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LOCAL INITIAL SEGMENTS OF THE TURING DEGREES

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Abstract. Recent results on initial segments of the Turing degrees are presented, and some conjectures about initial segments that have implications for the existence of nontrivial automorphisms of the Turing degrees are indicated.

§1. Introduction. This article concerns the algebraic study of the upper semilattice of Turing degrees. Upper semilattices of interest in this regard tend to have a least element, hence for convenience the following definition is made.

Definition 1.1. A unital semilattice (usl) is a structure $\mathcal{L} = (L, *, e)$ satisfying the following equalities for all $a, b, c \in L$.

1. $a * (b * c) = (a * b) * c$,
2. $a * b = b * a$,
3. $a * a = a$, and
4. $a * e = a$.

A bounded unital semilattice (busl) is a structure $\mathcal{L} = (L, *, e, z)$ such that $(L, *, e)$ is an usl and such that the following equality holds for all $a \in L$:

5. $a * z = z$.

The Turing degrees $\mathcal{D} = (\mathcal{D}, \cup, \boldsymbol{0})$ where $\boldsymbol{0}$ is the degree of the recursive sets and $\boldsymbol{a} \cup \boldsymbol{b}$ is the join (least upper bound) of the degrees $\boldsymbol{a}, \boldsymbol{b}$ is an example of an usl. Of course semilattices have natural orderings; that is $\boldsymbol{a} \leq \boldsymbol{b} \leftrightarrow \boldsymbol{a} \cup \boldsymbol{b} = \boldsymbol{b}$, and $\boldsymbol{a} \leq \boldsymbol{b}$ just in case there are sets $A, B \subseteq \omega$ of degree $\boldsymbol{a}, \boldsymbol{b}$, respectively, such that A can be computed using an oracle for B. Note that $\mathcal{D}$ has size $2^{\aleph_0}$ and has the countable predecessor property, i.e., for no $\boldsymbol{a} \in \mathcal{D}$ are there uncountably many $\boldsymbol{b} < \boldsymbol{a}$. The jump operator $\boldsymbol{a} \mapsto \boldsymbol{a}'$ is included in the language under consideration below. Note however that in any ideal where the jump is invariant or definable, such as $\mathcal{D}$ itself [20], jump can be removed from the language.

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An *initial segment* of $\mathcal{D}$ is a subset of $\mathcal{D}$ which is downward closed. An *ideal* of $\mathcal{D}$ is a set $\mathcal{I} \subseteq \mathcal{D}$ such that there exists an usl $\mathcal{L}$ and an usl homomorphism $\varphi : \mathcal{D} \to \mathcal{L}$ such that $\boldsymbol{a} \in \mathcal{I} \leftrightarrow \varphi(\boldsymbol{a}) = e$.

PROPOSITION 1.2. *Equivalently, $\mathcal{I}$ is a nonempty initial segment and closed under join.*

PROOF. In one direction, let $\mathcal{L}$ be the set of equivalence classes of the relation E defined by $\boldsymbol{a}\, E\, \boldsymbol{b} \leftrightarrow \boldsymbol{a} \cup \boldsymbol{i} = \boldsymbol{b} \cup \boldsymbol{i}$ for some $\boldsymbol{i} \in \mathcal{I}$, and let $*$ be defined by $\varphi(\boldsymbol{a}) * \varphi(\boldsymbol{b}) = \varphi(\boldsymbol{a} \cup \boldsymbol{b})$. To check that this is well-defined, suppose $\boldsymbol{a}_k, \boldsymbol{b}_k$ are in $\mathcal{D}$, and $\varphi(\boldsymbol{a}_k) = \varphi(\boldsymbol{b}_k)$, for $k = 0, 1$. Then there exists $\boldsymbol{i}_k$ in $\mathcal{I}$ such that $\boldsymbol{a}_k \cup \boldsymbol{i}_k = \boldsymbol{b}_k \cup \boldsymbol{i}_k$ for $k = 0, 1$. Let $\boldsymbol{i} = \boldsymbol{i}_0 \cup \boldsymbol{i}_1$. Then $\boldsymbol{a}_0 \cup \boldsymbol{b}_0 \cup \boldsymbol{i} = \boldsymbol{a}_1 \cup \boldsymbol{b}_1 \cup \boldsymbol{i}$, so $\varphi(\boldsymbol{a}_0 \cup \boldsymbol{b}_0) = \varphi(\boldsymbol{a}_1 \cup \boldsymbol{b}_1)$, as desired. In the other direction, $\mathcal{I}$ is nonempty since it contains $\boldsymbol{0}$, and if $\boldsymbol{a} \leq \boldsymbol{b} \in \mathcal{I}$ then $\boldsymbol{a} \cup \boldsymbol{b} = \boldsymbol{b}$ so $\varphi(\boldsymbol{a}) = \varphi(\boldsymbol{a}) * e = \varphi(\boldsymbol{a}) * \varphi(\boldsymbol{b}) = \varphi(\boldsymbol{a} \cup \boldsymbol{b}) = \varphi(\boldsymbol{b}) = e$. ⊣

Spector [22] found the first nontrivial usl isomorphism type of an ideal of $\mathcal{D}$; namely, an ideal with two elements. This was extended step by step, to all finite lattices by Lerman [11], to all countable usls by Lachlan and Lebeuf [10] and to all size $\leq \aleph_1$ usls with the countable predecessor property by Abraham and Shore [1]. But Groszek and Slaman [5] showed that it is consistent with ZFC that the same statement with $2^{\aleph_0}$ in place of $\aleph_1$ is false.

