy (Subsection 4.2) and the quasiminimal excellent strategy (Subsection 5.3).

The first order strategy aims to show that the structure $\mathbb{G}$ is strongly minimal in the geometry ($\alpha = 1$ see 4.1.1) case and at least $\aleph_1$-categorical otherwise. (It is often almost strongly minimal; see [6].) When $\alpha = 1$, for every M and for every singleton a, $d_M(a) \leq 1$. The key idea is to force, by controlling the primitive extensions, that if $d_M(a) = 0$, then a is algebraic. From this it is easy to deduce strong minimality.

DEFINITION 4.1.12. *Let $A, B \in \boldsymbol{K}_0$. We say A is* primitive *over B if $\delta(A/B) = 0$ and for any A' with $B \subset A' \subset A$, $\delta(A/A') < 0$.*

In the *ab initio* case (see Example 4.1.1) one needs to also minimize the base B; in the bicolored field case this falls out from the general theory of canonical bases.

The following description (accurate in the *ab initio* case) oversimplifies the statement in e.g., the bicolored field case, but expresses the spirit of the argument. Suppose $A/B \in \boldsymbol{K}_0$ is primitive, let M be $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}})$-homogeneous and let $\chi_M(A/B)$ denote the number of copies of A over B in M. To guarantee $\aleph_1$-categoricity of the generic, one studies the subclass $\boldsymbol{K}^{\mu}$ of $\boldsymbol{K}_0$ where for each primitive A/B,

$$\chi_M(A/B) \leq \mu(A/B)$$

for a given function μ from primitive pairs into the integers. If the generic model for $\boldsymbol{K}^{\mu}$ is ω-saturated, categoricity follows easily. Baldwin and Holland [5] provide a sufficient condition for the ω-saturation of the generic. Another approach is to show that the types which are directly controlled by the geometry do in fact determine the entire theory. Hrushovski [25] summarized the goal of this strategy as the production of a Robinson theory—essentially a universal theory with the amalgamation property. (Hrushovski gave a syntactic condition equivalent to amalgamation by [2].) Then [25] proves that (on the existentially closed models of a Robinson theory) all existential formulas are equivalent to a (possibly infinite) disjunction of quantifier-free formulas. Definition 4.1.13 makes the connection with $(\boldsymbol{K}, \prec_{\boldsymbol{K}})$. See also [26].

DEFINITION 4.1.13. *Assume* $\boldsymbol{K}$ *has* δ*-formulas for minimal intrinsic extensions over subsets. Form the language* L^+ *by adding a relation symbol* $R_{AB}(\boldsymbol{x})$ *for each pair* (A, B) *where* B *is a minimal intrinsic extension of* A. *For any of our theories,* T^0 *(see Definition* 4.1.2*),* T^0_{nat} *is the* L^+*-theory extending* T^0 *which asserts*:

$$[\exists \boldsymbol{y}\phi_{AB}(\boldsymbol{x}, \boldsymbol{y})] \leftrightarrow R_{AB}(\boldsymbol{x}).$$

We denote the natural expansion of an L*-structure* N *to* L^+ *by* N^+ *and the collection of expansions of models in a class* $\boldsymbol{K}$ *by* $\boldsymbol{K}^+$.

If the theory of the existentially closed in L^+ models of $\boldsymbol{K}^+$ is first order axiomatizable then it admits quantifier elimination. Thus one technique for determining ω-stability (or strong minimality) is just to study the quantifier free L^+-types. In fact, as we briefly describe in the next subsection most of the published work uses two other techniques.

4.2. The first order case. In the first order situation, the first step is:

I. Show $\boldsymbol{K}_0 = \{A : \delta(A) \geq 0\}$ is first order axiomatizable.

Now the aim is to construct a complete first order theory. One approach is to show

II. the generic model is saturated and take its theory.

An alternative strategy is to show one of

IIIa Show $\boldsymbol{K}^{ec}$, the class of L-structures that are existentially closed for $\boldsymbol{K}_0$ is first order.

IIIb Show $\boldsymbol{K}^{ec,+}$, the class of L^+-structures that are existentially closed for $\boldsymbol{K}_0^+$ is first order.

IIIc Show that the class of $(\boldsymbol{K}_0, \prec_{\boldsymbol{K}_0})$-homogeneous universal models is first order axiomatizable.

