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IDENTITY OF PROOFS BASED ON NORMALIZATION AND GENERALITY

KOSTA DOŠEN

Abstract. Some thirty years ago, two proposals were made concerning criteria for identity of proofs. Prawitz proposed to analyze identity of proofs in terms of the equivalence relation based on reduction to normal form in natural deduction. Lambek worked on a normalization proposal analogous to Prawitz's, based on reduction to cut-free form in sequent systems, but he also suggested understanding identity of proofs in terms of an equivalence relation based on generality, two derivations having the same generality if after generalizing maximally the rules involved in them they yield the same premises and conclusions up to a renaming of variables. These two proposals proved to be extensionally equivalent only for limited fragments of logic.

The normalization proposal stands behind very successful applications of the typed lambda calculus and of category theory in the proof theory of intuitionistic logic. In classical logic, however, it did not fare well.

The generality proposal was rather neglected in logic, though related matters were much studied in pure category theory in connection with coherence problems, and there are also links to low-dimensional topology and linear algebra. This proposal seems more promising than the other one for the general proof theory of classical logic.

§1. General proof theory. When a question is not in the main stream of mathematical investigations, and is also of a certain general, conceptual, kind, it runs the danger of being dismissed as "philosophical". Before the advent of recursion theory, many mathematicians would presumably have dismissed the question "What is a computable function?" as a philosophical question (or perhaps even as a psychological, empirical, question). It required something like the enthusiasm of a young discipline on the rise, which logic was in the first half of the twentieth century, for such a question to be embraced as legitimate, and seriously treated by mathematical means—with excellent results.

An outsider might suppose that the question "What is a proof?" should be important for a field called *proof theory*, and then he would be surprised to find that this and related questions, one of which will occupy us here,

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are exactly of the kind to be dismissed as "philosophical". These questions are of the *conceptual* kind, and not of the *technical* kind, but a great deal of mathematics—a great deal of the best mathematics—is of the conceptual kind.

By the beginning of the 1970s, Prawitz, inspired by ideas of Kreisel, inaugurated the field of *general proof theory* with this and related questions (see [44], Section I.2; see also the next section of this survey). The field exists, I believe, but its borders are not well marked, and the name Prawitz gave to it is not widespread. (Some authors speak nowadays of *structural* proof theory, but I am not sure this kind of proof theory, which seems to amount to *basic* proof theory, is Prawitz's general proof theory.) We need, however, a name for that part of proof theory whose foundations are in Gentzen's thesis [21], and which differs from the proof theory whose subject grew out of consistency proofs for formalized fragments of mathematics inaugurated also by Gentzen. This other kind of proof theory, whose foundations are in Hilbert's program and Gödel's results, and which was dominant throughout the twentieth century, was called *reductive* by Prawitz.

General proof theory, although it is seemingly more philosophical, is closer to applications than reductive proof theory. It is in the period when logic became increasingly intertwined with computer science, in the last decades of the twentieth century, that general proof theory started gaining some ground. Two matters mark this relative, still limited, success: one is the connection with the typed lambda calculus, based on the Curry-Howard correspondence, and the other is the connection with category theory. A third area of proof theory, distinct from general proof theory and reductive proof theory, emerged at roughly the same period in association with complexity theory (see [46]). Hilbert's name is usually tied to reductive proof theory, but from the very beginning of the century he was also interested in some questions of general proof theory that touch upon complexity theory (see [52]).

§2. The Normalization Conjecture. To keep up with the tradition, throughout the present survey we speak of "proof", though we could as well replace this term by the term "deduction", which might be more precise, since we have in mind deductive proofs from assumptions (including the empty collection of assumptions).

The question "What is a proof?", as the basic question of the field of general proof theory, was introduced by Prawitz in [44] (Section I) with additional specifications. Let us quote the relevant passage in extenso:

> Obvious topics in general proof theory are:
>
> 2.1 The basic question of defining the notion of proof, including the question of the distinction between different kinds of proofs such as constructive proofs and classical proofs.

2.2 Investigation of the structure of (different kinds of) proofs, including e.g., questions concerning the existence of certain normal forms.
2.3 The representation of proofs by formal derivations. In the same way as one asks when two formulas define the same set or two sentences express the same proposition, one asks when two derivations represent the same proof; in other words, one asks for identity criteria for proofs or for a "synonymity" (or equivalence) relation between derivations.
2.4 Applications of insights about the structure of proofs to other logical questions that are not formulated in terms of the notion of proof. (p. 237)

We are here especially interested in Prawitz's topic 2.3, namely, in the question of identity criteria for proofs. Prawitz did not only formulate this question very clearly, but also proposed a precise mathematical answer to it.

Prawitz considered derivations in natural deduction systems and the equivalence relation between derivations that is the reflexive, transitive and symmetric closure of the immediate reducibility relation between derivations. Of course, only derivations with the same assumptions and the same conclusion may be equivalent. Prawitz's immediate reducibility relation is the one involved in reducing a derivation to normal form. (As it is well known, the idea of this reduction stems from Gentzen's thesis [21].) A derivation reduces immediately to another derivation (see [44], Section II.3.3) when the latter is obtained from the former either by removing a *maximum formula* (i.e. a formula with a connective α that is the conclusion of an introduction of α and the major premise of an elimination of α), or by performing one of the *permutative reductions* tied to the eliminations of disjunction and of the existential quantifier, which enables us to remove what Prawitz calls *maximum segments*. There are some further reductions, which Prawitz called *immediate simplifications*; they consist in removing eliminations of disjunction where no hypothesis is discharged, and there are similar immediate simplifications involving the existential quantifier, and "redundant" applications of the classical absurdity rule. Prawitz also envisaged reductions he called *immediate expansions*, which lead to the expanded normal form where all the *minimum formulae* are atomic (minimum formulae are those that are conclusions of eliminations and premises of introductions).

Prawitz formulates in [44] (Section II.3.5.6) the following conjecture, for which he gives credit (in Section II.5.2) to Martin-Löf, and acknowledges influence by ideas of Tait:

Conjecture. Two derivations represent the same proof if and only if they are equivalent.

We will call this conjecture the *Normalization Conjecture*.

Prawitz considered the right-to-left direction of the Normalization Conjecture as relatively unproblematic. He found it more difficult to unearth facts that would support the other direction of the conjecture. At the same time (in the same book where Prawitz's paper [44] appeared), Kreisel considered how one could justify by mathematical means a conjecture such as the Normalization Conjecture, and in particular the left-to-right direction of the conjecture (see [27], pp. 114-117, 165).

The Normalization Conjecture is an assertion of the same kind as Church's Thesis: we should not expect a formal proof of it. (Kreisel speaks in [27], p. 112, of the "informal" character of the conjecture.) The Normalization Conjecture attempts to give a formal reconstruction of an intuitive notion. (Like Church's Thesis, the Normalization Conjecture might be taken as a kind of definition. It is, however, better to distinguish this particular kind of definition by a special name. The Normalization Conjecture, as well as the Generality Conjecture, which we will consider in the next section, might be taken as a case of *analysis* in the sense of [5].)

