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TWO SPACES LOOKING FOR A GEOMETER

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Abstract. In this talk I will introduce two *spaces*: the first space is the usual n-dimensional vector space with the unusual feature that n is a non-integer; the second space is composed of the linear matrices acting on the previous space (physicists are particularly interested in studying the limit as n goes to zero). These two *spaces* are not known to most mathematicians, but they are widely used by physicists. It is possible that, by extending the notion of space, they can become well defined mathematical objects.

§1. Introduction. In the last thirty years physicists have commonly used *spaces* having non-integer dimensions. These *spaces* do not have a clear mathematical meaning and it not evident in which sense the name space is not abusive. On the other hand it would be very interesting if one could properly define these objects that are commonly used by physicists. It is possible this task may be achieved by generalizing the notion of space.

There are many examples of the previous construction. Here we will concentrate our attention on two different cases:

- The usual space R^n, when n is no longer a positive integer, but it is a real number [18].
- The space of $n \times n$ matrices, for n real. In this case we are mainly interested in studying the limit as $n \to 0$ [6].

The strategy followed by physicists in defining these strange objects consists in picking some properties of *bona fide* spaces for positive integer n and in making the analytic continuation to non-integer n, without paying too much attention to the definition of the objects they are using.

A clarification of the mathematical structures involved would be extremely interesting because these spaces are extremely useful and their introduction had a strong heuristic value: it allowed us to obtain many results that later on have been proved using conventional techniques. It is unlikely that this task can be done without generalizing the notion of space in an appropriate way.

§2. The case of R^n with non-integer n. One of the physical motivations for introducing spaces with non-integer-dimension is the following. There are

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some physical systems whose properties cannot be computed directly in the case of the three dimensional space R^3; however one can formally compute these properties in the space R^n (that has dimension n) for n near to some specific value (e.g., $n = 4 - \varepsilon$ [18], [12]).

The procedure used by physicists for introducing non-integer dimensional spaces is very simple. The properties we would like to compute can be expressed in terms of some integrals. In some cases the integrals can be computed for any dimension and their value can be extended to a meromorphic function of the dimensions [7].

Let us consider a very simple example: the volume of the d-dimensional unit sphere is given by

$$V(d) = \frac{\pi^{d/2}}{\Gamma(d/2+1)}. \tag{1}$$

The previous formula for $V(d)$ can be extended to an entire function of d for real (and complex) d. Can one say that for generic real d the volume of the unit sphere is given by $V(d)$?

If a unit sphere in a space with a non-integer dimension would exist, it is likely that its volume would be given by eq. (1). Unfortunately a space with a non-integer dimension is not defined and therefore the question of the volume of the unit sphere is ill-posed.

Now we could try to reverse the question. We would like to say that a space with non-integer dimension d is characterized by the property that the volume of the unit sphere is given by eq. (1). Obviously one integral is not sufficient to identify a space and other integrals must be defined (for example the volume of two or more intersecting spheres). On the other hand this construction is likely to be empty; for example for $d = -3$, $V(d) = -2\pi$ is negative and this does not make sense if the integration measure is positive.[1]

The program may be the following; one can define geometrical objects (e.g., spheres. lines, planes) only giving some rules on how to compute the measure of their intersections, without making any reference to the coordinates of the space where they are embedded. If the construction is sufficiently rich, we could recover most of the geometrical results without having to introduce the space. Nothing forbids some geometrical objects having a negative measure, as long as they do not correspond to real objects in a real space.

In the following I will first present a tentative construction of the space with non-integer dimensions along these lines.

2.1. The construction of a non-integer dimensions space. The first objects I would like to define are the functions of one (or more) vectors that belong

[1] As we shall see later, negative dimension spaces can be interpreted as positive dimensional spaces where some of the coordinates are anticommuting and the integration measure $d^d x$ is not positive definite [15].

to a space with non-integer dimensions. This is a standard construction in physics and it has been discussed with full mathematical rigour by Etingof [7].