If any of Condition II or IIIa, IIIb, IIIC holds, we find a compete ω-stable theory.

If the generic model is saturated then IIIb holds. But there are cases where IIIb holds but the generic is not saturated. (e.g., the Shelah-Spencer random graph [9]). And [1] provides a 'toy' example where δ maps into the integers but the generic is not ω-saturated.

Poizat has introduced the study of an intermediate stage; construct various expansions of fields with infinite rank by the Hrushovski construction [42], [43]. This exercise helps to illuminate the situation in a simpler case than actually finding $\aleph_1$-categorical structures as in [24], [3], [5], [61]. We briefly compare the ω-stable case, $\boldsymbol{K}$, and the $\aleph_1$-categorical case, $\boldsymbol{K}^\mu$, in three situations.

Ab initio: Hrushovski proved $\aleph_1$-categoricity of $\boldsymbol{K}^\mu$ in the seminal [24]. Poizat [14] simplified the argument to $\boldsymbol{K}$ where the structure constructed has infinite rank. Holland [22] proved that the strongly minimal sets were model complete. Baldwin and Holland [6] have shown that Baldwin's projective plane [3] is model complete after adding some constants.

Bicolored fields: Expand $\mathbb{C}$ by an infinite unary predicate. For $\boldsymbol{K}$, the infinite rank case, Steps IIIc and II are fairly straightforward [42]. Note that IIIa fails although IIIb follows from II. In the finite rank case, $\boldsymbol{K}^\mu$, the harder Step II is done by Baldwin-Holland [5]. For this, it is essential that the function μ be finite-to-one. Baldwin and Holland [6] have shown that the infinite rank bicolored field is not model complete, while the finite rank bicolored fields are. It is easy to check that bicolored fields are set-determined. So we get a homogeneous model even if μ is not finite to one. It is shown in [5] that the generic need not be saturated.

Bad fields: A *bad field* is a field, possibly with additional predicates, whose first order theory has finite Morley rank such that there is a proper definable infinite subgroup of the multiplicative group. The strategy for constructing one has long been apparent. Expand $\mathbb{C}$ by naming an infinite (torsion-free) subgroup of the multiplicative group; try to use the techniques above to find the desired complete theory. Condition I was proved by Poizat [43] and independently by Baldwin-Holland (Marker) (unpublished), using the Zilber-Hrushovski true CIT (see Theorem 5.1.2). Conditions IIIc and II are sketched by Poizat [43] for the infinite rank case $\boldsymbol{K}$. Holland has independent work [23] which yields a complete proof. For finite rank, $\boldsymbol{K}^\mu$, much remains open. In particular, the thumbtack lemma will play an important role although Poizat seems to have avoided it in the infinite rank case.

4.3. Analytic models of the Hrushovski construction. Zilber [59] suggested the following problem, which we noted in the first paragraph of this paper.

REMARK 4.3.1 (A thesis of Zilber). Hrushovski models can be obtained as pseudoanalytic structures.

The notion of *pseudoanalytic* is not well-defined. The strongest version is *a reduct of an 'analytic' expansion of the complex numbers*. But this is too strong and as we note below several weakenings have been proposed.

The term Hrushovski model is deliberately vague; it means an object constructed by the general method of Hrushovski. Zilber originally conjectured that every strongly minimal set was 'trivial', 'vector space like' or 'field like'. Hrushovski refuted this conjecture. This thesis is an attempt to regain the conjecture by widening the notion of field-like. The first widening is to replace first order definable in the complex numbers by the more general, interpretable in an analytic expansion of the complex field.

Thesis 4.3.1 rests on an article of faith: 'natural = canonical'. Again, canonical is read as categorical in some reasonable syntax. The Hrushovski constructions yield categorical objects, thus they must be representable in a natural object. Conversely, complex exponentiation is natural; ergo it must be canonical and so categorical. See Section 5 for progress on verifying this converse.

Thesis 4.3.1 is false with the given interpretation of pseudoanalytic. It is certainly impossible to realize strongly minimal sets as structures whose definable relations are analytic subsets of the complex numbers. (Analytic functions cannot be finite-to-one.) This leads to a second widening. Zilber has notions about how to weaken 'analytic expansion' to 'pseudoanalytic expansion'; some of these are reported in [41]. Roughly, the idea is that the strongly minimal structure can be found as a restriction of an analytic structure to a collection of infinitesimal neighborhoods.

