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REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

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In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

J. MICHAEL DUNN and GARY M. HARDEGREE. ***Algebraic methods in philosophical logic.*** Oxford logic guides, no. 41. Clarendon Press, Oxford University Press, Oxford, New York, etc., 2001, xv + 470 pp.

The publication date for this book has a surprising, symbolic dimension—the book will be unavoidably assessed as being the *summa* of the logical knowledge from the turn of the twentieth century. Indeed, the book covers a significant part of contemporary algebraic and philosophical logic and presents many of the most valuable results obtained in these areas during the last decades. On the other hand, the work is a kind of opening balance-sheet since it shows new research perspectives and outlines problems that, with certainty, will be in the focus of logicians' attention in the coming years. As the authors write: "this book tries to represent a reasonable portion of the intersection of algebraic logic and philosophical logic, but still contains only a fraction of the results" (p. viii). The main topic of the book, appearing in almost all chapters, is the problem of identification and algebraic description of models appropriate for the study of various logics as well as the problem of representation and duality for the classes of algebras that constitute the algebraic semantics for these deductive systems. From the point of view of metalogic, the primary goal of this theory is to provide a general framework for specifying and proving completeness theorems for propositional logics. This cluster of problems is central to classical algebraic logic while in the so-called abstract algebraic logic (AAL), which is *not* the proper topic of the book, the focus is on the algebraization process itself. The term "algebraization" receives in AAL a

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very concrete meaning. AAL also isolates and classifies various forms of algebraizability using the properties of the Leibniz operator. The well-known representation theorems for Boolean algebras (BA), the Stone duality theory for BA, and the relationship between these results and completeness theorems for classical logic constitute the core of classical algebraic logic. The book provides a uniform semantic approach to a variety of non-classical logics, including modal logic and intuitionistic logic, and develops a general framework for representation and duality theorems for the classes of algebras naturally associated with these logics. The theory presented in the book is based on the abstract treatment of "logical principles … involving relationships between pairs of logical operators that may be seen under the algebraic abstractions of residuation and Galois connections" (p. 394). These relationships are abstracted into an algebraic structure called a "gaggle." (By an euphonious accident, "gaggle" is the pronunciation of the acronym for "general Galois logics.") The above semantic approach is thus a far-reaching extension of the purely Boolean paradigm of representation and duality since it provides a general account of relations holding between representation results and completeness theorems. This theory originates from the classical representation results for Boolean algebras with operators due to Jónsson and Tarski. As is nowadays well known, these remarkable results anticipated the Kripke-style semantics and *de facto* they constitute the natural relational semantics for (multi)modal logics. A deep analysis of the work of Jónsson and Tarski combined with investigations into algebraic semantics for relevant logics has resulted in a number of new concepts, important in representation theory, such as a distributoid, a tonoid, and a gaggle, already mentioned. The (equational) classes of distributoids and gaggles subsume Boolean models and modal algebras as special cases and they enable one to represent more involved algebraic structures. Considerations concerning representation for the above concepts, showing the relationship between them and many logical systems and their fragments such as intuitionism or modal or relevance logic, form the content of final chapters of the book. The basic conceptual issues are expounded in Chapters 12 and 13. They form the most important and valuable part of the book. The results presented there principally belong to the first of the authors since these chapters are based on Dunn's earlier publications which appeared in various logical journals. These chapters also offer a further research perspective by discussing various possible directions in developing representation and duality theories for the classes of algebras associated with logical systems.

The readers are required to have a very modest algebraic and logical preparation enabling them to read and understand the content of preliminary parts of the book before passing to more advanced topics in final chapters. This requirement to a large extent defines the structure of the book. Chapters 1 and 2 have a preparatory character. They contain a lucid, easily digestible lecture of basic facts from universal algebra and from some selected (but important from the perspective of the book) branches of algebra such as ordered sets, lattices, ordered algebras, tonoids, and the theory of filters and ideals. The general tendency is to speak about difficult matters in plain terms. This task is excellently accomplished in the book.

The next chapters are devoted to metalogical considerations. After a short exposition of purely syntactic issues (here the algebra of sentences is assumed to be an absolutely free algebra), the book turns to the foundations of semantics. Fundamental to the semantic framework developed in the book is "the distinction between two kinds of semantic functions, which we call *interpretations* and *valuations*. … An interpretation assigns to every sentence a meaning … whereas a valuation assigns to every sentence a truth value" (p. 155). Algebraically, an interpretation is required to be a homomorphism, whereas a valuation is simply a function from the language into the (two-element) set of truth values. The above semantics is therefore compositional (with respect to meanings) and bivalent. (But truth values do not compose algebraically.) The semantic constructions are based on the notions of a logical matrix and of a logical atlas. (An atlas corresponds roughly to the notion of a generalized or ramified matrix in Wójcicki's sense.) Usually truth values are assigned in a manner that respects the meaning of sentences; such valuations are called *admissible*. This leads to the notions of a *locally* evaluated interpreted language and a *globally* evaluated interpreted language, as well as to various derived concepts. The authors take the view that "consequence" (what follows from what) is the central term of logic. Accordingly, the term "logic" itself is viewed in a threefold manner. In the first meaning, it is a formal *assertional* logic, where the focus is on the set of logical theses. It is also viewed as a formal *asymmetric* consequence operation satisfying the familiar Tarski-style conditions with further amendments due to Łoś and Suszko. Finally, the word "logic" is being understood as a formal *symmetric* consequence relation. The last notion is essentially due to D. Scott and it provides a natural abstract framework for various Gentzen-style (and multiple-conclusion) formalizations of logics. It is remarkable that every symmetric consequence relation, apart from being complete with respect to a set of truth-valuations, has the absoluteness property—it is determined by exactly *one* class of valuations.

