THE BULLETIN OF SYMBOLIC LOGIC
Volume 8, Number 4, Dec. 2002

REVIEWS

This issue of the BULLETIN is the last one for which Herbert Enderton is the Coordinating Editor of the reviews. Beginning with the 2003 volume, Alasdair Urquhart will be the Managing Editor of the reviews. The other reviews editors are Geoffrey Hellman, Thomas Jech, Wolfram Pohlers, and Philip Scowcroft,

Authors and publishers are requested to send, for review, copies of books to the *Association for Symbolic Logic, Box 742, Vassar College, 124 Raymond Avenue, Poughkeepsie, NY 12604, USA*.

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The journal of symbolic logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 49***5*** or 280***1*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

MARK BALAGUER. ***Platonism and anti-platonism in mathematics.*** Oxford University Press, Oxford and New York 1998, x + 217 pp.

There are two major challenges for the philosophy of mathematics that present severe difficulties for any attempt to defend Platonism or anti-Platonism regarding mathematical objects. Defenders of Platonism are faced with Benacerraf's challenge, to explain how we can have knowledge of abstract mathematical objects in the absence of any causal link between the mathematical realm and the physical realm. In contrast, defenders of anti-Platonism, given that they cannot take mathematical statements *read at face value* to be true, must tackle the Quine–Putnam indispensability argument, which holds that the indispensability of mathematics to natural science implies that we are committed to the truth of mathematical claims.

What is remarkable about Mark Balaguer's ***Platonism and anti-platonism in mathematics*** is that he attempts to solve both of these problems, providing a version of Platonism (namely full-blooded Platonism, or FBP) that he claims to be immune to the Benacerraf challenge, as well as a version of anti-Platonism (namely fictionalism) that he defends against the Quine–Putnam argument. Furthermore, Balaguer argues, FBP (which holds that all logically possible mathematical objects exist) is the only defensible version of Platonism, and fictionalism is the only defensible version of anti-Platonism (since "realist" anti-Platonist

1079-8986/02/0804-0003/$5.50

views such as Mill's empiricism are indefensible, while anti-realist anti-Platonist positions must agree with fictionalism regarding the truth values of mathematical claims taken at face value).

Rather than claim victory for either FBP or fictionalism, though, Balaguer argues that we cannot find any non-question-begging argument to choose between the two. This is because FBP and fictionalism agree on everything they say about mathematical practice (e.g. they agree that, in mathematics, "all consistent purely mathematical theories are, from a metaphysical or ontological point of view, equally 'good' " (p. 152)), they agree on the truth conditions for mathematical statements (they both adopt a "realistic semantics for mathematese" (p. 153)), and they present the same account of the applicability of mathematics (taking mathematics to enter into scientific theories only as a descriptive aid). The only disagreement between the two views is over the existence of mathematical objects. But if this is the *only* point of disagreement (so that the disagreement over ontology does not lead to fundamental disagreements elsewhere), then Balaguer's claim is that we have no way of resolving this disagreement, since we have no way of accessing the (alleged) mathematical realm, and so no way of discovering whether such a realm does or does not exist. From this discussion, Balaguer draws the strong metaphysical conclusion that there is no fact of the matter regarding the existence of mathematical objects, since our usage fails to provide any possible-world-style truth conditions for the claim that there are objects that do not exist in space-time.

An immediate response to Balaguer's epistemic conclusion (that we cannot find any non-question-begging argument to choose between FBP and fictionalism) would be to employ Ockham's razor. The mere existence of an anti-Platonist alternative to FBP that can account for everything that FBP accounts for regarding our mathematical practice should be enough to force the rejection of FBP, since it is ontologically cavalier to believe in the existence of every possible mathematical object (as FBP counsels), when an alternative theory is available that assumes that none of these objects exist. Balaguer argues, however, Ockham's razor cannot be applied here, since Ockham's razor "comes into play only in adjudicating between two explanations of an agreed-upon collection of facts" (p. 145). In the case of FBP and fictionalism, the FBP-ist will argue that FBP explains more facts than fictionalism explains, since it accounts for such facts as the fact that $2 + 2 = 4$. The fictionalist, of course, will disagree with this evaluation, since according to fictionalism, it is not a fact that $2 + 2 = 4$ (since there are no mathematical objects to be related in this way), and so fictionalism explains all of the facts that need explaining.

Balaguer's presentation of this deadlock between fictionalism and FBP as resting on a disagreement over whether to consider mathematical statements as factual stems from his presentation of the two positions as both presenting a standard semantics for mathematical statements. Both FBP and fictionalism claim that for '$2 + 2 = 4$' to be true, there must exist abstract objects for the singular terms '2' and '4' of the mathematical statement to refer to. Since the fictionalist thinks there are no abstract objects, '$2 + 2 = 4$' is evaluated as false, when taken at face value, although the sense in which this statement expresses a truth is preserved in the claim that it is true in the story of arithmetic. The FBP-ist, however, thinks that every logically possible mathematical object exists, and so since '$2 + 2 = 4$' is consistent, there are objects about which it tells the truth.

There is a difficulty, however, with Balaguer's claim that the existence of every possible mathematical object is enough to guarantee the truth of mathematical statements taken at face value. Since mathematical statements include singular terms that purport to pick out unique objects, the FBP-ist has difficulties not with existence, but with uniqueness. Since the mathematical realm posited by FBP is a full one, Balaguer notes that in general there will be more than one collection of objects that satisfy the claims of a given mathematical theory. This non-uniqueness does not worry Balaguer, however, since he argues that, for '3 is prime' to be true, "it needs to be the case that (a) there is at least one standard model of arithmetic, and (b) '3 is prime' is true in all of the standard models of arithmetic" (p. 90). While this seems to be a reasonable way for the FBP-ist to deal with the non-uniqueness problem, taking mathematical statements to be claims about what is true in all standard models is surely not to take them at face value. The existence, in the FBP-ist's picture, of more than one standard model of arithmetic means that, if one does interpret the singular terms used in the statements of arithmetic at face value, the FBP-ist should agree with the fictionalist that these statements, taken literally, are false.

The real disagreement between FBP and fictionalism is thus not over the truth values of mathematical claims taken at face value. Their disagreement is rather regarding how to reinterpret these straightforward falsehoods so that there is some sense in which they can be said to be true. And here the fictionalist proposal (that such statements are true in the sense that they are true according to our mathematical stories) might well be seen to win out over the FBP-ist alternative, for reasons of ontological economy, since the FBP-ist must rely on the fictionalist's notion of a story to explain what counts as a standard model of our theories, while it is unclear what explanatory value is gained by the FBP-ist's additional claim that our stories should have at least one instantiation.

While Balaguer's strong epistemological and metaphysical conclusions might thus be a little hasty, ***Platonism and anti-platonism in mathematics*** is an impressive work. Balaguer presents forceful arguments for the viability of both FBP and fictionalism, and against the feasibility of any substantially different Platonist or anti-Platonist position. In the light of the challenges of Benacerraf and Quine, any defence of Platonism *or* anti-Platonism that answers these problems would be welcome. To provide plausible defences of both, as Balaguer has done, is an admirable achievement.

MARY LENG

St. John's College, Cambridge CB2 1TP, U.K. mcl33@cam.ac.uk.

TOMASZ PLACEK. ***Mathematical intuitionism and intersubjectivity. A critical exposition of arguments for intuitionism.*** Synthese library, vol. 279. Kluwer Academic Publishers, Dordrecht, Boston, and London, 1999, xii + 218 pp.

Dividing the twentieth century roughly into three parts, the author discusses the philosophical justification of intuitionism given by three prominent representatives of the intuitionist program: Brouwer, Heyting, and Dummett. He investigates their philosophical account in the light of the following three issues: (1) the often-heard accusation that since on the intuitionist view mathematics consists of mental constructions, intuitionism leads to solipsism and the incommunicability of mathematical proofs; (2) the justification of the need for revising classical logic and mathematics; and (3) the question whether it is possible to formulate a satisfactory general intuitionist theory of the meaning of mathematical statements.

The book contains an introduction, followed by three chapters devoted respectively to Brouwer, Heyting, and Dummett. The reviewer feels that it was indispensable to read the whole book carefully in order to understand fully and to appreciate the clear concluding chapter, in which the author succinctly summarizes his findings.

The chapter devoted to Brouwer is the longest of the book. This reflects the author's sympathy for Brouwer's philosophy of mathematics. In his view, Brouwer has produced key elements in the solution of two of the three issues under investigation. Brouwer's response to (1) is that the ideal mathematician should not be seen as an empirical subject but as a transcendental subject. By the nature of a transcendental subject, it is necessary that any two of them carry out mathematical constructions in essentially the same way. Brouwer also saw that the best argument for revising classical logic lies in the notion of choice sequence. (This central aspect of the author's argumentation is discussed at length in the review by Mark van Atten, ***History and philosophy of logic***, vol. 21 (2000), pp. 166–168.) Since choice sequences are always partially undetermined, they provide a strong argument for abandoning bivalence and (a version of) the law of excluded third. The author finds this argument significantly stronger than the argument based on hitherto unresolved and perhaps unresolvable mathematical problems, which he regards as in the final analysis indecisive (pp. 120–121). With respect to (3), Brouwer's theory is not as convincing. The author argues that Brouwer advocated a directive theory of meaning, according to which expressions do not refer but merely serve as a means for a speaker (writer) to impose his goals on his listeners (readers). But this theory of meaning was never fully developed by Brouwer.

In contrast to Brouwer, Heyting does not fully escape (1), since he identifies the ideal mathematician with an (idealized) psychological subject. It seems harder to justify philosophically that psychological subjects must carry out mathematical constructions in essentially the same way. Heyting's answer to (2) is substantially the same as that of Brouwer. Heyting makes a distinctive contribution to (3) by means of his proof interpretation of intuitionist logic. This proof interpretation was implicit in Brouwer's work (e.g. in his argument against bivalence based on unresolved problems) but was also inspired by and is related to phenomenological views of Husserl and Becker on meaning. Nevertheless, Heyting, like Brouwer, never worked out a full-fledged intuitionist theory of mathematical meaning. This is the task that Dummett set himself, by exploring the constraints that a general theory of meaning should meet. As a result of these constraints, Dummett argues that a need for revision (i.e., an answer to (2)) follows from these general considerations. The author finds Dummett's arguments to this effect unconvincing, however.

The book as a whole is a sustained attempt to find out how the best philosophical case for intuitionism can be made, and whether this case is ultimately convincing. Technical results are explained and appealed to only insofar as they are directly relevant to the philosophical considerations at issue. The author is exceptionally open-minded and intellectually honest about the issues under investigation. He emphasizes the strengths of the philosophical arguments of the intuitionists, but is by no means blind to the junctures where the intuitionist program is vulnerable. This allows the author to arrive at a fair and balanced judgment about the strength of the intuitionist position today. Without hiding his sympathy for the intuitionist case, he arrives at the conclusion that the debate over the intuitionist program is presently still essentially unresolved.

For these reasons, the book deserves to be classified as one of the most valuable recent contributions to the debate between classical and intuitionist mathematics. Nevertheless, this review ends with two critical remarks.

First, the author at times expresses himself in too loose a fashion. To concentrate on one example, the author writes: "Had Brouwer not noticed that a source of paradoxes of Cantor's set theory lies in the unrestricted comprehension axiom, which is the heart of Cantor's method of introducing mathematical objects . . . , he would not have been a mathematical genius" (p. 40). Now it is common knowledge in Cantor scholarship that the founder of transfinite set theory was acutely aware of the threat of the paradoxes and did not use the unrestricted comprehension principle. Moreover, more than a century after its creation, no one has found a contradiction in Cantor's theory. The quoted passage seems flatly to contradict this. When one reads the whole book (especially p. 196), however, one is strongly inclined to conclude charitably that the author does not mean what he literally says here. Nevertheless, he should have expressed himself more carefully.

The second complaint is more substantial. Heyting held that classical mathematics needs a (Platonist) metaphysics to support it, whilst intuitionism remains philosophically neutral (p. 104–106). In the course of the book under review, it becomes clear that the second part of this assertion is incorrect—although the author might have highlighted this point more than he did. It was mentioned before that in order to escape the threat of incommunicability of mathematics, the transcendental subject is brought to the fore (p. 43). Moreover, in order to justify the concept of choice sequence, the concept of radical human freedom is invoked (p. 126). These concepts can hardly be regarded as philosophically neutral or even unmetaphysical, and the author is aware of this. The author does not indicate, however, that the first part of Heyting's assertion is also debatable. This is due to the fact that, in the reviewer's opinion, the author is insufficiently sensitive to the pluriformity of the (classical) opposition. He is too exclusively focused on the prototypical Cantorian set theorist as the defender of classical mathematics. This is witnessed, for example, in the following passage: "The upshot . . . is that so long as the classical mathematician abstains from invoking Platonist assumption[s], he has no convincing arguments against the intuitionist's challenge to bivalence and to some forms of reasoning of classical mathematics" (p. 122). From the context in which this quotation appears, it is clear that the Platonist assumptions that the author has in mind here include not only the subject-independent existence of mathematical objects, but also the existence of actually infinite, even non-denumerable sets such as the Platonist continuum (p. 122). But it is not clear that a defender of classical mathematics must subscribe to these strong metaphysical claims. For example, there is the predicativist program which goes back at least to (the pre-intuitionist) Hermann Weyl and which is in our days vigorously pursued by Feferman, Friedman, and Simpson, among others. It has been very successful in proof-theoretically reducing much of classical mainstream mathematics to a classical arithmetical basis. This arithmetical basis can be interpreted as asserting merely the existence of the natural numbers as a potentially infinite collection, which can in turn be taken, if one so desires and is speculatively inclined, to be generated by exactly the process of two-ity from which all mathematical objects are constructed according to Brouwer. Now Brouwer and Heyting could still appeal to the argument based on unresolved mathematical problems to protest that classical arithmetic cannot be interpreted as being about the natural numbers conceived as a *potential* infinity. But such a move would accord ill with a main thesis of the book under review that was alluded to earlier, i.e., that the argument based on choice sequences rather than the argument based on unresolved problems is decisive in the question of the validity of the law of bivalence.

LEON HORSTEN

Center for Logic, Institute for Philosophy, University of Leuven, Kardinaal Mercierplein 2, B-3000 Leuven, Belgium. leon.horsten@hiw.kuleuven.ac.be.

GRIGORI MINTS. ***A short introduction to intuitionistic logic.*** The university series in mathematics. Kluwer Academic/Plenum Publishers, New York etc. 2000, ix + 131 pp.

The book successfully aims at being short, but still contains some pearls of the subject. It leads up to rather interesting results, including, for example, a proof of (a special case of) the coherence theorem. Part of the reason why the author managed to keep the text short is that he has given new or at least modified proofs of several well-known theorems. To achieve all this, the author was able to make use of decades of experience as a leading researcher in intuitionistic logic and proof theory. The emphasis of the text, and also the criterion on what to select, are possible applications in computer science.