However, the infinite rank ω-stable models built by the Hrushovski construction might have analytic models. In particular, there is one fully worked out exemplar of this conjecture, which is obtained by adding a generic unary function to the complex numbers. Consider the language $L\colon +, \cdot, 0, 1, H(x)$. Koiran [32], building on Wilkie [58], defines the limit theory of generic polynomials as the model completion of the class $\boldsymbol{K}_0$ arising from

$$\delta(x_1, \dots, x_n) = \mathrm{d_f}(x_1, \dots, x_n, H(x_1), \dots, H(x_n)) - |(x_1, \dots, x_n)|,$$

where $\mathrm{d_f}$ denotes transcendence degree. He proves:

THEOREM 4.3.2 (Koiran). *The limit theory of generic polynomials has a model* $(\mathbb{C}, +, \cdot, f)$ *where* f *is an analytic function.*

The function f is

$$f(x) = \Sigma_{i=1}^{\infty} \frac{x^i}{a_i}$$

where for every $\ell \geq 1$, $|a_{i+1}| \geq |a_i|^{i^\ell}$, for all sufficiently large i. Intuitively, the functions obtained by truncating f at x^n approximate generic degree n polynomials.

From the general study of the first order Hrushovski construction outlined earlier in this section we can see that the limit theory of generic polynomials is an ω-stable first order theory, but with infinite rank. Zilber pointed out that the analytic model M is quasiminimal excellent and so the $L_{\omega_1,\omega}(Q)$ -theory of M is categorical. Thus among the models with power $\aleph_1$, we can choose with $L(Q)$ one which is canonical (categorical); this model has many small definable sets.

§5. Complex exponentiation $(\mathbb{C}, +, \cdot, \exp)$**.** Zilber's general program is to formalize complex exponentiation as a well-behaved (ideally categorical) theory in some logic. There are several choices of what one takes as 'complex exponentiation'. They differ in the choice of logical vocabulary to describe 'complex exponentiation' and in the logic in which the theory Σ is expressed. We considered in Subsection 3.2 one weak approximation, which was formalized in $L_{\omega_1,\omega}$: covers of the multiplicative group. In Subsections 5.2 and 5.3 we report two further versions of the program; the first considers raising to real powers, the second full complex exponentiation. Zilber's work on this program involves several algebraic advances which we recount in 5.1. In this entire section we restrict to characteristic 0.

5.1. The necessary algebra. There are several algebraic results/conjectures which are needed for this program. We discussed the thumbtack lemma, which says that divisibly closed multiplicatively closed subgroups are finitely determined, as Theorem 3.2.5. We will have a further applications of it below. Two further assertions concern the rank of the intersection of tori with varieties.

DEFINITION 5.1.1. 1. *In this context, a* torus *is a variety in* k^n *given by equations of the form* $y_1^{m_1} \cdot \cdots \cdot y_n^{m_n} = c$ *where the* m_i *are integers.*
2. *The torus is* basic *if* $c = 1$.
3. *Let* $W \subseteq C^n$ *be an algebraic variety defined over* $\mathbb{Q}$, $T \subseteq (C^*)^n$ *a torus, and* S *an infinite irreducible component of* $W \cap T$. *We say* S *is an* atypical component *if*

$$\dim S > \dim W + \dim T - n.$$

In his paper, Conjectures on the Intersection of Tori [62], Zilber proves one theorem and makes two more ambitious conjectures.

THEOREM 5.1.2 (true CIT). [Zilber] *Given a variety* $W \subseteq \mathbb{C}^{n+k}$ *defined over* $\mathbb{Q}$, *there is a finite set* A *of nonzero elements of* $\mathbb{Z}^n$ *such that given any coset* $T \subseteq (\mathbb{C}^*)^n$ *of a torus and any* $\boldsymbol{b} \in \mathbb{C}^k$, *if* S *is an atypical component of* $W(\boldsymbol{b})$ *(where* $W(\boldsymbol{b}) = \{\boldsymbol{a} \subseteq \mathbb{C} \colon \boldsymbol{ab} \in W\}$*) then for some* $\boldsymbol{m} \in A$ *and some* γ *from* $\mathbb{C}$, *every element of* S *satisfies* $\boldsymbol{x}^{\boldsymbol{m}} = \gamma$.

