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DIFFERENTIAL CALCULUS AND NILPOTENT REAL NUMBERS

ANDERS KOCK

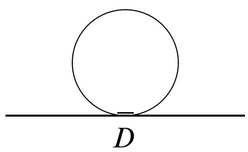
Do there exist real numbers d with $d^2 = 0$ (besides $d = 0$, of course)?

The question is formulated provocatively, to stress a formalist view about existence: existence is consistency, or better, coherence.

Also, the provocation is meant to challenge the monopoly which the number system, invented by Dedekind et al., is claiming for itself as THE model of the geometric line. The Dedekind approach may be termed "arithmetization of geometry".

We know that one may construct a number system out of synthetic geometry, as Euclid and followers did (completed in Hilbert's Grundlagen der Geometrie, [2, Chapter 3]): "geometrization of arithmetic". (Picking two distinct points on the geometric line, geometric constructions in an ambient Euclidean plane provide structure of a commutative ring on the line, with the two chosen points as 0 and 1).

Starting from the geometric side, nilpotent elements are somewhat reasonable, although Euclid excluded them.[1] The sophist Protagoras presented a picture of a circle and a tangent line; the apparent little line segment D which tangent and circle have in common, are, by Pythagoras' Theorem, precisely the points, whose abscissae d (measured along the tangent) have $d^2 = 0$. Protagoras wanted to use this argument for destructive reasons: to refute the science of geometry.[2]



A couple of millenia later, the Danish geometer Hjelmslev revived the Protagoras picture. His aim was more positive: he wanted to describe Nature *as it was*. According to him (or extrapolating his position), the Real

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[1]The geometric number system constructed by Euclid is a *field*. Geometrically expressed: through two non-equal points passes a unique line.

[2]According to Hermann Weyl, Hume presented a similar "refutation".

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Line, the Line of Sensual Reality, had many nilpotent infinitesimals, which we can see with our naked eyes.[3]

He called his geometry "Natural Geometry", or "The Geometry of Reality". The ensuing number system is not a field, because it considers nilpotent elements; in fact, the ring of dual numbers $R[\varepsilon]$ is a model of the geometric line in his system. The study of geometry, based on this and related rings, has been developed further by a school of, mainly German, geometers, cf. e.g., [7].

Hjelmslev is *not* one of the forefathers of the "synthetic calculus" and "synthetic differential geometry", as we understand it and which we shall describe briefly below. Rather, he is brought in here for contrast.

First a mathematical contrast. Even though uniqueness of the line connecting two points may fail, in Hjelmslev's conception, he maintains the existence of *at least one* connecting line, for any two points in the plane. Algebraically, this implies that given two elements on the number line, one of them divides the other.[4] In particular, the preorder relation given by divisibility is *linear*, and the nilpotents (=infinitesimals) are not only small, but their smallness has a quantitative (linearly ordered) character. This is incompatible with the synthetic calculus we are expounding here, see below.

Secondly, the identification of a mathematical structure with physical objects leads to the idea that the laws of logic have limited use. According to Hjelmslev, there are in the theory " ... conflicts [with reality] which testify to the strong limitation of formal logic. And here, as in all other fields of human cognition, observation of reality must be the highest judge" ([6, p. 55]).

This is a denial of the right and duty of mathematical theories to make those formalizations, abstractions, idealizations, simplifications, extrapolations, etc. which make the theory beautiful and hence teachable and useful. The appearance of nilpotent elements in synthetic calculus is a formalism, which is coherent and simple; but it is not something one ascertains or refutes by observation of reality, or in Hjelmslev's words [4], by "genaue Untersuchungen (Experimente und Wahrnehmungen)".

Before I embark on expounding synthetic calculus, let me remind you of an analogous use of a formalism. When we teach calculus, we use as an example *population growth* as something which may be modelled by certain differential equations. Clearly, "observations of reality", as Hjelmslev talks about, will reveal that populations are counted by integers — to which differential calculus does not apply. The formalist view I am advocating, is

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[3]Hjelmslev was teaching draughtmanship at the Polytechnic High School in Copenhagen, and knew, and taught, that the line you can draw between two points on the paper is *not* unique if the two points are too close to each other.