To make the material more accessible, the basic techniques are presented for propositional logic first; extensions and modifications necessary to cover also predicate logic are treated in a (rather short) second part of the book.

Unfortunately, the usefulness of the book is somewhat hampered by the fact that almost all cross-references are only approximately right; somehow there must have been a problem with the automated referencing of the text processing system. Another minus is that the author only very occasionally gives historical notes. Generally the reader is referred to other texts for such information. A particularly unfortunate misprint is the omission of the cut formula in the left premiss of the cut rule on page 55.

After an extremely short (one-page) Chapter 1 consisting of notational conventions, Chapter 2 presents Gentzen's natural deduction system NJp for intuitionistic propositional logic.

The exposition here is rather standard; emphasis is given to what the author calls the ADC-method (for "analysis and direct chaining"). The final two pages of the chapter contain a very nice and short proof of the derivability of $\neg\neg\tau$ in NJp for every tautology τ; the proof is by means of a cleverly chosen auxiliary statement, which can be proved inductively. In another short (two-page) Chapter 3, the Gödel–Gentzen negative translation is treated, together with an again rather nice proof of Glivenko's theorem. Chapter 4 presents the Brouwer–Heyting–Kolmogorov interpretation of intuitionistic propositional logic, in the $\to, \wedge, \vee$-language. The Curry–Howard isomorphism is described in detail. Chapter 5 shows that it preserves β-equality, and presents Turing's short proof of (weak) normalization. The usual consequences from the existence of β-normal forms are drawn, e.g. properties of the main branch of a derivation, and as corollaries simple forms of the disjunction property and Harrop's theorem are obtained. The non-derivability of $p \vee \neg p$ and of $\neg\neg p \to p$ are proved as consequences. Chapter 6 deals with the coherence theorem, a subject one clearly expects in a text written by this author. First the subformula property is proved, where polarities are taken into account. Then—since this is required by the intended application to category theory—η-conversion is treated briefly, and the existence of $\beta\eta$-normal forms is proved, simply by first β- and then η-normalizing. One has only to observe that η-conversions preserve β-normal forms. The coherence theorem (i.e., that every balanced sequence has a unique deduction up to $\beta\eta$-equivalence) is proved in Section 6.3, for simplicity for the implicational fragment only. Its possible extension to products and hence its relation to Cartesian closed categories is hinted at, with appropriate references to the literature. Chapter 7 presents Kripke models, and illustrates the use of frame conditions to characterize superintuitionistic logics. Chapter 8 introduces a multi-conclusion propositional system LJpm equivalent to NJp, that is more suitable for proof search. Following Fitting's well-known text (***Intuitionistic logic model theory and forcing***, JSL XXXVI 166), soundness and completeness for LJpm is proved. Algebraic models (based on pseudo-Boolean or Heyting algebras) are introduced, and soundness and completeness with respect to them is proved. Moreover, Jaskowski's original completeness proof by means of the Lindenbaum algebra is mentioned (in the form of an exercise). A topological completeness proof is presented in detail in Chapter 9. Chapter 10 deals with proof search for a tableau system, again yielding a soundness and completeness proof and in addition decidability of intuitionistic propositional logic. Chapter 11 introduces Gentzen's system LJp. After proving its equivalence with LJpm by the obvious method, a more useful translation of LJpm into LJp is given. It makes use of permutations of inferences along the lines of Kleene, and preserves the structure of derivation much better. The short Chapter 12 derives (a simple form of) the interpolation theorem from the existence of normal derivations, with Beth's definability theorem as a corollary.

Part II extends in Chapters 13, 14, and 15 the material from Chapters 2–5, 7, and 8 to first-order predicate logic. In the final Chapter 16 the problem of proof search is considered. Rather than extending the approach for LJpm carried out in Chapter 11 (which would lead to technical difficulties), a technique due to Kripke is used instead.

To summarize, this text is a very welcome addition to the proof theory literature. It provides a short and well-balanced introduction to the subject, aimed at students with a somewhat moderate background in mathematical logic and mathematics generally, but with some interest in intuitionistic logic, proof theory, and its applications in and relations to computer science. For those with a deeper interest, the introduction provides well-selected hints on what to read next. I can highly recommend it as a text for introductory courses in the subject.

HELMUT SCHWICHTENBERG

Mathematisches Institut der Universität München, Theresienstr. 39, D-80333 München, Germany. schwicht@mathematik.uni-muenchen.de.

FRANK O. WAGNER. ***Simple theories.*** Mathematics and its applications, vol. 503. Kluwer Academic Publishers, Dordrecht, Boston, and London, 2000, xi + 260 pp.

It is a brave act to write a book in an emerging field. The book under consideration here, ***Simple theories*** by Frank Wagner, is precisely that. In some sense, it is not accurate to call the study of simple theories an emerging area since it has been emerging for about thirty years. In 1980, Shelah wrote the paper *Simple unstable theories* (***Annals of mathematical logic***, vol. 19 (1980), pp. 177–203) in which the definition of simple theories appears, but in his 1978 book ***Classification theory and the number of non-isomorphic models*** (JSL XLVII 694), he had already begun the abstract study of dividing and forking. He also introduced a notion of rank called degree or D-rank; if D-rank is ordinal-valued in a theory it is now known as a supersimple theory. Although the next study of simple theories for their own sake after Shelah's *Simple unstable theories* was B. Kim's 1996 dissertation (University of Notre Dame), the generalization of stability theory to unstable settings such as smoothly approximable structures and pseudo-finite fields had progressed throughout the intervening years. Nevertheless, with Kim's proof of symmetry of dividing in simple theories, interest in the field grew enormously, and publishing a book meant to bring the reader to the forefront of research in the area only four years later is quite courageous.

So what is simplicity theory? At its most general, it is the study of the combinatorial notion of dividing, and simple theories are exactly those in which dividing is well behaved. What makes this project somewhat less artificial is the fact that dividing in a simple theory provides a notion of dependence and dimension which, in particular examples, is often best thought of as a generalization of algebraic dependence and dimension from algebraically closed fields. With the success of applications of stability theory in recent years (for instance, Hrushovski, *The Mordell–Lang conjecture for function fields*, JSL LXIII 744), the hope for simple theories lies in the potential for broader applications by keeping a firm handle on some of the motivating examples.

The first chapter of ***Simple theories*** contains a brief introduction, indicates what a reader must know before venturing further, and has a list of examples. Each chapter, including the first, ends with extensive bibliographic remarks. These are quite helpful for those who wish to delve deeper into a subject or who want to look at the original treatment. The book proper starts in Chapter 2 where the book's main topic, simplicity, is introduced. In any subject, there is a core of fundamentals that anyone who wants to work in an area must know and in the study of simple theories, these fundamentals revolve around the notion of dividing. A type $p(x, B)$ over a small parameter set B inside a saturated model M is said not to divide over some subset A of M if for all A-indiscernible sequences $\langle B_i : i \in I\rangle$ where $\mathrm{tp}(B_i) = \mathrm{tp}(B)$ for all $i \in I$, the conjugates of p, $\{p(x, B_i) : i \in I\}$, are mutually consistent. The author defines a simple theory to be one in which the notion of dividing is symmetric, i.e., $\mathrm{tp}(a/b)$ does not divide over c iff $\mathrm{tp}(b/a)$ does not divide over c. Other equivalent definitions have been used; Shelah's original definition involved the finiteness of certain ranks, the $D(-, \varphi, k)$-ranks. Requiring that dividing, or a derived notion, forking, have local character, i.e., every type $p(x, B)$ does not divide (fork) over a set $A \subseteq B$ of size at most $|T|$, is also an equivalent definition found in the literature. It seems now that requiring forking to have local character provides the smoothest path through the basics but nonetheless, all the material in Chapter 2 must be internalized by anyone working in the field. Two of the most practical tools are highlighted with their own sections: the independence theorem and a criterion for recognizing when a theory is simple in terms of the presence of an abstract dependence relation. The chapter concludes with a section on stable theories. One of the philosophical positions that the author takes is that stability theory should be seen as a particular instance of simplicity. Although this is literally true—stable theories are particular simple theories—this position hides some of the motivation for many of the constructs in simple theories: they worked well in stable theories and so, attempting to generalize them to simple theories may and does prove useful.

Chapter 3 addresses one of the most novel aspects of simple theories. In the development of stable theories, the role of the canonical base was crucial. If we work in a saturated model M of a stable theory then, roughly speaking, one says that two stationary types are parallel if they have a common non-forking extension. The canonical base of a stationary type acts as a code for its parallelism class, and it is instructive to note that in stable theories, the canonical base is most naturally found in M^{eq}, the expansion of the universe by sorts for equivalence classes of definable equivalence relations. In simple theories, canonical bases exist for the appropriate analogue of parallelism class but one must include objects known as hyperimaginaries: quotients of possibly infinite sequences by type-definable equivalence relations. Chapter 3 is devoted to the development of the logic associated with hyperimaginaries, including the notion of dividing for hyperimaginaries, and a proof that canonical bases exist in simple theories as hyperimaginaries. One lingering open question in simple theories is whether hyperimaginaries are, in fact, needed. A theory eliminates hyperimaginaries if each hyperimaginary is interdefinable with a sequence of imaginaries. It is not known if simple theories eliminate hyperimaginaries.

Chapter 4 is devoted to the study of groups appearing in simple theories. One of the lessons from geometric stability theory is to pay attention to the groups that can appear in a structure. The author includes much of what was known to date in this chapter but here I have a small complaint. One of the motivations for studying groups in simple theories is the attempt to generalize the binding group theorem and the group configuration theorem from stable theories. Although the author refers to this goal in the introduction and the preface—where he notes that these conjectures remain open in the simple case—he does not tie together the pieces found in Chapter 4 to help the reader understand their possible impact. For example, one section discusses generically given hyperdefinable groups without noting their relevance to the generalization of the group configuration theorem, though the author almost certainly believes that the former provide the key to the latter.

The final two chapters of the book deal with somewhat more advanced topics. Chapter 5 is devoted to supersimple theories, those theories in which all types do not divide over a finite subset of their domain. In these theories, dimension-carrying types are ubiquitous; that is, every type is not orthogonal to a type for which the set of realizations, together with dividing, forms a pre-geometry, and so it makes sense to talk of the dimension of a set. This chapter contains the very challenging proof that supersimple theories eliminate hyperimaginaries. As mentioned above, the problem whether this is true for an arbitrary simple theory is still open. Chapter 6 is devoted to three separate topics, each interesting in its own right. The first section deals with several open problems that have positive answers when the simple theory has only countably many types over the empty set. In the second section, attention is given to ω-categorical theories. Most interesting is a very concise section on Hrushovski's amalgamation construction. A key example for the field, the existence of an SU-rank-1, non-1-based theory, is given as an example in this section without much fanfare. The final section deals with means by which other simple theories can be created from known ones; these include the addition of a generic predicate and a generic automorphism.

This book is very good for active researchers who work in simple theories as well as advanced graduate students who want a good source for all the fundamentals. Someone who is coming to the field for the first time would be best to enlist the help of a guide. Barring that, a few additional sources might be called for: keep a copy of Pillay's ***Geometric stability theory*** (JSL LXIII 1190) handy to compare analogous situations in stable theories and take seriously the list of examples (and references) found in Section 1.3 of this book—it is important to keep them in mind while working through many of the details. The author keeps errata and addenda on his website www.desargues.univ-lyon1.fr/home/wagner; it contains several small corrections to the book but more importantly, the current status of many of the open problems in the book can be found there.

BRADD HART

Department of Mathematical Sciences, McMaster University, Hamilton, ON L8S 4K1, Canada. hartb@mcmaster.ca.

M. HOLZ, K. STEFFENS, and E. WEITZ. ***Introduction to cardinal arithmetic.*** Birkhäuser advanced texts. Birkhäuser Verlag, Basel, Boston, and Berlin, 1999, vii + 304 pp.

The book under review, henceforth referred to as HSW, is the first introductory-level text to have been written since the development of Shelah's pcf theory that includes an introduction to that theory. The authors aim their text at beginners in set theory. They start literally from the axioms and prove everything they need. The result is an extremely useful text and reference book which is also very pleasant to read. Approximately half the text, occupying seven of the nine chapters, is devoted to pcf theory. Pcf theory has had such an impact on the study of cardinal arithmetic that a text on the subject must now cover it. This is a tall order: the theory continues to evolve and is expounded in numerous substantial works of Shelah. Shelah presented some of the theory, as it was up to 1989, in ***Cardinal arithmetic*** (JSL LXII 1035), henceforth referred to simply as ***Cardinal arithmetic***. The pcf theory in HSW does not go beyond what is covered in ***Cardinal arithmetic***. Various people have presented portions of this theory, elaborating on and occasionally simplifying Shelah's arguments, in the form of published or unpublished papers. Much of Chapters 3–7 of HSW is adapted from the dissertation of E. Weitz, *Untersuchungen über die Grundlagen der pcf-Theorie von Saharon Shelah* (Hannover, 1996). The authors have also drawn some of their material on pcf theory from the paper by the reviewer and Menachem Magidor, *Shelah's pcf theory and its applications* (BSL VIII 307) and papers by Thomas Jech, *Singular cardinal problem: Shelah's theorem on* $2^{\aleph_\omega}$ (BSL VIII 308) and *A variation on a theorem of Galvin and Hajnal* (***Bulletin of the London Mathematical Society***, vol. 25 (1993), pp. 97–103). Among the many useful contributions made in HSW toward rendering Shelah's arguments more accessible is an extended discussion, in the final chapter, of Shelah's pp function and its relationship to the Galvin–Hajnal theorems. The first two-thirds of the text (four chapters) contains many exercises of varying levels of difficulty. There are relatively few exercises in the remaining sections on the more advanced pcf theory. The main weakness that the reviewer finds in the book is the scarce commentary on the limitations of the theorems. For example, Chapter 7 deals with sets of regular cardinals a such that $|\mathsf{a}| < \min(\mathsf{a})$ and $\mathrm{pcf}(\mathsf{a})$ has no regular limit points. It does not, however, mention that it is unknown whether the second condition can ever fail in the presence of the first. When it is shown that for any interval of regular cardinals a such that $|\mathsf{a}| < \min(\mathsf{a})$ we have $|\mathrm{pcf}(\mathsf{a})| \leqq |\mathsf{a}|^{+3}$, no comment is made about the unusual exponent.

The recent history of cardinal arithmetic (mainly attempts to prove bounds on cardinal exponentiation) has been well described in this journal by Thomas Jech in the article *Singular cardinals and the pcf theory*, ***Bulletin of symbolic logic***, vol. 1 (1995), pp. 408–424. Very little will be said here about this historical context and the reader is referred to Jech's article for more details. (Note: Theorems 9.5 and 9.6 of that article state more than what is known at present. See the description of Chapters 5–7 below for the correct statements.) It should also be mentioned that Shelah maintains a useful "*Analytical guide*" to his books and papers. A version of it occurs at the end of ***Cardinal arithmetic*** (pp. 435–460) and updates are posted occasionally to Shelah's archive at Rutgers (www.math.rutgers.edu/~shelah/all/E12.pdf).