Local initial segments are initial segments of the degrees below a given degree, for example a degree above $\boldsymbol{0}''$ or a nonzero r.e. degree.

By an *automorphism* of $\mathcal{D}$ is meant an usl automorphism of $\mathcal{D}$. Since the jump operator is definable in the language $\{\cup\}$ [20], every automorphism is jump-preserving. Slaman and Woodin [21] showed that if π is an automorphism of $\mathcal{D}$ and $\boldsymbol{x} \geq \boldsymbol{0}''$ then $\pi(\boldsymbol{x}) = \boldsymbol{x}$.

Another proof of this fact using Shore's coding with exact pair technique was presented in [16]. Historically, initial segments were used to obtain partial results toward the rigidity of the Turing degrees. The work of Slaman and Woodin rendered this use obsolete; however recent results may revive this application of initial segment theory.

§2. Results and applications.

This section starts with a presentation of a new proof of rigidity above $\boldsymbol{0}''$ that uses the results on ideals of $\mathcal{D}$ of [8].

DEFINITION 2.1. Let $\boldsymbol{a}$ be a Turing degree. An usl $\mathcal{U}$ is called $\Sigma^0_1(\boldsymbol{a})$-presentable if there exist countable structures $\mathcal{L} = (L, \leq, \vee)$ and $\mathcal{L}' = (\omega, \leq', \vee')$ such that $\mathcal{U}$ is isomorphic to $\mathcal{L}$ and such that the following conditions hold.

1. $\leq'$ is a transitive and reflexive binary relation on ω which is r.e. in $\boldsymbol{a}$,
2. $\vee'$ is a binary operation on ω which is recursive in $\boldsymbol{a}$,
3. L is the set of equivalence classes of the equivalence relation on ω given by $a \leq' b$ & $b \leq' a$,
4. $\leq$ is the relation on L induced by $\leq'$, and
5. $\vee$ is the operation on L induced by $\vee'$.

For $n \geq 0$, $\mathcal{U}$ is called $\Sigma^0_{n+1}(\boldsymbol{a})$-presentable if it is $\Sigma^0_1(\boldsymbol{a}^{(n)})$-presentable, where $\boldsymbol{a}^{(n)}$ denotes the nth jump of $\boldsymbol{a}$. $\mathcal{U}$ is called $\boldsymbol{a}$-presentable if there exist such $\mathcal{L}$, $\mathcal{L}'$ with $\leq'$ recursive in $\boldsymbol{a}$, and $\mathcal{L}$ isomorphic to $\mathcal{L}'$.

Lemma 2.2. *$[\boldsymbol{a}, \boldsymbol{b}]$ is $\Sigma^0_3(\boldsymbol{b})$-presentable, for any Turing degrees $\boldsymbol{a} \leq \boldsymbol{b}$.*

Proof. Let $B \in \boldsymbol{b}$, $A \in \boldsymbol{a}$, choose $e \in \omega$ such that $A = \{e\}^B$, the eth Turing functional applied to B, and let

$$C = \{i \mid \{i\}^B \text{ is total and } \{e\}^B \leq_T \{i\}^B\}.$$

The set C is $\Sigma^0_3(B)$ by a standard argument, so $C = \{h(n) \mid n < \omega\}$ for some injective $h \leq_T B''$; define $i \leq' j \leftrightarrow \{h(i)\}^B \leq_T \{h(j)\}^B$.

If f_i for $i \in \{0, 1\}$ are recursive functions such that every element of ω is in the range of exactly one f_i, then a representative of the Turing degree $\boldsymbol{a} \cup \boldsymbol{b}$ is given by $A \oplus B = \{f_0(x) \mid x \in A\} \cup \{f_1(x) \mid x \in B\}$. By abuse of notation, define a recursive function $\oplus$ such that for $e_0, e_1 \in \omega$, $X \subseteq \omega$ and $i \in \{0, 1\}$,

$\{e_0 \oplus e_1\}^X(f_i(x)) = \{e_i\}^X(x)$. Clearly if the $\{e_i\}^X$ are (characteristic functions of) subsets of ω then $\{e_0 \oplus e_1\}^X = \{e_0\}^X \oplus \{e_1\}^X$. Now define $i \vee' j = h^{-1}(h(i) \oplus h(j))$ and observe that the conditions of Definition 2.1 obtain. ⊣

