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BARWISE: ABSTRACT MODEL THEORY AND GENERALIZED QUANTIFIERS

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CONTENTS

§1. Introduction. After the pioneering work of Mostowski [29] and Lindström [23] it was Jon Barwise's papers [2] and [3] that brought abstract model theory and generalized quantifiers to the attention of logicians in the early seventies. These papers were greeted with enthusiasm at the prospect that model theory could be developed by introducing a multitude of extensions of first order logic, and by proving abstract results about relationships holding between properties of these logics. Examples of such properties are

κ-compactness. *Any set of sentences of cardinality $\leq \kappa$, every finite subset of which has a model, has itself a model.*

Löwenheim-Skolem Theorem down to κ. *If a sentence of the logic has a model, it has a model of cardinality at most κ.*

Interpolation Property. *If ϕ and ψ are sentences such that $\models \phi \to \psi$, then there is θ such that $\models \phi \to \theta$, $\models \theta \to \psi$ and the vocabulary of θ is the intersection of the vocabularies of ϕ and ψ.*

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Lindström's famous theorem characterized first order logic as the maximal $\aleph_0$-compact logic with Downward Löwenheim-Skolem Theorem down to $\aleph_0$. With his new concept of absolute logics Barwise was able to get similar characterizations of infinitary languages $L_{\kappa\omega}$. But hopes were quickly frustrated by difficulties arising left and right, and other areas of model theory came into focus, mainly stability theory. No new characterizations of logics comparable to the early characterization of first order logic given by Lindström and of infinitary logic by Barwise emerged. What was first called soft model theory turned out to be as hard as hard model theory.

Mostowski [29] characterized first order logic model theoretically among extensions of first order logic obtained by adding one so called simple unary generalized quantifier (see below). Lindström first in [22] extended Mostowski's characterization to non-unary generalized quantifiers and then in [23] to extensions of first order logic satisfying only some fairly mild conditions. For these results Mostowski's proofs used methods going hardly beyond what is needed in predicate logic with only unary predicates. Lindström took advantage of the full power of first order model theory of the time. Barwise [2] extended Lindström's methods to the then new infinitary model theory, combining them with ideas from the emerging generalized recursion theory. His main accomplishment was a characterization of infinitary languages $L_{\kappa\omega}$ and their fragments.

A question, asked, among others, by Feferman [13], Friedman [15] and Shelah [33], right from the beginning of the study of abstract logic and generalized quantifiers was

Open Problem. Is there a proper $\aleph_0$-compact extension of first order logic which has the Interpolation Property?

Different attempts to solving this question dominate also this review of Barwise's work in the area of abstract model theory generalized quantifiers.

The volume [8] edited by Barwise and Feferman is a good handbook for the developments in abstract model theory and generalized quantifiers through the 80's.

§2. Generalized quantifiers. At least two kinds of extensions of first order logic were already well-known in the 50's, namely infinitary logics $L_{\kappa\lambda}$ and higher order logics. In both cases mainly negative results were known: results of incompactness, incompleteness, etc. With the appearance of generalized quantifiers in the early 60's extended logics with compactness and completeness theorems emerged.

Mostowski defined generalized quantifiers as follows: A (simple unary) generalized quantifier is a collection Q of pairs (I, X), $X \subseteq I$, satisfying the condition

$$\big((I, X) \in \mathrm{Q}, |X| = |Y| \text{ and } |I - X| = |J - Y|\big) \Rightarrow (J, Y) \in \mathrm{Q},$$

for example:

$$\mathrm{Q}_n = \{(I, X) \colon |X| \geq \aleph_n\}\,.$$

Such a quantifier Q can be thought of as a logical operation by adding the following rules to the rules of ordinary first order logic:

- If $\phi(x, \bar{y})$ is a formula, then so is $Qx\phi(x, \bar{y})$.
- $\mathfrak{A} \models Qx\phi\,(x, \bar{a}) \Leftrightarrow (A, \{b\colon\ \mathfrak{A} \models \phi\,(b, \bar{a})\}) \in \mathrm{Q}$.

Let us denote the resulting extension of first order logic by $L_{\omega\omega}\,(Q)$. Mostowski proved that $L_{\omega\omega}\,(Q_0)$ cannot be (effectively) axiomatized. He argued as follows: Let θ be the sentence

$$\forall x \neg Q_0 y\,(y < x)$$

of $L_{\omega\omega}\,(Q_0)$. Let P denote the ordinary first order Peano axioms. Now for any first order sentence ϕ of number theory we have

$$(\mathbb{N}, +, \cdot, 0, 1, <) \models \phi \Leftrightarrow (\forall\,\mathfrak{A} \models \mathrm{P})\,(\mathfrak{A} \models (\theta \to \phi))\,.$$

Since there is no arithmetical method to decide the left hand side there cannot be any complete and arithmetical provability predicate for $L_{\omega\omega}\,(Q_0)$ either. Barwise [3] extended this argument to its full power by showing that inside $L_{\omega\omega}\,(Q_0)$ hides in implicit form an infinitary logic, namely the smallest admissible fragment L_{HYP}.

Mostowski asked whether the extension

$$L_{\omega\omega}\,(Q_1)$$

of first order logic, for which the above argument clearly fails, is axiomatizable. A positive answer was provided by Vaught [38] using an indirect argument. Keisler [20] gave a simple explicit axiomatization based on the principle that a countable union of countable sets is countable. Shelah [33] extended $L_{\omega\omega}\,(Q_1)$ to stationary logic $\mathrm{L}_{\omega\omega}(\mathrm{aa})$ (see Section 9 below). Barwise, in co-operation with Kaufmann and Makkai [9, 10] showed that stationary logic has a natural explicit axiomatization, much like Keisler's for $\mathrm{L}_{\omega\omega}(Q_1)$.

Mostowski also gave a model theoretic characterization of the first order quantifiers: Any extension obtained from first order logic by adding a simple unary generalized quantifier, which satisfies the condition

Every sentence with an infinite model has a model in every infinite cardinality.

is equivalent to first order logic.