Kreisel suggests (in [27], p. 116) that the right-to-left direction of the Normalization Conjecture may be seen as analogous to a *soundness* assertion, while the left-to-right direction is analogous to a *completeness* assertion. Derivations are the syntactical matter, while intuitive proofs are their semantics. One could, however, take an exactly opposite view: proofs are the given syntactical matter, which we model by derivations.

Martin-Löf examined the Normalization Conjecture in [40] (Section 2.4, p. 104), giving credit to Prawitz for its formulation. He maintained that there is little hope of establishing this conjecture unless "represent the same proof" on the left-hand side is replaced by "are definitionally equal", or identity of proofs on the same side is replaced by *provable* identity; in both cases, the matter is treated further within the realm of his intuitionistic theory of types of [39].

The Normalization Conjecture was formulated by Prawitz at the time when the Curry-Howard correspondence between derivations in natural deduction and typed lambda terms started being recognized more and more (though the epithet "Curry-Howard" was not yet canonized). Prawitz's equivalence relation between derivations corresponds to beta-eta equality between typed lambda terms, if immediate expansions are taken into account, and to beta equality otherwise.

When speaking of the Curry-Howard correspondence, one should, however, bear in mind that it covers really well only the conjunction-implication fragment of intuitionistic propositional logic. In the presence of disjunction, the typed lambda calculus is extended with a ternary variable-binding operation that serves to code disjunction elimination. Such lambda calculuses exist, but they are not particularly appealing. Negation, i.e. the absurd constant proposition, brings its own complications. (With variables of the type of the absurd constant proposition one derives every equation.)

Nevertheless, the fact that we have an alternative formal representation of proofs with typed lambda terms, and that equality between these terms agrees so well with Prawitz's equivalence of derivations, is quite remarkable. This fact lends support to the Normalization Conjecture, as Church's Thesis is supported by the fact that alternative formal representations of computable functions cover the same functions of arithmetic.

Besides derivations in natural deduction and typed lambda terms, where according to the Curry-Howard correspondence the latter can be conceived just as codes for the former, there are other, more remote, formal representations of proofs. There are first Gentzen's sequent systems, which are related to natural deduction, but are nevertheless different, and there are also representations of proofs as arrows in categories. Equality of arrows with the same domain and codomain, i.e. commuting diagrams of arrows, such as we find in category theory, should now correspond to identity of proofs via a conjecture analogous to the Normalization Conjecture.

The fact proved by Lambek (see [31] and [32], Part I; see also [7]) that the category of typed lambda calculuses with functional types and finite product types, based on beta-eta equality, is equivalent to the category of cartesian closed categories, and that hence equality of typed lambda terms amounts to equality between arrows in cartesian closed categories, lends additional support to the Normalization Conjecture. Equality of arrows in bicartesian closed categories (i.e. cartesian closed categories with finite coproducts) corresponds to equivalence of derivations in Prawitz's sense in full intuitionistic propositional logic.

In category theory, the Normalization Conjecture is tied to Lawvere's Thesis that all the logical constants of intuitionistic logic are characterized by adjoint situations. Prawitz's equivalence of derivations, in its beta-eta version, corresponds to equality of arrows in various adjunctions tied to logical constants (see [33], [6], Section 0.3.3, [8] and [7]). Adjunction is the unifying concept for the reductions envisaged by Prawitz.

Even the fact that equality between lambda terms, as well as equality of arrows in cartesian closed categories, were first conceived for reasons independent of proofs is remarkable. This tells us that we are in the presence of a solid mathematical structure, which may be illuminated from many sides.

Prawitz formulated the Normalization Conjecture having in mind mainly intuitionistic logic. It was meant to cover classical logic too. One may, however, formulate an analogous conjecture for all logics with a normalization procedure analogous to Prawitz's procedure for intuitionistic logic. Such are in particular substructural logics. Such are also modal extensions of various logics. It is not evident that Lawvere's Thesis applies to all the logical constants of all these logics; the adjunctions in question may be hidden, or perhaps there is no adjunction, but only something related to it (for example, something like an adjunction but lacking the unit or the counit natural transformation).

It may be asked whether the Normalization Conjecture, or another answer to the question "When are two representations of proofs equal?", brings us closer to answering the basic question "What is a proof?". If our answers to the first question about identity criteria for proofs cut across various representations of proofs, so that when one representation is translated into another they agree about the notion of identity of proof, it seems that we could answer the basic question "What is a proof?" by taking one of our representations and declaring that the equivalence classes of this representation determine the notion we are seeking. A proof is the equivalence class of one of its representations. This does not settle, however, which representation we should choose. It presumably cannot be any representation, but one that neither abolishes important distinctions in the structure of the proof, nor introduces irrelevant features. Anyway, it seems that the Normalization Conjecture brings us closer to answering the basic question.

The answer to the question "What is a computable function?", which is given by Church's Thesis, did not involve such a factoring through an equivalence relation, but the related question "What is an algorithm?" may require exactly that. This last question was raised in recursion theory fairly recently by Moschovakis in [41], where the fundamental interest of identity criteria for algorithms is stressed (see Section 8 of that paper). It is rather natural to consider analogous the questions "What is a proof?" and "What is an algorithm?". As a matter of fact, many logicians conceive proofs just as a kind of algorithm for generating theorems. What should distinguish, however, this particular kind of algorithm is the role logical constants play in it.

While Moschovakis approaches the question of identity criteria for algorithms in a mathematical spirit, proof-theorists nowadays most often ignore the question of identity criteria for proofs, or they dismiss it as "philosophical", sometimes expressing doubt that a simple answer will ever be given to this question. The answer Prawitz gave to this question with the Normalization Conjecture is certainly not philosophical if "philosophical" means "vague" or "imprecise". Prawitz's answer, which is rather simple, is perhaps not conclusive, but it is a mathematical answer. All the more so if we take into account its connection with the lambda calculus and fundamental structures of category theory.

§3. The Generality Conjecture. At the same time when Prawitz formulated the Normalization Conjecture, in a series of papers ([28], [29], [30] and [31]) Lambek was engaged in a project where arrows in various sorts of categories were construed as representing proofs. The domain of an arrow corresponds to the collection of assumptions joined by a conjunction connective, and the codomain corresponds to the conclusion. With this series of papers Lambek inaugurated the field of categorial proof theory.

The categories Lambek considered in [28] and [29] are first those that correspond to his substructural syntactic calculus of categorial grammar (these are monoidal categories where the functors $A \otimes \ldots$ and $\ldots \otimes A$ have right adjoints). Next, he considered monads, which besides being fundamental for category theory, cover proofs in modal logics of the S4 kind. In [30] and [31] Lambek dealt with cartesian closed categories, which is Prawitz's ground, since these categories cover proofs in the conjunction-implication fragment of intuitionistic logic. He also envisaged bicartesian closed categories, which cover the whole of intuitionistic propositional logic.

Lambek's insight is that equations between arrows in categories, i.e. commuting diagrams of arrows, guarantee cut elimination, i.e. composition elimination, in an appropriate language for naming arrows. (In [6] it is established that for some basic notions of category theory, and in particular for the notion of adjunction, the equations assumed are necessary and sufficient for cut elimination.) Since cut elimination is closely related to Prawitz's normalization of derivations, the equivalence relation envisaged by Lambek should be related to Prawitz's.