The book provides a wealth of bibliographical and historical notes. The extensive bibliography almost covers the existent literature. Also the remarks on the history of the subject are well balanced and accurate. Some complementary remarks are natural in this context. The cancellation property (§7.8) for *asymmetric* consequence logics was already studied by Łoś and Suszko, *Remarks on sentential logics*, JSL XL 603 (in a simplified version appropriate for compact consequences), and then in full generality by Wójcicki in his booklet, ***Investigations into methodology of sentential calculi (I)***, Institute of Philosophy and Sociology of the Polish Academy of Sciences, Warsaw 1971. They proved that a formal asymmetric logic is characterized by a single matrix if and only if the logic has the cancellation property; see Wójcicki, ***Theory of logical calculi***, JSL LV 1324, §3.2. The Kripke-style semantics for the minimal orthologic was already defined by Dishkant, *Semantics of the minimal logic of quantum mechanics*, ***Studia logica***, vol. 30 (1972), pp. 23–30 (but in a rather intricate way). In a reprinted version of this paper (***The logico-algebraic approach to quantum mechanics***, Volume II, D. Reidel, 1979, pp. 17–29), he remarks that Kripke frames with symmetric accessibility relations have the same logical power, i.e., they define the orthologic, the result proved by Goldblatt. The history of Leibniz congruences goes back to the 1949 work of Łoś, ***O matrycach logicznych***, JSL XVI 59. Wójcicki calls them "the largest matrix congruence." The suggestive term "Leibniz congruence" was coined by Blok and Pigozzi. They defined the notion of the Leibniz *operator* Ω and started systematic investigations of it. Theorem 7.13.6 as it stands characterizes the class of *protoalgebraic* asymmetric consequence logics, and not algebraizable ones. In order to characterize compact algebraizable deductive systems in terms of Ω one should add the conditions of injectivity and continuity of the operator Ω. Theorem 7.5.13, a result of the present reviewer, was independently proved by Zygmunt. This theorem additionally requires that the symmetric consequence relation be compact. Theorem 7.7.3 is not adequately stated. The version presented in the book is rather confusing and hard to understand. The first sentence of Remark 7.7.2 seems to be false; the inclusion signs should be reversed there. In Definition 7.13.2 the phrase *there exists* should be replaced by *there exists a class K of algebras and*. Theorem 3.6.13 is tautological when compared with Definition 3.6.1. Asymmetric consequence logics with (directly) supporting congruences (p. 215) are called *self-extensional* by Wójcicki (JSL LV 1324). Also limit cases of some definitions need a careful inspection in various contexts. For example, Definition 3.18.2 requires that every lattice filter be non-empty. This is in conflict with Theorem 3.18.5 stating that the intersection of every non-empty collection of filters is a filter. This conflict disappears if the aforementioned family is assumed to be finite or the lattice has the unit element. All these observations do not detract from the merit of the book. Their aim is merely to pick up some minor but bothersome shortcomings or errors which appear in some places.

The book has a coherent structure—consecutive chapters form a closely interrelated group. The subject matter includes many philosophical, semantic, algebraic, and logical discussions about what the enterprise of algebraically oriented philosophical logic is about as well as formal analyses based on the framework developed in the book. The book is not, however, dominated by formal results—the emphasis is conceptual rather than technical. Formal considerations are reduced to the absolute minimum. Instead the focus is on comments and remarks presenting the essence of the definitions and theorems.

The style of the book is lucid, elegant, and economical. The book is of great educational importance. It should be recommended to everyone who would like to go deep into the foundations of contemporary logic. But also professional logicians will find a great interest in the chapters dealing with representation and duality theory for algebras and, last but not least, in many historical or bibliographical notes spread through this excellent book.

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Martin Zeman. ***Inner models and large cardinals.*** de Gruyter Series in Logic and its Applications, vol. 5. Walter de Gruyter and Co., Berlin, 2002, xii + 369 pp.

Gödel's Constructible Universe L is the prototype for the kind of inner models that are the subject of the book. Jensen's fine structure theory for L is the kind of analysis of these models that one is after. The author's purpose is to expose techniques for constructing these sorts of inner models with measurable cardinals and even more powerful large cardinals. Let us begin here with a few words on the meaning of the terms *inner models* and *large cardinal* that are used in the title.