In the usual integer dimensional case there is no difficulty in defining a mapping from R^n to the real numbers. If n is a non-integer there is no reasonable way to define a generic point x that belongs to this space, so we cannot define a function $F(x)$ and the question of computing

$$\int d^n x\, F(x) \tag{2}$$

does not make sense if n is non-integer.

An alternative approach consists in giving some *minimal* definition of the function $F(x)$ and defining $\int d^n x\ F(x)$ as a linear functional on the appropriate space without entering into the definition of x.

Let us consider a simple example of this strategy. We consider a function $F(x)$ of the form

$$F(x) = f(x^2). \tag{3}$$

In this case, for integer dimensions, we have that

$$\int d^n x\ F(x) = nV(n) \int_0^\infty ds\, s^{n-1} f(s^2), \tag{4}$$

The previous expression defines a linear functional over the functions f that is well defined if $f(s)$ goes to zero sufficiently fast when s goes to infinity. The extension to non-integer n presents no difficulty and one obtains a meromorphic function of n if the function f is C^∞ at the origin.

The previous case may seem rather peculiar; however we can extend it to the case where the function F depends on the vectors x only via scalar products with other vectors (y). The whole construction should be invariant under the rotation group $O(n)$, if we rotate the vectors x and y.

The simplest cases are functions of two vectors and three vectors that respectively have the following form:

$$G(x, y) = g(x^2, y^2, x \cdot y), \tag{5}$$

and

$$H(x, y, z) = h(x^2, y^2, z^2, x \cdot y, x \cdot z, y \cdot z). \tag{6}$$

The integral over x of the previous functions is obviously well defined for integer n and we will show that the final formulae are defined also for non-integer n.

Let us consider the following expression

$$\int d^n y\, g(y^2, x \cdot y), \tag{7}$$

evaluated for $x^2 = r^2$. In the usual case, going to radial coordinates, one finds that

$$\int d^n y\, g(y^2, x\cdot y) = nV(n-1)\int_0^\infty ds\, s^{n-1}\int_{-\pi}^{\pi} d\theta \sin(\theta)^n f(s^2, sr\cos(\theta)). \quad (8)$$

The previous expression makes sense also for non-integer n. In many cases it is a meromorphic function of n and this allows us to define the r.h.s. of equation (8) also in the region where the integral is not convergent [7].

This construction can be generalized to more complex cases; however an alternative and simpler construction, that gives the same results, is the following. We suppose that the function g has a generalized Laplace transform and that it can be written as:

$$g(y^2, x\cdot y) = \int_0^\infty d\alpha \int_0^\infty d\beta\, G(\alpha,\beta)\exp(-\alpha y^2 - \beta x\cdot y). \quad (9)$$

Using standard Gaussian integrals one finds that

$$\int d^n y\, g(y^2, x\cdot y) = \int_0^\infty d\alpha \int_0^\infty d\beta\, G(\alpha,\beta)\left(\frac{\pi}{\alpha}\right)^{n/2}\exp\left(\frac{\beta^2 r^2}{4\alpha}\right). \quad (10)$$

In the same way we may easily compute integrals of the form

$$\int d^n y\, g(y^2, y\cdot x_1, \dots, y\cdot x_k). \quad (11)$$

Generally speaking a function of k vectors depends on $k(k+1)/2$ invariants.[2] In other words f is a *bona fide* function of one variable, g of three variables and h of six variables. Only in our perverse imagination the arguments of the functions F, G and H are vectors in a non-integer dimensional space. The crucial point where the dimension of the space enters is when we attach a meaning to each term of the expression *integral of a function over the whole space*.