Third proof of rigidity above $\boldsymbol{0}''$. Suppose π is an automorphism of $\mathcal{D}$ and $\boldsymbol{x} \in \mathcal{D}$. Then $(\mathcal{D}, \boldsymbol{x}) \cong (\mathcal{D}, \pi\boldsymbol{x})$; in particular, if one defines $[\boldsymbol{a}, \boldsymbol{b}] = \{\boldsymbol{y} \in \mathcal{D} \mid \boldsymbol{a} \leq \boldsymbol{y} \leq \boldsymbol{b}\}$, then $[\boldsymbol{0}, \boldsymbol{x}] \cong [\boldsymbol{0}, \pi\boldsymbol{x}]$. Also, using the invariance of the jump under π, if one defines $I_{\boldsymbol{x}} = \{$ usl isomorphism types of initial segments $[\boldsymbol{0}, \boldsymbol{g}]$ with $\boldsymbol{g}'' \leq \boldsymbol{x}$ $\}$, then it follows that $I_{\boldsymbol{x}} = I_{\pi\boldsymbol{x}}$.

Now define $S_{\boldsymbol{x}} = \{$ $\Sigma_1(\boldsymbol{x})$-presentable usls $\}$. Assuming that $\boldsymbol{x} \geq \boldsymbol{0}''$, it is shown in [8] that $I_{\boldsymbol{x}} = S_{\boldsymbol{x}}$. It is also possible to show that for any Turing degrees $\boldsymbol{x}, \boldsymbol{y}$, if $S_{\boldsymbol{x}} = S_{\boldsymbol{y}}$ then $\boldsymbol{x} = \boldsymbol{y}$ by constructing appropriate usls, along the lines of the use of Slaman-Woodin sets in [16]. Now if $\boldsymbol{x} \geq \boldsymbol{0}''$ then by invariance of the jump also $\pi\boldsymbol{x} \geq \boldsymbol{0}''$, so $S_{\boldsymbol{x}} = I_{\boldsymbol{x}} = I_{\pi\boldsymbol{x}} = S_{\pi\boldsymbol{x}}$, so $\boldsymbol{x} = \pi\boldsymbol{x}$, completing the proof. ⊣

The inclusion $I_{\boldsymbol{x}} \subseteq S_{\boldsymbol{x}}$ holds for any degree $\boldsymbol{x}$. Namely, $[\boldsymbol{0}, \boldsymbol{g}]$ is $\Sigma_1(\boldsymbol{g}'')$-presentable, hence since $\boldsymbol{g}'' \leq \boldsymbol{x}$ also $\Sigma_1(\boldsymbol{x})$-presentable.

For the inclusion $S_{\boldsymbol{x}} \subseteq I_{\boldsymbol{x}}$ one constructs, given a $\Sigma_1(\boldsymbol{x})$-presentable usl $\mathcal{L}$, a function $g : \omega \to \omega$ and functions $g_k : \omega \to \omega$ for each $k \in \mathcal{L}$, such that $[\boldsymbol{0}, \boldsymbol{g}] = \{\boldsymbol{g}_k \mid k \in \mathcal{L}\}$ and $\mathcal{L} \models k \leq m$ just in case $g_k \leq_T g_m$, and such that $\boldsymbol{g}'' \leq \boldsymbol{x}$. This is done by breaking the problem up into the following requirements:

1. For each e, if $\{e\}^g$ is total then there exists k such that $\{e\}^g \equiv_T g_k$. This is accomplished using an elaboration of Spector's splitting tree technique.
2. If $k \not\leq m$, then $g_k \not\leq_T g_m$. This is accomplished using a diagonalization argument, see e.g., [12].

3. It can be determined recursively in $\boldsymbol{x}$ and uniformly in e whether or not $\{e\}^g$ is total. This is accomplished using e-total trees, a well-known technique, see e.g., [12].
4. If $k \le m$ then $g_k \le_T g_m$.

The requirements under item 4 are satisfied using a representation of $\mathcal{L}$ as a set of equivalence relations, much as in [10]. The methods of that paper can be used to obtain the analogous result for $\boldsymbol{x}$-presentable instead of $\Sigma_1(\boldsymbol{x})$-presentable usls. The main obstacle faced in extending their result was to find a suitable usl representation. Instead of strengthening the already quite complicated representations of [10], a natural representation was found based on a paper of Pudlák [17], a paper that has been celebrated for quite different reasons in the past among lattice theorists.

The proof sketched above uses the theorem of Slaman-Woodin that all automorphisms preserve the double jump, so it does not replace the Slaman-Woodin manuscript. Perhaps the proofs and results above are more interesting in relation to rigidity above $0'$, which might be solvable using initial segments.