[4]Take *line* to be a set described by an equation $ax + by = c$ with a or b invertible. If now u and v are arbitrary numbers, there is a line $ax + by = 0$ connecting (u, v) to $(0, 0)$. If a is invertible, $u = (-b/a)v$, so v divides u; similarly, if b is invertible, u divides v.

that applying calculus here, i.e., replacing integers by continuous numbers, is a typical good formalization, allowing a much simpler theory/calculus to be employed. The continuum is simpler than the discrete.

Now to the formalism of synthetic calculus. This has been expounded in many texts [9, 17, 12, 1], so I shall be brief. We follow Euclid in abstracting as a decisive abstract general relationship in geometry: that the line R has the structure of a commutative ring (once two distinct points 0 and 1 have been chosen). Unlike Euclid, but like in the formalism of Hjelmslev, this ring structure is not that of a field. In fact, there are "sufficiently many" elements d with $d^2 = 0$ (cf. the above picture). Let D be the set of these. To say that there are sufficiently many such d's, or that D is big enough, is rendered precise by the following axiom:[5]

for every function $f : D \to R$, there are unique elements $a, b \in R$ so that

$$f(d) = a + b \cdot d \qquad \text{for all } d \in D, \tag{1}$$

or "every function $D \to R$ extends uniquely to an affine function $R \to R$". Putting $d = 0$ in the axiom shows that the unique a mentioned there is in fact $f(0)$; whereas the unique b mentioned deserves a new name, we call it $f'(0)$, so that with this notation

$$f(d) = f(0) + d \cdot f'(0) \qquad \text{for all } d \in D,$$

and this determines $f'(0)$. More generally, from the axiom follows that for any function $f : R \to R$, and for each $x \in R$, there is a unique $f'(x)$ such that "1st order Taylor expansion" holds:

$$f(x + d) = f(x) + d \cdot f'(x) \qquad \text{for all } d \in D,$$

(apply, for each x, the axiom to the function $d \mapsto f(x + d)$).

A fair amount of differential calculus follows then purely algebraically — chain rule, Leibniz rule, etc. — as well as Taylor series calculus, and some differential geometry, cf. the literature quoted.

We now make the comparison with Hjelmslev's geometry. Applying the axiom (1) twice, one gets that any function $f : D \times D \to R$ is uniquely of the form

$$f(d_1, d_2) = a + b_1 \cdot d_1 + b_2 \cdot d_2 + c \cdot d_1 d_2 \tag{2}$$

(or in function space notation: from $R^D \cong R \times R$ (which follows from the axiom), one gets

$$R^{D \times D} \cong (R^D)^D \cong (R \times R)^D \cong R^D \times R^D \cong (R \times R) \times (R \times R),$$

so $R^{D \times D} \cong R^4$). However, in Hjelmslev's formalism, the term $d_1 d_2$ in (2) vanishes, since one of the d_i's divides the other, and has square zero. And if

[5] "Kock-Lawvere axiom", cf. [17, 12].

$d_1 d_2$ vanishes, c cannot be uniquely determined. So his formalism is incompatible with ours. Put geometrically, in terms of points in the coordinate plane: in Hjelmslev's formalism, two points can always be connected by *at least* one line; this is not so in our formalism. Another way of formulating the difference between Hjelmslev's number system and ours goes as follows: call two numbers x and y *neighbours* if $x - y \in D$. In Hjelmslev's formalism, the neighbour relation is *transitive*, or equivalently, D is stable under addition. This is not so in our formalism: arithmetically, if $d_1 \in D$ and $d_2 \in D$ (i.e., have square zero), $(d_1 + d_2)^2 = 2d_1 d_2$, so $d_1 + d_2 \in D$ only if $d_1 d_2 = 0$, and in Hjelmslev geometry, this is so, but in Synthetic Differential Calculus not.[6] Fortunately so! Pairs of such infinitesimals d_1, d_2 with $d_1^2 = 0$ and $d_2^2 = 0$, without $d_1 d_2 = 0$, occur crucially in the work of Sophus Lie under the name of (a pair of) *independent infinitesimals* (these are crucial for instance in the synthetic construction of Lie brackets of two vector fields, which Lie in essence gives).