We notice that the integral of a positive function is not *a priori* positive, as can be seen from formula (8). Indeed the volume of the unit sphere is no longer a positive function for negative dimensions. For example negative values for the integral (11) may arise if the k vectors x_i $(i = 1, \dots, k)$ form an orthonormal basis and the value of k is larger than the dimension of the space. In other words we can always decompose an n dimensional space into the product of a usual k dimensional space (with integer k) with a space of

[2]For integer n and sufficiently large k the $k(k+1)/2$ invariants are not independent quantities, but this fact does not concern us directly, because we are considering a generic dimensional space.

dimension $n-k$. If $n-k$ is negative the corresponding measure may no longer be positive definite. In this case, if we denote $y_k = x_k \cdot y$, we get:

$$\int d^n y\, g(y^2, y_1, y_2, \ldots y_k) \tag{12}$$
$$= (n-k)V(n-k)\int dy_1\, dy_2 \ldots dy_k\, dr\, r^{n-k-1} g(r^2 + \sum_{i=1,k} y_i^2, y_1, y_2 \ldots y_k),$$

where the integral is done on the region $r \geq 0$ and it should be understood in the distribution sense as analytic continuation from the convergence region. In the familiar case $k = n$ we recover the usual formula:

$$\int d^n y\, g(y^2, y_1, y_2, \ldots y_k)$$
$$= \int dy_1\, dy_2\, \ldots dy_k\, g(\sum_{i=1,k} y_i^2, y_1, y_2 \ldots y_k). \tag{13}$$

In a similar way we can define derivatives, Laplacians and so on. Generally speaking we can study vector-valued functions, under the constraint that they transform in a covariant way under the rotation group: e.g.,

$$x_\mu f(x^2). \tag{14}$$

In other words we study our space using the filter of rotational invariance: we consider only rotational invariant (or covariant) functions and we define a set of transformations (e.g., integrals and derivatives) that depend analytically on a parameter n; for integer values of n these transformations coincide with the usual ones for functions defined on an n-dimensional space.

In general the geometrical meaning of this construction is unclear. The only case where there is a simple geometrical interpretation is the case of negative integer dimensions. For example in the case of $n = -2$ we have that

$$\int d^{-2}x\, F(x) = -2\pi \int d\theta\, d\overline{\theta}\, F(\theta), \tag{15}$$

where θ is an anticommuting variable. Indeed in the case where $F(x) = f(x^2)$ both integrals give

$$-2\pi \frac{df}{dx^2}\bigg|_{x^2=0}. \tag{16}$$

Negative even dimensional spaces can be interpreted as spaces with anticommuting (Grassmann) coordinates as was noticed a long time ago [15].

§3. Zero-dimensional matrices.

3.1. The replica method. The interest in zero-dimensional matrices has been triggered by the so called replica method [6] for disordered systems.

In many problems of statistical mechanics of random systems we have to compute the quantity

$$f = \lim_{N\to\infty} f(N), \tag{17}$$

where

$$-f(N) = N^{-1} \int d\mu(J)\ \ln(Z_J(N)), \tag{18}$$

Here the partition function $Z_J(N)$ is usually a positive quantity (defined in some complicated way) and $d\mu(J)$ is a normalized measure $(\int d\mu(J))$. For example in the Sherrington-Kirkpatrick model for spin glasses the partition function is given by

$$Z_J(N) = \sum_{\{\sigma\}} \exp\left(\beta \sum_{i,k=1,N} J_{i,k}\sigma_i\sigma_k\right), \tag{19}$$

where the sum over $\{\sigma\}$ denotes the sum over the 2^N configurations of the N variables σ_i that may take only the values ± 1. The measure $d\mu(J)$ is a Gaussian measure over the $N(N+1)/2$ components of the symmetric J matrix whose elements are identically distributed independent Gaussian variables with zero average and variance N^{-1}.

In the usual approach it is convenient to define the quantity

$$-\Phi(N,n) = (Nn)^{-1} \ln\left(\int d\mu(J)\,(Z_J(N)^n\right). \tag{20}$$

As it can be trivially seen, if the appropriate integrals are convergent, $\Phi(N,n)$ is an analytic function of n, that is regular around $n = 0$, and

$$f(N) = \lim_{n\to 0} \Phi(N,n). \tag{21}$$

This baroque construction has been introduced because, in some problems for integer n, one can perform the integration over the variables J and one can prove the following exact formula [6], [16]:

$$\exp(-Nn\Phi(N,n)) = \int dQ \exp(-NF_n(Q)), \tag{22}$$

where the integral is done over all $n\times n$ matrices $Q_{a,b}$ $(a,b = 1,\dots,n)$ such that its diagonal elements are equal to 1 (i.e., $Q_{a,a} = 1$). Using the method of the point of maximum one can rigorously prove that for integer n

$$\phi(n) \equiv \lim_{N\to\infty} \Phi(N,n) = F_n(Q^*), \tag{23}$$

where the matrix Q^* minimizes the function $F_n(Q)$.

