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REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The Journal of Symbolic Logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

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In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

H. ANDRÉKA, S. GIVANT, and I. NÉMETI. ***Decision problems for equational theories of relation algebras.*** Memoirs of the American Mathematical Society, vol. 126, no. 604. American Mathematical Society, Providence, March 1997, xiv + 126 pp.

This monograph contains an Introduction and four chapters. The Introduction does an excellent job of explaining all the main results. Chapter I, entitled Preliminaries, summarizes what is needed from the theory of relation algebras, universal algebra, recursion theory, and model theory. There are three major results, one in each of the remaining three chapters.

In Chapter II the authors extend Tarski's theorem that the variety RRA of representable relation algebras has an undecidable equational theory. An equivalent form of Tarski's theorem was announced in 1941. Tarski's original proof used Church's theorem on the undecidability of validity in first-order logic. It was shown much later by J. D. Monk (*On representable relation algebras*, ***The Michigan Mathematical Journal***, vol. 11 (1964), pp. 207–210) that RRA is *not* finitely axiomatizable, but by early 1942 Tarski had also proved that the variety RA of relation algebras, which *is* finitely axiomatized, also has an undecidable equational theory. Tarski was able to show that very many undecidable first-order theories (including most set theories) can be formalized as equational theories of relation algebras. From this the undecidability of the equational theory of relation algebras follows immediately. This result was not mentioned in print until 1948, and Tarski's proof was not published until 1987 (see A. Tarski and S. Givant, ***A formalization of set theory without variables***, American Mathematical Society, 1987), although a different proof appeared earlier

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in R. D. Maddux, ***Topics in relation algebras***, Dissertation, University of California, Berkeley, 1978. The equational theory of relation algebras is probably the first example of a finitely based undecidable equational theory that was recognized as such and explicitly announced in the literature. However, it was eventually realized that a finitely axiomatized undecidable equational theory can be built from any finitely presented semigroup with an unsolvable word problem—just regard the generators as unary constant-valued operations and regard the relations on the generators as equational axioms. The existence of finitely presented semigroups with unsolvable word problems was known from the mid-1940's. For more historical remarks on this topic see §8.7 of A. Tarski and S. Givant, ***A formalization of set theory without variables***, American Mathematical Society, 1987.

Tarski's undecidability result for RA actually shows that if K is any subclass of RA that contains the relation algebra of all binary relations on a countable set, then the equational theory of K is undecidable. Although quite general, this theorem leaves many open problems. Over the years, Tarski asked whether various classes of relation algebras have decidable equational theories. Three examples are group relation algebras (relation algebras that arise from groups in a natural way), relation algebras that have representations on finite sets, and symmetric relation algebras. In the early 1970's Givant proved the undecidability of the equational theory of group relation algebras by observing that a finitely presented group with unsolvable word problem can interpreted in the equational theory of group relation algebras. Later in the 1970's, the class of relation algebras that have representations on finite sets was shown to have an undecidable equational theory by W. Schönfeld, *An undecidability result for relation algebras*, ***The Journal of Symbolic Logic***, vol. 44 (1979), pp. 111–115. Around 1989, P. Jipsen proved that the equational theory of the symmetric relation algebras is undecidable (see theorem 3.24 of P. Jipsen, ***Computer-aided investigations of relation algebras***, Dissertation, Vanderbilt University, Nashville, 1992). Jipsen used the undecidability of the equational theory of modular lattices and the fact that every modular lattice is a subreduct of some symmetric relation algebra (R. Maddux, *Embedding modular lattices into relation algebras*, ***Algebra Universalis***, vol. 12 (1981), pp. 242–246). Various undecidability results for modular lattices have been proved by R. Freese, L. Lipshitz, and A. Urquhart, who used (and generalized) J. von Neumann's notion of n-frame for complemented modular lattices, which is in turn derived from von Staudt's work on projective geometry. The authors have obtained the following stronger undecidability result by further generalizing and applying the notion of n-frame.

THEOREM (i). *Suppose* K *is a class of relation algebras such that either* (1) *for every* $n \in \omega$ *there is a simple algebra in* K *that has at least* n *elements below the identity element* $1'$, *or* (2) *for every* n *there is an algebra in* K *that contains a subset with at least* n *pairwise disjoint elements that forms a group under the operations of relative multiplication and conversion. Then there is no recursive set of equations that contains the equational theory of* RA *and is included in the equational theory of* K. *In particular, the equational theory of* K *is undecidable.*

This theorem subsumes Tarski's theorem, all the undecidability results mentioned above, and many more. Although condition (1) implies condition (2), condition (1) is somewhat easier to understand and apply in practice, and the implication from (1) to (2) is perhaps not immediately obvious. (Indeed, it was omitted from the first draft; see p. 50). In the course of proving Theorem (i), the authors show that semigroups can be interpreted wherever a 4-frame can be found, and that 4-frames can be found when large finite groups (or one infinite group) are present. Theorem (i) thus has strong historical roots in projective geometry, modular lattices, and coordinatization theory. It implies the undecidability of the equational theories of many classes of relation algebras, including representable relation algebras, symmetric relation algebras, and group relation algebras.

The main result in Chapter III also has a connection to projective geometry. R. C. Lyndon proved (*Relation algebras and projective geometries*, ***The Michigan Mathematical Journal***, vol. 8 (1961), pp. 21–28) that every projective geometry gives rise to a relation algebra that is representable if and only if the geometry can be embedded in another projective geometry with one more dimension. Certain Lyndon algebras, the ones that come from the "trivial" projective geometries that consist of a single line with 3 or more points, have some remarkable properties. These are the algebras that were used by J. D. Monk (*On representable relation algebras*, ***The Michigan Mathematical Journal***, vol. 11 (1964), pp. 207–210) to show that RRA is not finitely axiomatizable. For each $n \in \omega$, let $\mathcal{L}_n$ be the finite symmetric relation algebra having $n + 4$ atoms, namely, 1', $a_0, \ldots, a_{n+2}$, where relative multiplication, which is completely distributive over join in any relation algebra and hence is completely determined by its action on atoms, satisfies these conditions: for all $i, j, k \in \{0, \ldots, n+2\}$, the identity law $a_i ; 1' = a_i = 1' ; a_i$ holds, and $a_i ; a_j \geq a_k$ whenever i, j, k either coincide or are pairwise distinct. For each $X \subseteq \omega$, let L_X be the class of algebras $\mathfrak{A}$ with the property that, for some $n \in \omega$, either $\mathfrak{A}$ is isomorphic to a *proper* subalgebra of $\mathcal{L}_n$, or else $\mathfrak{A} \cong \mathcal{L}_n$ and $n \in X$. The map from X to the equational theory of L_X preserves degree of unsolvability. In particular, X is recursive if and only if the equational theory of L_X is decidable. This and other properties of the map $X \to \mathsf{L}_X$ are the subject of Chapter III.