True CIT is applied to show that certain algebraic geometric concepts can be expressed by first order axioms. That application is not strictly necessary for the study of pseudoexponentiation (Subsection 5.3); similar applications are essential for approaching the construction of a first order finite rank bad field. The study of real powers (Subsection 5.2) uses the following stronger conjecture on tori.

CONJECTURE 5.1.3 (full CIT). *For any variety* $W \subseteq \mathbb{C}^n$ *defined over* $\mathbb{Q}$, *there is a finite collection* $\tau(W) = \{T_1, \ldots, T_k\}$ *of proper basic tori in* $(C^*)^n$ *such that for any proper basic torus* $T \subseteq (C^*)^n$ *and any atypical component of* $W \cap T$,

$$S \subseteq T_i \text{ for some } T_i \in \tau(W).$$

A still more general version of CIT (replacing tori with semi-Abelian varieties) can be seen [62] as a generalization of both the Mordell-Lang and Manin-Mumford conjectures.

Finally, some aspects of the program depend on a conjecture in number theory that is almost fifty years old.

CONJECTURE 5.1.4 (Schanuels 's Conjecture:). *If $x_1, \dots, x_n$ are $\mathbb{Q}$-linearly independent complex numbers then $x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n}$ has transcendence degree at least n over $\mathbb{Q}$.*

5.2. Raising to powers. In [64] Zilber considers structures: (D, Ex, R) where D is an infinite dimensional vector space over a fixed countable field K of characteristic 0, R is a field of characteristic 0, Ex is a homomorphism of the additive group of D onto the multiplicative group R^* of the field.

He is mainly concerned with the case that D and R are both the complex numbers, K is the reals, and Ex is interpreted as exponentiation exp. Under this interpretation, for each real number a, the formula $(\exists z) z = \mathrm{Ex}(z) \wedge y = \exp(a \cdot z)$ defines a relation which is represented locally by an analytic function $y = z^a$.

The theory Σ is obtained by another first order application of the Hrushovski construction. The appropriate rank is given by:

$$\delta(X) = \mathrm{ld}_K(X) + \mathrm{d_f}(\mathrm{Ex}(X)) - \mathrm{ld}_Q(X),$$

where ld stands for linear dimension.

Here Ex is a unary function that is being axiomatized. Zilber gets positive solutions for steps I (using Lemma 5.1.2) and IIIb of the first order strategy and concludes that every completion of the theory is superstable. This gives an approach to the ideal of a well-behaved theory, with superstability rather than categoricity. Using Schanuel and Conjecture 5.1.3, and now interpreting Ex as exp, complex exponentiation, he gets the following result:

THEOREM 5.2.1 (Zilber). *Assume full CIT and Schanuel's Conjecture. The first order theory of the complex numbers with raising to all* real *powers allows quantifier elimination in an appropriate language and is superstable.*

5.3. Pseudo-exponentiation. A pseudo-exponential is a unary function from the additive group of a field to the multiplicative group that satisfies certain conjectural properties of complex exponentiation.

The most ambitious aim of the pseudoanalytic model program is to realize $(\mathbb{C}, +, \cdot, \exp)$ as a model of an $L_{\omega_1,\omega}$-sentence discovered by the Hrushovski construction. This program, the quasiminimal excellent strategy, has two parts.

Objective A. Model theory: Using a Hrushovski like dimension function, expand $(\mathbb{C}, +, \cdot)$ by a unary function f which behaves like exponentiation. Prove that there is an $L_{\omega_1,\omega}$-sentence Σ satisfied by $(\mathbb{C}, +, \cdot, f)$ that is quasi-minimal excellent.

Objective B. Algebra and analysis: Prove $(\mathbb{C}, +, \cdot, \exp)$ is a model of the sentence Σ found in Objective A.

In this section we outline the argument concerning complex exponentiation from [61] which obtains objective A outright and formulates precise algebraic conjectures sufficient for objective B. Section 3.2 concerns quasiminimal excellent classes without the Hrushovski construction; Sections 4.2 and 5.2 concerns the first order Hrushovski construction. Here the two methods are joined.

We follow Zilber and describe Σ by successively presenting classes, denoted by $\mathcal{E}$ with various decorations, culminating in $\mathcal{E}C^*_{st}$, which is the quasiminimal excellent class. This establishes Objective A. At the conclusion of this section we discuss the status of Objective B.