Forgetting physical motivations the mathematical problem is the following: given a function $F_n(Q)$ defined on $n \times n$ matrices one should go through the following steps:

1. To prove that the quantity $\Phi(N, n)$ defined in eq. (22) can be extended to an analytic function of n.
2. To prove that the limit $f(N) = \lim_{n \to 0} \Phi(N, n)$ exists.
3. To prove that the quantity $f(N)$ has a limit (f) when N goes to infinity.
4. To give a closed expression for the quantity f.

Of course everything depends on the form of the function $F_n(Q)$. For example in the case where

$$F_n(Q) = \frac{\beta}{2} \operatorname{Tr}(Q^2) \tag{24}$$

everything is trivial. One explicitly finds that

$$\Phi(N, n) = -\frac{n-1}{2} \ln\left(\frac{\beta}{2\pi}\right) \tag{25}$$

and all the issues on the dependence on n and N of $\Phi(N, n)$ can be easily settled.

If the function $\Phi(N, n)$ also has the representation eq. (20), points 1 and 2 are easy, point 3 is more difficult, but it can be solved in some interesting cases [9]. At the present time point 4 does not have a rigorous mathematical solution in the general case.

In order to give an idea of the problems physicists are interested in, it is convenient to write down some of the functions that have been studied in the literature.

$$\begin{aligned} F_n(Q) &= -\operatorname{Tr}[\ln(Q)] + \beta^2 \operatorname{Tr}(Q^2) \\ F_n(Q) &= -\operatorname{Tr}[\ln(Q)] + \beta^2 \sum_{a,b} (Q_{a,b})^3 \\ F_n(Q) &= -G_n(Q) + \beta^2 \operatorname{Tr}(Q^2) \end{aligned} \tag{26}$$

where in the last case (that corresponds to the SK model where we have the surprising relation $\Phi(N, n) = \Phi(n, N)$) we have that:

$$\exp(nG_n(Q)) = \prod_{a=1,n} \left[\sum_{\sigma_a = \pm 1}\right] \exp\left(\sum_{a,b} Q_{a,b}\sigma_a\sigma_b\right) \tag{27}$$

The physicists have used the following approach. For integer n the matrix Q^* that minimizes the function $F_n(Q)$ can be easily found and in this way one computes $\phi(n)$ for integer n. Usually $\phi(n)$ has a simple expression that can be analytically continued to $n = 0$. We would naturally guess that this analytic continuation gives the exact result.

However in general this is not the correct procedure. The function $\phi(n)$ is defined as the infinite N limit of $\Phi(N, n)$ and, although $\Phi(N, n)$ is analytic in n, the function $\phi(n)$ may be a non-analytic function of n (e.g., it may have a singularity in the interval 0-1) so that the analytic continuation of $\phi(n)$ is not the limit for N going to infinity of $\Phi(N, n)$. In some crucial cases it was proved that this happens (the naive approach gives inconsistencies) and something else should be tried [11].

The next approach consists in trying to use the method of the point of maximum directly for non-integer n, with all possible complications that may arise in dealing directly with functions defined in a non-integer dimensions case [11]. This approach is consistent (in the sense that the final results are not contradictory). Its results have not yet been proved to be correct and it goes under the name of spontaneously broken replica symmetry (the naive approach is usually called the replica symmetric one).

It is a great challenge to people working in probability theory to prove (or to disprove) that the results of the replica broken approach are correct [17].