Härtig [16] and Rescher [32] introduced the quantifiers

$$\begin{aligned}&\mathfrak{A} \models \mathrm{I}xy\phi\,(x, \bar{a})\,\psi\,(y, \bar{a}) \Leftrightarrow\\ &|\{b\colon\ \mathfrak{A} \models \phi\,(b, \bar{a})\}| = |\{b\colon\ \mathfrak{A} \models \psi\,(b, \bar{a})\}|\end{aligned}$$

and

$$\mathfrak{A} \models \mathrm{R}xy\phi(x,\bar{a})\,\psi(y,\bar{a}) \Leftrightarrow$$
$$|\{b\colon \mathfrak{A} \models \phi(b,\bar{a})\}| \leq |\{b\colon \mathfrak{A} \models \psi(b,\bar{a})\}|,$$

both of which went beyond what could be expressed with Mostowski's quantifiers. Lindström tells in [24] how he came, after unsuccessful attempts to view the quantifiers of Härtig and Rescher as examples of Mostowski's quantifiers, upon the following even more general concept, which has subsequently become the standard definition of generalized quantifiers:

DEFINITION 1. *Suppose L is a relational vocabulary and* Q *is a class of L-structures such that* Q *is closed under isomorphism. We add a new quantifier symbol* Q *to first order logic as follows: To simplify notation, let us assume that L consists of one unary predicate and one binary predicate only.*

$$\mathfrak{A} \models Qx, yz\phi(x,\bar{a})\,\psi(y,z,\bar{a}) \Leftrightarrow$$
$$(A, \{b\colon \mathfrak{A} \models \phi(b,\bar{a})\}, \{(b,c) : \mathfrak{A} \models \psi(b,c,\bar{a})\}) \in \mathrm{Q}.$$

Härtig's and Rescher's quantifiers correspond to the choices

$$\mathrm{I} = \{(A,X,Y)\colon |X| = |Y|\}$$
$$\mathrm{R} = \{(A,X,Y)\colon |X| \leq |Y|\}.$$

An example of a generalized quantifier in a vocabulary with a binary predicate, and one that plays a role in Barwise's study of absolute logics, is

$$\mathrm{WO} = \{(A,R)\colon R \subseteq A^2 \text{ well orders its field}\}.$$

We return of generalized quantifiers in Section 9.

§**3. Abstract logic.** Lindström's definition of abstract logics was merely a list of general properties the definable model classes of any abstract logic should have. No mention was made of the syntax of the logic. In this way Lindström achieved extreme generality. Barwise liked to emphasize the role of syntax even in abstract model theory. This is how Barwise defined abstract logics in [2]:

Let L be a vocabulary and let the concept of L-structure be the usual one. A *name changer* is a bijection π from L onto another vocabulary L' mapping n-ary predicate symbols to n-ary predicate symbols, n-ary function symbols to n-ary function symbols, and constant symbols to constant symbols. Associated with a name changer π is a natural operation on structures, mapping an L-structure $\mathfrak{A}$ onto an L'-structure $\mathfrak{A}_\pi$.

DEFINITION 2. *An* **abstract logic for a vocabulary** *L is a pair $(L^*, \models^*)$, where L^* is a set of objects called sentences of L^* and $\models^*$ is a relation between L-structures and sentences of L^*. We call $\models^*$ the satisfaction relation of L^*. The satisfaction relation is assumed to obey the following basic Isomorphism Condition: If L is a vocabulary and ϕ is an L^*-sentence then for all $\mathfrak{M}$:*

(1) *If* $\mathfrak{M} \models^* \phi$ *and* $\mathfrak{M} \cong \mathfrak{N}$, *then* $\mathfrak{N} \models^* \phi$.

A **system of logics** *is an operation* $*$ *which associates every countable vocabulary* L *with an abstract logic for* L *such that the following conditions are satisfied*:

(2) *If* $K \subseteq L$, *then* $K^* \subseteq L^*$ *and for every* $\phi \in K^*$ *and every* L*-structure* $\mathfrak{M}$, $\mathfrak{M} \upharpoonright K \models^* \phi$ *if and only if* $\mathfrak{M} \models^* \phi$.
(3) *If* $\pi \colon L \to K$ *is a name changer then for every* L^**-sentence* ϕ *there is a* K^**-sentence* ϕ_π *such that* $\mathfrak{M} \models^* \phi$ *if and only if* $\mathfrak{M}_\pi \models^* \phi_\pi$.

In many results other assumptions are used, such as closure under conjunction, negation and first order quantification.

This differs insignificantly from Lindström's definition in [23]. Lindström identifies a sentence with the class of its models. Thus an abstract logic for a vocabulary L will be a collection of classes of L-structures, each closed under isomorphism, and an abstract logic will be a collection of L-classes of structures for various L, each closed under isomorphisms. Corresponding to the above conditions (2) and (3) there are obvious conditions on reducts and changing the vocabulary.

Since Barwise wanted to put definability conditions on the logics, with the usual inductive definition of syntax and semantics in mind, he had to be explicit about syntax. Later in [3] he went further and used category theoretic concepts to emphasize the functorial nature of syntax. He considered the category $\mathcal{C}$ of all vocabularies with interpretations (by means of terms) of vocabularies as morphisms. Such morphisms induce canonically operations on structures corresponding to what is usually meant by interpretation of a structure in another. The **syntax** of an abstract logic $\mathcal{L}^*$ is a functor $*$ on some subcategory of $\mathcal{C}$ to the category of classes. Elements of L^* are called sentences. The functor is supposed to satisfy an *Occurrence Axiom*, which says, roughly, that for every sentence ϕ there is a smallest vocabulary L such that $\phi \in L^*$. The **semantics** of $\mathcal{L}^*$ is a relation $\models$ such that the *Isomorphism Axiom* (Condition (1) of Definition 2) is satisfied. The syntax and semantics of $\mathcal{L}^*$ are tied together by the *Translation Axiom* which is like Condition (3) of Definition 2.