Another proof can be given which does not require the new results on initial segments but is more roundabout:

FOURTH PROOF OF RIGIDITY ABOVE $\boldsymbol{0}''$: The construction of Lachlan-Lebeuf with e-total trees and relativization tells us that every $\boldsymbol{x}$-presentable busl is an initial segment $[\boldsymbol{0}, \boldsymbol{g}]$ of the degrees below $\boldsymbol{x}$, for any $\boldsymbol{x}$ above $\boldsymbol{0}''$, with $\boldsymbol{g}'' \le \boldsymbol{x}$. On the other hand, any such initial segment must be $\Sigma_1(\boldsymbol{x})$-presentable, and if $\boldsymbol{y}$ is not below $\boldsymbol{x}$ then one can construct an usl which is $\boldsymbol{y}$-presentable but not $\Sigma_1(\boldsymbol{x})$-presentable. Hence $[\boldsymbol{0}, \boldsymbol{x}]$ = the set of Turing degrees $\boldsymbol{z}$ such that every $\boldsymbol{z}$-presentable busl is an initial segment $[\boldsymbol{0}, \boldsymbol{g}]$ below $\boldsymbol{x}$ with $\boldsymbol{g}'' \le \boldsymbol{x}$ and the theorem follows. ⊣

Now as mentioned the result on automorphisms has been proved twice before, and two more proofs were presented above. The question is whether any of the available proofs shed light on the open problem: does there exist an automorphism π of $\mathcal{D}$ and a degree $\boldsymbol{x} \ge \boldsymbol{0}'$ such that $\pi\boldsymbol{x} \ne \boldsymbol{x}$? In other words, the problem asks if $\boldsymbol{0}''$ can be replaced by $\boldsymbol{0}'$. To pursue this problem using constructions of ideals $[\boldsymbol{0}, \boldsymbol{g}]$, one would probably need to look at ideals of $[\boldsymbol{0}, \boldsymbol{0}']$ and, by relativization $[\boldsymbol{d}, \boldsymbol{d}']$ for $\boldsymbol{d} \in \mathcal{D}$. A first result in that direction is the following.

THEOREM 2.3. *If $\boldsymbol{x} \ge \boldsymbol{0}''$ or $\boldsymbol{x} = \boldsymbol{0}'$ and $\mathcal{L}$ is a $\Sigma_2(\boldsymbol{x})$-presentable usl then there exists $\boldsymbol{g} < \boldsymbol{x}$ such that $\mathcal{L} \cong [\boldsymbol{0}, \boldsymbol{g}]$.*

PROOF SKETCH. First suppose $\boldsymbol{x} \ge \boldsymbol{0}''$. The argument of Lachlan and Lebeuf [10] shows that the theorem holds with $\boldsymbol{x}$-presentable in place of $\Sigma^0_2(\boldsymbol{x})$-presentable. Using a finite injury argument as in [12] Exercise VIII.1.16 this result is strengthened to $\boldsymbol{x}'$-presentable. The point is that if the injury is finite then the construction of any tree in a chain $T_0 \supseteq T_1 \supseteq \cdots$ can start over again finitely often, but keeping the part of g built so far, so there is no injury to g. This can be done because it does not matter where on T_n the root of T_{n+1} is put. Finally, the use of Pudlák representations, discussed in the next section, allows the strengthening from $\boldsymbol{x}'$-presentable to $\Sigma^0_1(\boldsymbol{x}')$-presentable. If $\boldsymbol{x} = \boldsymbol{0}'$ then we borrow the techniques from the construction of Lerman ([12], Chapter XII) and again use Pudlák representations to deal with the high complexity of the usl. ⊣

Definition 2.4. For $n \geq 1$ we say that $\boldsymbol{g}$ is generalized low$_n$, or GL$_n$, if $\boldsymbol{g}^{(n)} = (\boldsymbol{g} \cup \boldsymbol{0}')^{(n-1)}$.

Corollary 2.5. If $\boldsymbol{x} \geq \boldsymbol{0}''$ or $\boldsymbol{x} = 0'$ then the following are equivalent for any countable busl L which happens to be a lattice:

(i) L is $\Sigma^0_2(\boldsymbol{x})$-presentable.
(ii) L is isomorphic to $[\boldsymbol{0}, \boldsymbol{g}]$ for some $\boldsymbol{g} < \boldsymbol{x}$.

In particular, the following are equivalent:

1. $\mathcal{L}$ is a Σ_3-presentable bounded lattice.
2. There exists $\boldsymbol{g} < \boldsymbol{0}'$ such that $\mathcal{L} \cong [\boldsymbol{0}, \boldsymbol{g}]$ and $[\boldsymbol{0}, \boldsymbol{g}]$ is a lattice.

Proof. By Jockusch and Posner [6] if $[\boldsymbol{0}, \boldsymbol{g}]$ is a lattice then $\boldsymbol{g}$ is GL$_2$, $\boldsymbol{g}'' = (\boldsymbol{g} \cup \boldsymbol{0}')'$, and so $[\boldsymbol{0}, \boldsymbol{g}]$ is $\Sigma_2(\boldsymbol{g} \cup \boldsymbol{0}')$-presentable, and since both $\boldsymbol{g}$ and $\boldsymbol{0}'$ are below $\boldsymbol{x}$, also $\Sigma_2(\boldsymbol{x})$-presentable. ⊣

The first partial results toward Corollary 2.5 were obtained by Spector [22] and Sacks [18] who found minimal degrees below $\boldsymbol{0}''$ and $\boldsymbol{0}'$, respectively. Next, it is natural to consider initial segments below arbitrary nonzero r.e. degrees or low initial segments.