The inner models considered in the book are constructed in stages that are indexed by ordinals. Stage β of the construction results in the definition of a structure denoted J^E_β. Such structures are called *premice*. There are other conditions in the definition of premice some of which we will describe shortly. The universe of J^E_β is also denoted J^E_β as is common. The predicates of this structure are set membership and a sequence $\langle E_\alpha \mid \alpha \leq \beta \rangle$. If $\beta = \alpha + 1$, then the universe of J^E_β is the rudimentary closure of the structure J^E_α. Every definable subset of the structure J^E_α is an element of J^E_β but there are other sets there as well. The limit stages of the construction are handled by taking unions. The resulting proper class structure is denoted $L[E]$.

Notice that so far we have said nothing about the choice of E_β nor the restrictions on this choice other than $E_\beta \subseteq J^E_\beta$. That is where large cardinals axioms come in. For the most part, such axioms assert the existence elementary embeddings from the universe of sets V to some other transitive class. A goal of the construction described above is for $L[E]$ to be a model of ZFC with large cardinals. Towards this goal, one requires that E_β codes an elementary embedding from J^E_β to J^F_γ for some $\gamma \geq \beta$. There are several additional conditions that J^E_β must satisfy in order to be a premouse. A key clause is Mitchell's coherence condition, which says that, in the situation above, $F \restriction \beta = E \restriction \beta$ and F_β is trivial, i.e., F_β codes the identity function from J^F_β to J^F_β. We will not discuss here the further conditions.

The most important tool of inner model theory is iteration trees, which result from taking iterated ultrapowers of premice but not necessarily in a linear fashion. The complexity of the possible tree structures is tied to the level of large cardinals under consideration. At successor stages of an iteration, one cannot take for granted that the associated ultrapowers are wellfounded. Nor can one assume that at limit stages there exists a cofinal branch in the tree structure whose corresponding direct limit model is wellfounded. These are always key issues. Generally speaking, premice are called *iterable* if there is a strategy for avoiding illfoundedness of this sort, and iterable premice are called *mice*. The casual reader can get a sense of how iteration trees arise as follows. It is easy to see that if $X \prec J^E_\beta$, then $X \simeq J^{\overline{E}}_{\overline{\beta}}$ for some $\overline{E}$ and $\overline{\beta} \leq \beta$. The special case of this fact in which $J^E_\beta = J_\beta$ is known as the condensation property of L and is used to prove fundamental results such as $\diamond$ holds in L. However, the general form is not particularly useful since $\overline{E}$ and E may have little to do with each other. To obtain a useful general condensation theorem, one compares $J^{\overline{E}}_{\overline{\beta}}$ and J^E_β by co-iterating them until they are lined up; see sections 4.4 and 5.5 of the book.

The author takes up the subject of $L[E]$ models at several levels of the large cardinal hierarchy. In Chapters 1 through 3, he presents Jensen fine structure in a form that is not limited to a particular large cardinal. The definition of premouse he gives at the beginning of Chapter 4 allows for any number of measurable cardinals but rules out the existence of a measure U over a cardinal κ with the property that $\{\alpha < \kappa \mid \alpha \text{ is measurable}\} \in U$. In other words, the Mitchell order of U is exactly zero. As we said before, what turns premice into mice is the ability to form iterated ultrapowers without encountering illfounded models. Measures of order zero are well within the range where linear iterations are the only kind that matter. In Chapter 7, the author presents the theory of K at this level of the large cardinal hierarchy.

In fact, linear iterations suffice up to the level of strong cardinals. In Chapter 8, the author makes a first-order change in the definition of premouse so as to allow the extenders on the sequence E to witness that a cardinal is strong in $L[E]$. He leaves the second-order iterability condition basically unchanged since linear iterations are still enough.

The existence of *overlapping extenders* is a technical condition on the sequence E that marks where we can no longer make do with linear iterations. Overlapping extenders is the subject of Chapter 9. There the author gives an abridged account of how inner models with Woodin cardinals are constructed.

The methods and theorems presented in this book have a complicated history. Most of the important post-1980 ideas found in the book are due to Dodd, Jensen, Martin, Mitchell and Steel. The author does an accurate job of explaining the distribution of credit and citing sources at the end of each chapter.

Inner model theory is a subject that involves many technical calculations. Accounts that emphasize the big picture can be extremely useful but run the risk of leaving readers with doubts about their ability to provide the remaining details themselves. Ironically, accounts that give every last detail can seem even less convincing to readers. This book strikes a good balance that is different from that found in any other available source. Readers, especially the uninitiated, will also benefit from the gradual approach that the author takes: he develops fine structure and the theory of K in the simple setting of measures of order zero, then turns to issues of indexing in the context of strong cardinals, and finally, for more powerful large cardinals, he focuses on iteration trees. The book is well thought out and excellently written; it is a welcome and valuable addition to the literature.

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BENEDIKT LÖWE AND PHILIP WELCH. *Set-theoretic absoluteness and the revision theory*. ***Studia Logica***, vol. 68 (2001), pp. 21–41.