3.2. The replica symmetry. We have already seen that symmetry considerations play a crucial role in defining quantities on non-integer dimensional spaces. It is therefore reasonable to investigate the symmetry properties of our problems. Moreover the presence of a symmetry group in a function F allow us to divide the set of its stationary points into orbits under the action of the symmetry group.

Let us introduce the usual permutation group S_n (i.e., the replica group) and let us denote by $\pi(a)$ (for $a = 1, \ldots, n$) the element where a is carried by the permutation π. The permutation group (that is a subgroup of the orthogonal group) acts in a canonical way on the matrices Q, permuting both row and columns:

$$Q^{\pi}_{a,b} = Q_{\pi(a),\pi(b)}. \tag{28}$$

In the cases of interest the permutation group is a symmetry of the problem, i.e., $F(Q^{\pi}) = F(Q)$. This symmetry is related to the procedure used in the construction of the function F; it is clearly exact in the examples we have considered.

If a function F has a symmetry, the point (or the points) left invariant by the symmetry is a natural candidate for a minimum: it is automatically a stationary point, i.e., a solution of the equation

$$\frac{\partial F}{\partial Q_{a,b}} = 0. \tag{29}$$

A matrix that is left invariant by the action of the permutation group must have all off-diagonal elements equal:

$$Q_{a,b} = q \quad \text{for} \quad a \neq b. \tag{30}$$

(The diagonal elements of Q are equal to one by construction.)

However the symmetric solution of equation (29) does not need to be a minimum: it may be a maximum, or a saddle. In order to establish the nature of the symmetric solution we have to compute the eigenvalues of the Hessian

$$\mathcal{H}_{a,b;c,d} = \frac{\partial^2 F}{\partial Q_{a,b} Q_{c,d}}. \tag{31}$$

In one of the most interesting cases (the Sherrington-Kirkpatrick model for spin glasses [16], i.e., the last of our examples) one finds that the eigenvalues of the Hessian are all positive for positive integer n, their analytic continuation is still positive for $n > n_c$, but some of them change sign at n_c where $0 < n_c < 1$ [4]. By changing the parameter n a minimum becomes a saddle point and another stationary point becomes the true minimum: this a very usual phenomenon (e.g., a bifurcation or spontaneous symmetry breaking) in an unusual setting.

3.3. Breaking the replica symmetry. If the symmetric point is not a minimum, we can look for a minimum that is not symmetric. In this case physicists say that the symmetry group is spontaneously broken. The subgroup that leaves invariant the true minimum (the little group) is the surviving (or unbroken) symmetry group.

Our task is to find a new candidate for a minimum. Different attempts have been made. The most promising candidate [11], [13] is given by the matrix Q, whose off-diagonal elements can be written as

$$\begin{aligned} Q_{a,b} &= q_1 \qquad \text{if} \quad I(a/m) = I(b/m), \\ Q_{a,b} &= q_0 \qquad \text{if} \quad I(a/m) \neq I(b/m), \end{aligned} \tag{32}$$

where $I(x)$ is the integer part of x and m is an integer that divides n. Obviously if $q_1 = q_0 = q$ we recover the previous case. The symmetry group that leaves invariant this matrix is (for $q_1 \neq q_0$) the semidirect product of $S_{n/m}$ with the direct product of n/m copies of S_m and it is obviously a subgroup of S_n.

The value of $F(Q)$ evaluated at this saddle point gives a function $F(q_0, q_1, m)$, that can be written in a relatively simple form, depending on the model (in the worst case it involves a few integrals).

Now a natural question is which values of m are allowed. Originally m should be an integer that divides n, but, when n is no longer an integer, there is apparently no reason for imposing that m remains an integer. It is clear that we are in a no man's land and any prescription is arbitrary. We will present the *standard* prescription, that may justified *a posteriori* by its success in giving sensible results, that are quite likely the correct ones. We will follow the rules:

- Non-integer values of m are allowed as soon as n is no longer a positive integer.