Actually, Lambek dealt with cut elimination in categories related to his syntactic calculus and in monads. For cartesian closed categories he proved in [31] another sort of result, which he called *functional completeness*. Lambek's functional completeness is analogous to the functional completeness (often called *combinatory completeness*) of Schönfinkel's and Curry's systems of combinators, which enables us to define lambda abstraction in these systems. With the help of functional completeness, via the typed lambda calculus, Lambek established that his equivalence relation between derivations in conjunctive-implicative intuitionistic logic and Prawitz's beta-eta equivalence amount to the same thing, as we mentioned in the preceding section.

Lambek's work is interesting for us here not only because he worked with an equivalence relation between derivations amounting to Prawitz's, but also because he envisaged another kind of equivalence relation. Lambek's idea is best conveyed by considering the following example. In [30] (p. 65) he says that the first projection arrow $\pi^1_{p,p} : p \wedge p \to p$ and the second projection arrow $\pi^2_{p,p} : p \wedge p \to p$, which correspond to two derivations of conjunction elimination, have different *generality*, because they generalize to $\pi^1_{p,q} : p \wedge q \to p$ and $\pi^2_{p,q} : p \wedge q \to q$ respectively, and these two arrows do not have the same codomain; on the other hand, $\pi^1_{p,q} : p \wedge q \to p$ and $\pi^2_{q,p} : q \wedge p \to p$ do not have the same domain. The idea of generality may be explained roughly as follows. We consider generalizations of derivations that diversify variables without changing the rules of inference. Two derivations have the same generality when every generalization of one of them leads to a generalization of the other, so that in the two generalizations we have the same assumptions and conclusion (see [28], p. 257). In the example above this is not the case.

Generality induces an equivalence relation between derivations. Two derivations are equivalent if and only if they have the same generality. Lambek does not formulate so clearly as Prawitz a conjecture concerning identity criteria for proofs, but he suggests that two derivations represent the same proof if and only if they are equivalent in the new sense. We will call this conjecture the *Generality Conjecture*.

The left-to-right direction of the Generality Conjecture seems pretty intuitive. The other direction is less clear, and if it happens to be true, that would be something like a discovery. With the Normalization Conjecture, the situation was different: the right-to-left direction seemed unproblematic, and the left-to-right direction was less clear.

Lambek's own attempts at making the notion of generality precise (see [28], p. 316, where the term "scope" is used instead of "generality", and [29], pp. 89, 100) need not detain us here. In [30] (p. 65) he finds that these attempts were faulty.

The simplest way to understand generality is to use graphs whose vertices are occurrences of propositional letters in the assumptions and the conclusion of a derivation. We connect by an edge occurrences of letters that must remain occurrences of the same letter after generalizing, and do not connect those that may become occurrences of different letters. So for the first and second projection above we would have the two graphs

$$\begin{array}{cccc} & p & \wedge & p \\ \pi^1_{p,p} & & \diagdown & \\ & & p & \end{array} \qquad\qquad \begin{array}{cccc} & p & \wedge & p \\ \pi^2_{p,p} & & \diagup & \\ & & p & \end{array}$$

When the propositional letter p is replaced by an arbitrary formula A we have an edge for each occurrence of propositional letter in A.

The generality of a derivation is such a graph. According to the Generality Conjecture, the first and second projection derivations from $p \wedge p$ to p represent different proofs because their generalities differ.

One defines an associative composition of such graphs, and there is also an obvious identity graph with straight parallel edges, so that graphs make a category, which we call the *graphical category*. If on the other hand it is taken for granted that proofs also make a category, which we will call the *syntactical category*, with composition of arrows being composition of proofs, and identity arrows being identity proofs (an identity proof composed with any other proof, either on the side of the assumptions or on the side of the conclusion, is equal to this other proof), then the Generality Conjecture may be rephrased as the assertion that there is a faithful functor from the syntactical category to the graphical category.

If our syntactical category is precisely determined in advance, by relying on the Normalization Conjecture, or perhaps in another way, then the Generality Conjecture is something we can prove or disprove formally. If on the other hand this syntactical category is left undetermined, then the Generality Conjecture is in the same position as the Normalization Conjecture and Church's Thesis, and cannot be proved or disproved formally. We can only try to see how well it accords with our other intuitions, and in particular with the intuition underlying the Normalization Conjecture.

For example, it may be taken that the syntactical category in conjunctive logic (without implication), both intuitionistic and classical (these two conjunctive logics do not differ), is a cartesian category $\mathcal{K}$ freely generated by a set of propositional letters (a category is cartesian when it has all finite products). Conjunctive logic here includes the true constant proposition. That this syntactical category may indeed be taken as the category of proofs in conjunctive logic is in accordance with the Normalization Conjecture in its beta-eta version, but it is not excluded that we have chosen this category for other reasons. The graphical category $\mathcal{G}$ in this particular case may be taken to be the category opposite to the category of functions on finite ordinals. The finite ordinals correspond to the number of occurrences of propositional letters in a formula, and functions go from the conclusions to the assumptions; that is why we have the opposite of the category of functions. The Generality Conjecture is the assertion that there is a faithful functor from $\mathcal{K}$ to $\mathcal{G}$; that is, a functor G from $\mathcal{K}$ to $\mathcal{G}$ such that for f and g with the same domain and the same codomain we have

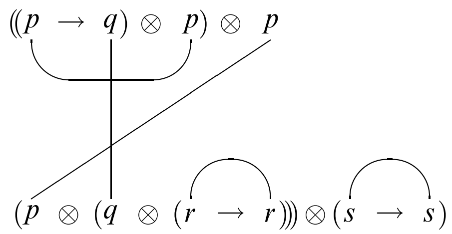
$$(*) \qquad f = g \textit{ in } \mathcal{K} \quad \textit{if and only if} \quad G(f) = G(g) \textit{ in } \mathcal{G}.$$

From left to right, the equivalence $(*)$ follows from the functoriality of G, and from right to left, it expresses the faithfulness of G. In this particular case, the Generality Conjecture understood as $(*)$ can be proved formally (see [42], [12] and references therein concerning earlier statements and proofs of equivalent results).

Note that graphs are not representations of proofs, as derivations in natural deduction were. A graph may abolish many important distinctions; in particular, the images of different connectives, like conjunction and disjunction, may be isomorphic in the graph. The Generality Conjecture answers the question of identity criteria for proofs, and brings us closer to answering the basic question "What is a proof?", by involving representations of proofs more plausible than graphs, like those we find in natural deduction, or in sequent systems, or in the syntactical categories. A proof would be the equivalence class of such a plausible representation with respect to the equivalence relation defined in terms of generality, i.e. in terms of graphs.

Understood in the sense of $(*)$, the Generality Conjecture is analogous to what in category theory is called a *coherence theorem*. Coherence in category theory was understood in various ways (we cannot survey here the literature on this question; for earlier works see [38]), but the paradigmatic coherence results of [37] and [26] can be understood as faithfulness results like $(*)$, and this is how we will understand coherence. (The coherence questions one encounters in the theory of weak n-categories, which we will mention at the end of the next section, are of another, specific, kind.)