BENEDIKT LÖWE. *Revision sequences and computers with an infinite amount of time*. ***Journal of Logic and Computation***, vol. 11 (2001), pp. 25–40.

Belnap and Gupta (JSL LX 1314) have generalized the Revision Theory of Truth to a general theory of circular definitions. In the first paper *Set-theoretic absoluteness and the revision theory of truth*, on which I shall focus, the authors discuss some observations and results on this theory. The paper contains only proof sketches; more details and proofs are given in the second paper *Revision sequences and computers with an infinite amount of time* and especially in a yet unpublished paper by Welch *On revision operators* (`http://www.maths.bris.ac.uk/~mapdw`, to appear in the ***Journal of Symbolic Logic***). Therefore this is a review of all three papers.

In Revision Theory one starts with a certain extension E_1 of a distinguished predicate $\dot{x}$ in the language of first-order arithmetic (or another language with enough coding power). A new extension E_2 for that predicate is obtained by considering the set of objects defined by some fixed formula Φ (or a scheme of a certain kind) if $\dot{x}$ is interpreted by E_1 and if the remaining vocabulary is interpreted by the standard model of arithmetic. This revision step can be iterated. Since some numbers may flip in and out of the extension of the predicate $\dot{x}$, the operation Γ generating new extensions is—in contrast to inductive operators—not inclusive, that is, neither $\Gamma(X) \subseteq X$ nor $X \subseteq \Gamma(X)$ holds. Therefore one cannot simply take unions or intersections at limits. Different proposals for the definition of the extension of $\dot{x}$ at limit levels have been made; these different methods are called *limit rules*. In general, the revision process will not come to a halt—unlike usual inductive definitions. That is, the extension of $\dot{x}$ will not become constant from some level on. Nevertheless, some extensions will reappear again and again in certain revision sequences, that is, sequences of extensions of $\dot{x}$. Revision Theorists are mainly interested in those elements falling under all cofinally recurring extensions in every revision sequence. These elements are called *categorically stable*. In general, different rules for generating extensions at limit levels yield different recurring sets.

In *Revision sequences and computers with an infinite amount of time* Löwe posed the question to what extent different limit rules affect circular definability. He also observed that results on the limit rule in the theory of Infinite Time Turing Machines obtained by Welch (*Eventually infinite time Turing degrees: Infinite time decidable reals*, JSL LXV 1193) might be applied to solve this problem.

Answering this question Welch shows in his yet unpublished paper on revision operators that—independently of the employed limit rule—for certain operators Γ, the categorically stable elements form a Π^1_2-complete set. This result applies to a large class of limit rules which contains all limit rules discussed in the literature. The set of numbers that are in every cofinally recurring extension of a revision sequence associated to a circular definition are said to be *revision-theoretically definable* by that circular definition. According to a further result by Welch, one always ends up with the same class of revision-theoretically definable sets independently of which limit rule is actually applied (again as long as the limit rule meets certain sensible conditions). Thus, in a sense, the discussion on the different strategies for treating limit levels is pointless from the standpoint of definability.

As is well known and demonstrated in detail, e.g., in McGee's ***Truth, vagueness and paradox*** (Hackett, 1991), Kripke's theory of truth (JSL L 1068) and positive inductive definability are deeply related. The Revision Theory of Truth is linked to the theory of circular definitions in an analogous way. But whereas the theory of positive inductive definitions is very well understood, only very little was known about the mathematics of the general theory of *circular* definitions. Welch's results fill this gap.

The main example of a revision operator is the Tarski operator: If the the distinguished predicate $\dot{x}$ is interpreted by E_1 the next extension E_2 is defined as the set of all true sentences (or rather their codes) if $\dot{x}$ is interpreted by E_1. The above-mentioned results imply that the sentences common to all recurring extensions of the truth predicate in every revision sequence, also known as the (categorically) stable truths, form a Π^1_2-complete set of numbers. This extends a result due to John P. Burgess (JSL LI 663) who proved this for a particular class of rules.

Welch uses his insights on definability by circular definitions in order to produce a truth-theoretic problem that is independent of ZFC: One starts with an expansion of the standard model of arithmetic by the set of all stable truths T^0 (relative to some limit rule) as an interpretation of $\dot{x}$. The set T^1 of all stable truths of this expanded model is complete Π^1_2 in the complete Π^1_2 set of stable truths T^0—that is T^1 is complete $\Pi^1_2(T^0)$. This process can be iterated through all finite levels, and sets of higher and higher Δ^1_2-degrees are obtained. This process is not unnatural given the standards of modern theories of truth; in fact Belnap and Gupta have considered iterations through all ordinal levels in chapter 6 of their book. Using a result from Friedman's *Minimality in the Δ^1_2-degrees* (***Fundamenta Mathematicae***, vol. 81, 1974, pp. 183–192), Welch shows in his unpublished paper that the problem whether the sequence of Δ^1_2-degrees so obtained has a least upper bound turns out to be independent from ZFC (see also Welch's *On Gupta-Belnap revision theories of truth, Kripkean fixed points, and the next stable set*, BSL VII 345). Also this result does not depend on the particular limit rule (given the usual minimal restrictions on possible limit rules). As far as the reviewer knows, this is the first problem in the theory of truth that turned out to be independent from ZFC and whose validity is affected by large cardinals and combinatorial principles stronger than ZFC.