Let us look more carefully at each of the chapters of the book.

The book assumes little knowledge of set theory. It works in Zermelo–Fraenkel set theory with the axiom of choice (ZFC). A few results are proved without the axiom of choice, and the axiom of choice is shown in ZF to be equivalent to the well-ordering principle and to Zorn's lemma, but otherwise the axiom of choice is assumed throughout. As mentioned above, the text begins by describing the axioms of ZFC and develops from scratch all the elementary set theory it needs, including the theory of ordinals and cardinals. This basic material is contained in the first chapter, which also contains the classical results of König, Hausdorff, and Tarski concerning infinite sums and products of cardinals, and sections on the singular cardinals problem, normal ideals (Fodor's lemma and Solovay's theorem on partitioning stationary subsets of a regular cardinal), and partition calculus (Ramsey's theorem and the Erdös–Rado theorem). This very substantial chapter occupies a third of the book.

The second chapter contains results on obtaining bounds for κ^λ in terms of the values of κ^μ for $\mu < \lambda$ when λ is a singular cardinal of uncountable cofinality. This material is essentially the content of the article by Fred Galvin and András Hajnal, *Inequalities for cardinal powers* (JSL XLVI 864), with the exception of some estimates due to Shelah of Galvin–Hajnal norms. It includes Silver's theorem on GCH at singular cardinals of uncountable cofinality, and the result that $2^{\aleph_\eta} < \aleph_{(2^{|\eta|})^+}$ when $\aleph_\eta$ is a strong limit cardinal of uncountable cofinality.

The rest of the book, the third chapter through to the ninth, develops Shelah's pcf theory. This theory deals with the possible cofinalities of reduced products $(\prod \mathsf{a}/D, <_D)$ where a is a set of regular cardinals, D is an ultrafilter on a and the order is given by $f <_D g$ if $\{\alpha \in \mathsf{a} : f(\alpha) < g(\alpha)\} \in D$. The assumption $|\mathsf{a}| < \min(\mathsf{a})$ is essential for most of the results. The authors call a set satisfying this assumption *progressive*. (Curiously, given the importance of the concept, they seem to be the first to have assigned it a name.) The basic results of pcf theory depend on understanding when a sequence $\langle f_\alpha : \alpha < \delta\rangle$ of ordinal functions, all defined on a common domain A, has a least upper bound for the order $<_I$, where I is an ideal on A and $f <_I g$ means $\{a \in A : f(a) \geqq g(a)\} \in I$. The relevant results and their elementary consequences are given in the third chapter.

The fourth chapter develops enough model theory to be able to handle the elementary submodel arguments needed in the rest of the text. As was the case for the basic set theory in the first chapter, the reader is not assumed to have any previous knowledge of the material. The existence of certain club-guessing sequences is shown as well.

In Chapter 5, it is shown that for each λ, the ideal $J_{<\lambda^+}(\mathsf{a})$ is generated over $J_{<\lambda}(\mathsf{a})$ by adding a single set b_λ. ($J_{<\lambda}(\mathsf{a})$ is the ideal of all sets $\mathsf{b} \subseteq \mathsf{a}$ such that $\mathrm{cf} \prod \mathsf{a}/D < \lambda$ whenever D is an ultrafilter on a and $\mathsf{b} \in D$.) The smoothness property (also called transitivity) of the generators, i.e., the property that $\mu \in b_\lambda$ implies $b_\mu \subseteq b_\lambda$ (which, for example, can be arranged for $\langle b_\lambda : \lambda \in N \cap \mathrm{pcf}(\mathsf{a})\rangle$ where N is an elementary submodel of some $H(\Theta)$, $\mathsf{a} \subseteq N$, and $|N| \leqq \min(\mathsf{a})$—see the final section of S. Shelah, *Further cardinal arithmetic*, ***Israel journal of mathematics***, vol. 95 (1996) pp. 61–114), is not proved. This useful property is sometimes used to prove some of the basic properties of the pcf function, but the authors manage without it.

In Chapter 6, the following case of the localization theorem from Chapter 7 is proved: there is no sequence $\langle \kappa_i : i < |\mathsf{a}|^+\rangle$ in $\mathrm{pcf}(\mathsf{a})$ such that for each $i < |\mathsf{a}|^+$, $\max \mathrm{pcf}\{\kappa_j : j < i\} < \kappa_i$. The notion of a control function is introduced and results on "nice" models are given.

Chapter 7 deals with the question whether $\mathrm{pcf}(\mathrm{pcf}(\mathsf{a})) = \mathrm{pcf}(\mathsf{a})$. It is not known whether this holds for every progressive set a of regular cardinals, but there are partial results. For example, it holds if $\mathrm{pcf}(\mathsf{a})$ has no regular limit points, in particular if $|\mathrm{pcf}(\mathsf{a})| < \min(\mathsf{a})$. The case of the localization theorem established in the previous chapter is used to establish a more general form of the theorem: if a is a progressive set of regular cardinals, $\mathsf{b} \subseteq \mathrm{pcf}(\mathsf{a})$ has no regular limit points, and $\lambda \in \mathrm{pcf}(\mathsf{b})$, then there is a set $\mathsf{c} \subseteq \mathsf{b}$ such that $|\mathsf{c}| \leqq |\mathsf{a}|$ and $\lambda \in \mathrm{pcf}(\mathsf{c})$. Under the same assumption on a, it is also shown that if $\mathsf{c} \subseteq \mathrm{pcf}(\mathsf{a})$ is an interval of regular cardinals (i.e., the set of *all* regular cardinals between two ordinals), then $|\mathsf{c}| \leqq |\mathsf{a}|^{+3}$.

Chapter 8 contains an application of pcf theory to the construction of Jónsson algebras and applications by Todorcevic of pcf theory to partition calculus. The notion of a

Jónsson algebra is explained in detail. There is also an application to the calculation of the cofinality of partial orders of the form $([\lambda]^\kappa, \subseteq)$, where $[\lambda]^\kappa$ is the set of all subsets of λ that have cardinality κ. Specifically, it is shown that for any infinite cardinal κ and any non-zero $\alpha < \kappa$, $\mathrm{cf}([\kappa^{+\alpha}]^\kappa, \subseteq) = \max \mathrm{pcf}\{\kappa^{+(\beta+1)} : \beta < \alpha\}$. This is one example from a rich collection of theorems in ***Cardinal arithmetic*** and subsequent works of Shelah concerning the relationship between covering numbers and pcf's. These results are quite useful in cardinal arithmetic and related areas. In addition to Shelah's original works, a good source for further theorems relating cofinalities and pcf's is an elegant recent article on pcf theory by Uri Abraham and Menachem Magidor, *Cardinal arithmetic*, which is available at www.cs.bgu.ac.il/~abraham/math.html and which will appear in the ***Handbook of set theory***, to be published by Kluwer.

Chapter 9 covers Shelah's pp function. For singular cardinals λ, $\mathrm{pp}(\lambda)$ is the supremum of all cardinals of the form $\mathrm{cf}(\prod \mathsf{a}/D)$, where a is a cofinal set of regular cardinals in λ, a has cardinality $\mathrm{cf}(\lambda)$, and D is an ultrafilter on a that contains all final segments of a. In ***Cardinal arithmetic***, Shelah proposes that $\mathrm{pp}(\lambda)$ is the "true" $\lambda^{\mathrm{cf}\lambda}$. One advantage it has over $\lambda^{\mathrm{cf}\lambda}$ is that its value is much harder to change by forcing. For example, the results of Chapter 7 imply that $\mathrm{pp}(\aleph_\omega) < \aleph_{\omega_4}$. Versions of the Galvin–Hajnal theorems from Chapter 2 are established for the pp function.

A couple of the proofs in the text contain minor gaps. (1) On page 109, the proof of Lemma 2.2.1 (the case of the free set lemma where the domain is a set A of regular cardinality κ and the set mapping K satisfies $|K(a)| < \lambda < \kappa$ for all $a \in A$) seems to go wrong at the end. The $\subseteq$-maximal set $\mathcal{P}$ of pairwise disjoint popular sets at the end of the proof could be simply $\{A\}$ (if each $K(a)$ is non-empty). This is easily repaired, but the entire argument can be worded more succinctly: denoting $X \cup \bigcup\{K(x) : x \in X\}$ by $H[X]$, for $\xi < \lambda$, inductively choose a maximal independent subset X_ξ of $A \setminus \bigcup_{\eta<\xi} H[X_\eta]$. Then $|X_\xi| = \kappa$ for some $\xi < \lambda$, for otherwise $\bigcup_{\xi<\lambda} H[X_\xi]$ has size $< \kappa$ and for any $a \in A$ not in this set the maximality of each X_ξ and the fact that $a \notin K(x)$ for any $x \in X_\xi$ gives $K(a) \cap X_\xi \neq \emptyset$ for all $\xi < \lambda$, contradicting $|K(a)| < \lambda$. (2) The proof of Theorem 9.2.4, on page 277, introduces a club-guessing sequence for $S = \Sigma_{\mu,\aleph_0}$, using Lemma 4.5.3 to justify its existence. But Lemma 4.5.3 assumes that S is contained in a set of the form $\Sigma_{\kappa,\mu}$ where μ is uncountable. The proof of Lemma 4.5.3 for $\mu = \aleph_0$ is similar but somewhat harder. In ***Cardinal arithmetic*** the required result is Claim 2.3(2) of Chapter III. It can also be found in a useful preprint of Menachem Kojman entitled *The A,B,C of pcf: a companion to pcf theory, part I*, available at the time of writing of this review at www.cs.bgu.ac.il/~kojman. This preprint also contains several other club-guessing principles and develops the basic results of pcf theory using the coherent sequences that are discussed in ***Cardinal arithmetic*** in §0 of the *Analytical guide* on page 436.

MAXIM R. BURKE

Department of Mathematics and Computer Science, University of Prince Edward Island, Charlottetown PE, C1A 4P3, Canada. burke@upei.ca.

STEVO TODORCEVIC. ***Topics in topology.*** Lecture notes in mathematics, vol. 1652. Springer, Berlin, Heidelberg, New York, etc., 1997, viii + 153 pp.

Faithfully to its title, this book is an author's personal selection of several diverse topics in topology. It grew out of a set of lecture notes from a graduate course given by Todorcevic at the University of Toronto during the fall of 1995. The book consists of four independent chapters, each covering a different "topic" in general topology.

The first chapter is the longest and most extensive one. It is entitled *Compact sets in function spaces* and it considers $C_p(X)$, the space of continuous functions with respect to the topology of pointwise convergence on a Polish space X. The goal of Chapter I is to give a complete proof of a dichotomy result of Bourgain–Fremlin–Talagrand about sequences in $C_p(X)$. Each section of the first chapter is a building block for a proof of this theorem, interesting in its own right. Eberlein's theorem on sequentiality of $C_p(X)$ is given in §2, and Namioka's theorem on separate continuity appears in §4. Pták's criterion for the existence of convex means giving a small measure to each set in some prescribed family of finite subsets of $\mathbb{N}$ is used to give a proof of Mazur's theorem about closed convex hulls of continuous functions in §3. Rosenthal's l_1-dichotomy that both predated and inspired the Bourgain–Fremlin–Talagrand theorem appears in §5.

Besides the topological results on the structure of $C_p(X)$, a significant portion of Chapter I is devoted to combinatorics. Infinitely-dimensional Ramsey theory is formally introduced in §6, but some ideas leading to it, in particular lemmas of Rosenthal and of Nash-Williams and Galvin, appear already in §5. Local infinitely dimensional Ramsey theory with respect to selective coideals (or "happy families," as originally named by Mathias) is given in §9. The heart of Chapter I is in §12. To a sequence $\{f_n : n \leqq \omega\}$ of continuous functions in $C_p(X)$, Todorcevic assigns the ideal $\mathcal{I} = \{A \subseteq \omega : f_\infty \notin \overline{\{f_n : n \in A\}}\}$. This ideal is both analytic and selective, and hence susceptible to Mathias-style combinatorics. It seems to be the first example of a selective ideal naturally occurring outside of Ramsey theory. Chapter I concludes with §13 which contains a proof of the Bourgain–Fremlin–Talagrand theorem.

Of particular interest to set theorists may be the fact that the combinatorial approach together with forcing and absoluteness methods was utilized further in Todorčević, *Compact subsets of the first Baire class* (***Journal of the American Mathematical Society***, vol. 12 (1999), pp. 1179–1212), which gives a classification of separable compact sets of Baire class-1 functions on a Polish space (so-called Rosenthal compacta). The results of this paper were obtained during the preparation of the book under review.

While in Chapter I combinatorial methods play a prominent role in a proof of a topological result, in Chapter II the situation is reversed. Some elegant ideas from topological dynamics are employed to give short and transparent proofs of some of the key results from Ramsey theory, such as Hindman's and van der Waerden's theorems. The eighteen pages of Chapter II provide a good self-contained introduction to this well-developed and lively subject. A reader interested in pursuing it further is referred to an excellent and comprehensive book by N. Hindman and D. Strauss, ***Algebra in the Stone–Čech compactification***, Walter de Gruyter, 1998.

Chapter III is the second longest, and it deals with compact and compactly generated groups, as studied by the Russian general topology school. As in Chapter I, the reader is given a tour of selected topics. §18 gives a fresh proof of Tkachenko's theorem that all σ-compact groups satisfy the countable chain condition. Michael's selection theorem (§19) is used to prove Haydon's theorem about inverse limits that are absolute extensors (§21), while this in turn is used to prove the main result of Chapter III in §23. This is a theorem of Ivanovskiĭ, Vilenkin, and Kuzminov that all compact groups are *dyadic*, that is, continuous images of some Cantor cube. Pestov's theorems about compactly generated groups are given in §24 and the construction of a free topological group is given in §25. It is used to provide an example of a group with countable chain condition that does not support a strictly positive measure, thus going back to the topic that started the chapter.

Chapter IV is entitled *Hyperspaces*. It is mostly concerned with compact metric spaces and their *exponential spaces*. The exponential space of X, $\exp(X)$, is the space of all closed subsets of X with its Vietoris topology. If X is compact metric then so is $\exp(X)$, the metric being the familiar Hausdorff metric. §30 gives Marjanovic's classification of spaces of the form $\exp(X)$ for X zero-dimensional, compact, and metric, as well as some of its surprising applications: There are exactly nine zero-dimensional compact metric spaces X such that $X \approx \exp(X)$. Also, there are two non-homeomorphic zero-dimensional compact metric spaces with homeomorphic squares.