The coherence result of [26] proves the Generality Conjecture for the multiplicative conjunction-implication fragment of intuitionistic linear logic (modulo a condition concerning the multiplicative true constant proposition, i.e. the unit with respect to multiplicative conjunction), and, inspired by Lambek, it does so via a cut-elimination proof. The syntactical category in this case is a free symmetric monoidal closed category, and the graphical category is of a kind studied in [18]. The graphs of this graphical category are closely related to the *tangles* of knot theory. In tangles, as in braids, we distinguish between two kinds of crossings, but here we need just one kind, in which it is not distinguished which of the two crossed edges is above the other. (For categories of tangles see [57], [54] and [24], Chapter 12.) Tangles with this single kind of crossing are like graphs one encounters in Brauer algebras (see [1] and [55]). Here is an example of such a tangle:



Tangles without crossings at all serve in [6] (Section 4.10; see also [9]) to obtain a coherence result for the general notion of adjunction, which according to Lawvere's Thesis underlies all logical constants of intuitionistic logic, as we mentioned in the preceding section. In terms of combinatorial low-dimensional topology, the mathematical content of the general notion of adjunction is caught by the Reidemeister moves of planar ambient isotopy. An analogous coherence result for self-adjunctions, where a single endofunctor is adjoint to itself, is proved in [17]. Through this latter result we reach the theory of Temperley-Lieb algebras, which play a prominent role in knot theory and low-dimensional topology, due to Jones' representation of Artin's braid groups in these algebras (see [25], [36], [43] and references therein).

In [17] one finds also coherence results for self-adjunctions where the graphical category is the category of matrices, i.e. the skeleton of the category of finite-dimensional vector spaces over a fixed field with linear transformations as arrows. Tangles without crossings may be faithfully represented in matrices by a representation derived from the orthogonal group case of Brauer's representation of Brauer algebras (see also [55], Section 3, and [23], Section 3). This representation is based on the fact that the Kronecker product of matrices gives rise to a self-adjoint functor in the category of matrices, and this self-adjointness is related to the fact that in this category, as well as in the category of binary relations between finite ordinals, finite products and coproducts are isomorphic.

In [42] there are interesting coherence results, which extend [37], for the multiplicative conjunction fragments of substructural logics. Further coherence results for intuitionistic and classical logic will be considered in the next section.

In the light of the connection with combinatorial aspects of low-dimensional topology, which we mentioned above, the following quotation from [27] sounds prophetical:

> *Remark.* At the risk of trying to explain *obscurum per obscurius*, I should point out a striking analogy between the problem of finding (α) a conversion relation corresponding to identity of proofs and (β) an "equivalence" relation such as that of *combinatorial equivalence* in topology corresponding to a basic invariant in our geometric concepts. (p. 117)

(The sequel of Kreisel's text suggests he had in mind equivalence relations that cut across various representations of proofs, i.e. are insensitive to the peculiarities of particular formalizations.)

From a logical point of view, the Generality Conjecture may be understood as a completeness theorem. The left-to-right direction of the equivalence $(*)$ is *soundness*, and the right-to-left direction is *completeness* proper. (This terminology is opposite to that suggested by Kreisel for the Normalization Conjecture, which we mentioned in the preceding section.)

Like the Normalization Conjecture, the Generality Conjecture gives a precise mathematical answer to the question about identity criteria for proofs, and, like the former conjecture, it has ties with important mathematical structures.

§4. The two conjectures compared. The Normalization Conjecture and the Generality Conjecture agree only for limited fragments of logic. As we said in the preceding section, they agree for purely conjunctive logic (without implication), with or without the true constant proposition $\top$. Conjunctive logic is the same for intuitionistic and classical logic. Here the Normalization Conjecture is taken in its beta-eta version. By duality, the two conjectures agree for purely disjunctive logic, with or without the absurd constant proposition $\bot$. If we have both conjunction and disjunction, but do not yet have distribution of conjunction over disjunction, and have neither $\top$ nor $\bot$, then the two conjectures still agree (see [14]). And here it seems we have reached the limits of agreement as far as intuitionistic and classical logic are concerned, provided the graphs involved in the Generality Conjecture correspond to relations between the domain and the codomain. With more sophisticated notions of graphs, matters may stand differently, and the area of agreement for the two conjectures may perhaps be wider, but it can be even narrower, as we will see below.

Coherence fails for the syntactical bicartesian categories, in which besides nonempty finite products and coproducts we have also an empty product and an empty coproduct, i.e. a terminal object $\top$ and an initial object $\bot$ (an object in a category is terminal when there is exactly one arrow from any object to it, and it is initial when there is exactly one arrow from it to any object). In order to regain coherence we must pass to special bicartesian categories where the first projection from the product of $\bot$ with itself $\pi^1_{\bot,\bot} : \bot \times \bot \to \bot$ is equal to the second projection $\pi^2_{\bot,\bot} : \bot \times \bot \to \bot$, and analogously for the first and second injection from $\top$ to the coproduct of $\top$ with itself, namely $\top + \top$ (see [13]). Here $\times$ corresponds to conjunction and $+$ to disjunction. The assumption concerning these two projections holds in the category **Set**, but the assumption concerning the two injections does not. Both assumptions hold in some other important bicartesian categories: the category of pointed sets, and its subcategories of commutative monoids with monoid homomorphisms and of vector spaces over a fixed field with linear transformations. They hold also in the category of matrices mentioned in the preceding section. Anyway, in the presence of $\top$ and $\bot$ the two conjectures do not agree, since normalization does not deliver the assumption concerning the two injections.

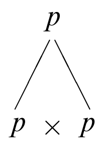
The graphical category with respect to which we have coherence for syntactical categories with nonempty finite products and coproducts is the category of binary relations between finite ordinals (see [14]). In that category, $\times$ and $+$, which correspond to conjunction and disjunction respectively, are isomorphic.

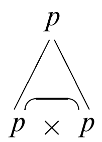
The intuitive idea of generality for conjunctive-disjunctive logic is not caught by arbitrary binary relations, but by binary relations that are *difunctional* in the sense of [47] (Section 7). A binary relation R is difunctional when $RR^{-1}R \subseteq R$. The composition of two difunctional relations is not necessarily difunctional, so that we do not have at our disposal the category of difunctional relations, with respect to which we could prove coherence. It can be shown, however (with the help of the Cut Elimination of [14]), that the image under the faithful functor of an arrow from our syntactical category with nonempty finite products and coproducts is a difunctional relation.