Theorem 2.6. *If $\boldsymbol{a} \in \mathcal{D}$ and L is a $\Sigma^0_2(\boldsymbol{a})$-presentable busl then $L \cong [\boldsymbol{a}, \boldsymbol{g}]$ for some $\boldsymbol{g}$ satisfying $\boldsymbol{g}' = \boldsymbol{a}'$.*

Proof sketch. Lerman ([12] Corollary XII.5.10) obtains the analogous result for L being $\boldsymbol{a}'$-presentable rather than $\Sigma^0_1(\boldsymbol{a}')$-presentable, but using Pudlák representations there is no essential distinction. ⊣

Conjecture 2.7 (Lowness conjecture). *If $\boldsymbol{a} \in \mathcal{D}$, $\boldsymbol{x} \geq \boldsymbol{a}''$ or $\boldsymbol{x} = \boldsymbol{a}'$ and $\boldsymbol{g} \leq \boldsymbol{x}$, if $[\boldsymbol{a}, \boldsymbol{g}]$ is a lattice and $\boldsymbol{g} \in GL_1(\boldsymbol{a})$, i.e., $\boldsymbol{g}' = \boldsymbol{g} \cup \boldsymbol{a}'$, then $[\boldsymbol{a}, \boldsymbol{g}]$ is $\Sigma^0_1(\boldsymbol{x})$-presentable.*

A stronger version of the conjecture: If $\boldsymbol{a} \in \mathcal{D}$, $\boldsymbol{x} \geq \boldsymbol{a}''$ or $\boldsymbol{x} = \boldsymbol{a}'$ and $\boldsymbol{g} \leq \boldsymbol{x}$, if $[\boldsymbol{a}, \boldsymbol{g}]$ contains no 1-generic relative to $\boldsymbol{a}$ and $\boldsymbol{g} \in$ GL$_1(\boldsymbol{a})$, i.e., $\boldsymbol{g}' = \boldsymbol{g} \cup \boldsymbol{a}'$, then $[\boldsymbol{a}, \boldsymbol{g}]$ is $\Sigma^0_1(\boldsymbol{x})$-presentable.

Lemma 2.8. *If Conjecture 2.7 is true then every automorphism of $\mathcal{D}$ is the identity above $\boldsymbol{0}'$.*

Proof. Let $\boldsymbol{x} \geq \boldsymbol{0}'$ and by the Friedberg jump inversion theorem let $\boldsymbol{a}$ be such that $\boldsymbol{x} = \boldsymbol{a}'$. Consider the usl isomorphism types of intervals of the form $[\boldsymbol{a}, \boldsymbol{g}]$ with $\boldsymbol{g}' = \boldsymbol{a}'$ that are lattices. By Theorem 2.6 and Conjecture 2.7, these are exactly the $\Sigma_1^0(\boldsymbol{x})$-presentable busls, and this class determines $\boldsymbol{x}$ uniquely. ⊣

The usual constructions of initial segments below $\boldsymbol{0}''$ using forcing with recursive perfect trees yield degrees $\boldsymbol{g}$ that are hyperimmune-free (HIF). This means that every function recursive in $\boldsymbol{g}$ is dominated by a recursive function. In general, $\boldsymbol{g}$ is HIF relative to $\boldsymbol{a}$ if every function recursive in $\boldsymbol{g} \cup \boldsymbol{a}$ is dominated by a function recursive in $\boldsymbol{a}$.

LEMMA 2.9. *Conjecture* 2.7 *is true for any* $\boldsymbol{g}$ *which is HIF relative to* $\boldsymbol{a}$.

PROOF. It can be readily shown that HIF degrees $\boldsymbol{g}$ satisfy $\boldsymbol{g}'' = \boldsymbol{g}' \cup \boldsymbol{0}''$. If a degree $\boldsymbol{g}$ below $\boldsymbol{x} \geq \boldsymbol{0}''$ is both HIF and GL_1, then $\boldsymbol{g}'' = \boldsymbol{g}' \cup \boldsymbol{0}'' = (\boldsymbol{g} \cup \boldsymbol{0}') \cup \boldsymbol{0}'' = \boldsymbol{g} \cup \boldsymbol{0}'' \leq \boldsymbol{x}$, so $[\boldsymbol{0}, \boldsymbol{g}]$ is $\Sigma_1^0(\boldsymbol{x})$-presentable, and this argument relativizes to $\boldsymbol{a} \in \mathcal{D}$. And for any $\boldsymbol{a} \in \mathcal{D}$, there are no HIF degrees in the interval $(\boldsymbol{a}, \boldsymbol{a}']$ (see [15]), so the lemma is trivial for $\boldsymbol{x} = \boldsymbol{a}'$. ⊣

On the other hand, there are HIF degrees $\boldsymbol{g}$ below $\boldsymbol{x}$ with $[\boldsymbol{0}, \boldsymbol{g}]$ of any $\Sigma_2^0(\boldsymbol{x})$-presentable isomorphism type; in fact these degrees can all be forced to be not GL_1.