Chapter IV presents a very interesting relation algebra. It is one-generated, infinite, simple, representable, and has a decidable equational theory. Its Boolean algebraic reduct is the complete atomic Boolean algebra whose atoms are a_i and $\breve{a}_i$ (the converse of a_i) for every $i \in \omega$, together with 1' (the identity element for relative multiplication). The algebra thus has a countable set of atoms. Relative multiplication is determined by the condition that, for all $i, j, k \in \omega$, $a_i ; a_j \geq a_k$ if and only if $i + j \geq k$. Although this relation algebra is quite easy to describe, the proof of the decidability of its equational theory is by far the longest and most intricate in the paper. It is a detailed analysis of the structure of the algebra, reminiscent in some ways of proofs of decidability by elimination of quantifiers.

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COLIN MCGINN. ***Logical properties: identity, existence, predication, necessity, truth.*** Clarendon Press, Oxford 2000, vi + 114 pp.

This short book comprises 'a series of studies of what we might call logical ontology—the ontological character and standing of certain (broadly) logical notions' (p. 14). Absent is 'the kind of formalistic fetishism and scholasticism that has characterized too much of philosophical logic in recent years' (p. vi). McGinn tells us that his goal is to bring philosophy back into philosophical logic, and it is in service of this goal that he has kept the book relatively free of formulae. In what follows, I summarize McGinn's positive accounts of the logical properties, and then conclude by making some general comments.

Identity. McGinn defends four theses about identity. They are as follows. (1) Identity is unitary, which is the Fregean thesis: identity admits of no qualification or variation, it doesn't come in degrees or in different types. (2) Identity is indefinable. So, Leibniz's law, $\forall P \forall x \forall y((P(x) \leftrightarrow P(y)) \leftrightarrow (x = y))$, is not a definition of identity. (3) The logical law of identity, $\forall x(x = x)$, is fundamental, i.e., it is presupposed in all other logical laws. (4) Identity is a genuine relation, contra the *Tractatus* thesis that identity is a pseudo-relation. McGinn claims that (1)–(4) are not terribly controversial and he spends little time defending them.

Existence. McGinn considers two analyses of the concept of existence: the quantifier and the predicative accounts. The first unpacks the concept of existence in terms of the existential quantifier, while the second account holds that 'exists' is a predicate and that it expresses a property just as other predicates do. McGinn favors the predicative account of existence because 'there are no good objections to this view and the alternative to it is full of difficulties' (p. 50). Furthermore, McGinn points to uses of 'some' in contexts in which existence is not implied (e.g., 'some of the things you are talking about don't exist', 'some cities are imaginary'). McGinn thinks that these and other examples indicate that 'some' has existential import only as a matter of conversational implicature. Hence, the predicative account rather than the quantifier account is the best analysis of ordinary language existential claims.

In any case, the existential quantifier is analyzable in terms of the predicate of existence. Consider, 'Eskimos exist'. The quantifier account turns this sentence into '$\exists x$ Eskimo(x)'. The existential quantifier does double duty: it conveys the existence of things that satisfy the open formula 'Eskimo(x)', and it specifies how many things satisfy 'Eskimo(x)'. McGinn thinks that it is a mistake to conflate two distinct linguistic functions into '$\exists x$'. Rather, 'Eskimos exist' should be understood as '$\exists x$ (x is an Eskimo and x exists)' where 'x' ranges over the collection of both purely intentional objects and existent (non-intentional) objects, i.e., the collection of things we think and talk about.

Necessity. McGinn complains that the possible worlds semantic treatment of modal terms fails to explain their meaning. An adequate analysis of 'possibly' in 'Socrates is possibly a wise man' awaits an analysis of 'possible' in 'Socrates is a wise man in some possible world'. Of course, one can take (and many do take) possible worlds semantics to offer merely a truth-conditional account of modal sentences and not an analysis of modality. McGinn is after the later. He argues for what he calls the copula modifier theory of modality. Briefly, what 'necessarily' modifies in 'Socrates is necessarily human' is not the predicate 'human' to form the modal predicate 'necessarily human' which is predicated of Socrates, but the copula modifying the way Socrates instantiates the property of being human. The sentence 'Socrates is necessarily human' means that the non-modal property *being human* inheres in Socrates necessarily as opposed to contingently. The instantiation relation is the only thing that gives rise to modal properties: the properties of being instantiated necessarily or contingently. The account handles not only *de re* uses of modal words but *de dicto* uses as well. For example, when we say that it is necessary that $2 + 2 = 4$ is true, we are asserting that the proposition, $2 + 2 = 4$, necessarily instantiates the property of truth. Truth can be possessed by propositions in the mode of necessity or contingency.

Truth. 'Snow is white' is true iff snow is white. Disquotation is the essence of truth. How does McGinn account for the disquotational aspect of truth? For example, how does McGinn explain the truth of the above biconditional? In brief, he believes that it is a substantive metaphysical claim that reveals an essential aspect of the property of truth. Truth is the only property of propositions that entails the fact that makes the propositions true. He remarks that 'truth is a property whose conditions can be stated without making reference to that property—moreover, it is the only property of which this can be said ... Truth is the self-effacing property, i.e., that unique property whose application conditions can be stated without reference to the property' (p. 104).

We may identify the application condition of the concept of a cube with the concept of a prism with four congruent sides. Here we give the application conditions of the first concept in terms of another concept that refers to the same property. Truth, however, can be defined without making reference to the property 'true' denotes—or using any predicate equivalent to it. In short, truth is disquotational because it is self-effacing. Even stronger, no other property sustains disquotation; truth is *the* self-effacing property.

Predication. The view to be criticized in this Chapter is the Quinean position that since a predicate has divided reference (it has as many referents as there are objects in its extension) and a name refers uniquely to a single object, predicates are not names. McGinn denies that predicates refer to what they are true of. Rather, predicates refer to properties (attributes) which are distinct from their extensions. Since predicates uniquely designate one thing, they are singular terms like names.

McGinn supports his view of the semantic functioning of predicates on the basis of three considerations. First, predicates, appropriately nominalized, may occupy subject positions in sentences (e.g., 'Baldness is beautiful'), and in such form they do not have multiple referents. Semantic uniformity demands that we take predicates to designate properties when they occupy the predicate position in a sentence. Second, McGinn claims that an adequate account of the meaning of a word must reflect what is known or grasped by the speaker who uses the term in a sentence. Let's call this the epistemological requirement. Clearly, an acquaintance with all things bald is not part of a speaker's understanding of, say, 'Michael Jordan is bald'. Hence, the referent of 'bald', contra Quine, can't be identified with all things bald. McGinn's approach satisfies the epistemological requirement: when a predicate is understood what is known is precisely the identity of the property expressed by the predicate, just as understanding a name involves knowledge of the identity of its bearer.