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MOTI GITIK AND MENACHEM MAGIDOR. *Extender based forcings*. ***The Journal of Symbolic Logic***, vol. 59 (1994), pp. 445–460.

MOTI GITIK AND WILLIAM J. MITCHELL. *Indiscernible sequences for extenders, and the singular cardinal hypothesis*. ***Annals of Pure and Applied Logic***, vol. 82 (1996), pp. 273–316.

MOTI GITIK. *Blowing up the power of a singular cardinal*. ***Annals of Pure and Applied Logic***, vol. 80 (1996), pp. 17–33.

MOTI GITIK AND CARMI MERIMOVICH. *Possible values for $2^{\aleph_n}$ and $2^{\aleph_\omega}$*. ***Annals of Pure and Applied Logic***, vol. 90 (1997), pp. 193-241.

MOTI GITIK. *Blowing up power of a singular cardinal—wider gaps*. ***Annals of Pure and Applied Logic***, vol. 116 (2002), pp. 1–38.

Moti Gitik is one of a small handful of the most prominent set theorists working today. His work exhibits a steady engagement with central and difficult problems of set theory, a persistence of endeavor over almost three decades, and above all a masterful virtuosity in the application of sophisticated techniques across a broad range. The papers under review represent his recent work on the Singular Cardinals Problem, which has been the main longitudinal focus of his research and has featured many of his most notable successes.

The Singular Cardinals Problem refers to the general problem of clarifying in ZFC the possibilities for the continuum function 2^κ for singular cardinals κ. For regular κ, it became quickly known from the work of William Easton in the 1960's at the advent of forcing that there are no restrictions on 2^κ other than those imposed by monotonicity ($\kappa \leq \lambda$ implies $2^\kappa \leq 2^\lambda$) and the Zermelo-König inequality ($\mathrm{cf}(2^\kappa) > \kappa$). In the Easton forcing models, for singular κ, 2^κ was the *least* possible cardinal respecting the above two restrictions, and the question quickly arose whether there were any other possibilities. The least possibility can be expressed in cardinal arithmetic as follows:

$$\text{For singular } \kappa,\ \kappa^{\mathrm{cf}(\kappa)} = \max(2^{\mathrm{cf}(\kappa)}, \kappa^+).$$

This became known as the Singular Cardinal Hypothesis (SCH), and the issue became whether this can fail, with the preoccupation being how 2^κ for singular κ is restricted by what the 2^α's are for $\alpha < \kappa$. The focal cases became whether it is possible to violate SCH in certain straightforward ways: Is there a singular κ which is a strong limit (i.e., $\alpha < \kappa$ implies

$2^\alpha < \kappa$) and yet $2^\kappa > \kappa^+$? In fact, can GCH hold below a singular cardinal and fail there? Finally, can GCH hold for the $\aleph_n$'s and fail at $\aleph_\omega$?

The investigation of the Singular Cardinals Problem would involve the necessary use of large cardinal hypotheses, and the theory would further motivate large cardinals as the natural hierarchy of propositions beyond ZFC. Having simple and basic formulations in basic cardinal arithmetic, the Singular Cardinals Problem became a driving, central problem of axiomatic set theory, leading to major advances in infinitary combinatorics and the investigation of forcing extensions and inner model theory. The first significant result combined work of Karel Prikry and Jack Silver.

In his 1968 dissertation Prikry showed how to change the cofinality of a measurable cardinal κ to ω in a cardinal preserving forcing extension having the same bounded subsets of κ (so that κ remains a strong limit cardinal). Prikry forcing may have seemed a curiosity at first, but it would soon be shown that large cardinal hypotheses are necessary for changing cofinalities in such a restrictive fashion. With U a normal κ-complete ultrafilter over κ, the Prikry forcing conditions are of form $\langle s, A\rangle$ where s a finite subset of κ, $A \in U$, and $\max(s) < \min(A)$. $\langle t, B\rangle$ is a stronger condition than $\langle s, A\rangle$ when $s \subseteq t$, $B \subseteq A$, and $t - s \subseteq A$. That is, a condition $\langle s, A\rangle$ specifies a finite initial segment s of an eventual ω-cofinal sequence, and extensions of s are to be drawn from a large set $A \in U$. Large cardinal properties lead to the crucial property that given any condition and any formula of the forcing language, there is a stronger condition with the same initial component s that decides the formula. This pivotal "Prikry property" shows that no new bounded subsets of κ are added. Prikry forcing would become paradigmatic for the later forcing arguments having to do with the Singular Cardinals Problem.