It follows from results of Shore [19] that if $\boldsymbol{g}$ is not GL_2 then the lemma is best possible, i.e., every presentation of $[\boldsymbol{0}, \boldsymbol{g}]$ has degree at least $\boldsymbol{g}^{(3)}$. On the other hand initial segment constructions lead to GL_2 degrees $\boldsymbol{g}$, and such initial segments can have arbitrary prescribed isomorphism type, hence the interval $[\boldsymbol{0}, \boldsymbol{g}]$ may even have a presentation of degree $\boldsymbol{0}$. The fact that non-GL_2 initial segments $[\boldsymbol{0}, \boldsymbol{x}]$ are complicated may be seen as evidence against Conjecture 2.7. However Shore's proof uses explicitly the fact that $[\boldsymbol{0}, \boldsymbol{x}]$ is not a lattice. Namely, the proof involves a coding of the set $X^{(3)}$, (where X is a set of degree $\boldsymbol{x}$) by

$$n \in X^{(3)} \leftrightarrow \boldsymbol{g}_n \leq \boldsymbol{y}_0 \,\&\, \boldsymbol{g}_n \leq \boldsymbol{y}_1$$

for certain degrees $\boldsymbol{g}_0, \boldsymbol{g}_1, \dots, \boldsymbol{y}_0, \boldsymbol{y}_1 < \boldsymbol{x}$, and the meet (greatest lower bound) $\boldsymbol{y}_0 \cap \boldsymbol{y}_1$ can be shown to not exist.

§3. An overview of the proofs.

Considering the theorem that every $\Sigma_1(\boldsymbol{x})$-presentable busl is isomorphic to $[\boldsymbol{0}, \boldsymbol{g}]$ for $\boldsymbol{g} < \boldsymbol{x}$ if $\boldsymbol{x} \geq \boldsymbol{0}''$, one has the Turing degrees $\mathcal{D}$ on one hand, and a countable usl L on the other, and it is desired to embed L into $\mathcal{D}$ as an initial segment. The first step is to break $\mathcal{L}$ up into countably many finite pieces. One way to do this would be to write $L = \cup_i L_i$ where each L_i is a finite busl substructure of L. However it is preferable to use a more general representation of L as a direct limit. That is, find finite busl substructures L_i of L and busl homomorphisms $\varphi_i : L_i \to L_{i+1}$ such that L is isomorphic to the disjoint union of the L_i modulo an equivalence relation $\approx$ generated by the relations $a \approx \varphi_i(a)$ for each $i < \omega$ and each $a \in L_i$. The advantage of using homomorphisms rather than embeddings (injective homomorphisms) is that if L is $\Sigma_1(\boldsymbol{x})$-presentable then a suitable sequence of L_i and φ_i for $i < \omega$ can be found recursively in $\boldsymbol{x}$.

To each L_i associate a countably infinite chain $\Theta_0(L_i) \subseteq \Theta_1(L_i) \subseteq \ldots$ of finite sets, where both the sequence and its union are referred to as $\Theta(L_i)$ or just Θ when no confusion is likely; note that this gives a doubly infinite array $\Theta^i_j = \Theta_j(L_i)$, $i, j < \omega$. On each Θ_j there are finite functions $f_k = f^{i,j}_k : \Theta_j \to \Theta_j$ for each $k \in L_i$, such that $f^{i,j+1}_k$ extends $f^{i,j}_k$ as a function. This is done in such a way that the associated equivalence relations given by $x \equiv_k y \leftrightarrow f_k(x) = f_k(y)$ have the property that $L \models k \leq m$ just in case $\equiv_k \supseteq \equiv_m$ (considering an equivalence relation as a set of pairs).

In the end a function $g : \omega \to \omega$ (with degree denoted in boldface, $\boldsymbol{g}$) is needed such that the interval $[\boldsymbol{0}, \boldsymbol{g}]$ in the Turing degrees is equal to $\{\boldsymbol{g}_k \mid k \in L\}$, where $g_k(x) = f_k(g(x))$. The way the functions f_k are set up, if $k \leq m$ in L then $g_k(x) = f_k(g_m(x))$. Hence in this case $g_k \leq_T g_m$, in fact $g_k \leq_{tt} g_m$. Note however that the reduction procedure using f_k is not a many-one reduction; this is no accident, as the many-one degrees form a distributive usl, and usls are not in general distributive.