There is another incompatibility. Namely, our formalism is incompatible with the law of excluded middle. With the law of excluded middle, one can for instance construct the Kronecker delta function δ

$$\delta(x) = 1 \text{ if } x = 0; \ \delta(x) = 0 \text{ if } x \neq 0,$$

and a Taylor expansion of this function δ from 0 leads to absurdity.

However, our formalism is compatible with intuitionistic logic. So, here we see intuitionistic logic, not as over-cautiousness and self-inflicted pain, but as something that permits a certain useful and strong theory to flourish — a theory which cannot coexist with the law of excluded middle.

The necessity of intuitionistic logic here is a trace of the origins of our formalism in category theoretic thinking (recall that intuitionistic logic applies to more categories than classical logic does); this brings us back to the question of *existence*. *Do* there exist numbers d with $d^2 = 0$?

Mathematical theories are constructions of the mind, "existence" of the objects of the theories is a matter of consistency of the theory, or better, of its coherence. Coherence is here construed as something more extrovert than mere inner logical consistency: I take it to mean that the theory is embedded in a network (cf. [11]) of other good theories, through relative interpretations, and that these theories directly or indirectly formalize and reflect aspects of the real world.

The network justifies the theories; the theories justify the objects they talk about; the objects justify their "elements". The synthetic calculus and synthetic differential geometry is a theory which has interpretations in topos theory, and it reflects aspects of the real world: namely the reasoning and the

[6]However, $d_1 + d_2$ has *cube* zero, so is a *second order* infinitesimal; the resulting notion of 1st order, second order, ... infinitesimals has predecessors in French Algebraic Geometry, with Grothendieck et al., cf. e.g., [18].

concepts used over at least three centuries — with or without the aritmetization of the continuum. This old reasoning is the essence of the existence of the d's of square zero.

To make the statement "the theory justifies the objects it talks about" more explicit, I would like to think of the theory as a theory about a *category* — the category whose objects are the objects of the theory, and whose morphisms are those transformations, constructions, etc., which are allowed by the theory.[7] Thus, any category of synthetic calculus and synthetic differential geometry deals with smooth objects, and smooth transformations — only smooth constructions are allowed; hence by the nature of smoothness, the law of excluded middle has no role.

Objects do not exist *per se*, but only by virtue of a *context*. They are social beings, objects, interacting *in a category*. There is a trace of this viewpoint in Euclid already: the line only exists in the context of the plane, and in the context of certain constructions and transformations that are "presupposed" (like ruler and compass constructions). Likewise, there is such a trace in Klein's Erlangen program.

So returning to the question "Do there exist enough real numbers d with $d^2 = 0$?"; the answer is Yes, in suitable contexts!

But you have to refute the monopoly which the arithmetically constructed continuum (Dedekind Cuts), has claimed for itself as the only mathematical model of the continuum. And you have to refute the "model of ZF set theory" as the only context of mathematics.[8]

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[7] A more mathematical formulation of this Network/Theory/Object/Element hierarchy is Lawvere's "Category of Categories as a Foundation of Mathematics", [13].

[8] Sometimes, this kind of reductionism is called "formalism", and sometimes, David Hilbert is blamed for it. The first is an illegitimate narrowing of the word "formalism"; the second is an equally illegitimate distortion of the evaluation of Hilbert, and it is completely incompatible with the view on geometry that he gives in the preface to the wonderful book [3] from 1932.

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DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF AARHUS
DK 8000 AARHUS C, DENMARK
E-mail: kock@imf.au.dk