The final consideration in support of McGinn is that if McGinn's view of predication is rejected in favor of Quine's, then we are stuck with semantic indeterminacy: there is no fact of the matter about the semantics of predicates. To see this, consider the following three accounts of the truth conditions for a simple subject-predicate sentence.

(1) 'G. W. Bush is a Republican' is true iff G. W. Bush is a member of the set of Republicans.
(2) 'G. W. Bush is a Republican' is true iff G. W. Bush has the property of being a Republican.
(3) 'G. W. Bush is a Republican' is true iff the extension of 'Republican' is a member of the set of extensions assigned to 'G. W. Bush'.

(1) approximates Quine's position, (2) McGinn's, and McGinn conjures up (3) to make the following case. Suppose one argues that an adequate account of the semantic functioning of predicates is constrained by Occam's razor and by the above epistemological requirement. There is nothing by way of ontology in (3) that already isn't there in (1), so Occam's Razor cannot decide between (1) and (3). Nor are (1) and (3) distinguished by the epistemological requirement (both fail according to McGinn). Furthermore, both satisfy the Tarski T schema. If we do not appeal to properties in the semantic account of predicates, then there is no basis for choosing between the competing extensional accounts of the semantic functioning of predicates represented by (1) and (3). This favors McGinn's account over Quine's, since semantic indeterminacy is undesirable.

Conclusion. I end by highlighting three related positions, two philosophical and one methodological, which emerge from the work in this book. One philosophical position, which we may call logical primitivism, holds that the logical concepts of identity, existence, predication, necessity, and truth are primitive. They cannot be defined in any illuminating non-circular way. So, for example, model-theoretic characterizations of modality should not be taken as analyses of modality. McGinn's motive for logical primitivism seems to be the second philosophical position that emerges from the discussion, the realist (anti-naturalist) view of logical properties.

According to this view, identity, existence, predication, necessity, and truth are logical (as opposed to, say, mental or physical) properties of things that are just as real as blueness, electrical charge or goodness. Logical properties, causally inert and sense-imperceptible, are objects of pure intellectual cognition. This mystifies our (*qua* physical beings) epistemic access to these causally inert properties. McGinn's response: 'It may be that the mystery stems from our conceptual and theoretical limitations. The "queer" is here, and we need to learn to live with it—even if we may not bring ourselves to love it' (p. 86).

Finally, McGinn's inquiry into the five logical properties assumes an uncommon methodological stance which, arguably, is dictated by the realist view of these properties. In general, since truth conditions by themselves underdetermine semantics, our conception of the semantic properties of terms should dictate the assignments of truth-conditions to the sentences in which the terms occur, not the other way around. Hence McGinn's contention that the quantifier is vastly over-rated as a tool of logical and linguistic analysis. We should not be seduced by the success of the reduction of identity, existential, and modal sentences to quantifications over properties, things, and possible worlds, respectively. Even if these reductions fix the truth-conditions for identity, existence, and modal sentences, the appeal to quantification distorts the semantic facts about '=', 'exists', and 'possibly' because it fails to yield adequate analyses of the concepts of identity, existence, and modality. More generally, McGinn believes that there are substantive facts about the semantic functioning of the syntactic correlates of the five logical properties, and that these facts are epistemically accessible to us. McGinn gets to these semantic facts by first doing metaphysics—getting an account of the nature of the logical properties—and then using the results to develop an account of the semantic functioning of the corresponding terms.

Thus, semantics cannot be the royal road to metaphysics, because the correct semantics is constrained by the correct metaphysics. Surely, on some level this is correct. We can't argue against the Parmenidean by claiming that there exists a counterexample to the argument: some are eternal, so all are eternal. Since the counterexample requires the possibility of there being more than one thing, the required semantics of the quantifiers presupposes that Parmenidean metaphysics is wrong. McGinn's work highlights the extent to which metaphysics penetrates and permeates the conceptual framework of logic. Inquiry into the central concepts of logic cannot ignore metaphysics. Even if one does not agree with all the conclusions of the book, one has to grant that *Logical Properties* makes a good case for bringing, as McGinn puts it, philosophy back into philosophical logic.

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SERGEI TUPAILO. *Realization of analysis into explicit mathematics*. ***The Journal of Symbolic Logic***, vol. 66 (2001), pp. 1848–1864.

The paper under review explores realizability interpretations of various intuitionistic subsystems of second-order arithmetic into Feferman's explicit mathematics. For the latter systems, this therefore provides an alternative method to establish their lower proof-theoretic bounds.

Explicit mathematics was introduced by Feferman (*A language and axioms for explicit mathematics*, JSL XLIX 308(1); *Constructive theories of functions and classes*, JSL LXIX 308(2)) in the seventies. Beyond its original aim to provide a basis for Bishop-style constructivism, the explicit framework has gained considerable importance in proof theory in connection with the proof-theoretic analysis of subsystems of second-order arithmetic and set theory. The most famous example in this connection is the reduction of the classical subsystem of second-order arithmetic based on Δ^1_2 comprehension and bar induction to the most prominent framework of explicit mathematics, T_0, achieved by Jäger (*A well-ordering proof for Feferman's theory T_0*, ***Archiv für mathematische Logik und Grundlagenforschung***, vol. 23 (1983), pp. 65–77) and Jäger and Pohlers (*Eine beweistheoretische Untersuchung von* $(\Delta^1_2\text{-}CA) + (BI)$ *und verwandter Systeme*, ***Sitzungsberichte der Bayerischen Akademie der Wissenschaften, Mathematisch-naturwissenschaftliche Klasse***, 1982, pp. 1–28).

The language of explicit mathematics, more precisely its operational or applicative core, allows for a very smooth and elegant formulation of abstract Kleene-style notions of realizability. Moreover, even realizability for second-order languages finds its natural place, since collections of (first-order) individuals, so-called classes or types, are directly named or represented by objects of first order, which in turn can act as arguments of abstract operations in the applicative basis of explicit mathematics.

The subsystems of intuitionistic second-order arithmetic that are shown in this article to be realizable into the system T_0 and its subsystems are based on schemes of arithmetic comprehension, choice, and replacement, as well as a principle of inductive generation. The strongest of these systems is the theory IARI introduced by Griffor and Rathjen (*The strength of some Martin-Löf type theories*, ***Archive for Mathematical Logic***, vol. 33 (1994), pp. 347–385), which has the same proof-theoretic strength as Δ^1_2 comprehension plus bar induction, hence exhausting all of T_0.