Some version of the splitting tree technique of Spector seems to be a necessary tool for constructing initial segments of the Turing degrees. Hence g should lie on trees $T_0 \supseteq T_1 \supseteq \cdots$. These trees will be *trees for* L_i for various i, for example T_{2i} and T_{2i+1} can be trees for L_i for each $i < \omega$. The domain of a tree for L_i will be

$$S(\Theta(L_i)) = \{\sigma \in \omega^{<\omega} \mid \forall x < |\sigma| \, (\sigma(x) \in \Theta_x(L_i))\}$$

and the range will be contained in $S(\Theta(L_0))$. In order to have T_{i+1} be a subtree of T_i, $\Theta(L_{i+1})$ must be embedded into $\Theta(L_i)$ in a certain sense. Because L_{i+1} may have many more elements than L_i, it is necessary to use an increasing function m_i and embed $\Theta_j(L_{i+1})$ into $\Theta_{m_i(j)}(L_i)$.

The trees T_{2i+1} can be used to satisfy the requirements (1),(2),(3) above. The trees T_{2i+2} serve only the purpose of going from L_i to L_{i+1}. For simplicity assume that only the trees with even subscripts are needed, so T_{2i+2} is referred to as "T_{i+1}" in the following. First, the root $T_{i+1}(\emptyset)$ is taken to be any $T_i(\sigma)$ with $|\sigma| > m_i(0)$. The reason is that then $T_{i+1}(\langle x \rangle)$ for $x \in \Theta_0(L_{i+1})$ can be defined as a string extending $T_i(\sigma * \langle x \rangle)$. The embedding of $\Theta(L_{i+1})$ into $\Theta(L_i)$ can be considered to be an inclusion map. If $T_{i+1}(\langle x \rangle) = T_i(\sigma * \langle x \rangle^{m_i(1)-m_i(0)})$ then the process can be continued in the same way at the next level of T_{i+1}.

In the construction of initial segments of the Turing degrees a notion of homogeneity arises (see for example Lerman [12]). With assistance from Pavel Pudlák and Ralph McKenzie, a lattice theoretic way to understand this notion was discovered. The most natural notion of homogeneity is slightly different and weaker than the notion of *weak homogeneity* in Lerman's book, but is still sufficient for the needs of initial segment constructions once a small addition to such constructions is made. The lattice theoretical meaning of this new notion is that a set of equivalence relations which form a lattice under inclusion has the homogeneity property just in case the set of equivalence relations is the congruence lattice of an algebra (for definitions and properties of congruence lattices see [3]), a fundamental concept of universal algebra.

The new notion of homogeneity is the following.

DEFINITION 3.1. Θ is called homogeneous if the following condition obtains.

Suppose for each $k \in L$, $a \equiv_k b \Rightarrow c \equiv_k d$. Then there exist $z_1, \ldots, z_n$ and maps $f_i : \Theta \to \Theta$ preserving all the $\equiv_k$, such that

1. $\{f_0(a), f_0(b)\} = \{c, z_1\}$,
2. $\{f_i(a), f_i(b)\} = \{z_i, z_{i+1}\}$, and
3. $\{f_n(a), f_n(b)\} = \{z_n, d\}$.

Note that only equality of sets is required here, as opposed to equality of ordered pairs.

DEFINITION 3.2. If A is a set, f is a binary operation on A and E is an equivalence relation on A, then E is a *congruence relation* if whenever $x\,E\,x'$ and $y\,E\,y'$ then $f(x, y)\ E\ f(x', y')$.

The set of all congruence relations on a given set may be ordered by inclusion if an equivalence relation is considered as a set of pairs. The resulting partial order is always a lattice. A classic example: the lattice of normal subgroups of a group G is isomorphic to the congruence lattice of G. The isomorphism sends the subgroup H to the relation $x\,E\,y \Leftrightarrow xy^{-1} \in H$.

The following observation may be credited to Mal'cev [13], [14].

LEMMA 3.3. *The congruence lattice of an algebra is homogeneous.*

PROOF SKETCH: Suppose $a, b, c, d \in \Theta$ and $a \equiv_k b \to c \equiv_k d$ for each $k \in L$. Then $\langle c, d\rangle$ is in the equivalence relation E generated by $\langle a, b\rangle$ within Θ. But since Θ is a congruence lattice, E is identical to the congruence relation on Θ generated by $\langle a, b\rangle$. So $\langle c, d\rangle$ is in the equivalence relation generated by pairs $\langle f(a), f(b)\rangle$ for f homomorphism of Θ. ⊣

REMARK 3.4. The following is an open problem in lattice theory which has a bearing on initial segment constructions. Call a finite structure in a finite language with no relation symbols a *finite algebra*. Say that a finite lattice is CLFA if it is isomorphic to the congruence lattice of some finite algebra. Is every finite lattice CLFA? In fact, given $n \in \omega$, let M_n be the lattice which has an antichain of size n, a top and a bottom. It is not known whether or not every M_n is CLFA. The class CLFA includes the finite lattices that have a homogeneous lattice table in the sense of Lerman [12]. Lachlan [9] makes a claim that is equivalent to saying that every CLFA is an ideal in $\mathcal{D}$, but does not give the construction as Lerman [11] had already shown that every finite lattice is an ideal in $\mathcal{D}$, using an infinite representation Θ. With finite Θ, it seems CLFA is all one can get. Whether Θ is finite or infinite corresponds to whether the trees have branching rates bounded by a constant, or by a recursive function.