Topics in topology contains no results due to the author, but many results are given new proofs and the topics discussed have been put in a modern context. Another distinguishing feature of this book is that it is completely self-contained. The proofs of all results are given into such detail that they should be accessible to a beginning graduate or an advanced undergraduate student. The book is suitable as a textbook for an advanced graduate course in topology. The treatment of each topic is supplemented by a discussion of further directions of research and a selected bibliography. There is a plenitude of exercises of varying difficulties. The exercises have been suitably chosen to illustrate the techniques introduced and also to provide a glimpse of further research topics.

The selected topics are presented in an original way, so that even a researcher already familiar with the results covered here may find reading this book worth while.

ILIJAS FARAH

Department of Mathematics, York University, Toronto, ON M3J 1P3, Canada. ifarah@mathstat.yorku.ca.

R. G. DOWNEY and M. R. FELLOWS. ***Parameterized complexity.*** Monographs in computer science. Springer, New York, Berlin, and Heidelberg, 1999, xv + 533 pp.

Parameterized complexity is a branch of complexity theory that gives a framework for a refined complexity analysis of hard algorithmic problems. The basic idea can best be explained by an example: Consider the problem of evaluating a sentence of a given language (a query) in a finite structure (a relational database). This problem usually has a high complexity (depending on the language, of course, but the problem is NP-complete even for the very basic primitive sentences—conjunctive queries). The main factor contributing to this complexity is the length of the query. In practice, however, queries are usually short, certainly much shorter than the size of the database. Thus when analyzing the complexity of the problem one should put much more emphasis on the size of the database than on the length of the query. An algorithm evaluating a query of length k in a database of size n in time $O(2^k \cdot n)$ is therefore much better than one performing the same task in time $O(n^{k/2})$, although both are exponential. Parameterized complexity provides a framework to pursue this distinction.

Parameterized complexity theory studies problems whose instances are *parameterized* by some function of the input whose values are natural numbers, such as the length of the query in the example. The idea is to choose the parameterization in such a way that it can be assumed to take small values for the instances one is interested in. A parameterized problem is *fixed-parameter tractable* if there is an algorithm solving it in time $f(k) \cdot n^c$, where n denotes the size of the input, k is the parameter, and where $f : \mathbb{N} \to \mathbb{N}$ is a computable function and $c > 0$ is a constant. The class of fixed-parameter tractable problems is denoted by FPT. Parameterized complexity theory provides methods for proving problems to be in FPT, but also gives a framework for dealing with apparently intractable problems.

Parameterized complexity theory has been developed in the last ten years; ideas from that theory have found their way into various areas of computer science, such as artificial intelligence, computational biology, and database theory. The basic definitions and concepts were introduced by R. G. Downey and M. R. Fellows, the authors of the monograph under review. They—sometimes with other authors—developed large parts of the theory and of its applications.

The book is divided into three parts. The first part is devoted to parameterized tractability. It presents various methods to show that a given problem is in FPT and applies these methods to a variety of problems. For example, Chapter 3 explains the method of bounded search trees and the method of problem kernel. Both can be used to show that Vertex Cover, parameterized by the size of the vertex cover, is in FPT. Chapter 6 and Chapter 7 present methods based on automata, bounded tree-width, and on Robertson and Seymour's graph minor theorem. Among others, Courcelle's theorem is proved stating that the evaluation problem for monadic second-order logic, parameterized by the tree-width of the structure, is in FPT. There is an interesting connection between the approximability of optimization problems of (classical) complexity theory and the fixed-parameter tractability of the corresponding parameterized problems; this theory is presented in Chapter 4.

Many interesting parameterized problems appear to be parametrically intractable. Part II of the book explores a framework in which parametric intractability can be demonstrated. For classical problems, the notion of NP-completeness is central to a nice and simple theory for intractable problems. Unfortunately, the world of parameterized intractability is more complex: there is a variety of seemingly different classes of parameterized intractability. The authors phrase this phenomenon in positive terms: "It seems to us that the situation is the following. Classical reductions are so coarse that most natural problems in, say, NP that are apparently not in P seem to be NP-complete. Parameterized reductions are sufficiently refined that instead of one large class of naturally intractable problems all of the same complexity, there seem to be many sets of natural combinatorial problems, all intractable in the parameterized sense, and yet of differing parameterized complexity" (p. 283).

The classes $W[1], W[2], \dots$ of the W-hierarchy are the most important intractable parameterized complexity classes. They are defined as the closure of a family of problems under (suitable) reductions, the problems being satisfiability problems for classes of circuits parameterized by the number of input nodes getting the value TRUE.

Chapters 10 and 11 are devoted to $W[1]$, considered to be the most important class of parameterized intractability. In particular, the halting problem for non-deterministic Turing machines, parameterized by the number of steps, and the problem Clique, parameterized by the size of the clique, are $W[1]$-complete. In Chapter 12 it is shown that the Dominating Set problem (parameterized by the size of the dominating set) is $W[2]$-complete. So, three important NP-complete problems, Vertex Cover, Clique, and Dominating Set, are complete for three parameterized complexity classes, FPT, $W[1]$, and $W[2]$, respectively.

The authors state: "It seems that the overwhelming majority of natural parameterized problems belong to no more than five or six ... classes" (p. 18). The class $W[P]$, dealt with in Chapter 12, is one of them. It is the closure of the problem of satisfiability of an arbitrary circuit, again parameterized by the number of input nodes set to TRUE. Parameterized analogues of PSPACE are considered in Chapter 14.

Part III of the book is about various structural issues concerning the parameterized complexity framework. Is the W-hierarchy proper? Could it be that $W[1] \neq \mathrm{FPT}$ while for all $t \geqq 1$, $W[t] = W[1]$? It is shown that the implication "$W[1] = \mathrm{FPT}$ implies $\mathrm{P} = \mathrm{NP}$" cannot be proved by any argument that relativizes (but note that $W[1] \neq \mathrm{FPT}$ implies $\mathrm{P} \neq \mathrm{NP}$). Moreover, if $W[1] = \mathrm{FPT}$ then it is possible to solve the classical problem 3Sat in time $2^{o(n)}$ for expressions having n variables.

The book covers a large amount of material and the exercises point to a variety of further results. Some parts of the books closely follow the corresponding original papers (often due to the authors themselves). For the reader this might lead to some confusion, since sometimes the notions or concepts are defined in a slightly different way in different papers.

During the last ten years parameterized complexity theory has matured, expanded, and sent out its tentacles into various areas. Because of that it is nice to have a book written by the pioneers of the area. It is remarkably comprehensive. The book is a valuable guide for any student of parameterized complexity theory and is *the* reference for any researcher in this field.

JÖRG FLUM

Institut für Mathematische Logik, Universität Freiburg, Eckerstraße 1, D-79104 Freiburg, Germany. joerg.flum@math.uni-freiburg.de.

Davide Sangiorgi and David Walker. ***The π-calculus: a theory of mobile processes.*** Cambridge University Press, Cambridge, New York, etc., 2001, xii + 580 pp.

One of the most important research topics within the theory of programming language semantics is that of process calculi, originating in the work of, in particular, 1991 Turing Award recipient Robin Milner. The theory of process calculi gives a precise mathematical account of the behaviour of computational systems that exhibit parallelism, analogous to the way in which the λ-calculus allows us to reason about computable functions.

A central observation made by Milner is that *communication* among computational agents is at the heart of the theory. Inspired by formal language theory and using the then-new methods of structural operational semantics, Milner introduced a number of important notions, among them the notion of bisimulation equivalence, now widely used in theoretical computer science and model theory.

In 1990, Robin Milner, Joachim Parrow, and David Walker described a new process calculus, a calculus of mobile processes which has since become known as the π-calculus. Interaction in the form of communication is still central to the calculus. But the new and important contribution of the π-calculus is the notion of *naming*; names play the part of both channel names and names of variables and are thus references to computational resources. Names may be communicated and may have a local scope, so the π-calculus is really a calculus of communicable references.

A tiny example of a π-calculus process is

$$x(y).\overline{y}w \mid (vz)\overline{x}z.z(u)$$

consisting of two parallel components. The right component has a local name z (indicated by the v-binder) which it emits on channel x. Next, the process will accept a name on the channel z and bind it to u. The left component will accept any name on the channel x and bind it to y; it will then output the (globally known) name w on the channel bound to y.

As a result of the first actions of the components, we get the process:

$$(vz)(\overline{z}w \mid z(u))$$

The scope of the name z has been extended to both components and the channel can thus be used for a communication between the components.

The present book is written by two of the most prominent researchers in the area and is a thorough and mathematically rigorous survey of central results in the theory of the π-calculus, based on the research results of the authors. Their explicit aim is, as stated in their preface, to show how one can "express systems precisely and reason about their behaviour rigorously."

While the treatment is extensive, the authors have made a conscious decision not to try to be all-encompassing. In particular, such topics as the study of modal logics for the description of π-calculus processes or the study of alternative semantics (denotational or non-interleaving) are not covered.

The book falls into seven parts. Part I of the book introduces the syntax and semantics of the π-calculus and the notion of bisimulation equivalence. In fact, bisimulation equivalence is less straightforward in the π-calculus, since not just one, but several inequivalent definitions can be formulated. Two of these, strong bisimilarity and strong barbed congruence, are introduced in this part of the book.

The π-calculus is really a family of calculi. Part II of the book is devoted to results concerning such calculi. In particular, a chapter examines how restrictions on the syntax influences the notions of bisimilarity. It turns out that the notions of bisimilarity, while generally distinct, coincide for some subcalculi, the most important being the so-called asynchronous π-calculus.

The notion of type in the λ-calculus inspires a notion of type in the π-calculus. Part III of the book examines typed π-calculi; a number of type systems are developed that can detect certain run-time errors. Part IV of the book continues this development by showing how one can use type systems for the π-calculus to reason about the behaviour of processes.

The remainder of the book describes how the π-calculus can be used as a meta-language for interpreting central ideas from programming language theory.

An alternative to the name passing ability of the π-calculus is *process passing*, i.e., transmitting an entire process along a channel. Process passing is central to describing, for instance, the World Wide Web where a WWW page may transfer a Java applet to the client machine reading the page.

Part V describes the relationship between process passing and name passing. The chapters of this part introduce and analyse HOπ (higher-order π), a calculus that allows name passing, and prove that in fact HOπ can be interpreted in the basic π-calculus.

Part VI relates the π-calculus to the λ-calculus, known to computer scientists as the theory underlying functional programming languages. In particular, this part of the book describes the interpretation of call-by-value and call-by-name reduction strategies of the λ-calculus in the π-calculus and extends the interpretations to typed settings. The equivalence on untyped λ-terms induced by the call-by-name encoding into the π-calculus corresponds to the equivalence on Lévy–Longo trees.

The π-calculus is a calculus of communicable references and is therefore well suited for the study of object-oriented programming languages, where references to resources play an important part. Part VII describes an encoding of a small class-based object-oriented programming language in the π-calculus and shows that one can use the encoding together with reasoning principles from the book to reason about properties of programs.

The book also contains an extensive bibliography, covering a large part of the existing π-calculus literature.

Sangiorgi and Walker have written an extremely valuable account of the π-calculus which should be appreciated by researchers in theoretical computer science and related areas of symbolic logic, as it covers the basic theory rigorously and in great depth. It is a valuable reference for those of us who are active researchers in the area. Readers who are primarily interested in a short introduction to the π-calculus with an emphasis on somewhat smaller examples should start with Milner's textbook ***Communicating and mobile systems: the π-calculus***, Cambridge University Press, 1999, as is also suggested by the authors of the present book.

HANS HÜTTEL

BRICS, Department of Computer Science, Aalborg University, Fredrik Bajersvej 7E, DK-9220 Aalborg Ø, Denmark. hans@cs.auc.dk.

SVEN OVE HANSSON. ***The structure of values and norms.*** Cambridge studies in probability, induction, and decision theory. Cambridge University Press, Cambridge, New York, etc., 2001, xiii + 314 pp.

"*Prolegomena*" (pp. 1–11) examines the virtues and dangers of formalization, defending the use of logic provided that a trade-off between simplicity and faithfulness be reached. As suggested by the title, the book contains two parts, the former on values (Chapters 2–8), the latter on norms (Chapters 9–13). There is also an "*Epilogue*" and then sixty pages devoted to the proofs of the statements ("theorems," "observations," or "corollaries") asserted in the first two parts.

Values are defined by means of the dyadic concepts of preference ("better," "worse," "equal in value to"), the monadic concepts ("good," etc.) being studied (Chapter 8) by virtue of the dyadic ones. A major distinction is considered regarding the objects of preferences: whether they are mutually exclusive (Chapters 2–4) or not, that is *combinative* (Chapters 5–7).

Chapter 2 presents the classic conditions for a preference relation over a set of alternatives (e.g. asymmetry of $>$), adding minimal principles for choice-guiding preferences (e.g. strong eligibility: There is at least one alternative A such that for all B, $A \geqq B$). Various conditions of restrictability are examined. A property of a preference relation, such as eligibility, is "said to hold restrictably for a set of alternatives if and only if it holds for all its nonempty subsets" (p. 27). Hansson also mentions other approaches to rational preference that he does not adopt.

Chapter 3 introduces a more general format for the representation of preference states (i.e., idealized states of mind with respect to value comparisons of a particular set of alternatives). They are represented by elements of a language $\mathcal{L}_{\mathcal{U}}$ of preference relations like $X \geqq Y$ and are closed under logical consequence. Similarly, preference models are defined. (At this stage of the book, the details of the formalism become rather complicated, although in my opinion perfectly justifiably so. I cannot enter into them here).

Chapter 4 deals with the important issue of changes in exclusionary preferences. As expected, we meet here the "AGM concepts," very well known for logic of belief (AGM = Alchouron, Gärdenfors, Makinson), *revision, contraction*, etc., with some specific modifications. The author concludes that "this is only a beginning in a subject that seems to have as many potential ramifications as belief change" (p. 56).

The next three chapters discuss combinative preferences (the contrary of exclusionary ones). A preliminary example explains the notion: "someone may express preferences for orchestral music over chamber music and also for Baroque over Romantic music" (p. 57). Comparison of how Baroque chamber music rates versus orchestral Romantic music requires considering relata of preference relations that are not mutually exclusive. New formal elements are used which involve sentential alternatives and a consequence relation. In this way two cases are distinguished, pairwise and decision-guiding preferences, as for the exclusionary preferences. In the first case (e.g. "It is better to have a cat than to have a dog"), an important piece of formalism is introduced, namely that of a *representation function* f for a set $\mathcal{A}$ of sentential alternatives. It permits us to construct a parallel between a simple relation $\geqq$ and the *represented* relation $\geqq_f$, and thus to study various properties of the latter. On this occasion, very fine analyses are given to situations already mentioned by famous predecessors, such as Halldén or Chisholm. Some other concepts are again introduced for the decision-guiding combinative preferences, such as those called "weighted." "If the weights are interpreted as probabilities, then this construction is an explication of 'probably better' " (p. 97). Chapter 7 ends with *extremal preferences* (they implement the *max* and *min* of sets of alternatives). Properties of them are examined. More or less expected properties do not hold, which shows the utility of the study.