Even if difunctionality is satisfied, it may still be questioned whether our intuitive idea of generality is caught by binary relations in the case of conjunctive-disjunctive logic. The problem is that if $w_p : p \to p \times p$ is a component of the diagonal natural transformation, and $\iota^1_{q,p} : q \to q + p$ is a first injection, then in categories with finite products and coproducts we have

$$(\mathbf{1}_q + w_p) \circ \iota^1_{q,p} = \iota^1_{q,p\times p}$$

where the left-hand side cannot be further generalized, but the right-hand side can be generalized to $\iota^1_{q,p\times r}$. The intuitive idea of generality seems to require that in $w_p : p \to p \times p$ we should not have only a relation between the domain and the codomain, as on the left-hand side below, but a graph as on the right-hand side:





(see [15], and also [16]). With such graphs we can still get coherence for conjunctive logic, and for disjunctive logic, taken separately, but for conjunctive-disjunctive logic the left-to-right direction, i.e. the soundness part, of coherence would fail. So for conjunctive-disjunctive logic the idea of generality with which we have coherence is not quite the intuitive idea suggested by Lambek, but only something close to it, which involves the categorial notion of natural transformation.

The fragments of logic mentioned above where the Normalization Conjecture and the Generality Conjecture agree all possess a property called *maximality*. Let us say a few words about this very important property.

For the whole field of general proof theory to make sense, and in particular for considering the question of identity criteria for proofs, we should not have that any two derivations with the same assumptions and conclusion are equivalent, i.e., it should not be the case that there is never more than one proof with given assumptions and a given conclusion. Otherwise, our field would be trivial.

This marks the watershed between proof theory and the rest of logic, where one is not concerned with proofs, but at most with consequence *relations*. With relations, we either have a pair made of a collection of assumptions and a conclusion, or we do not have it. In proof theory, such pairs are indexed with various proofs, and there may be several proofs for a single pair.

Now, categories with finite nonempty products, cartesian categories and categories with finite nonempty products and coproducts have the following property. Take, for example, cartesian categories, and take any equation in the language of free cartesian categories that does not hold in free cartesian categories. If a cartesian category $\mathcal{K}$ satisfies this equation, then $\mathcal{K}$ is a *preorder*; namely, all arrows with the same domain and codomain are equal. We have an exactly analogous property with the other sorts of categories we mentioned. This property is a kind of Post completeness. Any extension of the equations postulated leads to collapse.

Translated into logical language, this means that Prawitz's equivalence relation for derivations in conjunctive logic, disjunctive logic and conjunctive-disjunctive logic without distribution and without $\top$ and $\bot$, which in all these cases agrees with our equivalence relation defined via generality in the style of Lambek, is *maximal*. Any strengthening, any addition, would yield that any two derivations with the same assumptions and the same conclusion are equivalent.

If the right-to-left direction of the Normalization Conjecture holds, with maximality we can efficiently justify the left-to-right direction, which Prawitz found problematic in [44], and about which Kreisel was thinking in [27]. In the footnote on p. 165 of that paper Kreisel mentions that Barendregt suggested this justification via maximality. Suppose the right-to-left direction of the Normalization Conjecture holds, suppose that for some assumptions and conclusion there is more than one proof, and suppose the equivalence relation is maximal. Then if two derivations represent the same proof, they are equivalent. Because if they were not equivalent, we would never have more than one proof with given assumptions and a given conclusion. Nothing can be missing from our equivalence relation, because whatever is missing, by maximality, leads to collapse on the side of the equivalence relation, and, by the right-to-left direction of the conjecture, it also leads to collapse on the side of identity of proofs.

Prawitz in [44] found it difficult to justify the left-to-right direction of the Normalization Conjecture, and Kreisel was looking for mathematical means that would provide this justification. Maximality is one such means.

Establishing the left-to-right direction of the Normalization Conjecture via maximality is like proving the completeness of the classical propositional calculus with respect to any kind of nontrivial model via Post completeness (which is proved syntactically by reduction to conjunctive normal form). Actually, the first proof of this completeness with respect to tautologies was given by Bernays and Hilbert exactly in this manner (see [58]).

Maximality for the sort of categories considered above is proved with the help of coherence in [12] and [14] (which is established proof-theoretically, by normalization, cut elimination and similar methods). Maximality is proved for cartesian closed categories via a typed version of Böhm's theorem in [50], [48] and [11]. This justifies the left-to-right direction of the Normalization Conjecture also for the implicational and the conjunction-implication fragments of intuitionistic logic. The maximality of bicartesian closed categories, which would justify the left-to-right direction of the Normalization Conjecture for the whole of intuitionistic propositional logic is, as far as I know, an open problem. (A use for maximality similar to that propounded here and in [11] and [12] was envisaged in [56], it seems independently.)

In [6] (Section 4.11) it is proved that the general notion of adjunction is also maximal in some sense. The maximality we encountered above, which involves connectives tied to particular adjunctions, cannot be derived from the maximality of the general notion of adjunction, but these matters should not be foreign to each other.

The Normalization Conjecture and the Generality Conjecture do not agree for the conjunction-implication fragment of intuitionistic logic. We do not have coherence for cartesian closed categories if the graphs in the graphical category are taken to be of the tangle type Kelly and Mac Lane had for symmetric monoidal closed categories combined with the graphs we had for cartesian categories (see the preceding section). Both the soundness part and the completeness part of coherence fail for cartesian closed categories. The soundness part of coherence also fails for distributive bicartesian categories, and a fortiori for bicartesian closed categories. The problem is that in these categories distribution of conjunction over disjunction is taken to be an isomorphism, and graphs do not deliver that (see [14], Section 1).

The problem with the soundness part of coherence for cartesian closed categories may be illustrated with typed lambda terms in the following manner. By beta conversion and alpha conversion, we have the following equation:

$$\lambda_x \langle x, x\rangle \lambda_y y = \langle \lambda_y y, \lambda_z z\rangle$$

for y and z of type p, and x of type p^p (which corresponds to p *implies* p). The closed terms on the two sides of this equation are both of type $p^p \times p^p$. The type of the term on the left-hand side cannot be further generalized, but the type of the term $\langle \lambda_y y, \lambda_z z\rangle$, can be generalized to $p^p \times q^q$. The problem noted here does not depend essentially on the presence of surjective pairing $\langle \ldots, \ldots \rangle$ and of product types; it arises also with purely functional types. This problem depends essentially on the multiple binding of variables, which we have in $\lambda_x \langle x, x\rangle$; that is, it depends on the structural rule of contraction.

This throws some doubt on the right-to-left direction of the Normalization conjecture, which Prawitz, and Kreisel too, found relatively unproblematic. It might be considered strange that two derivations represent the same proof if, without changing inference rules, one can be generalized in a manner in which the other cannot be generalized.

In [19] (p. 234) Feferman expresses doubt about the right-to-left direction of the Normalization Conjecture because he finds that a proof π of $A(t)$ ending with the elimination of the universal quantifier from the premise $\forall x A(x)$ is different from the proof π' of $A(t)$ obtained from π by removing

$\forall xA(x)$ as a maximum formula. The proof π contains more information than π'. In [45] Prawitz replied that he takes a proof not "as a collection of sentences but as the result of applying certain operations to obtain a certain end result" (p. 249). It seems Feferman's objection is at the level of *provability*, and not at the level of *proofs*. A related objection due to Kreisel, and some other objections to the Normalization Conjecture, may be found in [53] (Section 5.3).