Barwise compares in [3] his category theoretic approach with that of Lindström as follows:

> Lindström avoids syntactic considerations altogether since he deals directly with classes of structures, rather than with the sentences which define them. We find this approach unsatisfactory on two grounds. In the first place, it seems contrary to the very spirit of model theory where the primary object of study is the relationship between syntactic objects and the structures they define. Secondly, it fails to make explicit that the closure conditions on the classes of structures (like formation of indexed unions and its inverse) arise

out of natural syntactic considerations, considerations which seem implicit in the very idea of a model-theoretic vocabulary.

It is undoubtedly true that Barwise's category theoretic approach captures essential features of the interaction between syntax and semantics. This approach has certainly not yet been fully exploited. It may be that we have to know much more about extensions of first order logics in general before the fine points that Barwise's approach brings forward can flourish.

One challenge any attempt to develop a theory of syntax for model-theoretically defined languages has to face is the so called Δ-operation, arising as follows: A model class is said to be $PC(L^*)$ if it is the class of reducts of elements of an L^*-definable model class. If a model class and its complement (among structures of the same vocabulary) are $PC(L^*)$, we say that the model class is $\Delta(L^*)$. We can view $\Delta(L^*)$ as an abstract logic in a natural sense, and it indeed satisfies all the required properties. Moreover, if L^* is κ-compact (or axiomatizable), then so is $\Delta(L^*)$, and if L^* satisfies the Löwenheim-Skolem Theorem down to κ, then so does $\Delta(L^*)$.

The Interpolation Property implies $\Delta(L^*) = L^*$. Thus $\Delta(L^*)$ is an attempt to approach the Interpolation Property without losing such properties as $\aleph_0$-compactness. This lead to the question, does $\Delta(L^*)$ itself satisfy the Interpolation Property? Also, the question arose, whether we can build up some kind of real syntax for $\Delta(L^*)$, if we know L^* has a nice syntax. In particular, does $\Delta(\mathrm{L}_{\omega\omega}(Q_1))$ have a (nice) syntax? (For recent partial results on this question, see [17] and [36]). Friedman (see [18]) proved that $\Delta(\mathrm{L}_{\omega\omega}(Q_1))$ does *not* have the Interpolation Property, answering a question Keisler has asked. On the other hand, Barwise proved that $\Delta(\mathrm{L}_{\omega\omega}(Q_0))$ *does* have the Interpolation Property. The story is the following: Mostowski had proved in [30] that if L^* is a logic extending first order logic in which $(\omega, <)$ is definable and which has the Interpolation Property, then for every recursive ordinal α there is a sentence the class of countable models of which (coded as a subset of the Baire space ω^ω) is not a Borel set of class α. Barwise generalized this to: If L^* is a logic extending first order logic in which $(\omega, <)$ is definable, then $L_{\mathrm{HYP}} \subseteq \Delta(L^*)$. This gave:

Theorem 3 (Barwise [3]). $\Delta(\mathrm{L}_{\omega\omega}(Q_0)) = L_{\mathrm{HYP}}$.

Barwise had already in [1] proved that L_{HYP} has the Interpolation Property, so it follows that also $\Delta(\mathrm{L}_{\omega\omega}(Q_0))$ has it. Barwise and independently Makowsky [27] extended this to generalizations of Q_0 involving an arbitrary set X of integers, leading to a characterization of $L_{\omega_1\omega}$. No other result about the Δ-operation competes with the beauty and simplicity of the early result Theorem 3.

§4. Back-and-forth properties. Lindström generalized Mostowski's model-theoretic characterization of first order quantifiers to the context of his own more general concept of generalized quantifiers using an adaption of the back-and-forth technique, which he had previously rediscovered. This technique became a favorite of Barwise, too.

Let L be a finite relational vocabulary. It is easy to prove by induction on k that there are, up to logical equivalence, only finitely many first order L-formulas of quantifier rank $\leq k$ with the free variables $x_1, \dots, x_n$. Let Fml^n_k denote the finite set of these formulas. Let us consider two models $\mathfrak{M}$ and $\mathfrak{N}$ of the vocabulary L. For any finite sequence $\bar{x}$ of elements of M and another sequence $\bar{y}$ (of the same length n) of elements of N we write

$$(\mathfrak{M}, \bar{x}) \equiv^n_k (\mathfrak{N}, \bar{y})$$

if the sequence $\bar{x}$ satisfies in $\mathfrak{M}$ the same elements of Fml^n_k that the sequence $\bar{y}$ satisfies in $\mathfrak{N}$. In the special case that $n = 0$ the set Fml^0_k consists of sentences. In this case the relation $\equiv^0_k$ is an equivalence relation on L-structures with finitely many equivalence classes, each definable by a sentence in Fml^0_k.

Lemma 4. *A class of L-structures is first order definable if and only if for some k it is the union of equivalence classes of $\equiv^0_k$.*

Proof. Suppose a class K of L-structures is first order definable by ϕ with quantifier rank k. Clearly K is closed under $\equiv^0_k$ and therefore is the union of equivalence classes of $\equiv^0_k$. Conversely, every equivalence class of $\equiv^0_k$ is first order definable by a sentence in Fml^0_k. Therefore also the union of some of these finitely many classes is first order definable. ⊣

The relation $\equiv^n_k$ has the following important *back-and-forth* property:

1. If $k > 0$, $(\mathfrak{M}, \bar{x}) \equiv^n_k (\mathfrak{N}, \bar{y})$ and a is an element of M, then there is an element b of N such that $(\mathfrak{M}, \bar{x}, a) \equiv^{n+1}_{k-1} (\mathfrak{N}, \bar{y}, b)$.
2. If $k > 0$, $(\mathfrak{M}, \bar{x}) \equiv^n_k (\mathfrak{N}, \bar{y})$ and b is an element of N, then there is an element a of M such that $(\mathfrak{M}, \bar{x}, a) \equiv^{n+1}_{k-1} (\mathfrak{N}, \bar{y}, b)$.