The present paper explores realizability in explicit mathematics of the relatively simple language of second-order arithmetic. This keeps the amount of technical details at a minimum and demonstrates the method most distinctly. The paper can be seen as an important preparatory work for two subsequent papers by Tupailo, dealing with a much more involved realizability interpretation of constructive set theory CZF, possibly augmented by large set axioms, into explicit mathematics (*Realization of constructive set theory into explicit mathematics: a lower bound for impredicative Mahlo universe*, ***Annals of Pure and Applied Logic***, to appear; *On non-wellfounded constructive set theory*, to appear in the proceedings of the Ninth Annual CSLI Colloquium on Logic, Language and Computation; both papers are available at `http://greta.cs.ioc.ee/~sergei/publ.html`). The former of these two papers provides a relative lower bound of the impredicative Mahlo universe in explicit mathematics.

In summary, the author offers an interesting, concisely written paper, which directly relates strong subsystems of intuitionistic analysis to explicit mathematics, without going through ordinal-theoretic proof theory.

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F. A. MULLER. *Sets, classes, and categories*. ***British Journal for the Philosophy of Science***, vol. 52 (2001), pp. 539–573.

Muller skillfully expounds the challenge for set theoretic foundations for category theory: Simple categorical methods, with evidently low proof theoretic strength, get formalized by using very large sets. The standard device, a *Grothendieck universe*, is a set U which itself satisfies all the ZF axioms. The category **Top** of all topological spaces, for example, is taken as the category of all topological spaces that exist in U. That is a set, though not in U, and there is a set of all functors $\mathbf{F}\colon \mathbf{Top} \to \mathbf{Set}$ and much more. The categorical theorems used in practice survive almost entire.

As Muller says, this is "not just a bit abundant [it is] mindbogglingly, flabbergastingly abundant"(550). A universe amounts to an inaccessible cardinal. Muller offers a more elegant approach using his strengthened form of Ackermann's theory of sets and classes.

He motivates his theory and proves numerous theorems on its expressive power. It has classes of every finite rank over the class of all sets, and so is more agile than Gödel-Bernays. Muller argues that his sets and classes reflect the true difference between logical and combinatorial collections, and his sets are sharply delineated in Cantor's sense while his classes are not (though they have extensionality). He shows his theory equiconsistent with ZF.

As to ZF-style foundations, the reviewer proposes a related strategy within ZF, defining 'small' sets as elements of $V(\omega+\omega)$. All usual mathematics apart from category theory is done in small sets. Define the category **Top** as all small topological spaces and continuous maps and similarly for similar categories. Both approaches are decidedly deflationary compared to even one universe. Further deflation would be desirable if it still justifies the simple categorical methods.

As to categorical foundations, Muller's opening pages confuse the Eilenberg-Mac Lane axioms which define the general notion of *category* with axioms for the category of sets and axioms for the category of categories. The latter aim to describe two different specific categories. Muller is clearly not interested in them. But he takes exactly the right line on ZF-style foundations in this reviewer's opinion: not detours avoiding large categories, nor large cardinals realizing them, but elegant deflationary ways to formalize them.

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ARNOLD BECKMANN. *Proving consistency of equational theories in bounded arithmetic*. ***The Journal of Symbolic Logic***, vol. 67 (2002), pp. 279–296.

One of the oldest outstanding problems in bounded arithmetic is whether the hierarchy of theories S^i_2 collapses. It is known if this hierarchy collapses then in fact the polynomial time hierarchy collapses. Since the theories S^i_2 are defined in a way reminiscent to the theories $I\Sigma_k$ and the latter are known to be separable by Gödel-style arguments, it seems reasonable to try Gödel-style arguments to separate the theories S^i_2. So far this approach has not been successful and there have been results indicating the approach is unlikely to work. In particular, P. Pudlak (*A note on bounded arithmetic* ***Fundamenta Mathematicæ*** vol. 136 (1990) pp. 85–89) has shown that $S_2 = \cup_i S^i_2$ does not prove the bounded consistency of S^1_2. By bounded consistency of a theory T, we mean the statement that T does not prove $0 = 1$ with a proof involving only formulas with bounded quantifiers. Due to this kind of results G.Takeuti (*Open Problems* ***Arithmetic, Proof theory, and computional complexity*** edited by P.Clote and J. Krajíček. pp. 1–9) conjectured that S_2 cannot prove the consistency of the equational fragment $S^{-\infty}_2$ of S_2 which allows only equations of the form $s = t$ for closed terms and natural rules based on recursive definitions of the base symbols. The paper under review proves this latter conjecture is false and shows in fact that S^1_2 can prove the consistency of a reasonably broad class of equational theories.

More precisely, the paper defines what it means for a set of equations Ax involving the functions symbols $\mathcal{F}$ to be a *nice set of recursive axioms*. For Ax to be nice, $\mathcal{F}$ must contain a finite list of 0 or 1-ary constructors (at least one 0 and one 1-ary), $\mathcal{C}$, such that the closed terms (free algebra) over $\mathcal{C}$ can be used as the numerals of Ax. Then for each $f \in \mathcal{F}\backslash\mathcal{C}$ and each $c \in \mathcal{C}$ there must exist exactly one equation $s = t$ in Ax such that s has the form $f(c, x_1, x_2 \dots, x_n)$ if c has arity zero or has the form $f(c(x_0), x_1, x_2 \dots, x_n)$ if c has arity one. Lastly, the only rules in Ax must be of the previously defined type. Given such an Ax, the paper defines the theory $\mathsf{EqT}(Ax)$ to be the theory consisting of closed instances of Ax and those equations derivable from Ax by using either the definition of equality as an equivalence relation or by using the compatibility of function symbols with equality. By fixing a 0-ary constructor c and a 1-constructor c', both in $\mathcal{C}$, $0 = 1$ can be defined as $c = c'(c)$, and $\mathsf{EqT}(Ax)$ is consistent if it does not contain a proof of this. The main result of the paper is then that

$$S^1_2 \vdash \mathsf{Con}(\mathsf{EqT}(Ax)).$$

The class of nice sets of recursive axioms mentioned above is quite general. The author gives several interesting examples. The running example throughout the paper is the theory that uses 0 and S to define numerals and then has equational axioms to define the symbols $+$, '$\cdot$', and $\widetilde{\mathsf{exp}}$, denoting addition, multiplication and a kind of exponentiation. In addition to this example, the author shows how a nice set of recursive axioms can be given for the primitive and μ-recursive functions over $\mathbb{N}$, and how the system PV without the substitution and induction rules satisfies the definition of nice. Here PV is theory defined by Cook (*Feasibly constructive proofs and the propositional calculus* ***Seventh annual ACM symposium on theory of computing*** pp. 83–97) for reasoning about polynomial time computable functions.