In 1971 Jack Silver, showed with "reversed" (better: iterated or "upward") Easton forcing how to get a measurable cardinal κ satisfying $2^\kappa > \kappa^+$. This then followed by Prikry forcing to change the cofinality of κ to ω led to the first example of a failure of SCH. Silver had to start with a hypothesis substantially stronger than measurability.

Proceeding to later developments, in 1974 Silver established a surprising, groundbreaking result in ZFC, which had as a consequence that if κ is singular of *un*countable cofinality and GCH held below κ, then it also holds at κ. Ronald Jensen was motivated by this to establish in the same year the Covering Theorem for L, easily the most important result in set theory in the 1970's. It has as a consequence that if SCH fails, then $0^\#$ exists. Thus, the failure of SCH has large cardinal consequences. Soon Anthony Dodd and Jensen formulated the "core" models and through this new inner model theory established that the failure of SCH is at least as strong as the existence of a measurable cardinal.

Beyond the Prikry-Silver result, what about the failure of SCH at small singular cardinals? Menachem Magidor showed in the late 1970's that if there is a supercompact cardinal (resp. a huge cardinal) there is a forcing extension in which $\aleph_\omega$ is strong limit (resp. GCH holds below $\aleph_\omega$) and yet $2^{\aleph_\omega} = \aleph_{\omega+2}$. These initial forays first broached the possibilities at $\aleph_\omega$, but depended on very substantial large cardinal hypotheses.

With some large cardinal hypotheses implying the consistency of the failure of SCH and in turn that failure implying the consistency of weaker large cardinals, research soon focused on finding a natural large cardinal hypothesis that would be *equiconsistent* with the failure of SCH. Such an equiconsistency result would be a remarkable achievement that would exactly gauge the consistency strength of a basic combinatorial assertion in cardinal arithmetic. The candidates for a possible equiconsistency result were in place, their naturalness emerging as the methods became refined.

William Mitchell had already in the mid-1970's extended the Dodd-Jensen core model work to stronger large cardinal hypotheses defined in terms of the order of a measurable cardinal: For normal ultrafilters U, V over a measurable cardinal κ, define $V < U$ *iff* V belongs to the transitized ultrapower of the universe via U. This Mitchell order turns out to be well-founded, so for such U, define $o(U) = \sup\{o(V) + 1 \mid V < U\}$. Finally, for such κ, define $o(\kappa) = \sup\{o(U) + 1 \mid U$ is a normal ultrafilter over $\kappa\}$. The strongest possibility in this Mitchell order for measurable cardinals is $o(\kappa) = \kappa^{++}$, and Mitchell had developed the "core model for sequences of measures" to get at ramifications of this consistency strength. The Mitchell order hierarchy can be generalized to higher orders in a natural way by going into extenders, generalizations of ultrafilters, with $o(\kappa) = \infty$ corresponding to what is known as a strong cardinal.

In work in the 1980's, Mitchell showed that the failure of SCH led to the consistency of cardinals high in his order. Then in 1988 Hugh Woodin dramatically weakened the large cardinal hypothesis needed to get a measurable κ satisfying $2^\kappa > \kappa^+$, and hence the failure of SCH using the Prikry-Silver scheme, to a proposition about enhanced measurable cardinals. He also showed that from his proposition one could get Magidor's conclusion that $\aleph_\omega$ could be the least cardinal where GCH fails. It is at this point that Gitik's refinements produced an equiconsistency.

First, Gitik [*The negation of the singular cardinal hypothesis from $o(\kappa) = \kappa^{++}$.* ***Annals of Pure and Applied Logic***, vol. 43 (1989), pp. 209–234] showed how to get the consistency of Woodin's proposition just from the existence of a κ satisfying $o(\kappa) = \kappa^{++}$. Then, Gitik [*The strength of the failure of the singular cardinal hypothesis.* ***Annals of Pure and Applied Logic***, vol. 51 (1991), 215–240] applied Shelah's recently developed pcf theory to Mitchell's core model analysis to establish that the consistency of the existence of a κ satisfying $o(\kappa) = \kappa^{++}$ is actually necessary. Hence, the strength of the failure of the SCH was exactly gauged at $o(\kappa) = \kappa^{++}$. Gitik's virtuosity here was not only exhibited in getting the forcing direction to Woodin's proposition, but in combining Mitchell's sophisticated core model analysis with Shelah's elegant combinatorial analysis.

However successful the work on SCH and however elegant the equiconsistency result in revealing a intimate connection between powers of singular cardinals and large cardinals defined in terms of a hierarchy of ultrafilters, there still remained the question of the optimality of methods and the possibilities for the continuum function. For example, Woodin's model for GCH below $\aleph_\omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$ from what turned out to be the optimally weak large cardinal hypothesis required a delicate construction to arrange GCH below and an ingenious idea to get $2^{\aleph_\omega} = \aleph_{\omega+2}$. How about getting $2^{\aleph_\omega} > \aleph_{\omega+2}$?