Part I comes to an end with a logical study of monadic value predicates, mainly "good" and "bad." General properties are defined, like *positivity* (H is said to be $\geqq'$-positive when Hp & $q \geqq' p \rightarrow Hq$, where $\geqq'$ denotes a combinative preference relation). Some definitions for the predicates are proposed (e.g. "canonical good," $G_C\, p \longleftrightarrow (\forall q)(q \geqq'^{*} p \rightarrow q >' \neg q)$, where $\geqq'^{*}$ is the ancestral preference relation constructed on $\geqq'$) and compared with others already known.

Part II starts with Chapter 9 and is devoted to deontic logic. In his introduction, the author notes: "Unfortunately, most deontic logics, including standard deontic logic, have exactly this all-or-nothing approach to moral requirements" (p. 132). Usually, they allow no degree in the stringency of norms. Indeed, one of his aims is to remedy that. To begin with, *situationist* deontic logic is considered, i.e., "logic of that fraction of normative discourse that refers only to one moral appraisal of one situation, in one perspective" (p. 138). Intuitively, predicates O (obligatory), P (permitted), etc., must satisfy various requirements such as *contranegativity, positivity*, etc., with respect to preference relations; also, various theses are expected, e.g. *agglomeration* (Op & $Oq \rightarrow O(p$ & $q)$). Very precise and complete analysis is done about this matter. Many "reductions" of norms to preferences are indicated, for example $Op \longleftrightarrow (\forall s)(\neg p \geqq'^{*} s \rightarrow \neg s \geqq' s)$.

Limitations of situationist deontic logic are overcome first by introducing deontic statements that refer to some situation that does not actually obtain (Chapter 11: *Conflicts and counterfactuals*, hypothetical deontic statements), and secondly by referring to situations and alternative sets in general (Chapters 12 and 13 about *deontic rules*). In the first case, a special implicative connective is introduced (the dyadic conditional operator $O(p/q)$ being judged problematic). Perhaps, Chapter 12 is the most satisfying one. Formalism is here rather clear, normative rules are only sentences of the form $p \Rightarrow \delta$, with p belonging to a descriptive language and δ belonging to a set of normative sentences like $O\alpha$. There are two kind of rules, unrestrained and restrained: for the latter, "we require that the outcome be logically consistent. We may also want it to be obeyable, which is a stronger requirement than consistency" (p. 193). The analysis can be applied to some elaborate "realistic" cases, which makes it interesting.

Finally, Chapter 13 tries to use the whole formalism for the characterization and classification of legal relations such as rights, claims, and powers. This is a classic, "quasi-obliged" issue in this sort of study, including the comparison with Hohfeld's famous typology. I think that Hansson's study is very clear, and the reader will not be surprised by his conclusion: "As compared to the various typologies, this [Hansson's] framework provides us with greater expressive power, but—as is so often the case—only at the price of loss in terms of simplicity" (p. 222). That judgment properly qualifies the general features of the book. It is a dense and rich study, with an abundant set of formal terms, definitions, and theorems. It is a valuable study also for its numerous examples drawn from real life. Density and richness make it a difficult read— and make an understandable review difficult too.

PATRICE BAILHACHE

Département de Philosophie, Université de Nantes, B.P. 81 227, F-44312 Nantes cedex 3, France. patrice.bailhache@free.fr.

WOLFGANG BURR. *Fragments of Heyting arithmetic*. ***The journal of symbolic logic***, vol. 65 (2000), pp. 1223–1240.

During the last quarter of the previous century, a large amount of research has been done on the subsystems of classical first-order Peano arithmetic PA. The progress has been witnessed in a number of monographs, among which the work by P. Hájek and P. Pudlák (***Metamathematics of first-order arithmetic***, JSL LX 1317) is perhaps the most comprehensive one. In comparison, the fragments of Heyting arithmetic HA—the intuitionistic counterpart of PA—largely remain an unexplored territory. The paper under review is an attempt to remedy this situation and to get an insight into this fascinating area.

The main fragments of PA are defined by restricting the formulas in the schemata of induction or collection to the classes Π_n or Σ_n of the arithmetical hierarchy. A difficulty in the intuitionistic case is that the prenex normal form theorem does not hold and, correspondingly, natural (exhaustive) hierarchies of arithmetical formulas are missing.

A memorable theorem of A. Visser, further improved by K. Wehmeier (*Fragments of HA based on* Σ_1*-induction*, BSL VII 532), states that the fragment of HA based on the induction schema for all prenex formulas is Π_2-conservative over the induction schema restricted to Π_2-formulas only. Thus, intuitionistically, prenex induction is much weaker than full induction, and hence it does not provide a meaningful classification of the fragments of HA. Rather, one would expect that a reasonable classification of arithmetical formulas by their logical complexity should take into account the nestings of implications on a par with the quantifier alternations.

To answer this concern, W. Burr proposes a family of formula classes Φ_n as the proper intuitionistic analogues of the classes Π_n (for $n > 1$). These classes satisfy natural closure conditions, are classically equivalent to the classes Π_n, and $\bigcup_{n>1} \Phi_n$ exhausts all arithmetical formulas. Moreover, Burr shows that the induction schema restricted to the class Φ_n proves the same Π_2-sentences as the classical fragment of PA defined by the Π_n-induction schema, $I\Pi_n$. In other words, the Visser–Wehmeier theorem does not hold for the classes Φ_n. For his result, Burr uses an interesting variation of the so-called Friedman–Dragalin translation due to T. Coquand and M. Hofmann. In my opinion, the isolation of the classes Φ_n is a very basic and important contribution of the paper to intuitionistic arithmetic.

It is worth noticing that the question of the intuitionistic analogues of the classes Σ_n is left open. In fact, even for the classes Π_n, it is not excluded at present that there can be more than one natural counterpart of these classes in intuitionistic arithmetic. Only further research and "experimentation" can settle this question. By Burr's results, however, we now have at least one good candidate.

In the remaining part of the paper some additional facts are established. The next result is an improvement of the Visser–Wehmeier theorem. One defines another hierarchy of classes of formulas Θ_n, essentially by closing the classes Φ_n under existential (as well as universal) quantification. This makes the class Θ_2 already contain all prenex formulas. Burr shows, by means of Gödel's Dialectica interpretation, that the restriction of the induction schema to the class Θ_n proves the same Π_2-sentences as the classical fragment $I\Pi_n$. For $n = 2$ this statement implies the Visser–Wehmeier theorem.

Finally, collection principles in intuitionistic arithmetic are considered. For the standard formulation of the collection principle

$$\forall x \leqq a\, \exists y\, \varphi(x, y) \;\rightarrow\; \exists z\, \forall x \leqq a\, \exists y \leqq z\, \varphi(x, y),$$

where φ is any arithmetical formula, the author shows, using a result of U. Kohlenbach, that the provably total functions are bounded by polynomials. This contrasts with the behavior of this schema within classical logic: classically, it is equivalent to the full induction. The author also verifies that the contrapositive formulation of the collection schema over intuitionistic logic proves the law of the excluded middle, that is, it implies not only HA, but the whole of PA.

The paper is written in a nice and clear style. Although it does not develop significant new "methods," it does introduce an important new definition. It will predictably be often quoted in the future papers on intuitionistic arithmetic.

LEV BEKLEMISHEV

Department of Philosophy, Utrecht University, Heidelberglaan 8, 3584 CS Utrecht, The Netherlands. lev.beklemishev@phil.uu.nl.

THOMAS STRAHM. *Polynomial time operations in explicit mathematics*. ***The journal of symbolic logic***, vol. 62 (1997), pp. 575–594.

ANDREA CANTINI. *Feasible operations and applicative theories based on $\lambda\eta$*. ***Mathematical logic quarterly***, vol. 46 (2000), pp. 291–312.

More than two decades ago, Solomon Feferman (*A language and axioms for explicit mathematics*, JSL XLIX 308) introduced systems of explicit mathematics, a feature of which is a core of axioms for the *partial* combinatory logic of self-applicative operations. Feferman systems were originally motivated by foundational issues, and they typically included primitive recursive arithmetic. Influenced by the seminal work of Samuel Buss (***Bounded arithmetic***, JSL LVI 759), Thomas Strahm's paper defines two related systems of explicit mathematics, with a special unary predicate $\mathbb{W}$ for the binary words, whose provably total functions are exactly the polynomial-time computable functions. The main burden of the article lies in reducing (in the spirit of reductive proof theory) these systems to a known feasible theory, i.e., to a theory whose provably total functions are the polynomial-time computable functions. The chosen theory is Ferreira's Σ^b_1-NIA (a variant of Buss's S^1_2) augmented by the scheme of collection for bounded formulas. The reductions use a Σ_1-definable fixed-point construction in order to define the application operator of the partial combinatory logic.

In the final section of his paper, Strahm asks whether the axiom of totality, postulating that every term of the language is defined, raises the computational strength of his systems. The answer would be negative if one could prove the Church–Rosser theorem concerning Strahm's combinatory logic *within* a feasible theory. This seems notoriously out of reach, since the known proofs of the Church–Rosser theorem use the totality of the exponential function. In the second paper under review, Andrea Cantini is able to circumvent this approach and answer Strahm's question (in the negative). In a simplified form, Cantini's argument can be described in the following manner. First, he uses partial cut elimination in order to circumscribe himself to proofs in which cuts occur only on positive existential formulas and negations thereof. Secondly, he is able to witness inductively the positive existential statements of a proof by polynomial-time "controlled" terms of the combinatory logic. These combinatorial terms, when provably in $\mathbb{W}$, automatically reduce (in the sense of combinatorial reduction) to binary-word terms. He therefore obtains polynomial-time computable witness functions for the provably total functions of Strahm's theories with the axiom of totality. Cantini's method of witnessing also involves coding an equational combinatory calculus into binary words, from which he is able to obtain the following extra *intensional* information: the above-mentioned reductions are polynomial-time computable functions in terms of the given.

In his recent *Habilitationsschrift* (*Proof-theoretic contributions to explicit mathematics*, Universität Bern, 2001), Thomas Strahm greatly simplifies and extends Cantini's results. Strahm's extensions run in two different directions. On the one hand, he is able to deal inductively with positive formulas *tout court* (i.e., he also permits universal quantifications over individuals—observe that universal quantifications over $\mathbb{W}$ are not allowed, since $\mathbb{W}$ occurs negatively in such formulas). On the other hand, by suitably modifying the induction axioms of the theories, he studies theories associated with, for example, polynomial-space and linear-space computable functions. It should be noted, however, that Strahm's simplified methods do not yield the intensional information discussed in the previous paragraph.

FERNANDO FERREIRA

Departamento de Matemática, Faculdade de Ciências de Lisboa, Rua Ernesto de Vasconcelos, Bloco C1, 3, P-1749-016 Lisboa, Portugal. ferferr@cii.fc.ul.pt.

THOMAS STRAHM. *First steps into metapredicativity in explicit mathematics*. ***Sets and proofs, Invited papers from Logic Colloquium '97—European meeting of the Association for Symbolic Logic, Leeds, July 1997***, edited by S. Barry Cooper and John K. Truss, London Mathematical Society lecture note series, no. 258, Cambridge University Press, Cambridge, New York, and Oakleigh, Victoria, 1999, pp. 383–402.

The paper under review presents the proof-theoretic investigation of a metapredicative subsystem EMU of explicit mathematics. As the author explains in the opening lines, *metapredicativity* refers to "the analysis and study of formal systems whose proof-theoretic strength is beyond the Feferman–Schütte ordinal Γ_0 but which are nevertheless amenable to purely predicative methods." On the other hand, theories of explicit mathematics were introduced in the mid-seventies by Feferman (*A language and axioms for explicit mathematics*, JSL XLIX 308) as logical systems for representing Bishop's constructive mathematics (***Foundations of constructive analysis***, JSL XXXVII 744). Subsequently, they have been also regarded as formal frameworks for studying functional programming and its associated type-theoretic structures.

In this respect, there appears a third notion playing a central role in this paper, directly inspired by constructive type-theoretic foundations *à la* Martin-Löf: the notion of universe. Roughly, universes are types of types (or, more properly, of type names), generated under accepted type constructors, and the characteristic feature of the system EMU is that it includes a uniform operation u for constructing universes, which assigns a universe uc with $c \in uc$ to every type name c (uniformly in c).

Besides universe construction, EMU includes principles establishing that (i) the underlying structures are always (partial) combinatory algebras; (ii) the collection of types is closed under elementary comprehension and join (a uniform version of disjoint sum); (iii) there exists a "naming relation" $\mathfrak{R}$ correlating objects and types and allowing for a uniform version of the type existence axioms; (iv) universes are closed under elementary comprehension and join.

The main result is a sharp estimate of the proof-theoretic ordinal $|\mathsf{EMU}|$ of EMU. The first step is to establish the lower bound by proving in the theory the well-foundedness of each initial segment of the standard primitive recursive ordering of type $\varphi 1 \varepsilon_0 0$ (φ being a ternary generalization of the Veblen function used for characterizing predicative well-orderings). For proving this theorem, it is crucial to build up a hierarchy of universes for each α below ε_0, and the definition essentially applies the uniform universe constructor u (see the strategy already applied for the theory $\widehat{\mathsf{ID}}_{<\varepsilon_0}$ in Jäger, Kahle, Setzer, and Strahm, *The proof-theoretic analysis of transfinitely iterated fixed point theories*, ***The journal of symbolic logic***, vol. 64 (1999), pp. 53–67).

The upper bound is computed by means of a two-step reduction to $\widehat{\mathsf{ID}}_{<\varepsilon_0}$. EMU-derivations are embedded into a corresponding Tait calculus with ω-rule and then quasi-normalized by partial cut elimination (so that cuts can be reduced to have a suitable Σ-form, i.e., only existential type quantifiers occur, and no reference to the naming relation is made). The preparatory step is followed by a somewhat delicate asymmetric interpretation, which can be formalized by iterated fixed point construction (up to suitably high ordinals below ε_0) in the system $\widehat{\mathsf{ID}}_{<\varepsilon_0}$. Refinements are also possible when restricted forms of number-theoretic induction are present; in particular, it turns out that if number-theoretic induction is restricted to types, then the resulting subsystem $\mathsf{EMU}\restriction$ has the strength of predicative analysis.

To sum up, Strahm's paper is a valuable extension of work by Jäger, et al. (op. cit.); Jäger and Strahm, *Upper bounds for metapredicative Mahlo in explicit mathematics and admissible set theory* (***The journal of symbolic logic***, vol. 66 (2001), pp. 935–958); and Strahm, *Autonomous fixed point progressions and fixed point transfinite recursion* (BSL VII 535). It contributes to the advancement of the project of metapredicative proof theory.

Andrea Cantini

Dipartimento di Filosofia, Università di Firenze, via Bolognese 52, I-50139 Firenze, Italy. cantini@philos.unifi.it.