The proof of the main result uses a blend of a term rewriting argument together with an approximation scheme. Write $u \longrightarrow^1_{Ax} v$ to mean that there is some axiom $s = t$ in Ax and a ground substitution σ such that $s\sigma$ is a ground term and v is the result of replacing exactly one occurrence of $s\sigma$ in u by $t\sigma$. Define $u \longleftrightarrow^1_{Ax} v$ iff $u \longrightarrow^1_{Ax} v$ or $v \longrightarrow^1_{Ax} u$. Let $\longleftrightarrow^*_{Ax}$ and be the transitive closure of $\longleftrightarrow^1_{Ax}$. The article shows that S^1_2 can prove that if $u = v$ is provable in $\mathsf{EqT}(Ax)$ then $u \longleftrightarrow^*_{Ax} v$ and that if $u \longleftrightarrow^*_{Ax} v$ and u and v are numerals then the Gödel codes for u and v are the same. Thus, $S^1_2 \vdash \mathsf{Con}(\mathsf{EqT}(Ax))$ because if $0 = 1$ is provable in $\mathsf{EqT}(Ax)$ then $0 \longleftrightarrow^*_{Ax} 1$ and hence 0 and 1 must have the same code which is a contradiction.

The result that S^1_2 proves if $u = v$ is provable in $\mathsf{EqT}(Ax)$ then $u \longleftrightarrow^*_{Ax} v$ is proven by a straightforward induction on the $\mathsf{EqT}(Ax)$ proof. The second result that S^1_2 proves that if $u \longleftrightarrow^*_{Ax} v$ and u and v are numerals then the Gödel codes for u and v are the same is the heart of the argument. To do this a new symbol $*$ is introduced and $u \lhd v$ is defined to mean roughly that v can be obtained from u by replacing some of the subterms of v by $*$. It is obvious that if $u \lhd v \lhd u$ then $u = v$. Using now terms over $\mathcal{F} \cup \{*\}$, a notion of evidence is defined which captures schematically (replacing non-relevant things by $*$) what happens in a given reduction sequence. A notion of one term approximating another given some evidence is also defined. The definition entails that if w approximates t then $t \lhd w$. It is shown that S^1_2 can prove that if u and v approximate the same term according to some evidence then either $u \lhd v$ or $v \lhd u$. By induction on the reduction sequence of $u \longleftrightarrow^*_{Ax} v$, S^1_2 proves that there is an evidence such u approximates v. By a similar induction, one can show with the same evidence that v approximates u. Thus, we get $u \lhd v \lhd u$, and hence, $u = v$.

The paper under review is very well written with many examples. There was only one typo that caused this reviewer some minor confusion: In the example on page 288 right after Definition 4.6, the f's in the given table should be replaced with $\widetilde{\mathsf{exp}}$'s. This same mix-up of f and $\widetilde{\mathsf{exp}}$ also occurs in the equation right after the third paragraph of page 287. Both the main result of the paper and the proof technique are interesting. The paper gives some hope to the program that consistency statements might eventually be used to separate non-trivial bounded arithmetic theories. By showing some consistency results provable in these theories it may also help as a guide in finding statements provable in one theory but not in another.

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JEREMY AVIGAD. *Update procedures and the 1-consistency of arithmetic*. ***Mathematical Logic Quarterly***, vol. 48 (2002), pp. 3–13.

The paper under review is a neat piece of work on Hilbert's epsilon substitution method. It has been noted in W. Tait, *The substitution method* (JSL XXX 175), that one can view the ε-substitution method as the problem of finding solutions to systems of equations of a certain kind. The paper singles out a new class of continuous functionals, named *update procedures*, to reveal the mathematical/computational contents behind the method in W. Ackermann, *Zur Widerspruchsfreiheit der Zahlentheorie*, ***Mathematische Annalen***, vol. 117, pp. 162-194 (1940), W. Tait, *Functionals defined by transfinite recursion* (JSL XXX 155) and JSL XXX 175.

A finite partial function σ on N is a finite set of pairs defining a partial function. Let $\hat{\sigma}$ denote the total function obtained from σ by setting $\hat{\sigma}(x) = 0$ for x not in the domain of σ. Let $F(g, \vec{f})$ be a continuous functional mapping number-theoretic functions into $(\mathrm{N} \times \mathrm{N}) \cup \{\emptyset\}$. A functional is said to be *elementary* if it is elementary recursive.

For each fixed $\vec{f}$, F defines an operation $\widetilde{F}$ on partial functions σ as follows. If $F(\hat{\sigma}, \vec{f}) = \emptyset$, then $\widetilde{F}$ does nothing on σ. Otherwise let $F(\hat{\sigma}, \vec{f}) = \langle a, b\rangle$. Then $\widetilde{F}(\sigma, \vec{f})$ denotes the modification of σ that maps a to b, and agrees with σ otherwise. Starting with the empty function $\sigma^0 = \emptyset$, F yields a sequence $\{\sigma^i\}$ with $\sigma^{i+1} = \widetilde{F}(\sigma^i, \vec{f})$. If the sequence increases monotonically, $\sigma^i \subseteq \sigma^{i+1}$, then $g = \bigcup_{i\in\omega} \sigma^i$ defines a partial function, and the continuity of F implies the existence of a *finite fixed point* $\sigma^i = \widetilde{F}(\sigma^i, \vec{f})$.

A continuous functional $F(g, \vec{f})$ is an *update procedure* in g if for any σ, τ, a, b, c and $\vec{f}, \vec{h}$

$$F(\hat{\sigma}, \vec{f}) = \langle a, b\rangle \ \ \& \ \ \sigma \oplus \langle a, b\rangle \subseteq \tau \ \ \& \ \ F(\hat{\tau}, \vec{h}) = \langle a, c\rangle \Rightarrow b = c. \tag{1}$$

In other words, if F once sets the value of $\sigma(a)$ to b, it does not change it, no matter how the other arguments vary.

Any update procedure F produces increasing sequences. This is a very natural condition guaranteeing the monotonicity of the sequences. Moreover it is not hard to see that fixed points depend continuously on the parameters $\vec{f}$. Namely there is a continuous functional G (call it the *solution* functional for F) returning a finite fixed point $G(\vec{f}) = \widetilde{F}(G(\vec{f}), \vec{f})$.

A *system of nested update procedures* is given by a sequence of continuous functionals $F_1(f_1, \dots, f_n), \dots, F_n(f_1, \dots, f_n)$ such that for each i and fixed $f_1, \dots, f_{i-1}$, the functional $(f_i, f_{i+1}, \dots, f_n) \mapsto F_i(f_1, \dots, f_n)$ is an update procedure in f_i. Each F_i defines the operation $\widetilde{F_i}$ for fixed $\{f_j\}_{j\neq i}$. A *finite fixed point* of such a system is a sequence of finite partial functions $\sigma_1, \dots, \sigma_n$ which are simultaneous fixed points for $\widetilde{F_i}$: $\sigma_i = \widetilde{F_i}(\{\widehat{\sigma_j}\}_{j<i}, \sigma_i, \{\widehat{\sigma_j}\}_{j>i})$ for any i. We can find inductively a fixed point for any such system: first take a solution G for F_n. Next pick a fixed point $\{\sigma_i\}_{i<n}$ for the system of update procedures $(f_1, \dots, f_{n-1}) \mapsto F_i(f_1, \dots, f_{n-1}, \hat{g})$ with $g = G(f_1, \dots, f_{n-1})$. Then $\{\sigma_i\}_{i<n} \cup \{G(\hat{\sigma}_1, \dots, \hat{\sigma}_{n-1})\}$ is a desired one.