- The interesting region for m is $1 \leq m \leq n$ for $n > 1$ and $n \leq m \leq 1$ for $n < 1$.
- We look for a stationary point that satisfies the condition that all the eigenvalues of $\mathcal{H}$ are positive in order to find the minimum of $F(Q)$. In the previously defined interesting region ($n \leq m \leq 1$ for $n < 1$) this condition implies that $F(q_0, q_1, m)$ is a maximum as function of q_0 and q_1. This confusing situation is due to the fact the if the number of dimensions is negative, the two conditions ($\mathcal{H}$ has non-negative eigenvalues and $(x, \mathcal{H}x) \geq 0$) are not equivalent.
- By uniformity (i.e., m should be treated the same as the q's)[3] we decide that the function $F(q_0, q_1, m)$ should also be maximized with respect to m. The final prescription consists in finding the maximum with respect all the three arguments.

This prescription is called one step replica symmetry breaking. In certain models it is believed to give the exact results. i.e., in the cases for which one obtains a non-negative Hessian (an alternative approach to these models is sketched in the appendix). In other cases (e.g., in the SK model) one does not obtain a non-negative value of the Hessian although the absolute value of the negative eigenvalue decreases by a large factor (around 10). The prescription we have introduced is apparently able to compute in an exact way the quantity f in some models. In the other models, where it does not give a consistent result, it is a step toward the correct solution, that will be found in the next section.

3.4. Full replica symmetry breaking. As we have mentioned before the one step replica symmetry scheme does not produce consistent results in the case of the SK model. If we pay attention to the scheme that we have used to break the permutation group, the statements of the previous section correspond to saying that in the limit $n \to 0$ the unbroken subgroup of S_n is the semidirect product of the group S_0 with the direct product of zero copies of the group S_m. In other words the group S_0 contains itself as a subgroup and it is therefore an infinite group (it also contains infinitesimal elements [5]). This fact may not be surprising if we ponder the fact that the group S_0 is obtained as the analytic continuation of all the S_n groups, with arbitrary n.

One can generalize the formula eq. (32) to the case where there are $k+1$ parameters q_i $(i = 0 \cdots k)$ and k parameters m_i $(i = 0 \cdots k)$. The previous formulae correspond to the case $k = 1$. We present the explicit construction in the case $k = 2$. We assume the n is a multiple of m_2 and m_2 is a multiple of m_1. The off-diagonal elements of the matrix Q, can be written as

[3]One should minimize (or maximize) with respect to all the parameters if there is a variational principle [9].

$$
\begin{aligned}
Q_{a,b} &= q_2 && \text{if} \quad I(a/m_2) = I(b/m_2), \\
Q_{a,b} &= q_1 && \text{if} \quad I(a/m_2) \neq I(b/m_2) \\
& && \text{and} \quad I(a/m_1) = I(b/m_1), \\
Q_{a,b} &= q_0 && \text{if} \quad I(a/m_1) \neq I(b/m_1),
\end{aligned}
\tag{33}
$$

It is remarkable that after a permutation one could write write such a matrix under the form $Q_{a,b} = Q(a-b)$, where the function $Q(a)$ is a simple function of a [14]. Indeed this happens if we set

$$
\begin{aligned}
Q(a) &= q_2 && \text{if} \quad n/m_2 \quad \text{divides} \quad a, \\
Q(a) &= q_1 && \text{if} \quad n/m_1 \quad \text{divides} \quad a \\
& && \text{and} \quad n/m_2 \quad \text{does not divide} \quad a, \\
Q(a) &= q_0 && \text{if} \quad n/m_1 \quad \text{does not divide} \quad a,
\end{aligned}
\tag{34}
$$

It is not clear if this translation invariant formulation, that is at the heart of the p-adic approach [14], has a deeper meaning (up to now it has been used only at the technical level).

Also in this case we first compute the function F for integer n, m_1 and m_2 and we analytically continue the result up to $n = 0$.[4] At the end of the computation we no longer require that the m's are integers. The final expression is maximized as a function of the q's and of the m's, that turn out to be in the region $0 < m_1 < m_2 < 1$. In the case where the one step solution is not good (i.e., negative Hessian), also the two step solution is not good (the Hessian is still negative).