The Normalization Conjecture has, for the time being, the following advantage over the Generality Conjecture. It applies also to predicate logic, whereas the latter conjecture has not yet been investigated outside propositional logic.

When we compare the two conjectures we should also say something about their computational aspects. With the Normalization Conjecture, we have to rely on reduction to a unique normal form in the typed lambda calculus in order to check equivalence of derivations in the conjunction-implication fragment of intuitionistic propositional logic. Nothing more practical than that is known, and such syntactical methods may be tiresome. Outside of the conjunction-implication fragment, in the presence of disjunction and negation, such methods become uncertain.

Methods for checking equivalence of derivations in accordance with the Generality Conjecture, i.e. methods suggested by coherence results, often have a clear advantage. This is like the advantage truth tables have over syntactical methods of reduction to normal form in order to check tautologicality. However, the semantical methods delivered by coherence results have this advantage only if the graphical category is simple enough. And when we enter into categories suggested by knot theory, this simplicity may be lost. Then, on the contrary, syntax may help us to decide equality in the graphical category.

The Normalization Conjecture has made a foray in theoretical computer science, in the area of typed programming languages. It is not clear whether one could expect the Generality Conjecture to play a similar role.

The reflexive and transitive closure of Prawitz's immediate reducibility relation, i.e. Prawitz's reducibility relation, may be deemed more important than his equivalence relation, which we have considered up to now. This matter leads outside the topic of our survey, which is about *identity* of proofs, but it is worth mentioning. We may "categorify" the identity relation between proofs, and consider not only other relations between proofs, like Prawitz's reducibility relation, but maps between proofs. The proper framework for doing that seems to be the framework of weak 2-categories, where we have 2-arrows between arrows; or we could even go to n-categories, where we have $n+1$-arrows between n-arrows (one usually speaks of *cells* in this context). Composition of 1-arrows is associative only up to a 2-arrow isomorphism, and analogously for other equations between 1-arrows. Identity of 1-arrows is replaced by 2-arrows satisfying certain conditions, which are called *coherence conditions*. This notion of coherence is related to the coherence we have considered above, but it is specific, and need not be the same. Reducibility between arrows, however, does not necessarily give rise to an isomorphism.

In the context of the Generality Conjecture, we may also find it natural to consider 2-arrows instead of identity. The orientation would here be given by passing from a graph with various "detours" to a graph that is more "straight", which need not be taken any more as equal to the original graph.

With all this we would enter into a very lively field of category theory, interacting with other disciplines, mainly topology (see [34], [35] and papers cited therein). The field looks very promising for general proof theory, both from Prawitz's and from Lambek's point of view, but, as far as I know, it has not yet yielded to proof theory much more than promises.

§**5. The Normalization Conjecture in classical logic.** The Normalization Conjecture fares rather well in intuitionistic logic, but not so well in classical logic. The main difficulty in classical logic is tied to the fact that in every bicartesian closed category, for every object A there is at most one arrow with domain A and the initial object $\perp$ as codomain. Assuming the Normalization Conjecture, this would mean that in intuitionistic propositional logic for every formula A there is at most one proof with assumption A and the absurd constant proposition $\perp$ as conclusion, i.e. there cannot be more than one way to reduce A to the absurd.

In [32] the discovery of that fact is credited to Joyal (p. 116), and the fact is established (on p. 67, Proposition 8.3) by relying on a proposition of Freyd (see [20], p. 7, Proposition 1.12) to the effect that if in a cartesian closed category $Hom(A, \perp)$ is not empty, then $A \cong \perp$; that is, A is isomorphic to $\perp$. Here is a simpler proof of the same fact.

PROPOSITION 1. *In every cartesian closed category with an initial object $\perp$ we have that $Hom(A, \perp)$ is either empty or a singleton.*

PROOF. In every cartesian closed category with $\perp$ we have $\pi^1_{\perp,\perp} = \pi^2_{\perp,\perp} : \perp \times \perp \to \perp$, because $Hom(\perp \times \perp, \perp) \cong Hom(\perp, \perp^{\perp})$. Then for $f, g : A \to \perp$ we have $\pi^1_{\perp,\perp} \circ \langle f, g \rangle = \pi^2_{\perp,\perp} \circ \langle f, g \rangle$, and so $f = g$. ⊣

In [32] (p. 67) it is concluded from Proposition 1 that if in a bicartesian closed category for every object A we have $A \cong \neg\neg A$, where the "negation" $\neg B$ is $\perp^B$ (which corresponds to B *implies* $\perp$), then this category is a preorder. If classical logic requires $A \cong \neg\neg A$ for every proposition A, then the proof theory of that logic is trivial: there is at most one proof with given assumptions and a given conclusion.

If the requirement $A \cong \neg\neg A$ is deemed too strong, here is another proposition, which infers triviality from another natural requirement.

PROPOSITION 2. *Every cartesian closed category with an initial object $\bot$ in which we have a natural transformation whose components are $\zeta_A : \neg\neg A \to A$ is a preorder.*

PROOF. Take $f, g : \neg\neg A \to B$, and take the canonical arrow $\varepsilon'_A : A \to \neg\neg A$, which we have by the cartesian closed structure of our category. Then we have $\neg\neg(f \circ \varepsilon'_A) = \neg\neg(g \circ \varepsilon'_A)$ by Proposition 1, and from

$$\zeta_B \circ \neg\neg(f \circ \varepsilon'_A) = \zeta_B \circ \neg\neg(g \circ \varepsilon'_A),$$

by the naturality of ζ, we infer

$$f \circ \varepsilon'_A \circ \zeta_A = g \circ \varepsilon'_A \circ \zeta_A.$$

Since $\varepsilon'_A \circ \zeta_A = \mathbf{1}_{\neg\neg A}$ by Proposition 1, we have $f = g$.

Then, for $\top$ terminal, we have

$$\begin{array}{rcl} Hom(C, D) & \cong & Hom(\top, D^C) \\ & \cong & Hom(\neg\neg\top, D^C), \end{array}$$

since $\top \cong \neg\neg\top$, and $Hom(\neg\neg\top, D^C)$ is at most a singleton, as we have shown above. ⊣

Here is another proposition similar to Proposition 2.

PROPOSITION 3. *Every bicartesian closed category in which we have a dinatural transformation whose components are $\xi_A : \top \to A + \neg A$ is a preorder.*

PROOF. Take $f, g : \top \to A$. Then $\neg f = \neg g$ by Proposition 1, and from

$$(\mathbf{1}_A + \neg f) \circ \xi_A = (\mathbf{1}_A + \neg g) \circ \xi_A,$$

by the dinaturality of ξ, we infer

$$(f + \neg\mathbf{1}_\top) \circ \xi_\top = (g + \neg\mathbf{1}_\top) \circ \xi_\top.$$

Since $\xi_\top$ is an isomorphism, we obtain $f + \neg\mathbf{1}_\top = g + \neg\mathbf{1}_\top$, from which $f = g$ follows with the help of first injections. ⊣

The naturality of ζ in Proposition 2 is a requirement with proof-theoretical justification: it has to do with permuting cuts with rules for negation. Something similar applies to ξ.