Now the main result of the paper under review is that, over a weak arithmetic, 1-consistency of the first order arithmetic PA is equivalent to the statement: *Every system of nested elementary update procedures has a fixed point*.

First suppose the 1-consistency of PA. Then every α-recursive functional is total for any $\alpha < \varepsilon_0$. For solving a given system the author constructivizes the existence proof based on the continuity considerations as follows. If $F(g, \vec{f})$ is an α-recursive update procedure, then a solution functional G can be defined by ω^α-recursion. This means that if the height of the tree of F's unsecured sequences (the computation tree of F) is α, then the height of the tree of a solution is bounded by ω^α. The proof is concise and its idea is implicit in Tait's first paper on functionals.

In view of the proof it seems to me that ACA(=Arithmetical Comprehension Axiom) is equivalent to the solvability of arithmetical update procedures.

Second we need to show that any PA-provable Σ^0_1-sentence is true supposing that every system of nested elementary update procedure is solvable. The author first embeds PA into its Skolemized form, a purely universal theory. For a quantifier-free formula θ in the extended language let μ_θ denote the Skolem function for θ together with the μ-axiom:

$$\theta(x, \vec{y}) \to \theta(\mu_\theta(\vec{y}), \vec{y}) \wedge \mu_\theta(\vec{y}) \leq x.$$

Let S be a finite set of closed instances of μ-axioms. Say that a sequence $\{\sigma_i\}$ of finite partial functions is a *finite partial model* of S if every formula in S is true under the interpretation $\{\widehat{\sigma_i}\}$ of the Skolem functions μ_θ occurring in S. Then it is plain to see that the existence of finite partial models of μ-axioms implies, via Herbrand's theorem, the 1-consistency of PA.

Anyone who is familiar with Ackermann's paper notices that $\mu_\theta(\vec{y})$ corresponds to the epsilon matrix $\varepsilon x.\theta(x, \vec{y})$, and finite partial models to solving substitutions for critical formulas.

Next for a given finite set S of closed instances of μ-axioms the author associates a system of nested elementary update procedures so that any finite fixed point of the system gives a finite partial model of S. The update procedures are defined in a similar fashion as Ackermann's definition of the next ε-substitution to a nonsolving substitution. Thus the paper gives a lucid explanation of Ackermann's proof.

As the author states, the methods extend immediately to extensions of arithmetic with the transfinite induction principles for a wellfounded and elementary $\prec$. If one takes a $\prec$-minimal update in defining update procedures, then the resulting functionals are $\prec$-recursive. If one needs to avoid taking a $\prec$-minimal update and obtain elementary update procedures, then one can replace $\sigma^i \subseteq \sigma^{i+1}$ by the relation $\sigma^i \sqsubseteq \sigma^{i+1} :\Leftrightarrow \forall\langle a, b\rangle \in \sigma^i \exists\langle a, c\rangle \in \sigma^{i+1}(b \succeq c)$ in the monotonicity and replace the conclusion $b = c$ by $b \succeq c$ in the definition (1). Then $\{\sigma^i\}_{i\in\omega}$ yields a partial function g which is $\prec$-recursive: $\langle a, b\rangle \in g :\Leftrightarrow \exists i \forall j \geq i(\langle a, b\rangle \in \sigma^j)$.

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U. Berger, W. Buchholz, and H. Schwichtenberg. *Refined program extraction from classical proofs*. ***Annals of pure and applied logic***, vol. 114 (2002), pp. 3–25.

It is well-known how to extract computational content from minimal proofs; one simply uses the Curry-Howard correspondence to map derivations to λ-terms and argues that in the case that the original derivation was of a Π_2-formula, the resulting term in fact computes the function that formula asserts to exist. An active area of research has been to understand how (or even whether it is possible) to extract computational content from *classical* proofs. Of course the difficulty here is dealing with the stability schema $\neg\neg A \to A$. There are two main directions one can go to investigate this question.

The first follows from Griffin's observation in that the type of the stability schema corresponds to control operators (*A formulae-as-types notion of control*, ***Proceedings, Seventeenth annual ACM symposium on principles of programming languages, January 1990, San Francisco, California***, 1990, pp. 47–58). Such an operator allows for "non-local jumps" in a functional programming language, which in turn allow for the program to abort, implement catch/throw clauses for exception handling, etc. The power of this observation is that it allows one to translate directly from classical proofs; however, if one is interested in staying within a purely functional language, this approach does not do the trick.

The second direction uses the fact that classical logic is conservative over minimal logic for Π_2-sentences. Thus a classical existence proof of $\exists y.\varphi(x, y)$, (typically φ is $T(e, x, y)$, T some variant of the Kleene T-predicate) can be converted into a minimal proof of the same; a Curry-Howard correspondence (program extraction) is then used to obtain a λ-term in an appropriate calculus that computes the function that has been proved to be total. This approach has the feature that the computing term is a "standard" functional program, but suffers from the defect that the translation from classical to minimal logic typically results in a large number of double-negated atoms. Each of these generates computationally non-trivial subterms which nonetheless do not contribute in a non-trivial way to the value of the term as a whole.

The paper under review is a contribution to the second approach; in brief, the authors (1) show how to use the translation approach for a nicer class of formulas than Π_2, and (2) show how to perform the translation more carefully, so that the resulting programs are not unnecessarily complicated. They work in a variant Z of the $\to\forall\bot$-fragment of Heyting Arithmetic over finite types. A "classical existence proof" is a proof of $\forall y(\varphi(x, y) \to \bot) \to \bot$ in minimal logic. Of course, this formula is classically equivalent to $\exists y.\varphi(x, y)$, but performing program extraction on the classical proof itself is subject to the difficulties just mentioned. From now on, $\tilde{\exists} y.\varphi$ will be an abbreviation for $\forall y(\varphi \to \bot) \to \bot$.