Fraïssé [14] generalized this to the concept of a back-and-forth sequence, which came to play a central role in the study of infinitary logics. A **back-and-forth sequence of length** k **for** $\mathfrak{M}$ **and** $\mathfrak{N}$ is a sequence $\{E_i : i \leq k\}$ of binary relations between $\bar{x} \in M^{k-i}$ and $\bar{y} \in N^{k-i}$ such that

B1 $\emptyset\, E_i\, \emptyset$ for all $i \leq k$.
B2 If $\bar{x} E_i \bar{y}$, then the sequence $\bar{x}$ satisfies in $\mathfrak{M}$ the same elements of Fml^{k-i}_0 that the sequence $\bar{y}$ satisfies in $\mathfrak{N}$.
B3 If $j < i$, $\bar{x} E_i \bar{y}$ and a is an element of M, there is an element b of N such that $(\bar{x}, a) E_j (\bar{y}, b)$.
B4 If $j < i$, $\bar{x} E_i \bar{y}$ and b is an element of N, then there is an element a of M such that $(\bar{x}, a) E_j (\bar{y}, b)$.

Theorem 5 (Fraïssé). *$\mathfrak{M}$ and $\mathfrak{N}$ satisfy the same first order sentences of quantifier-rank $\leq k$ if and only if there is a back-and-forth sequence of length k for $\mathfrak{M}$ and $\mathfrak{N}$.*

PROOF. Let us first assume $\mathfrak{M} \equiv_k^0 \mathfrak{N}$. Then $\{\equiv_i^{k-i} : i \leq k\}$ is a back-and-forth sequence of length k for $\mathfrak{M}$ and $\mathfrak{N}$. On the other hand, if $\{E_i : i \leq k\}$ is a back-and-forth sequence of length k for $\mathfrak{M}$ and $\mathfrak{N}$, then it is easy to prove by induction that $\bar{x} E_i \bar{y}$ if and only if $\bar{x} \equiv_i^{k-i} \bar{y}$ for all $i \leq k$. ⊣

By putting together Lemma 4 and Theorem 5 we get a syntax-free characterization of first order logic, which proves quite useful in the forthcoming model-theoretic characterization.

LEMMA 6. *A class K of L-structures is first order definable if and only if there is a natural number k such that whenever $\mathfrak{A} \in K$ and there is a back-and-forth sequence of length k for $\mathfrak{A}$ and $\mathfrak{B}$, then $\mathfrak{B} \in K$.*

A **back-and-forth set** for $\mathfrak{M}$ and $\mathfrak{N}$ is a binary relation E between arbitrary sequences $\bar{x} \in M^{<\omega}$ and $\bar{y} \in N^{<\omega}$ of the same finite length such that

1. $\emptyset\, E\, \emptyset$
2. If $\bar{x} E \bar{y}$, and the length of $\bar{x}$ is n, then the sequence $\bar{x}$ satisfies in $\mathfrak{M}$ the same elements of Fml_0^n that the sequence $\bar{y}$ satisfies in $\mathfrak{N}$.
3. If $\bar{x} E \bar{y}$ and a is an element of M, there is an element b of N such that $(\bar{x}, a) E (\bar{y}, b)$.
4. If $\bar{x} E \bar{y}$ and b is an element of N, then there is an element a of M such that $(\bar{x}, a) E (\bar{y}, b)$.

If E is a back-and-forth set, then we get a back-and-forth sequence of any length by letting $\bar{x} E_i \bar{y}$ hold if and only if $\bar{x} E \bar{y}$. The simple "back-and-forth" proof of the following fundamental lemma is usually credited to Cantor:

THEOREM 7. *If $\mathfrak{M}$ and $\mathfrak{N}$ are countable and have a back-and-forth set, then $\mathfrak{M}$ and $\mathfrak{N}$ are isomorphic.*

§5. Lindström's Theorem. We defined above the concept of a back-and-forth sequence of length k for structures $\mathfrak{M}$ and $\mathfrak{N}$. In the following theorem we take advantage of a generalization of this concept. Let $(D, <)$ be any linear order. A sequence $\{E_i : i \in D\}$ is a *back-and-forth sequence of type $(D, <)$ for structure $\mathfrak{M}$ and $\mathfrak{N}$* if the above conditions (B1)-(B4) hold when "$i \leq k$" is replaced by "$i \in D$" and "$i < j$" is replaced by "$i <_D j$".

THEOREM 8 (Lindström [23] characterization of $\mathrm{L}_{\omega\omega}$). *$\mathrm{L}_{\omega\omega}$ is the largest $\aleph_0$-compact logic that satisfies the Löwenheim-Skolem theorem down to $\aleph_0$.*

PROOF. Suppose L is a finite vocabulary and $(L^*, \models^*)$ is an abstract logic for L. Assume that L^* is an $\aleph_0$-compact extension of first order logic satisfying the Löwenheim-Skolem theorem down to $\aleph_0$, and ϕ is an L^*-sentence, not equivalent to a first order sentence. (The assumption that L is finite can be eliminated but we omit this detail.) By Lemma 6, for any natural number k there are L-structures $\mathfrak{M} \models^* \phi$ and $\mathfrak{N} \not\models^* \phi$ and a back-and-forth sequence of length k for $\mathfrak{A}$ and $\mathfrak{B}$. Let $\pi : L \to L'$ be a name changer with $L \cap L' = \emptyset$, and $(\phi)_\pi$ the corresponding translation of ϕ into an $(L')^*$-sentence. Let D be a new unary predicate symbol and $<$ a new binary predicate symbol. Let K be a vocabulary which contains $L \cup L' \cup \{D, <\}$ together with some other necessary predicates (that we do not specify in this sketch). Let ψ be a K^*-sentence such that a K-structure $\mathfrak{N}$ is a model of ψ if and only if:

1. $\mathfrak{N} \restriction L \models^* \phi$
2. $\mathfrak{N} \restriction L' \not\models^* (\phi)_\pi$
3. $(D, <)^{\mathfrak{N}}$ is a linear order.
4. $\mathfrak{N} \restriction (K \setminus \{L \cup L'\})$ codes a back-and-forth sequence of type $(D, <)^{\mathfrak{N}}$ for $\mathfrak{N} \restriction L$ and $(\mathfrak{N} \restriction L')_{\pi^{-1}}$.