It is clear that this construction can go on for any number of steps we want in a recursive way. A detailed computation shows that we always find the Hessian is negative (for example in the SK model near the critical temperature the minimum eigenvalue of the Hessian is $-A(2k+1)^{-2}$ with positive A). It seems reasonable that the interesting case corresponds to the limit $k \to \infty$.

In order to better control the $k \to \infty$, it is important that for finite k and in the limit $n \to 0$ the interesting solutions of the stationary equation $\partial F/\partial Q_{a,b}$ are always in the region

$$0 \equiv m_0 < m_1 \ldots < m_k < 1 \equiv m_{k+1}. \tag{35}$$

One can thus associate to the matrix Q a piecewise constant function $q(x)$ such that

$$q(x) = q_i \quad \text{for} \quad m_i < x < m_{i+1}. \tag{36}$$

If we send k to infinity, one finds by an explicit computation that the function $q(x)$ becomes a continuous function. In this way we have a canonical way to associate to a function $q(x)$ a matrix in the *space* of zero by zero matrices.

[4]The group properties of this solution are discussed in detail in [1].

The introduction of the function $q(x)$ is not so arbitrary. Indeed simple expressions can be obtained for the S_n-invariants in the limit of n going to zero also for finite k. I present just two simple examples:

$$\lim_{n\to 0} n^{-1}\,\mathrm{Tr}(Q^2) = 1 - \int_0^1 dx\, q(x)^2,$$
$$\lim_{n\to 0} n^{-1}\,\mathrm{Tr}(Q^3) = 1 - 3\int_0^1 dx\, q(x)^2 + \frac{3}{2}\int_0^1 dx \int_x^1 dy\, q(x)q(y)^2 + \frac{1}{2}\int_0^1 dx\, xq(x)^3. \quad (37)$$

The last formula is related to the formula for the volume of the intersection of three p-adic spheres [14].

A more complex example is given by the quantity $G(0)$, defined in eq. (27): a non-trivial computation implies that

$$G(0) = f(0,0), \quad (38)$$

where the function $f(x,h)$ is defined in the strip $0 \le x \le 1$, $-\infty < h < \infty$ and it satisfies the following non-linear differential equation:

$$\frac{\partial f}{\partial x} = -\frac{1}{2}\frac{dq}{dx}\left(\frac{\partial^2 f}{\partial h^2} + x\left(\frac{\partial f}{\partial h}\right)^2\right). \quad (39)$$

with the boundary conditions

$$f(1,h) = \ln(\cosh(\beta h)). \quad (40)$$

In all the previous cases at the end of the computation the free energy becomes a functional of the function $q(x)$ ($F[q]$) and this functional has an explicit expression that in some cases may be rather complex. Maximizing the functional $F[q]$ (and depending on the temperature range this can be done analytically or numerically) one finds a consistent solution of many models (including the SK model) such that all the infinitely many eigenvalues of the Hessian[5] are non-negative. Moreover this solution satisfies all inequalities that one derives from general principles (e.g., positivity of the entropy).

It obvious that all this does not make sense from a strict mathematical point of view because the prescriptions we have used are somewhat arbitrary, although they look reasonable. On the other hand the replica method has been very successful as can be seen in the next section.

[5]The Hessian becomes now a rather complex integral operator and it has both an infinite point spectrum and a continuous spectrum.

§**4. The success of the replica method.** In the case of the Sherrington Kirkpatrick model the first success of the replica method, especially in the case of full replica symmetry breaking, is the existence of a consistent solution (a result that is not evident *a priori*).

It is also remarkable that in all the many infinite range models, where the replica method has been applied, the results are in very good agreement with the computer simulations and they coincide with the exact results, when available.