The first contribution of this paper (in Section 3) is the introduction of *definite* and *goal* formulas. A somewhat trivial example of a goal formula is $D_1 \to D_2 \to \cdots \to D_k \to P$

where the D_i are definite, P is atomic, the D_i are either quantifier-free or "relevant" (essentially, negated conjunctions). The full definition allows certain quantifications to be made. A likewise trivial example of a definite formula is $G_1 \to G_2 \to \cdots \to G_k \to P$ where the G_i are goals and if the G_i is relevant, so is $G_{i+1} \to \cdots \to G_k \to P$ (again, certain quantifications can be made). The point here is that one really wants to use lemmas in proofs of totality rather than work with Kleene T-predicates. For example, to prove the totality of $x \mapsto 2^x$, one really wants to introduce a fresh relation symbol R for its graph and prove $R(0,1) \to \forall nk(R(n,k) \to R(n+1,2k)) \to \exists y R(x,y)$. This formula is of the form $\vec{D} \to \exists y.G$ for definite $\vec{D}$ and goal G. From now on, D and G will refer to definite and goal formulas, respectively.

An overview of the main technical work of the paper is as follows. In Section 3 the authors prove that definite and goal formulas behave well under certain kinds of substitutions. Specifically, let Z^X be the system Z mentioned above extended with a free predicate symbol X, and for any formula F, let F^X be the result of replacing all occurrences of $\perp$ by X. Then (1) $Z^X \vdash D \to D^X$, and (2) $Z^X \vdash (\neg\neg G)^X$. In Sections 4 and 5, the authors define a program extraction to get a system T term $[\![M]\!]$ from a derivation M in minimal logic of $D \to \exists y G$ (they allow multiple lemmas and goals). Furthermore, this program extraction can be applied to the derivations of $D^X \to D$ and $(\neg\neg G)^X$ to obtain terms s and t that depend only on the shapes of D and G. They then use a modified realizability argument to show that $Z \vdash D \to G[y := [\![M]\!]ts]$, thereby concluding that the extracted term does in fact compute the desired function.

Now the existence statement to be proved usually arises as a translation of the corresponding classical statement; as mentioned above, the standard translation introduces a number of computationally extraneous subterms into the final term. The authors provide a more refined approach in Section 3.1 via their notion of *critical* predicate symbols for which, unfortunately, little intuition is given. Suppose one starts with a classical proof of $\forall C_1 \to \cdots \to \forall C_k \to \exists y B$, where the C_i and B are quantifier-free. Instead of double-negating every atom in sight, the authors note that by double-negating just the critical atoms different from $\perp$ one obtains definite assumptions C_i' and a goal formula B', and further that the given proof can be converted to a proof of the translated formula. The details are not provided here, but rather in an earlier paper of the first and third authors (*Program extraction from classical proofs*, ***Proceedings, Logic and computational complexity, October 1994, Indianapolis, Indiana***, Lecture Notes in Computer Science 960, Springer, 1994, pp. 77–97).

The last section of the paper is devoted to examples. They start with a proof of the existence of the Fibonacci numbers. The refined translation actually entails *no* double-negations, and the resulting program works by passing functions (instead of pairs), thereby actually taking advantage of the higher-type capabilities of the term language. They then turn to a proof of the well-foundedness of the usual order on the natural numbers: $\forall f \exists k\, (f(k) \leq f(k+1))$. The classical proof from the minimum principle chooses a minimal k in the range of the function f. However, the extracted program computes the least k such that $f(k) \leq f(k+1)$. The authors work out the classical proof in detail (especially that of the minimum principle), concluding with a Scheme program for the extracted term. They conclude with a discussion and classical proof of Dickson's Lemma, which asserts that, given k sequences of natural numbers and a fixed number ℓ, there is a sequence of ℓ indices such that all k sequences are nondecreasing on those indices. The method of the current paper is applied to Dickson's Lemma in a later paper of Berger, Schwichtenberg, and Seisenberger (*The Warshall algorithm and Dickson's lemma: two examples of realistic program extraction*, ***Journal of automated reasoning***, vol. 26 (2001), pp. 205–221).

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ALEXIS BÈS. *Decidability and definability results related to the elementary theory of ordinal multiplication*. ***Fundamenta Mathematica***, vol. 171 (2002), pp. 197–211.

The metamathematics of ordinal arithmetic has not received much attention of late, so it was a pleasure to read this fine paper on definability and decidability for the theory of ordinal multiplication $\times$ by itself. Of course, much is already known about these matters for $+$ and $\times$ together, and for $+$ alone. Bès shows that an ordinal $\alpha > 0$ is definable in the theory of ordinal multiplication if and only if it is $< \omega^{\omega^{\omega^{\omega}}}$ and is of form $\omega^{\beta_1} + \cdots + \omega^{\beta_k}$ with $\omega^{\omega} > \beta_1 > \cdots > \beta_k$, and that $\mathrm{Th}\langle\alpha; \times\rangle$ is decidable if and only if $\alpha < \omega^{\omega}$ ($\times$ is the three-place multiplication relation, suitably restricted).

Most of the definitions needed in the proofs are formulated in terms of the notions of left- and right-hand divisibility of ordinals, denoted by $|_l$ and $|_r$, respectively (e.g., $\alpha \mid_r \beta$ means that $\beta = \gamma\alpha$ for some γ). Bès considers the theory based just on these notions and shows that $\mathrm{Th}\langle\alpha; |_r, |_l\rangle$ is decidable if and only if $\alpha < \omega^{\omega}$. On the other hand, for any ξ, $\mathrm{Th}\langle\omega^{\omega^{\xi}}; |_r\rangle$ and $\mathrm{Th}\langle\omega^{\xi}; |_l\rangle$ are each decidable. The proofs are interesting applications of Feferman-Vaught generalized weak direct powers.

In the interests of brevity I will limit my outline of Bès' proofs to the definability and decidability results about the theory of ordinal multiplication. Let's begin by reviewing the prior situation: $\mathrm{Th}\langle\omega; +\rangle$ is decidable, and every natural number is definable in this theory (Presburger); all $\mathrm{Th}\langle 2^{\alpha}; +\rangle$ are decidable (Büchi); an ordinal α is definable in the theory of addition iff $\alpha < \omega^{\omega^{\omega}}$, and in the theory of both addition and multiplication, iff $\alpha < \omega^{\omega^{\omega^{\omega}}}$ (Ehrenfeucht). $\mathrm{Th}\langle\omega; \times\rangle$ is decidable (Skolem, Mostowski), but only 0 and 1 are definable in this theory: Given $m > 1$, let p be a prime factor of m, q be some other prime not a factor of m, and let f be the mapping on ω that simply interchanges p and q in the prime decomposition of its argument. Then f is an automorphism of $\langle\omega; \times\rangle$ and $f(m) \neq m$, so m is not definable. Bès has filled in the main blank in the picture: what happens with multiplication alone in the transfinite domain.