We know that ψ has models with $(D, <)^{\mathfrak{N}}$ arbitrarily long finite linear order. By applying the assumptions about L^* we can find a countable K-structure $\mathfrak{N}$ which is a model of ψ such that $(D, <)^{\mathfrak{N}}$ is non-well-founded. Let $d_0 >_D d_1 >_D \dots$ be an infinite descending chain in $(D, <)^{\mathfrak{N}}$. Let for any $\bar{x}, \bar{y} \in N^n$

$$\bar{x} E \bar{y} \iff \bar{x} E_{d_n} \bar{y}.$$

It is easy to see that E is a back-and-forth set for $\mathfrak{N} \restriction L$ and $(\mathfrak{N} \restriction L')_{\pi^{-1}}$. By Theorem 7, $\mathfrak{N} \restriction L \cong (\mathfrak{N} \restriction L')_{\pi^{-1}}$. This contradicts the assumption that the satisfaction relation of L^* is closed under isomorphism. $\dashv$

Let us look at the proof more closely. The role of the Löwenheim-Skolem theorem is to make Theorem 7 available. On the other hand, it is only needed to conclude that if $\mathfrak{M} \models^* \phi$ and there is a back-and-forth set for $\mathfrak{M}$ and $\mathfrak{N}$, then also $\mathfrak{N} \models^* \phi$. Barwise calls this property of L^* the **Karp property**. It is a consequence of the Löwenheim-Skolem theorem down to $\aleph_0$ (and equivalent to it under the assumption of Interpolation Property, as Barwise proved in [3]), and we may reformulate Theorem 8 as follows:

$L_{\omega\omega}$ is the largest $\aleph_0$-compact logic that has the Karp property.

What is the role of $\aleph_0$-compactness? We obtain a sentence $\psi \in L^*$ with the property that it has models $\mathfrak{N}$ with $(D, <)^{\mathfrak{N}}$ of any finite length, but ψ has no models with $(D, <)^{\mathfrak{N}}$ non-well-founded. Let us put this in an abstract form, following Barwise [3]: The **well-ordering number** $\mathrm{wo}(\psi)$ of a sentence ψ of any abstract logic for a vocabulary containing a unary predicate symbol D and a binary predicate symbol $<$, is the smallest (if any exist) ordinal γ such that if ψ has a model $\mathfrak{N}$ with $(D, <)^{\mathfrak{N}}$ well-ordered in type $> \gamma$, then ψ has a model $\mathfrak{N}$ with $(D, <)^{\mathfrak{N}}$ non-well-ordered. The **well-ordering number** $\mathrm{wo}(L^*)$ of an abstract logic L^* is the supremum of all $\mathrm{wo}(\psi)$, where $\psi \in L^*$. Any $\aleph_0$-compact logic L^* has, of course, $w(^*) = \omega$. We can again reformulate Theorem 8 as:

$L_{\omega\omega}$ is the largest logic with the Karp property, the well-ordering number of which is ω.

Lopez-Escobar [26, 25] proved that the well-ordering number of the infinitary logic $L_{\kappa\omega}$ is $< (2^\kappa)^+$. Thus $L_{\infty\omega}$ is what is called a **bounded** logic, i.e.,

$\mathrm{wo}(\psi)$ exists for each $\psi \in L_{\infty\omega}$ (even if $\mathrm{wo}(L_{\infty\omega})$ itself does not exist). The result of Lopez-Escobar raises the question whether there is an infinitary analogue of Theorem 8. Barwise studied this question in [2] and [3]. Before examining these results in detail, we review some preliminaries in infinitary logic.

§6. Infinitary back-and-forth properties. We refer to [21] for the definition of quantifier rank in the logic $L_{\infty\omega}$. Let Fml^n_γ be the set of formulas of $L_{\infty\omega}$ with quantifier rank $\leq \gamma$ and at most $x_1, \dots, x_n$ free. It is easy to prove by induction on γ that Fml^n_γ has, up to logical equivalence, at most $\beth_\gamma$ formulas. If γ is any ordinal we write

$$(\mathfrak{M}, \bar{x}) \equiv^n_\gamma (\mathfrak{N}, \bar{y})$$

if the sequence $\bar{x}$ satisfies in $\mathfrak{M}$ the same elements of Fml^n_γ that the sequence $\bar{y}$ satisfies in $\mathfrak{N}$. Thus $\equiv^0_\gamma$ is an equivalence relation in the class of all L-structures with at most $\beth_\gamma$ equivalence classes. Thus we have, analogously to Lemma 4:

Lemma 9. 1. *A class of L-structures is $L_{\infty\omega}$-definable if and only if for some γ it is the union of equivalence classes of $\equiv^0_\gamma$.*

2. *Suppose $\kappa = \beth_\kappa$. A class of L-structures is $L_{\kappa\omega}$-definable if and only if for some $\gamma < \kappa$ it is the union of equivalence classes of $\equiv^0_\gamma$.*

A **back-and-forth sequence of length** γ **for** $\mathfrak{M}$ **and** $\mathfrak{N}$ is a sequence $\{E^n_\alpha : \alpha \leq \gamma, n < \omega\}$ of binary relations between $\bar{x} \in M^n$ and $\bar{y} \in N^n$ such that

1. $\emptyset\, E^0_\alpha\, \emptyset$ for all $\alpha \leq \gamma$.
2. If $\bar{x} E^n_\alpha \bar{y}$, then the sequence $\bar{x}$ satisfies in $\mathfrak{M}$ the same elements of Fml^n_0 that the sequence $\bar{y}$ satisfies in $\mathfrak{N}$.
3. If $\alpha > \beta$, $\bar{x} E^n_\alpha \bar{y}$ and a is an element of M, there is an element b of N such that $(\bar{x}, a) E^{n+1}_\beta (\bar{y}, b)$.
4. If $\alpha > \beta$, $\bar{x} E^n_\alpha \bar{y}$ and b is an element of N, then there is an element a of M such that $(\bar{x}, a) E^{n+1}_\beta (\bar{y}, b)$.