Fix the language L to contain $+, \{\frac{1}{m} : m \in \omega\}, E, \{V(\boldsymbol{x}) :$ V a variety$\}$.

NOTATION 5.3.1. *Let $\mathcal{E}$ be the class of L-structures F which are reducts of algebraically closed fields. Naturally $+$ is $+$, the $\frac{1}{m}$ denote division by natural numbers, the $V(\boldsymbol{x})$ are the solution sets of varieties, and E is the graph of a surjective map* exp *from F to F^*, which is a homomorphism between the additive and multiplicative group.*

Note that we have the graph of multiplication but not the multiplication function; this allows us to consider partial maps which approximate our eventual exp. Denote by *sub* $\mathcal{E}$ the class of all substructures A of members of $\mathcal{E}$ such the domain $\boldsymbol{D}_A$ of exp is closed under addition and multiplication by rationals.

This construction varies from the first order case in several respects. One technical innovation is that the dimension function δ is defined relative to its ambient structure A. For X a finite subset of A,

$$\delta_A(X) = \mathrm{d_f}(X \cup \mathrm{span}(\exp X)) - \mathrm{ld}(X).$$

Here $\mathrm{d_f}$ is transcendence degree and ld is additive vector space dimension. More important, the actual 'amalgamation class' is restricted in three ways. The first is analogous to the earler examples; we restrict to those structures for which the function exp formally satisfies Schanuel's Conjecture. These restrictions use infinitary axioms.

NOTATION 5.3.2 (Formal Schanuel's Conjecture). *sub* $\mathcal{E}^0$ *is the class of $A \in$ sub $\mathcal{E}$ such that $\delta(A)$ is hereditarily nonnegative.*

That is, the assertion that

$$\delta_A(X) = \mathrm{d_f}(X \cup \mathrm{span}(\exp X)) - ld(X) \geq 0$$

amounts for $A = \mathbb{C}$ and X linearly independent to the Schanuel conjecture (for exp). At this point, this is only a requirement on an abstract function exp. *A priori* the axiom can be expressed in $L_{\omega_1,\omega}$; using the Holland-Poizat-Zilber variant on true CIT, the axiom can be made first order.

For $A \in sub\,\mathcal{E}$, $\ker_A$ denotes the kernel of the exponential map. That is, $\ker_A = \{a : a \in A \wedge \exp(a) = 1\}$. If $\ker_A$ is isomorphic to the integers we say it is *standard*; if $\boldsymbol{D}_A \,/\, \ker_A$, as a subgroup of k^* for algebraically closed k^*, contains all the torsion points we say A has *full kernel*. Now, the second restriction is given by:

NOTATION 5.3.3 (Z-standard). *$sub\,\mathcal{E}^0_{st}$ is the class of $A \in sub\,\mathcal{E}^0$ such that* $\ker_A$ *is both standard and full.*

Stating this condition is the first of several uses of $L_{\omega_1,\omega}$. The requirement that the kernel of the function f is always Z leads to the failure of homogeneity.

Let V be a variety in $2n$ variables and let $\mathrm{pr}_{\boldsymbol{x}}\, V$ denote the projection on $\boldsymbol{x}$, $\mathrm{pr}_{\boldsymbol{y}}\, V$ the projection on $\boldsymbol{y}$. A variety V contained in F^{2n}, which is definable over $A \cup f(C) \cup \ker(f)$, is *absolutely free of additive dependencies (of multiplicative dependencies)* if for any generic realization $\boldsymbol{a}$ of $\mathrm{pr}_{\boldsymbol{x}}\, V$ is additively (multiplicatively) linearly independent over $\mathrm{acl}(A)$. We say *free* for free absolutely free of both multiplicative and additive dependencies. By a *linear image* of a variety V in F^{2n}, we mean the result of applying a linear map additively in the first n-coordinates and multiplicatively in the second n. V is *normal* if (very roughly) for $k \leq n$, any linear image of V in F^{2k} has dimension at least k. A variety defined by a system of polynomial equations P is *non-overdetermined* if it is normal. The idea is that a generic variety in F^{2k} should have dimension k, (because exp links the additive and multiplicative group). If the variety itself, or some linear image of it has dimension less than k then it is over-determined.