Key to the development is an obscure but beautiful 1909 theorem of Jacobsthal, included in Sierpinski's book *Cardinal and Ordinal Numbers*, that just as with finite numbers, every transfinite ordinal has a prime decomposition. A prime is an ordinal $\geqslant 2$ which is not a product of two smaller ordinals. Transfinite primes can be shown to be ordinals of one of the forms $\omega^{\omega^{\xi}}$ or $\omega^{\gamma} + 1$ with $\gamma > 0$. Thus, there are limit primes and infinite successor primes. The prime factorization of an infinite ordinal is not unique in the same way as for finite ordinals, as the equation $\omega\omega = (\omega + 1)\omega$ illustrates. However, the factorization is unique if we simply add the requirement that all limit prime factors must appear first in non-increasing order. If an ordinal α has the Cantor Normal Form (CNF) $\omega^{\alpha_1} a_1 + \omega^{\alpha_2} a_2 + \cdots + \omega^{\alpha_k} a_k$, where $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ and the a_i are finite and > 0, and if the CNF of α_k is $\omega^{\mu_1} m_1 + \omega^{\mu_2} m_2 + \cdots + \omega^{\mu_n} m_n$, then the Prime Factorization (PF) of α is obtained from

$$(\omega^{\omega^{\mu_1}})^{m_1} (\omega^{\omega^{\mu_2}})^{m_2} \cdots (\omega^{\omega^{\mu_n}})^{m_n} a_k (\omega^{\alpha_{k-1} - \alpha_k} + 1) a_{k-1} \cdots (\omega^{\alpha_1 - \alpha_2} + 1) a_1$$

by replacing the finite numbers $a_1, \ldots, a_k$ by their prime factorizations.

Let T be the theory of ordinal multiplication, i.e., $T = \mathrm{Th}\langle\mathbf{On}; \times\rangle$ where $\mathbf{On}$ is the class of all ordinals. Ehrenfeucht showed $T = \mathrm{Th}\langle\omega^{\omega^{\omega^{\omega}}}; \times\rangle$. 0,1 are definable in T, but what else? Not all ordinals $< \omega^{\omega^{\omega^{\omega}}}$: given $\alpha = \omega^{\alpha_1} a_1 + \omega^{\alpha_2} a_2 + \cdots + \omega^{\alpha_k} a_k$ (CNF) with some $a_i > 1$, we use the same idea as in the finite domain, and define a function f on ordinals using the PF; f interchanges two different finite primes only one of which is a factor of α_i, and is the identity on infinite primes. Then f is an automorphism of $\langle\omega^{\omega^{\omega^{\omega}}}; \times\rangle$ and $f(\alpha) \neq \alpha$. Bès goes on to show that all other infinite ordinals $< \omega^{\omega^{\omega^{\omega}}}$ are definable in T—that is, strictly decreasing sums of powers of ω are definable, but nothing else.

The idea is to first show that every power of ω is definable, and then that sums of such powers $\omega^{\beta_1} + \omega^{\beta_2} + \cdots + \omega^{\beta_k}$ are as well. Let's consider this last step first. If $\alpha < \omega^{\omega^{\omega^{\omega}}}$ is a strictly decreasing sum of powers of ω, we can look at both its CNF and its PF:

$$\begin{aligned}\alpha &= \omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_k}\\ &= (\omega^{\omega^{\mu_1}})^{m_1}(\omega^{\omega^{\mu_2}})^{m_2}\cdots(\omega^{\omega^{\mu_n}})^{m_n}(\omega^{\alpha_{k-1}-\alpha_k}+1)\cdots(\omega^{\alpha_1-\alpha_2}+1)\end{aligned}$$

where the CNF of α_k is $\omega^{\mu_1}m_1 + \omega^{\mu_2}m_2 + \cdots + \omega^{\mu_n}m_n$. From this it follows that if each power of ω and each infinite successor prime (both $< \omega^{\omega^{\omega^{\omega}}}$, of course) are definable, then α is definable.

Now we turn to the definability of powers of ω. Unfortunately, there are a couple of errors in Bès' argument at this point; of the definitions given on page 201, two (for powers of ω and of ω itself) don't work. But the problem is easy to fix. The definitions of the set of primes as those ordinals with exactly two right-hand divisors and of the set of limit primes as those primes with more than two left-hand divisors are fine. The powers of ω are exactly those ordinals with a limit prime as a right-hand divisor. We note that although the full order relation is not definable in T, an important fragment of it is: $0 < \alpha \leqslant \omega^{\gamma}$ if and only if α is a left-hand divisor of ω^{γ}. ω is the least limit prime. The successor primes are those infinite primes that are not limit primes.

$\langle\omega^{\omega^{\omega}};+\rangle$ is isomorphic to $\langle\{\omega^{\gamma} : \gamma < \omega^{\omega^{\omega}}\};\times\rangle$ via the mapping $\gamma \mapsto \omega^{\gamma}$. Since every ordinal $< \omega^{\omega^{\omega}}$ is definable in $\mathrm{Th}\langle\omega^{\omega^{\omega}};+\rangle$, it follows that every ω^{γ} with $\gamma < \omega^{\omega^{\omega}}$ is definable in T. In particular, each individual limit prime $< \omega^{\omega^{\omega^{\omega}}}$ is definable. Each successor prime $\omega^{\gamma}+1$ is definable since it is the only successor prime between ω^{γ} and $\omega^{\gamma+1}$. Limit ordinals are the ones with ω as a left-hand divisor, successors are the others. Combining all these facts and examining the formula for the PF, we see that every infinite ordinal $< \omega^{\omega^{\omega^{\omega}}}$ which does not have a finite prime among its factors, i.e., every strictly decreasing sum of powers of ω, is definable.

Bès next considers the decidability of T, and answers this question in the negative by interpreting the two-letter word problem into T. Consider the alphabet $\Sigma = \{1,2\}$ and the mapping

$$m_0m_1\ldots m_n \;\mapsto\; (\omega^{m_0}+1)(\omega^{m_1}+1)\ldots(\omega^{m_n}+1)$$

where $m_0m_1\ldots m_n$ is a word over Σ^*. Via this mapping, the two-letter word problem is reduced to the decision problem for T.

Up to now, I have discussed $T = \mathrm{Th}\langle\omega^{\omega^{\omega^{\omega}}};\times\rangle$, whereas Bès' results apply to other theories $\mathrm{Th}\langle\alpha;\times\rangle$. In particular, he shows that ordinal definability in $\mathrm{Th}\langle\omega^{\xi};\times\rangle$ is exactly what one might expect—the ordinals definable there are those definable in T which are also $< \omega^{\xi}$. Further, the undecidability result applies to any theories $\mathrm{Th}\langle\alpha;\times\rangle$ with $\alpha \geqslant \omega^{\omega}$. But when $\alpha < \omega^{\omega}$, we do get decidability, which Bès shows by interpreting the relevant theories into $\mathrm{Th}\langle\omega;\times,0,1,2,\ldots\rangle$, a theory shown decidable by Mostowski.

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