The proof of the following result is almost identical to the proof of Lemma 5:

Theorem 10 (Karp [19]). *$\mathfrak{M}$ and $\mathfrak{N}$ satisfy the same $L_{\infty\omega}$-sentences of quantifier-rank $\leq \gamma$ if and only if there is a back-and-forth sequence of length γ for $\mathfrak{M}$ and $\mathfrak{N}$.*

§7. Characterizing infinitary logics. Let us return to the problem whether Theorem 8 has a generalization to infinitary logic. By combining Lemma 9 and Theorem 10 with the remarks we have already made, we obtain:

Theorem 11 (Barwise [3]). *Assume $\kappa = \beth_\kappa$. Then $L_{\kappa\omega}$ is the largest logic with the Karp property, the well-ordering number of which is $\leq \kappa$.*

COROLLARY 12 (Barwise [3]). *$L_{\infty\omega}$ is the largest bounded logic with the Karp property.*

Theorem 11 characterizes some infinitary logics, but there is the awkward assumption $\kappa = \beth_\kappa$. What about all $L_{\kappa\omega}$, where $\kappa < \beth_\kappa$, e.g., what if $\kappa = \aleph_n$? Barwise succeeded in characterizing also these logics by thinking of them in terms of *definability* constraints, as in generalized recursion theory, rather than *cardinality* constraints. This idea had already proved useful in his other work (see [21]).

On the other hand, we can leave $\kappa = \beth_\kappa$ as it is, but ask if the rather strong Karp property can be weakened. A combination of a Löwenheim-Skolem type property and the boundedness property (and $\kappa = \beth_\kappa$) is used in [37] to characterize, not $L_{\kappa\omega}$, but a new infinitary language between $L_{\kappa\omega}$ and $L_{\kappa\kappa}$, one with the Interpolation Property.

§8. Absolute logics.

Barwise wanted to give a generalized recursion theoretic definition of when we should call a logic, looking at it from outside, first order. Clear cases of first order logics were weak second order logic (with variables for finite sets), $\mathrm{L}_{\omega\omega}(Q_0)$ and the admissible fragment L_{HYP}. Intuitively a logic is, from outside, first order if the truth of a sentence in a structure should depend only on what kind of elements the domain M of $\mathfrak{M}$ has, not on what kind of subsets M has. This leads to the following definition:

DEFINITION 13. *Let T be a true set theory extending the Kripke-Platek axioms KP. An abstract logic L^* is* **absolute relative to** *T if there are Σ_1-predicates $R(x, y)$ and $S(x, y, z)$ and a Π_1-predicate $P(x, y, z)$ such that*

1. *For all ϕ: $\phi \in L^*$ if and only if $R(\phi, L)$.*
2. *For all $\phi \in L^*$ and all L-structures $\mathfrak{M}$, $\mathfrak{M} \models^* \phi$ if and only if $S(\mathfrak{M}, \phi, L)$.*
3. *The following is a theorem of T: For all languages z, all z-structures $\mathfrak{M}$ and all ϕ such that $R(\phi, z)$, $S(\mathfrak{M}, \phi, z)$ if and only if $P(\mathfrak{M}, \phi, z)$.*

An abstract logic is **strictly absolute** *if it is absolute relative to KP (or KP +Infinity).*

In specific results absolute logics are assumed to satisfy various closure conditions like closure under conjunction and other logical operations. In such cases the operations manifesting the closure are assumed to be absolute, too.

If L^* is an abstract logic and A is an admissible set, we use L^*_A to denote the sub-logic of L^* consisting of sentences which are elements of A. For $A = H(\kappa)$ we denote L^*_A by L^*_κ.

The logics $\mathrm{L}_{\omega\omega}$, $\mathrm{L}_{\omega\omega}(Q_0)$, $L_{\omega_1\omega}$ and $L_{\infty\omega}$ are, of course, absolute relative to *KP*. The weak second order logic is strictly absolute. The unbounded logic $L_{\infty\omega}(WO)$ is absolute relative to *KP* +Σ_1-separation. If we add the game quantifier

$$\forall x_1 \exists x_2 \forall x_3 \cdots \bigvee_{n<\omega} \phi_n(x_1, \ldots, x_{p_n}, \bar{y})$$

and its dual

$$\exists x_1 \forall x_2 \exists x_3 \cdots \bigwedge_{n<\omega} \phi_n(x_1, \ldots, x_{p_n}, \bar{y})$$

to $L_{\infty\omega}$ an interesting (also unbounded) logic, denoted by $L_{\infty G}$ emerges. This logic is absolute relative to $KP + \Sigma_1-$separation. The smallest admissible fragment L_{HYP} of $L_{\omega_1\omega}$ is an interesting absolute logic (see Theorem 3). The infinite quantifier logic $L_{\omega_1\omega_1}$ and second order logic L^2 are not absolute relative to any true first order set theory T. These logics would be absolute relative to a second order set theory but that is beside the point here as we plan to take advantage of results of first order set theory, such as:

THEOREM 14 (Levy Reflection Principle). *If $\phi(\bar{x})$ is a Σ_1-formula and $\kappa > \omega$ then $\forall \bar{x} \in H(\kappa)(\phi(\bar{x}) \rightarrow H(\kappa) \models \phi(\bar{x}))$.*

We can make some immediate observations about absolute logic L^* by means of Theorem 14: If $\phi \in L^*_{\kappa^+}$ has a model, then it has a model in $H(\kappa^+)$ and therefore a model of cardinality $\leq \kappa$. Thus $L^*_{\kappa^+}$ satisfies the Löwenheim-Skolem Theorem down to κ. We can prove the Karp Property almost as quickly: Suppose there is a back-and-forth set E for $\mathfrak{M}$ and $\mathfrak{N}$, but for some $\phi \in L^*$ we have $\mathfrak{M} \models^* \phi$ and $\mathfrak{N} \not\models^* \psi$. All this is Σ_1, so by Theorem 14 such objects $L, \phi, \mathfrak{M}, \mathfrak{N}, E$ must exist already in HC. But then $\mathfrak{M}$ and $\mathfrak{N}$ are countable, hence isomorphic by Lemma 7, a contradiction. We know from [3] that Interpolation Property together with the Karp Property imply Löwenheim-Skolem Theorem down to $\aleph_0$. Thus we may conclude that no absolute logic extending $L_{\omega_2\omega}$ can have the Interpolation Property.