Toshiyasu Arai. *Consistency proof via pointwise induction*. ***Archive for mathematical logic***, vol. 37 no. 3 (1998), pp. 149–165.

This is a beautiful and concisely written article about a nice application of the Gentzen–Takeuti method (see e.g. Takeuti, ***Proof theory***, JSL LVI 1094, for an exposition) to consistency proofs, which are a classical theme in proof theory. It is well known from Gödel's famous results that theories like PA do not prove their own consistency. Nevertheless if we augment PA by an appropriate transfinite induction scheme PR-TI(ε_0) (for primitive recursive predicates) then the extended system proves $\mathrm{Con}_{\mathrm{PA}}$, and this result can not be improved since PA proves PR-TI(α) for any $\alpha < \varepsilon_0$.

This short and handy description of Gentzen's achievements with respect to ε_0, the proof-theoretic ordinal of PA, is rather vague, and therefore the reader who is interested in a precise and more informative description of what is going on is referred to e.g. Rathjen's recent survey on proof theory and ordinal analysis, *The higher infinite in proof theory*, ***Logic Colloquium '95***, Lecture notes in logic, no. 11, Springer, 1998, pp. 275–304.

In a brief description, the main result of Arai's paper is that over elementary arithmetic (ERA), the scheme of pointwise transfinite induction up to $\psi_0(\varepsilon_{\Omega_{q+1}+1})$, the proof-theoretic ordinal of the theory ID_{q+1} of $(q+1)$-times iterated inductive definitions, is equivalent to $\{\mathrm{Con}^n(\mathrm{ID}_q) : n < \omega\}$. Here Con^n denotes an n-times iterated consistency predicate.

It is important to recognize that in the article under review the iterated consistency of ID_q is measured in terms of the proof-theoretic ordinal of the theory ID_{q+1} and not in terms of the proof-theoretic ordinal of ID_q. Such an assignment of ordinals to theories has been suggested by Girard and a result relating the Howard–Bachmann ordinal to PA has been obtained by U. Schmerl in *Number theory and the Bachmann/Howard ordinal*, ***Proceedings of the Herbrand Symposium, Logic Colloquium '81***, Studies in logic and the foundations of mathematics, vol. 107, North-Holland, 1982, pp. 287–298. It is now an immediate question to compare such ordinal assignments with the assignment of proof-theoretic ordinals but it is difficult to give a definite answer.

To give a more precise and more adequate description of the context in which the main result of the paper is stated, it is necessary to assume familiarity with ERA, ID_q, the Buchholz-style notation system $\mathrm{OT}_0(q)$ for the proof-theoretic ordinal of ID_q, and the Buchholz-style assignment $\cdot[\cdot]$ of fundamental sequences for the limits in $\mathrm{OT}_0(q)$ (cf. Buchholz, *A new system of proof-theoretic ordinal functions*, ***Annals of pure and applied logic***, vol. 32 (1986), pp. 195–207).

Thus we assume that $\lambda = \sup\{\lambda[x] : x < \omega\}$ for limits $\lambda \in \mathrm{OT}_0(q)$ and moreover we assume for technical reasons that $(\alpha+1)[x] = \alpha$ for $\alpha \in \mathrm{OT}_0(q)$. With respect to this specific context one can formulate the following inference rule $\mathrm{PI}(q+1)$—of pointwise transfinite induction along $\mathrm{OT}_0(q+1)$ (or $\psi_0\varepsilon_{\Omega_{q+1}+1}$)—in the language of ERA. From $A(0,x)$ and $\neg\alpha = 0 \rightarrow (A(\alpha[1], r(x)) \rightarrow A(\alpha, x))$ infer $A(\alpha, x)$ where α is a variable ranging over $\mathrm{OT}_0(q+1)$, A is an elementary recursive predicate, and r is an elementary recursive function.

This scheme may be considered as pointwise, since in the inductive assumption we refer to the special predecessor $\alpha[1]$ only and not to predecessors of the form $\alpha[s(\alpha)]$ where s is an elementary recursive function.

In the technical part of the paper, Arai shows that this pointwise induction rule is genuinely related to the Gentzen–Takeuti reduction procedure for the theories ID_q. For this purpose he introduces a specific notion of proof, an adequate reduction procedure r (on proofs of the empty sequent), and an ordinal assignment o (from such proofs into $\mathrm{OT}_0(q)$) such that the transition $\mathcal{P} \mapsto r(\mathcal{P})$ is elementary recursive and so that $o(r(\mathcal{P})) = o(\mathcal{P})[1]$. To achieve the last property he adds two new repetition rules to the Gentzen–Takeuti (and nowadays it would be fair to say the Gentzen–Takeuti–Arai) reduction procedure.

It seems that the proof of the main result relies heavily on the finitary character of the Gentzen–Takeuti–Arai approach to ordinal analysis and it is not clear whether a related result can be obtained by infinitary proof theory.

The present article leads to the borderline of the pointwise ordinal analysis. It is not clear to the reviewer how the consistency strength of stronger theories like KP*i* and KPM etc. can be characterized in terms of pointwise induction principles. This is in contrast to the traditional ordinal analysis which allows classifications of the consistency strengths of even stronger theories in terms of the usual proof-theoretic ordinal, and if we want to compare the concepts of proof-theoretic ordinal and pointwise ordinal of theories we are left with a huge number of interesting problems which have to be solved first.

Arai's paper is a major progress with respect to this line of investigation.

As a final remark, the problem at the end of the paper can be solved as follows:

$$I\Sigma_k \vdash \forall n \exists m \{D_0 D_1^{k-1}(\Omega \cdot n)[1]^m = 0\}.$$

ANDREAS WEIERMANN

Institut für Mathematische Logik und Grundlagenforschung der Westfälischen Wilhelms-Universität Münster, Einsteinstr. 62, D-48149 Münster, Germany.
weierma@math.uni-muenster.de.

Michael Rathjen. *The superjump in Martin-Löf type theory*. ***Logic Colloquium '98, Proceedings of the annual European summer meeting of the Association for Symbolic Logic, held in Prague, Czech Republic, August 9–15, 1998***, edited by Samuel R. Buss, Petr Hájek, and Pavel Pudlák, Lecture notes in logic, no. 13, Association for Symbolic Logic, Urbana, and A K Peters, Natick, Mass., 2000, pp. 363–386.

Using the concept of universes of types in Martin-Löf type theory, ML, Rathjen studies the notion of the superjump in constructive mathematics. After presenting a brief history of universes in type theory, he introduces a system of type theory MLF by means of a universe constructor. This constructor assigns to each operator F from families of sets to families of sets a universe closed under F. This gives rise to the superjump in type theory. Rathjen characterizes the system MLF mainly in terms of two systems of set theory, both based on the concept of Mahloness. The first system, KPM^r, is a system of classical admissible set theory with set foundation axiomatizing a recursively Mahlo universe. The second system is based on Aczel's constructive set theory CZF enriched by a rule (M). This rule is a reflection rule, which asserts that arbitrary true sentences are reflected on set-inaccessible sets. Additionally, Rathjen introduces a hierarchy of subsystems of $\mathsf{CZF} + (\mathsf{M})$ with n-inaccessible sets, CZF_n, exhausting the strength of $\mathsf{CZF} + (\mathsf{M})$. Moreover, he gives an interpretation of CZF_n within MLF and an interpretation of MLF within KPM^r, the latter utilizing realizability models. He also claims that all systems under discussion are of the same proof-theoretic strength.

At the end of Rathjen's paper there is a discussion about the boundaries of Martin-Löf type theory and a discussion about so-called old and new Martin-Löf type theory.

Michael Möllerfeld

Institut für mathematische Logik und Grundlagenforschung, Westfälische Wilhelms-Universität, Einsteinstr. 62, D-48149 Münster, Germany. mimoe@math.uni-muenster.de.

Solomon Feferman. *Computation on abstract data types. The extensional approach, with an application to streams*. ***Annals of pure and applied logic***, vol. 81 (1996), pp. 75–113.

To quote from the opening sentence: "This paper is a continuation of the work of Feferman [in a pair of 1992 papers] which initiated an approach through a form of *generalized recursion theory* (g.r.t.) to computation on *abstract data types* (ADTs). ... [W]e separate out the extensional part of the theory and show how it may be applied to computation on streams as an ADT. One of the main new contributions here is an explanation of how this is to be done for finite 'nonterminating' streams as well as infinite streams, and even more general partial ('gappy') streams."

For the present purpose, ADT's are simply classes of structures closed under isomorphism. A paradigm is provided by A-list structures, for an arbitrary set A. Such structures are characterised up to isomorphism relative to A (by the operations *Cons*, *Head* and *Tail*, and *nil*) as being the least structure containing *nil* and closed under *Cons*.

When one tries to construct, by analogy, the ADT of A-streams (i.e., possibly infinite sequences of elements of A), problems arise, for example in connection with the possibility of a similar characterisation up to isomorphism.

We work over many-sorted "functional structures" $\mathcal{A} = (A_0, A_1, \ldots, A_n, F_0, \ldots, F_m)$ where each F_k is a functional of type level $\leqq 2$ over the A_i's of specified arity, and A_0 is the Boolean set $\{\mathtt{tt}, \mathtt{ff}\}$. The *signature* Σ of $\mathcal{A}$ is $(n, \langle \overline{\sigma}_k, \overline{i}_k, j_k \rangle_{k \leqq m})$, i.e., a listing of n, m and the types of the F_k's.

We review notation and basic concepts. The basic type-one objects are *partial functions* on A, which means that one has to deal with the semantics of partially defined terms. So $t\downarrow$ means that t is defined; $t_1 = t_2$ means $t_1\downarrow \wedge\, t_2\downarrow \wedge\, t_1 = t_2$, and $t_1 \simeq t_2$ means $(t_1\downarrow \vee\, t_2\downarrow \Rightarrow t_1 = t_2)$. Also $f : B \xrightarrow{\sim} C$ means that f is a partial function from B to C, and $f : B \to C$ means that f is total. Let i, j, k, l range over the sort indices $1, \ldots, n$, and let $\overline{i}, \overline{j}, \ldots$ range over finite sequences of these. For $\overline{i} = i_1, \ldots, i_\nu$, write $\mathrm{lh}(\overline{i}) = \nu$, and let $A_{\overline{i}} = A_{i_1}, \ldots, A_{i_\nu}$, with typical elements $x = (x_1, \ldots, x_\nu)$. Let $\varphi, \psi, \ldots$ range over partial functions from $A_{\bar{\imath}}$ to A_j for some $\bar{\imath}, j$. A *type symbol of level one* has the form $\bar{\imath} \xrightarrow{\sim} j$; let σ, τ range over these. Then for $\sigma = (\bar{\imath} \xrightarrow{\sim} j)$, let $A_\sigma = \{\varphi \mid \varphi : A_{\bar{\imath}} \xrightarrow{\sim} A_j\}$. For $\overline{\sigma} = \sigma_1, \ldots, \sigma_\mu$, let $A_{\overline{\sigma}} = A_{\sigma_1} \times \cdots \times A_{\sigma_\mu}$, with typical elements $\varphi = (\varphi_1, \ldots, \varphi_\mu)$. Then for $\varphi, \psi \in A_{\overline{\sigma}}$, $\varphi \subseteq \psi$ means $\varphi_k \subseteq \psi_k$ for each k.

Functionals of type level two over A are partial maps, of the form $F : A_{\overline{\sigma}} \times A_{\bar{\imath}} \xrightarrow{\sim} A_j$. Write $F(\varphi, x)$ for $F(\varphi_1, \ldots, \varphi_\mu, x_1, \ldots, x_\nu)$. ($F$ actually has type level 1 if $\mu = 0, \nu > 0$, and 0 if $\mu = \nu = 0$.) The functional $F : A_{\overline{\sigma}} \times A_{\bar{\imath}} \xrightarrow{\sim} A_j$ is *monotonic* if

$$\forall \varphi, \psi \in A_{\overline{\sigma}} \forall x \in A_{\bar{\imath}} [F(\varphi, x)\downarrow \wedge \varphi \subseteq \psi \;\Rightarrow\; F(\varphi, x) = F(\psi, x)].$$

(This condition is vacuously satisfied if $\mathrm{lh}(\overline{\sigma}) = 0$.) For suitable combinations of types, each level-two monotonic F has an associated *least fixed point*. For example, if $F : A_\sigma \times A_{\bar{\imath}} \xrightarrow{\sim} A_j$, where $\sigma = (\bar{\imath} \xrightarrow{\sim} j)$, then F induces a total map $\hat{F} : A_\sigma \to A_\sigma$ where $\hat{F}(\varphi) = \lambda x.F(\varphi, x)$, with an associated least fixed point $\mathrm{LFP}(\hat{F})$ defined in the standard manner.

Feferman considers computation over Σ-structures of the above form, satisfying (i) $A_0 = \mathbb{B} = \{\mathtt{tt}, \mathtt{ff}\}$, (ii) each F_k has type level 2, 1, or 0, (iii) each F_k of type level 2 is monotonic, and (iv) the constants $\mathtt{tt}$ and $\mathtt{ff}$ are among the F_k's. He lists formal *schemata* for such structures:

I	(Initial function(al)s)	$\mathbf{F}(\varphi, x) \simeq F_k(\varphi, x)$
II	(Identity functions)	$\mathbf{F}(\varphi, x) = x$
III	(Application)	$\mathbf{F}(\varphi, x) \simeq \varphi(x)$
IV	(Conditional definition)	$\mathbf{F}(\varphi, x, b) \simeq [\text{if } b = \mathtt{tt} \text{ then } \mathbf{G}(\varphi, x) \text{ else } \mathbf{H}(\varphi, x)]$
V	(Structural)	$\mathbf{F}(\varphi, x) \simeq \mathbf{G}(\varphi_f, x_g)$
VI	(Individual substitution)	$\mathbf{F}(\varphi, x) \simeq \mathbf{G}(\varphi, x, \mathbf{H}(\varphi, x))$
VII	(Function substitution)	$\mathbf{F}(\varphi, x) \simeq \mathbf{G}(\varphi, \lambda y.\mathbf{H}(\varphi, x, y), x)$
VIII	(Least fixed point)	$\mathbf{F}(\varphi, x, y) \simeq [\mathrm{LFP}(\lambda\psi.\lambda z.\mathbf{G}(\varphi, \psi, x, z))](y)$

where the types must be coherent. Schema V abbreviates

$$\mathbf{F}(\varphi_1, \ldots, \varphi_\mu, x_1, \ldots, x_\nu) \simeq \mathbf{G}(\varphi_{f(1)}, \ldots, \varphi_{f(\mu')}, x_{g(1)}, \ldots, x_{g(\nu')}),$$

where $f : \{1, \ldots, \mu'\} \to \{1, \ldots, \mu\}$ and $g : \{1, \ldots, \nu'\} \to \{1, \ldots, \nu\}$.