It is very interesting that one can rederive most of the results of the replica method, without using replicas, but doing probabilistic arguments. It appears that the form of the matrix Q codes some information on the probability distribution of the equilibrium configurations of the system in a way that it would take too much time to explain [11], [13]. We only say that the organization of the pure states of the systems (called lumps by Talagrand [17] in this context) mirrors the form of the matrix Q: e.g., the ultrametric form of the matrix q, that is a remnant of its p-adic origin, implies that the distances among states satisfy the ultrametric inequality. This alternative probabilist approach is not evident and it was found a few years after the establishment of the replica approach. Moreover, in some models non-trivial rigorous results have been obtained that point toward the correctness of the results obtained via the replica method [17].

Most recently in a very clever paper Guerra has been able to prove in the SK model (and in similar models) the following inequality:

$$f \geq F[q] \quad \forall q(x). \tag{41}$$

This result is important because it shows that the functional $F[q]$, that is obtained via the replica method, is related to the true value (f) of the free energy.

The same results of the replica method may be obtained in the framework of off-equilibrium dynamics that has been shown to be equivalent to the replica method [3], [8]. Recent experimental results [10] beautifully confirm the theoretical predictions of this approach.

The existence of alternative routes clearly shows that the results make sense. In any case the replica method has a very strong heuristic value and it would be surprising if we could not assign a precise mathematical meaning to such a useful method.

Appendix. In this appendix we report an explicit computation in the case of one step replica symmetry breaking that may have some chance to be the starting point of a rigorous mathematical approach [2]. As we shall see, this result suggests that there is a correspondence from a class of 0×0 matrices with an infinite set of $n \times n$ matrices for any n.

At the technical level the idea is relatively simple. We write for negative s

$$\Gamma(-s)\overline{Z^s} = \int_0^\infty dt\, t^{-1-s}\overline{\exp(-tZ)} = \int_0^\infty dt\, t^{-1-s} \sum_{n=0,\infty} \frac{(-t)^n}{n!}\overline{Z^n}, \quad (42)$$

where everything has an implicit dependence on N and the bar denotes the average over J. If we continue the previous formula up to $s \to 0$ we obtain an expression for $\overline{\ln(Z^s)}$

If we can control the expression for $\overline{Z^n}$ with sufficient accuracy when $N \to \infty$, we have done our job. This task is not easy, because of possible cancellations due to the minus signs: the leading term of the sum may not be the sum of the leading terms for $\overline{Z^n}$ and therefore we have to control $\overline{Z^n}$ with a very high accuracy.

We have already seen that the quantity $\overline{Z^n}$ has an exact expression that is given by eq. (22). An interesting conjecture is that we obtain the asymptotically correct result for $\overline{Z^s}$, if inside eq. (42) we make the approximation

$$\overline{Z^n} \approx \sum_{Q^*} \exp(-NF_n(Q^*)), \quad (43)$$

where the sum is done over all the $n \times n$ matrices Q^* that are local minima of $F_n(Q)$. This approximation obviously gives the asymptotically dominant result for $\overline{Z^n}$, where only the global minimum counts; however it is not evident that it gives the correct result when it is used for computing $\overline{Z^s}$.

In the general case we do not know if the conjecture is true, how to compute all the mimima and if this conjecture does reproduce the result of the replica approach. However in some models where replica symmetry is broken at one step, the appropriate computations can be done [2] and one finds the same result obtained with the replica method.

There is indeed a class of models where for positive integer n all the minima of $F[Q]$ can be obtained in the following way: one divides the indices into M sets of size m_k ($k = 1, M$) where $\sum_k m_k = n$. One sets

$$Q_{a,b} = q_k, \quad (44)$$

if both a and b belong to the k^{th} set; otherwise one has $Q_{a,b} = 0$. In some cases [2] one finds that, if the leading contribution (that coming from the global minimum) for each n is used, the result coincides with the one coming form the replica symmetric approximation (that for a certain range of the parameters is not correct). One the other hand, if one considers the contribution coming from all possible minima (that are subdominant for given n), at the end one recovers the replica broken result where one has to find the maximum of the free energy for m in the region $0 \leq m \leq 1$.

We have no idea how the aforementioned conjecture can be proved and it would be interesting to test it in a wider context. We have mentioned it here because it hints that the matrices for non-integer n and m used in the replica method are related to families of matrices that are defined for all integer n.

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