THEOREM 15 (Barwise [2]). *If L^* is a strictly absolute logic and A is an admissible set, then L^*_A is contained in L_A.*

PROOF. The first observation is that it suffices to prove this for countable admissible sets. Why? Suppose the claim fails. Thus there is an admissible set A and a sentence $\phi \in L^*_A$ such that for all $\psi \in L_A$ there is a model $\mathfrak{M}$ of $\neg(\phi \leftrightarrow \psi)$. This can be written as a Σ_1-property of A. If an A with this property exists at all, one such exists in HC by Theorem 14.

The second observation is that Barwise proved in [1] that if A is a countable admissible set, then L_A satisfies the Interpolation Property, whence $\Delta(L_A) = L_A$. Thus it suffices to prove that if A is a countable admissible set, then L^*_A is contained in $\Delta(L_A)$. Suddenly the claim has become much easier. To find an explicit L_A-definition for a given $\phi \in L^*$ is like looking for a needle in a haystack, compared to writing an "implicit" $\Delta(L_A)$-definition, where new predicates can set things in their right places and provide extra tools.

Finally, suppose L is a finite vocabulary, L^* is a strictly absolute logic and $\phi \in A$ is an L^*-sentence. We take a new vocabulary K containing L and the new symbols $\varepsilon, \bar{\mathfrak{M}}, \bar{\phi}, \bar{L}$. It is possible to write down a sentence Φ of K_A such that the following conditions are equivalent for any infinite L-structure $\mathfrak{A}$:

1. $\mathfrak{A} \models^* \phi$.
2. There is an expansion $\mathfrak{M}$ of $\mathfrak{A}$ which is a model of Φ and which satisfies $S(\bar{\mathfrak{M}}, \bar{\phi}, \bar{L})$.
3. Every expansion $\mathfrak{M}$ of $\mathfrak{A}$ to a model of Φ satisfies $S(\bar{\mathfrak{M}}, \bar{\phi}, \bar{L})$.

The point of our assuming that L^* is strictly absolute rather than just absolute is the following: When we consider the expansions $\mathfrak{M}$ of $\mathfrak{A}$ as set-theoretical structures, we have no way of knowing that they are well-founded. Still we want to form the Mostowski collapse of $\mathfrak{M}$ in order to get e.g., from $\bar{\phi}^{\mathfrak{M}}$ to the real ϕ. Fortunately we have included in Φ an infinitary sentence guaranteeing that at least the transitive closure of $\bar{\phi}^{\mathfrak{M}}$ is well-founded in $\mathfrak{M}$. So we take the standard part $\mathfrak{M}_0$ of $\mathfrak{M}$, knowing that it is still a model of KP, and collapse $\mathfrak{M}_0$. It follows that ϕ is $\Delta(L_A)$-definable. ⊣

We get from Theorem 15 as a special case the promised characterization of infinitary languages $L_{\kappa\omega}$ for any κ:

COROLLARY 16 (Barwise [2]). *If L^* is a strictly absolute logic and $\kappa > \omega$, then L^*_κ is contained in $L_{\kappa\omega}$.*

COROLLARY 17. *$L_{\infty\omega}$ is the largest strictly absolute logic.*

What about logics that are absolute but not strictly absolute? Since absolute logics have the Karp property, we can infer from Corollary 12 that $L_{\infty\omega}$ is the largest bounded absolute logic. The logic $L_{\infty G}$ is absolute but not a sub-logic of even $L_{\infty\infty}$. Maybe all absolute logics are sublogics of $L_{\infty G}$. The problem comes with Interpolation: $\Delta(L_{\infty G}) \neq L_{\infty G}$. So we have to settle with the result $(L_{AG} = L_{\infty G} \cap A)$:

PROPOSITION 18 (Oikkonen [31]). *If L^* is an absolute logic and A is an admissible set, then L^*_A is contained in $\Delta(L_{AG})$.*

Burgess [11] developed further the theory of absolute logics using methods of descriptive set theory. For example, he showed that formulas of all absolute logics have similar approximations as do formulas of $L_{\infty G}$.

§9. New generalized quantifiers. Early work on generalized quantifiers was dominated by questions related to the so-called cardinality quantifiers Q_n. A lot of insight was gained about Q_0 and Q_1, but the rest have remained hard to tackle. For example, we have still the following innocent looking open problem:

Open Problem. Is $\mathrm{L}_{\omega\omega}(Q_2)$ $\aleph_0$-compact?

Chang [12] gave a positive answer using GCH. Shelah [35] has recently shown that $\mathrm{L}_{\omega\omega}(Q_1, Q_2)$ may fail to be $\aleph_0$-compact.