With each structure $\mathcal{A}$ and each $\mathbf{F}$ generated by these schemata is associated, in an obvious way, a function $\mathbf{F}^{\mathcal{A}}$ on $\mathcal{A}$ of the same type as $\mathbf{F}$. The LFP scheme is justified by the monotonicity condition (iii) above, as well as by the lemma that all the $\mathbf{F}^{\mathcal{A}}$ so defined are monotonic. (This, as well as many of the results below, is proved by induction on the schemata.)

Next it is shown that if $\mathcal{A}$ and $\mathcal{A}'$ are isomorphic Σ-structures, then $\mathbf{F}^{\mathcal{A}}$ and $\mathbf{F}^{\mathcal{A}'}$ correspond to each other under this isomorphism. Thus the schemata are invariant under isomorphism, and this justifies calling the $\mathbf{F}$'s *abstract computation procedures* (ACP's). Let $\mathrm{ACP}(\Sigma)$ be the set of ACP's generated by the schemata for signature Σ, and for a Σ-structure $\mathcal{A}$, let $\mathrm{ACP}(\mathcal{A})$ be the set of all $\mathbf{F}^{\mathcal{A}}$ for $\mathbf{F} \in \mathrm{ACP}(\Sigma)$.

In §4 Feferman states and proves the important *substructure theorem*: If $\mathcal{B}$ is a substructure of $\mathcal{A}$, then for each ACP $\mathbf{F}$, B is closed under $\mathbf{F}^{\mathcal{A}}$, and $\mathbf{F}^{\mathcal{A}} \restriction B = \mathbf{F}^{\mathcal{B}}$.

In §5 Feferman considers continuity. A functional F is said to be *continuous* if whenever $F(\varphi, x) \simeq y$, there exists finite $\widetilde{\varphi}$ such that $F(\widetilde{\varphi}, x) \simeq y$. (Note that for F of type level 1, F is automatically continuous.) The main theorem here states that if each F_k in $\mathcal{A}$ is continuous then for each ACP $\mathbf{F}$, $\mathbf{F}^{\mathcal{A}}$ is continuous.

In §6, Feferman considers, as a particular case, *first-order* structures. A structure $\mathcal{A} = (A_0, \ldots, A_n, F_0, \ldots, F_m)$ and its signature $(n, \langle \overline{\sigma}_k, \bar{\imath}_k, j_k \rangle_{k \leqq m})$ are said to be first-order if $\mathrm{lh}(\overline{\sigma}_k) = 0$ for each $k \leqq m$, i.e., each F_k is of type level $\leqq 1$. Thus each F_k is vacuously monotonic, and hence each ACP over $\mathcal{A}$ is monotonic. Moreover, schema VII is shown to be *redundant* for first-order structures. Furthermore, since each F_k is vacuously continuous, we have that if $\mathcal{A}$ is first-order, then $\mathbf{F}^{\mathcal{A}}$ is continuous for each ACP $\mathbf{F}$.

In §7 this theory is applied to computation on the (first-order) structure of the naturals, "the *ur*-structure for recursion theory": $\mathcal{N} = (\mathbb{N}, Sc, Pd, 0, Eq_0)$ where Sc and Pd are the successor and predecessor functions on $\mathbb{N}$, Eq_0 is the equality test, and $\mathbb{B}$ is identified with $\{0, 1\}$. Let $\Sigma_{\mathcal{N}}$ be the signature of $\mathcal{N}$. §7 proves the following three facts: (a) The functions of type level 1 in ACP($\mathcal{N}$) are exactly the partial recursive functions. (b) The functionals of type level 2 in ACP($\mathcal{N}$) are coextensive with the partial recursive functionals when restricted to total function arguments. (c) Every functional of type level 2 in ACP($\mathcal{N}$) is partial recursive on partial function arguments, but not conversely.

A counterexample to the reverse direction in (c) (shown in Platek's 1966 Stanford dissertation) is given by the non-deterministic "strong (parallel) or" functional OR^+, where $OR^+(\varphi_1, \varphi_2) \simeq 0 \Leftrightarrow \varphi_1(0) \simeq 0$ or $\varphi_2(0) \simeq 0$. Equivalence is, however, obtained by adding OR^+ to the ACP schemata on $\mathcal{N}$ (by a result of Sazonov).

Next Feferman considers structures formed from subsets of $\mathbb{N}$. A *partial recursive structure in* $\mathbb{N}$ is one of the form $\mathcal{A} = (A_0, \ldots, A_n, F_0, \ldots, F_m)$ where each $A_i \subseteq \mathbb{N}$, $A_0 = \mathbb{B} = \{0, 1\}$, and each F_k is the restriction to $\mathcal{A}$ of a partial recursive functional on $\mathbb{N}$ under which $\mathcal{A}$ is closed. The interesting result here is that for any such $\mathcal{A}$ of signature Σ and any $\mathbf{F}$ in ACP(Σ), $\mathbf{F}^{\mathcal{A}}$ is the restriction to $\mathcal{A}$ of a partial recursive functional on $\mathbb{N}$, and is continuous.

This result, a re-statement of a theorem in one of Feferman's earlier 1992 papers, shows that computation on ADT's can often be interpreted as ordinary computation on $\mathbb{N}$ (in the sense of recursion theory). For if the ADT K is an isomorphism type, and contains a partial recursive structure $\mathcal{A}$ on $\mathbb{N}$, then ACP's on any structure of K transfer under isomorphism to partial recursive functionals on the "implementation" $\mathcal{A}$ of K. This applies to ADT's K of structures such as "lists-of-A's," "finite-trees-of-A's," etc., which are generated from a countable A by finitely many finitary operations. Then if A is identified as a (not necessarily recursive) subset of $\mathbb{N}$, the lists-of-A (etc.) can be realised as a partial recursive structure.

In §§8–9 Feferman applies his general theory to (finite) list and (infinite) stream structures, respectively. First we consider the structure of A-lists, which is developed in the well-known way. Let LIST(A) be the collection of structures $\mathcal{L} = (A, L, Cons, Hd, Tl, nil, Eq_{nil})$ (implicitly containing the Boolean part $(\mathbb{B}, \mathtt{tt}, \mathtt{ff})$) satisfying the usual conditions, including a second-order version of list induction (minimality of L). Definition by list recursion can be derived as an ACP, by using the LFP scheme. That is, let $\mathcal{L}^p = (\mathcal{L}, \ldots)$ be an expansion of $\mathcal{L}$ in LIST. Suppose C is a subset of one of the basic sets in $\mathcal{L}^+$ and G, H are ACP's over $\mathcal{L}^+$ with $G \in C$ and $H : A \times L \times C \to C$. Then we can find an ACP F over $\mathcal{L}^+$ satisfying the three conditions: $F : L \to C$, $F(nil) = G$, and $F(Cons(a, l) = H(a, l, F(l))$ for all $l \in L$. Hence ACP's can be defined for many of the well-known list operations used in functional programming, such as *Map*, *Filter*, etc.

In the special case that $A \subseteq \mathbb{N}$, we have a partial recursive A-list structure $\mathcal{L}(A)$. In this case the ACP's determine partial recursive functionals, which are the restrictions of the corresponding functionals on $\mathcal{L}(\mathbb{N})$, by the substructure theorem.

In comparison with finite lists, the situation with streams (§9) is more subtle and more interesting, with bigger problems, and provides a real testing ground for Feferman's approach (which, I believe, passes the test handsomely). If one attempts to treat a stream structure (essentially a *second-order* object) in the *first-order* framework of §6, the axioms would be something like axioms (i)–(iii) listed below. The main problem here, of course, is that these (and similar) conditions do not determine $\mathcal{S}$ up to isomorphism. For S can be interpreted as the full set of all functions from $\mathbb{N}$ to A, as well as the subset of eventually constant functions, and so forth.

Secondly, the usual conditions do not guarantee closure under desired computation procedures. A further principle is needed corresponding to *minimality/induction/recursion* for lists. One well-known approach involves the dual concepts *maximality/co-induction/co-recursion*. A simple example of a co-recursion scheme is the following, which defines a function $F : C \to S$ from $G : C \to A$ and $H : C \to C$ (for some set C) by the equation $F(x) = \mathit{Cons}(G(x), F(H(x)))$, for $x \in C$. Feferman's solution (described below) is to work with a *second-order* stream structure which will solve both problems above: it guarantees isomorphism of the stream structure relative to A, and permits ACP's with co-recursive definitions of the above form (among others). Feferman argues that this cannot be achieved by adding a second-order condition dual to the induction scheme for lists, as the largest solution of a suitable monotone operator, since (as he shows by examples) this "largest solution" is highly sensitive to the universe that frames the problem. While appreciating the value of the co-inductive approach, he prefers another approach which provides an "internal" characterisation of the stream structure. This is given by a type-2 *Sim* functional (see below) which ensures that every total function from $\mathbb{N}$ to A can be simulated by an A-stream, and characterises stream structures up to isomorphism.

More general than the notion of *infinite* (*total*) *streams* on A is that of infinite or finite non-terminating, i.e., *potentially infinite* streams, which arise in both mathematical computations and physical phenomena. More general still is the notion of *partial* or "gappy" streams, which is what Feferman works with. Thus he considers structures of the form

$$\mathcal{S} = (A, S, \mathit{Cons}, \mathit{Hd}, \mathit{Tl}, \mathit{Sim}, \mathcal{N}),$$

where (i) $A \neq \emptyset$; (ii) $\mathit{Cons} : A \times S \to S$, $\mathit{Hd} : S \xrightarrow{\sim} A$, $\mathit{Tl} : S \to S$, $\mathit{Sim} : (\mathbb{N} \xrightarrow{\sim} A) \to S$; (iii) $\forall a \in A\, \forall s \in S\, [\mathit{Hd}(\mathit{Cons}(a, s)) = a$ and $\mathit{Tl}(\mathit{Cons}(a, s)) = s]$; (iv) $\forall \varphi \in (\mathbb{N} \xrightarrow{\sim} A)$ $\forall n \in \mathbb{N}\, [\mathit{Hd}(\mathit{Tl}^n(\mathit{Sim}(\varphi))) \xrightarrow{\sim} \varphi(n)]$. A new problem arises, however, with the use of partial streams: the functional *Sim* need not be monotonic. We only have monotonicity in the weaker sense $\varphi \subseteq \psi \Rightarrow \mathit{Sim}(\varphi) \subseteq_S \mathit{Sim}(\psi)$, where we define $(s)_n = \mathit{Hd}(\mathit{Tl}^n(s))$ and

$$s \subseteq_S s' \Longleftrightarrow_{\mathrm{df}} \forall n\, [(s)_n \downarrow \Rightarrow (s)_n = (s')_n].$$

This necessitates a more general definition of monotonicity on $\mathcal{A}$, based on chain-complete partial orderings $\subseteq_A$ (instead of equality) on the basic sets A, and then defined hereditarily for higher types based on these. The LFP functional can still be defined properly with this more general concept of monotonicity.

In §10 Feferman applies the above theory to recursive definitions on streams. He works with the collection P-STREAM ("P" for "partial") of stream structures $\mathcal{S}$ as above, together with the condition (v) $s, s' \in S \wedge \forall n \in \mathbb{N}\, [(s)_n = (s')_n] \Rightarrow s = s'$. In connection with the monotonicity definition, the relation $\subseteq_S$ is defined as above, and $\subseteq_A$ and $\subseteq_{\mathbb{N}}$ are taken to be equality on A and $\mathbb{N}$. This makes all the basic operations of $\mathcal{S}$ monotonic. Now the *Sim* functional characterises stream structures up to isomorphism. That is, whenever $\mathcal{S}' = (A', S', \mathit{Cons}', \mathit{Hd}', \mathit{Tl}', \mathit{Sim}', \mathcal{N})$ also satisfies (i)–(v) and $I : A \cong A'$, then we have $F : \mathcal{S} \cong \mathcal{S}'$ for $F(s) = \mathit{Sim}'(\lambda n.I((s)_n))$.

Feferman gives a number of examples of schemes for stream recursion, one of the simplest being the following: Let $\mathcal{S}^+ = (\mathcal{S}, \dots)$ be an expansion of $\mathcal{S}$ in P-STREAM. Suppose C is a subset of one of the basic sets in $\mathcal{S}^+$ and G, H are ACP's over $\mathcal{S}^+$ with $G : C \xrightarrow{\sim} A$ and $H : C \to C$. Then we can find an ACP F over $\mathcal{S}^+$ satisfying the conditions: (i) $F : C \to S$, (ii) $F(c) = \mathit{Cons}(G(c), F(Hc))$ for all $c \in C$, (iii) if $F' : C \to S$ satisfies (ii), then $F(c) \subseteq_S F'(c)$ for all $c \in C$.

Note that although F solves the fixed-point equation (ii), it is not the least fixed point, since that is the completely undefined function. Actually F is total, and is characterised by (iii) as the *least pointwise* among total solutions. In fact F is given by $F(c) = \mathit{Sim}(\lambda n.\psi(c, n))$ where $\psi : C \times \mathbb{N} \xrightarrow{\sim} A$ *is* the LFP of

$$\psi(c, n) \simeq \begin{cases} G(c) & \text{if } D(c) = \mathtt{tt} \wedge n = 0 \\ \psi(Hc,\ n-1) & \text{if } n > 0. \end{cases}$$

Various modified versions of this recursion scheme are given, and justified similarly. In some of them, the range of F may consist of streams that are total (as above) or potentially infinite (as in a filtering operation).

Using these schemes, Feferman is able to construct many of the well-known stream functionals found in the literature, such as *Map*, *Mesh*, *Filter*, and *Sieve*. He also argues for maintaining a clear distinction (conceptually and formally) between lists and streams; specifically, between lists of definite finite length, and (non-terminating but) possibly finite streams, such as may result from a filtering operation on infinite streams.

In §11 Feferman studies A-streams in the special case $A \subseteq \mathbb{N}$. From the substructure theorem it follows that $\mathcal{S}(A)$ is a substructure of $\mathcal{S}(\mathbb{N})$. Furthermore, the identification of $S(\mathbb{N})$ with $(\mathbb{N} \xrightarrow{\sim} \mathbb{N})$ induces an identification of the partial functions $\varphi : S(\mathbb{N})^{\nu_1} \times \mathbb{N}^{\nu_2} \xrightarrow{\sim} S(\mathbb{N})$ of type level 1 that preserve $\subseteq_{S(\mathbb{N})}$ with monotonic partial functions of type level 2 (or 1, if $\nu_1 = 0$) over $\mathbb{N}$. Hence monotonic partial functionals F of type level 2 over $\mathcal{S}(\mathbb{N})$ can be identified with monotonic partial functionals of type level 3 (or 2, if all arguments of F have numerical arguments only) over $\mathbb{N}$. Thus a recursion-theoretic interpretation of ACP's over $\mathcal{S}(\mathbb{N})$ can be given by a theory of *partial recursive functionals of finite type with hereditary partial continuous arguments*. Feferman sketches (promising full details at a later date) such a theory, involving formal neighborhoods, analogous to the treatment of finite type functionals of hereditarily *total* arguments in Kleene's *Countable functionals* (JSL XXVII 359) and Kreisel's *Interpretation of analysis by means of constructive functionals of finite types* (JSL XXXVI 169), and simpler than the approach in Ershov's *Vychislimye funktsionaly konechnykh tipov*, ***Algebra i logika***, vol. 11 (1972), pp. 367–437 (English translation in ***Algebra and logic***). It follows that the ACP's over $\mathcal{S}(\mathbb{N})$ are partial recursive in the usual sense.