A remarkable advance in method and results was made by Gitik and Magidor in their [*The Singular Cardinals Hypothesis revisited.* In: ***Set theory of the continuum***, edited by H. Judah, W. Just, and H. Woodin, Mathematical Sciences Research Institute publications, vol. 26, Springer-Verlag, New York 1992, pp. 243–279—hereafter, *SCH revisited*], an incisive paper that established the context for the papers under review. The Prikry-Silver two-stage approach, first making 2^κ large and then singularizing κ without adding any new bounded subsets or collapsing cardinals, had been the basic model for attacking the Singular Cardinal Problem. Gitik and Magidor showed how to add many subsets to a large cardinal κ while *simultaneously* singularizing it without adding any new bounded subsets or collapsing cardinals. Thus, it became much easier to arrange any particular continuum function behavior below κ, like achieving GCH below, while at the same time making 2^κ arbitrarily large. Moreover, the new method smacked of naturalness and optimality.

The new Gitik-Magidor idea was to add many new Prikry ω-sequences corresponding to κ-complete ultrafilters over κ while maintaining the basic properties of Prikry forcing. There is an obvious problem that if these Prikry sequences are too independent, information can be read from them that corresponds to new reals being added. The solution was to start from a sufficient strong large cardinal hypothesis and get a "nice system" of ultrafilters $\langle U_\alpha \mid \alpha < \lambda\rangle$, a system such that for many $\alpha \leq \beta < \lambda$ there is ground model function $f : \kappa \to \kappa$ such that: For all $X \subseteq \kappa$, $X \in U_\alpha$ *iff* $f^{-1}(X) \in U_\beta$. Having such a projection function is the classical way of connecting two ultrafilters together, and one writes that $U_\alpha \leq_{\mathrm{RK}} U_\beta$

under the Rudin-Keisler partial ordering. By this means one has the possibility of adding new subsets of κ, corresponding to different Prikry sequences, which are still dependent on each other so that no new bounded subsets need necessarily be added in the process. It is notable that in Gitik's longitudinal work, some of the ideas here had already occurred in his [*Changing cofinalities and the non-stationary ideal*. ***Israel Journal of Mathematics***, vol. 56 (1986), pp. 280–314]. Gitik and Magidor worked out how their new approach leads to what turns out to be optimal or near optimal consistency results, and incorporating collapsing maps as in previous arguments of Magidor and Shelah, they got models in which GCH holds below $\aleph_\omega$ yet $2^{\aleph_\omega}$ is large, indeed equal to $\aleph_{\alpha+1}$ for any prescribed countable ordinal α. Singularizing to get cofinality ω was the focus of the paper, but the ideas are presumably adaptable to get uncountable cofinalities.

Finally getting to the papers under review, they can be considered as part of Gitik's program for working out the possible new equiconsistency results in light of the methods and results of *SCH revisited*, akin to his equiconsistency result associating the failure of SCH with $o(\kappa) = \kappa^{++}$. Though there are exceptions and more information can be provided in special cases, the general pattern is: Getting a singular strong limit cardinal κ satisfying $2^\kappa \geq \lambda$ is equiconsistent to having a κ such that $o(\kappa) \geq \lambda$.

The first paper under review, joint with Magidor, is a continuation of their *SCH revisited* and refines some of its results. In *SCH revisited* the authors had started out with elementary embeddings hypotheses. For example, to adjoin λ new subsets to a large cardinal κ via a nice system they assumed the existence of an elementary embedding $j\colon V \prec M$ with critical point κ such that $V_{\kappa+\lambda} \subseteq M$. In the Mitchell hierarchy, having such an embedding entails that $o(\kappa) \geq \lambda + 1$. To reduce the consistency strength to what turns out to be the optimally minimal hypothesis $o(\kappa) = \lambda$, according to the definition of the Mitchell ordering one can only assume that there are many extenders whose Mitchell orders are less than albeit cofinal in λ. Gitik and Magidor in the present paper show how to weave together countably many nice systems corresponding to "overlapping" extenders to get, for example, the following optimal result: *If there is a κ satisfying $o(\kappa) = \kappa^{+\omega}$, then there is a forcing extension in which GCH holds below κ,* $\mathrm{cf}(\kappa) = \omega$, *and* $2^\kappa \geq \kappa^{+\omega}$. Gitik and Magidor work out other consistency results based on countably many nice systems, and moreover use a variant of their *SCH revisited* forcing to get the following result: *If there is a κ satisfying $o(\kappa) \geq \kappa^{+\alpha}$ where* $\kappa < \mathrm{cf}(\alpha) < \alpha$, *then there is a cardinal preserving forcing extension in which κ is measurable and* $2^\kappa = \kappa^{+\alpha}$. What the minimal hypotheses are for getting measurable cardinals κ so that 2^κ is large is connected to the Singular Cardinals Problem through the Prikry-Silver scheme.