Maybe other kinds of quantifiers are easier to tackle? Indeed, Shelah [33] introduced a host of new axiomatizable extensions of first order logic. A particularly nice new quantifier was the **cofinality quantifier**

$$\mathfrak{A} \models Q^{cf}_{\aleph_n} xy\, \phi(x,y,\bar{a}) \iff \{(c,d)\colon\ \mathfrak{A} \models \phi(c,d,\bar{a})\} \text{ is a linear order of its field of cofinality } \aleph_n.$$

What is interesting about $L_{\omega\omega}(Q^{cf}_{\aleph_n})$ is that it is, irrespectively of n and unlike $L_{\omega\omega}(Q_n)$, fully compact, i.e., κ-compact for all κ, and axiomatizable. As is characteristic of each new quantifier that was ever proposed, $L_{\omega\omega}(Q^{cf}_{\aleph_n})$ fails to have the Interpolation Property. In his search for new logics, Shelah [33] introduced the logic $L_{\omega\omega}(\text{aa})$. This logic has a generalized *second order* quantifier aa $s\,\phi(s)$ where s ranges over countable subsets of the domain. Intuitively aa $s\,\phi(s)$ says that almost all countable sets s satisfy $\phi(s)$. Naturally there are many candidates for interpreting "almost all", but the one that works here well is the following: A family which contains "almost all" countable subsets should at least *cover* every countable subset, i.e., if $A \subseteq M$ is countable, there should be $s \in X$ with $A \subseteq s$. In such a case we call X "unbounded". Another requirement is that X should be "closed" in the following sense: Whenever $s_0 \subseteq s_1 \subseteq \dots$ is an increasing ω-sequence of elements of X, also $\cup_{n<\omega} s_n$ should be in X. A set which is both unbounded and closed is called c.u.b. A family which meets every c.u.b. family is called *stationary*. The c.u.b. families form a normal filter on any set. Normality means that the following **Fodor's Lemma** holds: If X is a stationary family and f is a function on X such that $f(x) \in x$ for each $x \in X$, then there is a stationary $Y \subseteq X$ such that f is constant on Y. The interpretation of *aa* is thus:

$$\mathfrak{M} \models \text{aa}\, s\, \phi(s) \iff \{s\colon\ \mathfrak{M} \models \phi(s)\} \text{ contains a c.u.b. set.}$$

We can express both $Q_1 x\, \phi(x,\bar{y})$ and $Q^{cf}_{\aleph_0} xy\, \phi(x,y,\bar{x})$ by means of the new quantifier aa:

$$Q_1 x\, \phi(x,\bar{y}) \leftrightarrow \neg\, \text{aa}\ s\, \forall x(\phi(x,\bar{y}) \to s(x))$$

and, assuming $\phi(x,y,\bar{z})$ defines a linear order without last element,

$$Q^{cf}_{\aleph_0} xy\, \phi(x,y,\bar{z}) \leftrightarrow \text{aa}\ s\, \forall x(\exists y\, \phi(x,y,\bar{z}) \to \exists y(s(y) \wedge \phi(x,y,\bar{z}))).$$

THEOREM 19. [9, 10] *The logic $L(\text{aa})$ is complete relative to the axioms*

A0 aa $s\ \phi(s) \leftrightarrow$ aa $s'\ \phi(s')$
A1 $\neg$ aa $s(\text{false})$
A2 aa $s\ (s(x))$, aa $s'\ (s \subseteq s')$
A3 aa $s\ \phi \wedge$ aa $s\ \psi \to$ aa $s\ (\phi \wedge \psi)$
A4 aa $s\ (\phi \to \psi) \to$ (aa $s\ \phi \to$ aa $s\ \psi$)
A5 $\forall x$ aa $s\ \phi(x,s) \to$ aa $s\, \forall x(s(x) \to \phi(x,s))$

and the rule: From $(\phi \to \psi)$, *where s is not free in* ϕ, *infer* $(\phi \to \mathrm{aa}\ s\ \psi)$ *together with the usual axioms and rules of first order logic.*

Axiom A5 can be seen as a formulation of Fodor's Lemma giving the axioms a special air of naturalness. For some time there was great enthusiasm about this nice fragment of second order logic. Unfortunately even this logic fails to have the Interpolation Property [28]. There is even an implication in $\mathrm{L}_{\omega\omega}(Q_1)$ alone with no interpolant in $\mathrm{L}_{\omega\omega}(\mathrm{aa})$. However, one of the gems of the study of generalized quantifiers is the following relative Interpolation Property for $\mathrm{L}_{\omega\omega}(\mathrm{aa})$:

THEOREM 20. [34] *Stationary logic interpolates cofinality logic: If* ϕ *and* ψ *are sentences of* $\mathrm{L}_{\omega\omega}(Q^{cf}_{\aleph_0})$ *such that* $\models \phi \to \psi$, *then there is* $\theta \in \mathrm{L}_{\omega\omega}(\mathrm{aa})$ *such that* $\models \phi \to \theta$, $\models \theta \to \psi$ *and the vocabulary of* θ *is the intersection of the vocabularies of* ϕ *and* ψ.

In his paper [6] Barwise makes the observation that whenever a new $\aleph_0$-compact logic is proposed, it gives rise to an infinitary version with all the nice properties that the original admissible fragments enjoy. In this paper Barwise turns this observation into a theorem. He formulates an **Omitting Types Property** for an abstract logic L^* and proves that if L^* is an $\aleph_0$-compact logic and L^* satisfies Omitting Types Property, then $L^*_{\omega_1\omega}$ has many nice properties, e.g., a completeness theorem, boundedness theorem, and the admissible fragments are Σ_1-compact.

Barwise made also other contributions to the theory of generalized quantifiers, dealing with questions not directly related to issues discussed above. He applied approximations of branching quantifiers in model theory [4], and isolated monotonicity [7] and branching phenomena ([5]) among natural language quantifiers.

§**10. Conclusion.** When Barwise entered the abstract model theory scene around 1971, he quickly published the main ideas in a couple of very readable papers, making the area attractive to young logicians. He arrived at his concept of absolute logic by trying to characterize what does it mean that the semantics of a logic depends on the underlying set theory in first order way only. When this is the case, he saw, we can combine metamathematical methods and absoluteness arguments to prove theorems about the logic. He was right. The use of properties of (often non-standard) models of set theory to get model theoretic results, became a standard method. Subsequently abstract model theory got stuck with hard problems related to constructing uncountable structures with pre-described properties. Barwise turned to applications of generalized quantifiers in linguistics and computer science. The emergence of finite model theory and descriptive complexity theory led to sharply increased interest in infinitary logic. Will abstract model theory also experience a rebirth in the finite context?

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