So, with the Normalization Conjecture, the case for the general proof theory of classical logic looks pretty bleak. A number of natural requirements lead to collapse and triviality.

All these considerations involve an initial object $\bot$. In the absence of the initiality of $\bot$, matters stand better. Besides that, classical logic can be embedded by a double-negation translation not only in Heyting's intuitionistic logic, but also in Kolmogorov's minimal intuitionistic logic, which is essentially Heyting's negationless logic. Relying on this translation, we could still obtain a nontrivial proof theory for classical logic in the presence of the Normalization Conjecture.

The initiality of $\bot$, that is the requirement that there is just one proof with given conclusion and with $\bot$ as assumption, may perhaps be questioned. If $\bot$ is not devoid of structure, why could not we proceed in different ways to infer something from it? For example, if $\bot$ is taken to be $p \wedge \neg p$, we can infer p from it either by the first projection, or by passing to $p \wedge (\neg p \vee p)$ and then applying the disjunctive syllogism that enables us to infer B form $A \wedge (\neg A \vee B)$. Why must we take these proofs as identical? (We have here two derivations with different generality in Lambek's sense: the first generalizes to a derivation of p from $p \wedge \neg q$ and the second to a derivation of q from $p \wedge \neg p$; so, according to the Generality Conjecture, we have two different proofs.)

Still, rejecting the initiality of $\bot$ looks like a desperate measure, not in tune with the other intuitions underlying the Normalization Conjecture. And if we choose to embed classical logic in intuitionistic logic, classical logic would not be on its own, but would be taken only as a fragment of intuitionistic logic.

The problems we find with the Normalization Conjecture in classical logic might accord with the conception that intuitionistic logic is truly the logic of *provability*, whereas classical logic is the logic of something else: it is the logic of *truth* (and falsehood), and we should not be surprised to find that its general proof theory is trivial. If, however, we rely on the Generality Conjecture, matters may look different, as we will see in the next section.

§6. The Generality Conjecture in classical logic. The Generality Conjecture seems more promising for classical logic than the Normalization Conjecture. Anyway, it does not lead to collapse and triviality. The matter has not yet been fully explored, since apart from fragmentary coherence results, mentioned in Section 4 above, where the two conjectures agree, and apart from coherence results in substructural logics, the Generality Conjecture was rather neglected in logic.

The Generality Conjecture underlies the method of proof nets for multiplicative classical linear logic. The insight provided by [4] is that proof nets are based on a coherence result for $*$-autonomous categories, which in its turn is based on a coherence result for categories with the *linear distribution* $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee C$ (previously called *weak distribution*). Linear distribution is what we need for multiple-conclusion cut.

The distribution arrow of type $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$ defined in terms of linear distribution with the help of the diagonal arrow of type $A \rightarrow A \wedge A$ is not an isomorphism. This is an important difference with respect to bicartesian closed categories, where distribution is an isomorphism.

Graphs related to our graphs were tied to sequent derivations of classical logic in [2] and [3], but apparently without the intention to discuss the question of identity criteria for proofs.

The price we have to pay for accepting the Generality Conjecture in classical logic is that not all connectives will be tied to adjoint functors, as required by Lawvere's Thesis. For example, conjunction and disjunction will be tied to such functors in the conjunctive-disjunctive fragment of classical logic, but when we add implication or negation, in the whole of classical propositional logic, these adjunctions will be lost. The diagonal arrows of type $A \to A \wedge A$, for instance, will not make a natural transformation (this has to do with the failure of the equation involving $\langle x, x\rangle$, which we considered in Section 4).

But there might be gains in accepting the Generality Conjecture in classical logic, and we shall point briefly in the next section towards a prospect that looks interesting.

§7. Addition of proofs and zero proofs. Gentzen's multiple-conclusion sequent calculus for classical logic has a rule of *addition* of derivations, which is derived as follows:

$$\dfrac{\dfrac{\dfrac{f : A \to B}{\theta^R_C(f) : A \to B, C} \qquad \dfrac{g : A \to B}{\theta^L_C(g) : C, A \to B}}{cut(\theta^R_C(f), \theta^L_C(g)) : A, A \to B, B}}{f + g : A \to B}\;\textit{contractions}$$

Here $\theta^R_C(f)$ and $\theta^L_C(g)$ are obtained from f and g respectively by thinning on the right and thinning on the left, and $cut(\theta^R_C(f), \theta^L_C(g))$ may be conceived as obtained by applying to f and g a limit case of Gentzen's multiple-cut rule *mix*, where the collection of mix formulae is empty. (A related rule was considered under the name *mix* in linear logic.)

In a cut-elimination procedure like Gentzen's, $f + g$ is reduced either to f or to g (see [21], Sections III.3.113.1-2). If we have $f + g = f$ and $f + g = g$, then we get immediately $f = g$, that is collapse and triviality (cf. Appendix B.1 by Lafont of [22]). If we keep only one of these equations and reject the other, then to evade collapse we must reject the commutativity of $+$, but it seems all these decisions would be arbitrary. (For similar reasons, even without assuming the commutativity of $+$, the assumptions of [51], p. 232, C.12, lead to preorder.)

The Generality Conjecture tells us that we should have neither $f + g = f$ nor $f + g = g$. The addition of two graphs may well produce a graph differing from each of the graphs added. It also tells us that addition of proofs should be associative and commutative.

If we have addition of proofs, it is natural to assume that we also have for every formula A and every formula B a *zero proof* $0_{A,B} : A \to B$, with an empty graph, which with addition of proofs makes a commutative monoid. We may envisage having zero proofs $0_{A,B} : A \to B$ only for those A and B

where there is also a nonzero proof from A to B, but here we consider the more sweeping assumption involving every A and every B.

We should immediately face the complaint that with such zero proofs we have entered into inconsistency, since everything is provable. That is true, but not all proofs have been made identical, and we are here not interested in what is provable, but in what proofs are identical. If it happens—and with the Generality Conjecture it will happen indeed—that introducing zero proofs is conservative with respect to identity of proofs which do not involve zero proofs, then it is legitimate to introduce zero proofs, provided it is useful for some purpose. This is like extending our mathematical theories with what Hilbert called "ideal" objects; like extending the positive integers with zero, or like extending the reals with imaginary numbers.

What useful purpose might be served by introducing zero proofs? With addition of proofs, our graphical category in the case of conjunctive-disjunctive logic turns up to be a category of matrices, rather than the category of binary relations (see Sections 3 and 4 above), although the category of binary relations makes sense too, provided we accept that addition of proofs is idempotent (i.e. $f + f = f$, which means $+$ is *union* rather than addition of proofs). Composition becomes matrix multiplication, and addition is matrix addition. And in the presence of zero matrices, we obtain a unique normal form like in linear algebra: every matrix is the sum of matrices with a single 1 entry.

A number of logicians have sought a link between logic and linear algebra, and here is such a link. We have it not for an alternative logic, but for classical logic. We have it, however, not at the level of provability, but at the level of identity of proofs.