The second paper under review, with William Mitchell, is devoted to getting lower bounds in the Mitchell hierarchy for strong failures of SCH involving large powers for the singular cardinal. The paper provides a penetrating inner model analysis that in the main cases is sufficient together with the new forcing results to establish equiconsistencies. In Mitchell's seminal work in the 1970's on the core model for sequences of measures (ultrafilters), he had established a covering theorem for such a model, extending previous work of Dodd and Jensen. By this means he had achieved lower bounds for the failure of SCH, which were then improved by Gitik to the optimal $o(\kappa) = \kappa^{++}$ as discussed previously. Getting lower bounds for stronger failures of SCH, i.e., having a singular strong limit κ satisfying $2^\kappa \geq \lambda$ where $\lambda > \kappa^{++}$, would require applying covering theorems for the core model for extenders, developments in inner model theory that had emerged in the late 1980's.

Gitik and Mitchell in their paper first review the core model for extenders and the corresponding covering theorem. Such theorems are sensitive to having sequences of indiscernibles in the model, but there are considerable complications here because of the proliferation of such sequences. With K the core model for extenders, they then establish the main result: *Suppose that κ is a singular strong limit cardinal and $2^\kappa \geq \lambda$ where λ is not the successor of a cardinal of cofinality at most κ. If* $\mathrm{cf}(\kappa) > \omega$, *then* $o(\kappa) \geq \lambda$. *If* $\mathrm{cf}(\kappa) = \omega$, *then either*

$o(\kappa) \geq \lambda$ or there is an $n \in \omega$ such that $\{\alpha \mid K \models o(\alpha) \geq \alpha^{+n}\}$ is cofinal in κ. As in Gitik's previous work, Shelah's pcf theory is applied. The conclusion $o(\kappa) \geq \lambda$ provided the converse direction to the forcing arguments to complete the equiconsistency results, but there was a new wrinkle in the cofinality ω case. Remarkably, Gitik and Mitchell's analysis of indiscernibles is so careful that even this new wrinkle would become a substantial case.

In the third paper under review Gitik turns to "blowing up the power" of a cardinal which is already singular of cofinality ω. He starts with the optimal hypothesis as given by the new wrinkle in the Gitik-Mitchell analysis: *If there is a cardinal κ of cofinality ω such that for every $n \in \omega$, $\{\alpha \mid o(\alpha) \geq \alpha^{+n}\}$ is cofinal in κ, then there is a cardinal preserving forcing extension having the same bounded subsets of κ and satisfying* $2^\kappa = \kappa^{++}$. Gitik had already understood that the above hypothesis on κ led to a substantial new feature about indiscernibles and covering [*On hidden extenders.* ***Archive for Mathematical Logic***, vol. 35 (1996), pp. 349–369.] Building this into the work on countably many nice systems in the first paper under review, Gitik manages to establish his result by first working out a cardinal collapsing forcing and then taking a projection of it that does preserve cardinals.

The fourth paper under review is authored by Gitik together with his student Carmi Merimovich and is a generalized version of the latter's Masters of Science thesis. In *SCH revisited*, Gitik and Magidor from optimal large cardinal hypotheses had gotten a model in which GCH holds below $\aleph_\omega$ yet $2^{\aleph_\omega} = \aleph^{+m}$. Gitik and Merimovich here establish that any possible continuum function below can be realized. From optimal large cardinal hypotheses, they show that for any function $a\colon \omega \to \omega$ satisfying $n < a(n)$, there is a model satisfying $2^{\aleph_n} = \aleph_{a(n)}$ for every n while $2^{\aleph_\omega} = \aleph_{\omega+m}$. In order to do this they build on *SCH revisited* by incorporating added features into the forcing that control the resulting $2^{\aleph_n}$'s, and the complications are considerable.

In the last paper under review, Gitik improves the results of the third paper. That paper had addressed the new wrinkle discovered in the Gitik-Mitchell analysis in the second paper under review and shown that the new large cardinal hypothesis uncovered there also leads to the failure of SCH. Starting with that hypothesis, Gitik now gets higher powers for the singular cardinal: *If $\delta < \aleph_1$ and there is a cardinal κ of cofinality ω such that for every $n \in \omega$, $\{\alpha \mid o(\alpha) \geq \alpha^{+n}\}$ is cofinal in κ, then there is a cardinal preserving forcing extension having the same bounded subsets of κ and satisfying* $2^\kappa = \kappa^{\delta+1}$. Moreover, he is able to get a generalization. This paper is more systematic and self-contained, providing simplified versions of previous constructions. At the end, Gitik writes: "In a further paper we plan to extend the present techniques in order to handle arbitrary gaps between κ and 2^κ. The subject of consistency strength here [in this paper] was almost completed ignored. We hope to deal with this matter in a further paper as well."

Thus, Gitik will carry on with his enterprise with admirable confidence and perseverance. The prospect of further equiconsistency results in the offing is enticing and intriguing. There are still issues about the general possibilities for the continuum function below a singular cardinal and how large its power can possibly be in particular cases. The most important question is: Can $\aleph_\omega$ be a strong limit cardinal yet $2^{\aleph_\omega} \geq \aleph_{\omega_1}$? We await the further work of Gitik for the central insights it will likely provide about the Singular Cardinals Problem, large cardinals, and the limits of possibility in ZFC.

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