NOTATION 5.3.4 (Existential Closure). *$\mathcal{E}C^*_{st}$ is the members of $sub\,\mathcal{E}^0_{st}$ satisfying this condition*: *Any* free *and* non-overdetermined *irreducible system of polynomial equations*

$$P(x_1, \dots, x_n, y_1, \dots, y_n)$$

has a generic solution satisfying

$$y_i = f(x_i).$$

To demand a generic solution, $L_{\omega_1,\omega}$ is needed, so for the purposes of Objectives A and B, there is no gain in showing that the weaker condition omitting 'generic' is first order axiomatizable. But it is. The set of $\boldsymbol{z}$ such that $V(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ satisfies any of the following conditions is first order definable: exp-irreducible, absolutely additively free, absolutely multiplicatively free, normal. This fact depends on true CIT and the refining of it proved independently by Holland, Poizat, and Zilber. It turns out not to be essential here but the techniques will play a key role if a bad field is constructed.

We say F is *strongly exponentially algebraically closed* if for any exp-irreducible, additively and multiplicatively free, normal V defined over a finite $C \subset F$, there is a **generic** over C realization of V in F. So we can rephrase Assumption 5.3.4 as $\mathcal{E}C^*_{st}$ is the class of strongly exponentially algebraically closed structures.

Define a closure operation $\mathrm{cl}_F(A)$ from δ exactly as in Definition 4.1.4.

THEOREM 5.3.5. *$A \to \mathrm{cl}_F(A)$ in $F \in \mathcal{E}C_{\mathrm{st}}$ is a closure operation (with exchange) and for any A, $\mathrm{cl}(A)$ is a strongly exponentially algebraically closed subset of F.*

As in Section 3.1, We say a field F satisfies the *countable closure condition* if the closure of any countable subset is countable.

Now, using the thumbtack lemma Lemma 3.2.5 to get excellence as in Section 3.2, Zilber proves:

THEOREM 5.3.6. *$\mathcal{E}C^*_{st}$ (see 5.3.4) is $L_{\omega_1,\omega}$-axiomatizable and in fact quasiminimal excellent.*

The members of $\mathcal{E}C^*_{st}$ with countable closure are categorical in all uncountable powers. This class is $L_{\omega_1,\omega}(Q)$-axiomatizable.

Theorem 5.3.6 concludes the proof of Objective A. Objective B is given by the following theorem.

THEOREM 5.3.7. *If the Schanuel conjecture holds in $\mathbb{C}$ and if the strong exponential closure axioms hold in $\mathbb{C}$, then*

1. $(\mathbb{C}, +, \cdot, \exp) \in \mathcal{E}C^*_{st}$.
2. $(\mathbb{C}, +, \cdot, \exp)$ *has the countable closure property.*

The hypothesis of this theorem is a research program. Work on Schanuel's conjecture has continued for fifty years; Zilber's existential closure condition yield new and interesting number theoretic problems. Marker [38] uses the theory of Hadamard factorization and the Schanuel conjecture to prove strong exponential closure for the case: V is defined by $P(x, y) = Y - X$. His method do not obviously generalize to arbitrary $P(x, y)$ and open questions arise for any $P(x_1, x_2, y_1, y_2)$.

Note that $\mathcal{E}C^*_{st}$ is an AEC, indeed axiomatized in $L_{\omega_1,\omega}$. But unlike the covers example in Section 3.2, $L_{\omega_1,\omega}(Q)$ is needed to axiomaize the countable closure property.

This program leads to a more general question. Are there general conditions under which an AEC induced from a rank δ as in Definition 4.1.4 must be excellent?

§6. Summary. The work discussed in this paper involves many of the themes of the last 50 years. Shelah's work in 'pure' model theory tries to develop a structure theory. His attempts to generalize the Morley theorem to richer logics yield a number of partial results. In particular, the notion of excellence is isolated as a key to the structure theory of uncountable models while the notion of Abstract Elementary Class arises naturally in attempting to prove the categoricity theorem for $L_{\omega_1,\omega}(Q)$. Zilber is more concerned with classical mathematical structures. But in his attempt to identify canonical mathematical structures as those whose theory (in an appropriate logic) is categorical in all powers, he is led to the same logics that Shelah investigated and to specialized versions of some of the same theorems. Hrushovski used a special kind of Abstract Elementary Class to refute the trichotomy conjecture. In his attempt to regain the conjecture, Zilber uses an infinitary variant of the techniques. His investigation of complex exponentiation yields not only exciting model theory but new results and conjectures in algebraic geometry and complex analysis.

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