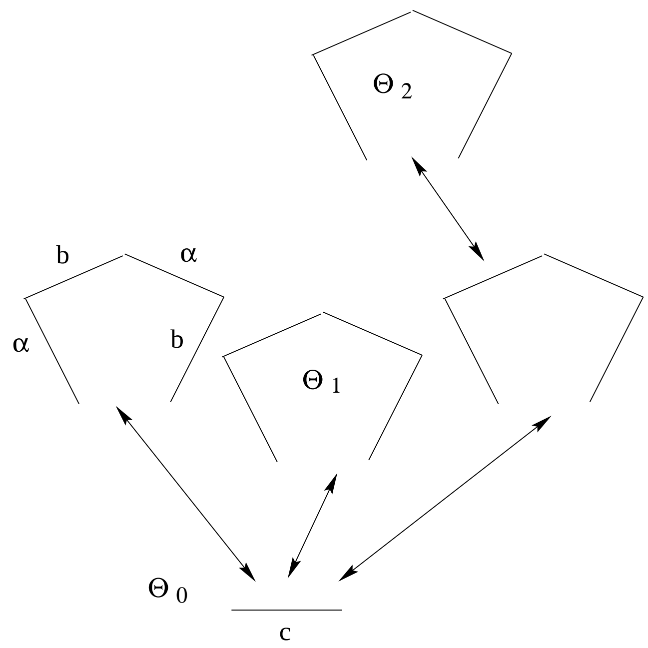

FIGURE 1. Pudlák's construction.

Once it is realized that the congruence lattice of an algebra is homogeneous, it is natural to consider Pudlák's paper [17] where a representation of a finite (or more generally, an algebraic) lattice as a congruence lattice is realized as a union of a chain $\Theta_0 \subseteq \Theta_1 \subseteq \cdots$. The question occurred to us whether Pudlák's construction also had the property that $\Theta(L_{i+1})$ could be embedded into $\Theta(L_i)$ in an appropriate sense. A slight modification of Pudlák's construction turned out to have this property. The resulting representation is significantly simpler than the one used by Lachlan and Lebeuf and in subsequent work. Instead of building all the properties needed of the representation into it by *brute force*, a representation is used which was created for a different and simpler purpose but which has these properties naturally. For an analogy, one can imagine a natural solution to Post's problem compared to the Friedberg-Muchnik solution.

Pudlák's construction of a representation $\Theta(L)$, where L is now a finite lattice, is that of a colored graph, where the colors are elements of L. In Θ_0 there is just a single edge labelled by an element of L other than 0. Then inductively, in Θ_{j+1}, for each edge of Θ_j labelled $a \in L$ there are four additional edges glued together so as to form a pentagon with edges labelled a, b, c, b, c, for any pair b, c such that $a \cup b \geq c$, as illustrated in Figure 1. Equivalence relations $\equiv_m$ for $m \in L$ are defined by: $x \equiv_m y$ if there is a path from x to y all of whose edges have colors $\leq m$.

To embed $\Theta(L_{i+1})$ into $\Theta(L_i)$, an apparent problem might be that the natural φ_i goes from L_i to L_{i+1} which is the opposite direction. There is a natural way to embed $\Theta(L_i)$ into $\Theta(L_{i+1})$; simply change the color of any edge labelled $a \in L_i$ to $\varphi_i(a)$. However, this embedding corresponds to the equivalence $m \leq k$ just in case $\equiv_m \subseteq \equiv_k$ for the Pudlák representation Θ and not just in case $\equiv_m \supseteq \equiv_k$. Instead the dual L_i^* of L_i should be considered, and $\Theta(L_i^*)$ used as representation of L_i. To embed $\Theta(L_{i+1}^*)$ into $\Theta(L_i^*)$, an usl homomorphism from L_{i+1}^* to L_i^* is needed, i.e., a map from L_{i+1} to L preserving meet $\cap$ and greatest element 1. Fortunately the following lemma is available.

LEMMA 3.5. *The category of finite usls L with usl homomorphisms φ is self-dual under the contravariant functor taking L to L^* and φ to φ^*, where φ is the* Galois adjoint [2] φ^* *of* φ *given by* $\varphi(a) \leq x \leftrightarrow a \leq \varphi^*(x)$.

The following is a partial reconstruction of the history of Pudlák's construction. Whitman [23] showed every lattice embeds in a partition lattice Jónsson [7] simplified Whitman's construction. Grätzer and Schmidt [4] characterized congruence lattices of algebras. Pudlák proved that Grätzer and Schmidt's proof could be simplified using a slight modification of Jónsson's construction.

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