The unique normal form suggested by linear algebra is not unrelated to cut elimination. In the graphical category of matrices, cut elimination is just matrix multiplication. And the equations of this category yield a cut-elimination procedure. They yield it even in the absence of zero proofs (provided $f + f = f$), and unlike cut-elimination procedures for classical logic in the style of Gentzen, the new procedure admits a commutative addition of proofs without collapse. So, in classical logic, the Generality Conjecture is not foreign to cut elimination, and it would not be foreign to the Normalization Conjecture if we understand the equivalence relation involved in this conjecture in a manner different from Prawitz's.

This need not exhaust the advantages of having zero proofs. They may be used also to analyze disjunction elimination. Without pursuing this topic very far, let us note that passing from $A \vee B$ to A involves a zero proof from B to A, and passing from $A \vee B$ to B involves a zero proof from A to B. If next we are able to reach C both from A and from B, we may add our two proofs from $A \vee B$ to C, and so to speak "cancel" the two zero proofs.

On an intuitive level, we should not imagine that zero proofs are faulty proofs. They are rather hypothetical proofs, postponed proofs, or something like the oracles of recursion theory.

§8. Propositional identity. Once we have answered the question of identity criteria for proofs, by the Normalization Conjecture, the Generality Conjecture, or in another satisfactory manner, armed with our answer we may try approaching other logical questions, as envisaged by Prawitz in his topic 2.4 of general proof theory (see the beginning of Section 2 above). Here we will sketch how we can apply this answer to settle the question of propositional identity (which was mentioned by Prawitz in topic 2.3).

Categorial proof theory suggests concepts that were not previously envisaged in logic. Such a concept is isomorphism between sentences, which is understood as in category theory. (A certain school of category theory recommends that we should drop identity and think always in terms of isomorphism.) A sentence A is *isomorphic* to a sentence B if and only if there is a proof f from A to B and a proof f^{-1} from B to A such that f composed with f^{-1} is equal to the identity proof from A to A and f^{-1} composed with f is equal to the identity proof from B to B. An identity proof is such that when composed with any other proof, either on the side of the assumptions or on the side of the conclusion, it is equal to this other proof. That two sentences are isomorphic means that they behave exactly in the same manner in proofs: by composing, we can always extend proofs involving one of them, either as assumption or as conclusion, to proofs involving the other, so that nothing is lost, nor gained. There is always a way back. By composing further with the inverses, we return to the original proofs.

Isomorphism between sentences is an equivalence relation stronger than mutual implication. So, for example, $A \wedge B$ is isomorphic to $B \wedge A$, while $A \wedge A$ only implies and is implied by A, but is not isomorphic to it. (The problem is that the composition of the proof from $A \wedge A$ to A, which is either the first or the second projection, with the diagonal proof from A to $A \wedge A$ is not equal to the identity proof from $A \wedge A$ to $A \wedge A$.)

It seems reasonable to suppose that isomorphism between sentences analyzes propositional identity, i.e. identity of meaning for sentences:

> Two sentences express the same proposition if and only if they are isomorphic.

This way we would base propositional identity upon identity of proofs, since in defining isomorphism of sentences we relied essentially on a notion of identity of proofs.

By relying on either the Normalization Conjecture or the Generality Conjecture, the formulae isomorphic in conjunctive logic are characterized by the equations of commutative monoids. (By duality, this solves the problem also for disjunctive logic.) By relying on the Normalization Conjecture, the formulae isomorphic in the conjunction-implication fragment of intuitionistic logic, i.e. objects isomorphic in all cartesian closed categories, have been characterized in [49] via the axiomatization of equations with multiplication, exponentiation and one, true for natural numbers. This fragment of arithmetic (upon which one comes in connection with Tarski's "high-school algebra problem") is finitely axiomatizable and decidable. (For an analogous result about the multiplicative conjunction-implication fragment of intuitionistic linear logic see [10].) The problem of characterizing isomorphic formulae in the whole intuitionistic propositional calculus, which corresponds to bicartesian closed categories, seems to be still open.

Leibniz's analysis of identity, given by the equivalence

> Two individual terms name the same object if and only if in every sentence one can be replaced by the other without change of truth value,

assumes as given and uncontroversial propositional equivalence, i.e. identity of truth value for sentences, and analyzes in terms of it identity of individuals. We analyzed above propositional identity in terms of isomorphism between sentences, a notion that presupposes an understanding of identity of proofs, and our analysis resembles Leibniz's to a certain extent.

Propositional equivalence, which in classical logic is defined by identity of truth value, amounts to mutual implication, and is understood as follows in a proof-theoretical context:

> A is equivalent to B if and only if there is a proof from A to B and a proof from B to A.

This relation between the propositions A and B, which certainly cannot amount to the stricter relation of propositional identity, does not presuppose understanding identity of proofs. We analyzed propositional identity by imposing on the composition of the two proofs from A to B and from B to A a condition formulated in terms of identity of proofs.

The question of propositional identity seems as philosophical as the question of identity of proofs, if not more so, but the answer we proposed to this question is rather mathematical, as were the answers given to the question of identity of proofs by the Normalization and Generality Conjectures.

§9. **Conclusion.** The question we have discussed here suggests a perspective in logic—or perhaps we may say a dimension—that has not been explored enough. Logicians were, and still are, interested mostly in provability, and not in proofs. This is so even in proof theory. When we address the question of identity of proofs we have certainly left the realm of provability, and entered into the realm of proofs. This should become clear in particular when we introduce the zero proofs of Section 7.

The complaint might be voiced that with the Normalization and Generality Conjectures we are giving very limited answers to the question of identity of proofs. What about identity of proofs in the rest of mathematics, outside logic? Shouldn't we take into account many other inference rules, and not only those based on logical constants? Perhaps not if the structure of proofs is taken to be purely logical. Perhaps conjectures like the Normalization and Generality Conjectures are not far from the end of the road.

Faced with two concrete proofs in mathematics—for example, two proofs of the Theorem of Pythagoras, or something more involved—it could seem pretty hopeless to try to decide whether they are identical just armed with the Normalization Conjecture or the Generality Conjecture. But this hopelessness might just be the hopelessness of formalization. We are overwhelmed not by complicated principles, but by sheer quantity.

An answer to the question of identity criteria for proofs could perhaps also be expected to come from the area of proof theory tied to complexity theory, which we mentioned at the end of the first section. There is perhaps an equivalence relation between derivations defined somehow in terms of complexity by means of which we could formulate a conjecture analogous to the Normalization and Generality Conjectures, but I am not aware that a conjecture of that kind has been proposed.

Although the problem of identity criteria for proofs is a conceptual mathematical problem, one might get interested in it for technical reasons as well. There is interesting mathematics behind this problem. But one should probably also have an interest in logic specifically. Now, it happens that many logicians are not interested in logic. They are more concerned by other things in mathematics. They dwell somehow within the realm of what is accepted as logic, but they deal with other mathematical matters. Logic is taken for granted. Some logicians, however, hope for discoveries in logic.

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MATHEMATICAL INSTITUTE
SERBIAN ACADEMY OF SCIENCES AND ARTS
KNEZ MIHAILOVA 35, P.F. 367, 11001 BELGRADE, SERBIA
E-mail: kosta@mi.sanu.ac.yu