In two appendices, Feferman considers the relationship between his formalism and two others: The first is Y. Moschovakis's FLR (formal language of recursion) (*Abstract recursion as a foundation for the theory of algorithms*, ***Computation and proof theory***, Springer, 1984, pp. 289–364, and *The formal language of recursion*, ***The journal of symbolic logic***, vol. 54 (1989), pp. 1216–1252), containing schemes for term formation, including explicit definition and simultaneous LFP recursion. Here Feferman proves equivalence of the systems. The second is Tucker and Zucker's system $\mu\mathrm{PR}^*(\mathcal{A})$ applied to structures $\mathcal{A}$ of infinite streams (*Computable functions on stream algebras*, ***Proof and computation***, NATO ASI series F, vol. 139, Springer, 1995, pp. 397–437) which differs from Feferman's notably in having (a) schemes for simultaneous primitive recursion on $\mathbb{N}$, and a least number operator, instead of an LFP operator, (b) for each data type A, a data type A^* of finite sequences of elements from A. Here Feferman has not proved, but conjectures, equivalence.

It remains to explain the word "extensional" in the title. As Feferman states in the *Introduction* (§1), the present paper shows how far one can go with a purely extensional theory of streams. A full treatment, however, sufficient for ordinary computational purposes, of ACP's on data types with non-trivial many-one representations—for example, reals represented by Cauchy sequences of rationals or streams of decimal digits ("infinite precision reals")—would require a more general "intensional" theory. This was attempted by Feferman in a 1992 paper, which, however, had some defects. Corrections are indicated in *Appendix C* of the present paper, and a full treatment of an intensional theory is promised in the future.

The paper is written in Feferman's usual lucid style. There are some minor slips: on page 102, line 3: replace "$K : C \to C$" by "$K : C \to S$"; lines 11 and 21: replace "Theorem 9" by "Theorem 11"; line 12: delete "$D(s) = \mathtt{tt}$ all s"; line 22, delete "$D(s, s') = \mathtt{tt}$ all s, s'"; page 103, line 1: replace "$H(s) = s^{\to}$" by "$H_0(s) = H_1(s) = s^{\to}$".

JEFFERY ZUCKER

Department of Computing and Software, McMaster University, Hamilton, ON L8S 4K1, Canada. zucker@mcmaster.ca.

SAHARON SHELAH and HUGH WOODIN. *Large cardinals imply that every reasonably definable set of reals is Lebesgue measurable*. ***Israel journal of mathematics***, vol. 70 (1990), pp. 381–394.

The paper gives a proof of the statement in its title, namely, that the existence of certain large cardinals, much weaker than a supercompact cardinal, implies that every set of reals in $L(\mathbb{R})$, the smallest transitive inner model of set theory that contains all the reals, is Lebesgue measurable, has the property of Baire, is Ramsey, and so forth. The program of deriving regularity properties of definable sets of real numbers from large cardinal hypotheses began with R. Solovay (*On the cardinality of* $\mathbf{\Sigma}^1_2$ *sets of reals*, JSL XXXIX 330) when he proved that the existence of a measurable cardinal implies that every $\mathbf{\Sigma}^1_2$ set of reals is Lebesgue measurable and has the Baire property, and culminated in the proof of projective determinacy by D. Martin and J. Steel [M-S] (*Projective determinacy*, JSL LVII 1132). The proof that the existence of not-so-large large cardinals outright implies that all sets of reals in $L(\mathbb{R})$ have all the classical regularity properties, like the Lebesgue measurability, and even that they are determined, is, without doubt, one of the greatest achievements of modern set theory. The major breakthrough occurred in 1984 when the authors of the paper, using a result of M. Foreman, M. Magidor, and S. Shelah [F-M-S] (*Martin's maximum, saturated ideals and non-regular ultrafilters, Part I*, JSL LVII 1131), proved that if there exists a supercompact cardinal, then all sets of reals in $L(\mathbb{R})$ are Lebesgue measurable, have the Baire property, are Ramsey, and so forth. The main goal of the paper under review is to give a proof of this theorem under the optimal large-cardinal hypothesis. The paper consists of an introduction followed by three sections with suggestive titles: *The dawn*, *The dusty road*, and *Happy end*.

Let NS_{ω_1} be the ideal of the non-stationary subsets of ω_1. NS_{ω_1} is a normal ideal, that is, its dual filter, the closed unbounded filter, is closed under diagonal intersections. An ideal $\mathcal{I}$ on ω_1 is saturated if the algebra $\mathcal{P}(\omega_1)/\mathcal{I}$ is $\aleph_2$-cc, i.e., every antichain has size at most $\aleph_1$. It is a well-known fact, going back to Solovay (*Real-valued measurable cardinals*, JSL XL 506), that if $\mathcal{I}$ is a normal saturated ideal on ω_1, forcing with $\mathcal{P}(\omega_1)/\mathcal{I}$ yields the existence of an elementary embedding of the universe V into a transitive class M that contains all the reals.

The dawn began in the spring of 1984 when Foreman, Magidor, and Shelah [F-M-S] proved that if κ is a supercompact cardinal, then there is a κ-cc forcing notion $\mathbb{P}$ of cardinality κ that does not collapse ω_1 and forces that NS_{ω_1} is saturated. Now suppose that κ is some inaccessible cardinal and there is a forcing notion $\mathbb{P}$ as above—i.e., of size κ and κ-cc and forcing that some normal ideal $\mathcal{I}$ on ω_1 is saturated—that does not add reals. Then if G is $\mathbb{P}$-generic over V, one can force over $V[G]$ with $\mathcal{P}(\omega_1)/\mathcal{I}$ to obtain, in the extension $V[G][H]$, an elementary embedding $j : V[G] \to M$ with all the reals of $V[G]$ contained in M. Notice that the whole forcing extension $\mathbb{P} * \mathcal{P}(\omega_1)/\mathcal{I}$ has size κ and, since $\mathcal{I}$ is saturated, it is κ-cc. Moreover, since $\mathbb{P}$ does not add reals, j restricted to $L(\mathbb{R})^V$ is an elementary embedding of $L(\mathbb{R})^V = L(\mathbb{R})^{V[G]}$ into $L(\mathbb{R})^M = L(\mathbb{R})^{V[G][H]}$. If, in addition, κ is weakly compact, then it is a well-known fact (cf. [F-M-S]) that, since $\mathbb{P} * \mathcal{P}(\omega_1)/\mathcal{I}$ is of size κ and κ-cc, one can force again over $V[G][H]$, without adding reals, in order to create a generic C over V for the Levy-collapse of κ to ω_1 so that $V[C]$ and $V[G][H]$ have the same reals. Thus the elementary embedding of $L(\mathbb{R})^V$ into $L(\mathbb{R})^{V[G][H]}$ yields an elementary embedding of $L(\mathbb{R})^V$ into $L(\mathbb{R})^{V[C]}$. Since by Solovay (*A model of set-theory in which every set of reals is Lebesgue measurable*, JSL XXXVIII 529) every set of reals in $L(\mathbb{R})^{V[C]}$ is Lebesgue measurable and has the Baire property, and by A. Mathias (*Happy families*, ***Annals of mathematical logic***, vol. 12 (1977), pp. 59–111) is Ramsey, by elementariness of the embedding so is every set of reals in $L(\mathbb{R})^V$. This is essentially the argument, due to Woodin, that proves Theorem 2.3 of the paper under review.

On the other hand, Lemma 2.4 shows that there is indeed a forcing notion $\mathbb{P}$ as above that does not add reals. This is the part of the argument that uses the machinery of [F-M-S]. For every $S \subseteq \omega_1$ stationary and co-stationary, there is such a $\mathbb{P}$ that forces that the ideal

$\mathrm{NS}_{\omega_1}|S = \{X \subseteq \omega_1 : X \cap S \in \mathrm{NS}_{\omega_1}\}$ is saturated. $\mathbb{P}$ is a revised countable-support iteration of length κ, κ supercompact, with a Laver function doing the bookkeeping. The idea is to destroy potential maximal antichains of NS_{ω_1} of size ω_2. This is accomplished by the sealing forcing, which shoots a club through the diagonal union of an initial segment of the antichain. So, whenever the partial ordering given by the Laver function is the forcing that seals a maximal antichain of $\mathrm{NS}_{\omega_1}|S$, and this forcing is semi-proper, then one forces with it. Otherwise, one collapses $2^{\aleph_1}$ to $\aleph_1$ with countable conditions. Now, it is a consequence of Shelah's general theory of forcing iterations (see Chapters V and XI of ***Proper and improper forcing***, BSL VI 83) that all maximal antichains of $\mathrm{NS}_{\omega_1}|S$ have been sealed along the iteration and so all of them have size at most $\aleph_1$. This uses the reflection properties of κ and the fact that in the final model a poset is semiproper iff it does not destroy stationary subsets of ω_1. Moreover, such an iteration does not add reals.

Thus, putting these two results together, the authors conclude that if there exists a supercompact cardinal κ, then $L(\mathbb{R})^V$ is elementarily embeddable into the $L(\mathbb{R})$ of the Levy-collapse of κ to ω_1 and, therefore, every set of reals in $L(\mathbb{R})^V$ is Lebesgue measurable, has the Baire property and is Ramsey.

The dusty road begins by wondering what is the right large-cardinal assumption: How large must large cardinals be in order for their mere existence to imply that every set of reals in $L(\mathbb{R})$ is Lebesgue measurable, and so forth? The conclusion (Conclusion 3.7) is that $\omega + 1$ many of what are now called Shelah cardinals suffice (Definition 3.5): κ is a Shelah cardinal if for every $f : \kappa \to \kappa$ there is an elementary embedding $j : V \to M$, M transitive, with critical point κ and such that $H(j(f)(\kappa)) \subseteq M$ and M is closed under sequences of length $< \kappa$. The point is that if κ is a Shelah cardinal, then for every stationary and co-stationary $S \subseteq \omega_1$ there is a κ-cc forcing notion of size κ that does not add reals and forces that $\mathrm{NS}_{\omega_1}|S$ is saturated (Lemmas 3.4 and 3.6). But then, as remarked above, it follows immediately from Woodin's argument that just one Shelah cardinal κ suffices. Conclusion 3.7 of the paper is, therefore, misleading.

The core of Section 3 is Lemma 3.2: suppose that $n \leqq \omega$, $S \subseteq \omega_1$ is stationary, $\langle \kappa_l : l \leqq n+1 \rangle$ is an increasing sequence of inaccessible cardinals, and $\langle \mathbb{P}_l : l \leqq n \rangle$ is a sequence of forcing notions that add no reals such that for every $l \leqq n$, $\mathbb{P}_l$ has cardinality $< \kappa_{l+1}$, does not destroy stationary subsets of ω_1 from V or $V^{\mathbb{P}_m}$, all $m < l$, and $\mathbb{P}_l$ forces that $\mathrm{NS}_{\omega_1}|S$ is saturated and $\kappa_l = \aleph_2$. Then, if $n < \omega$, every $\mathbf{\Sigma}^1_{n+2}$ set of reals is Lebesgue measurable, and so forth. And if $n = \omega$, this holds for every set of reals in $L(\mathbb{R})$. The conditions of the lemma do not look like large-cardinal hypotheses, so the point is to find some large-cardinal hypothesis that fits nicely within the hierarchy of large cardinals and implies the conditions. Claim 3.8 shows that these conditions are met whenever the property denoted by $(*)_{\kappa_l}$ holds for all cardinals κ_l, $l \leqq n$. The $\mathbb{P}_l$ are then the Levy-collapses of κ_l to ω_2. For a cardinal κ, the condition $(*)_\kappa$ says that if $\mathbb{P}$ is the Levy-collapse of κ to ω_1 and A is a maximal antichain of NS_{ω_1} in $V^{\mathbb{P}}$, then at every stage of the Levy-collapse, the remaining part of the collapse followed by the forcing that seals A is semiproper. In Lemma 3.6 it is shown that if κ is a Shelah cardinal, then a stronger condition $(*)^+_\kappa$ holds, namely, in the Levy-collapse of κ to ω_2, the forcing that seals any given antichain of NS_{ω_1} is semiproper, and this holds even after forcing with a forcing notion of size $< \kappa$. So, this explains the conclusion that $\omega + 1$ Shelah cardinals suffice for the Lebesgue measurability of sets of reals in $L(\mathbb{R})$, a hypothesis which, as we have already observed, is much stronger than what is really needed.

But the work of Section 3 is not wasted. In the last section, the dust settles and a weaker large-cardinal notion is introduced, that of a Woodin cardinal: κ is Woodin if for every $f : \kappa \to \kappa$ there is an elementary embedding $j : V \to M$, M transitive, with critical point some $\lambda < \kappa$, λ closed under f, and such that $H(j(f)(\lambda)) \subseteq M$. The main result (Theorem 4.1), which adapts the argument of Lemma 3.6, is that if a cardinal κ is Woodin, then $(*)_\kappa$ holds. Hence, if there are ω Woodin cardinals, then every $\mathbf{\Sigma}^1_n$ set of reals is Lebesgue measurable, has the property of Baire and is Ramsey, all $n < \omega$. And if there are $\omega + 1$ Woodin cardinals (actually ω Woodin cardinals plus a measurable cardinal above them all suffices), then the same holds for every set of reals in $L(\mathbb{R})$. Now this is indeed a *Happy end*, since by later results of Martin and Steel (see *Iteration trees*, reviewed below), the existence of n Woodin cardinals is consistent with the existence of a non-Lebesgue-measurable $\boldsymbol{\Sigma}^1_{n+2}$ set of reals.

Given the enormous importance of the theorems proved in the paper, it is unfortunate that it is written in a somewhat peculiar way and that the proofs omit many non-trivial details. Thus, the paper can only be fully understood by proficient set theorists who are already familiar with the subject matter. A further annoyance is that, apparently, the paper was not carefully edited before its publication: some references to the literature are wrong, several cross-references do not match, some non-standard notations are left undefined, and there are quite a number of typographical errors and small mistakes. Before going through the paper, the interested reader is therefore advised to consult section 3 of the aforementioned paper by Foreman, Magidor, and Shelah [F-M-S] and Chapters V and XI of Shelah's ***Proper and improper forcing***.

Further results than those contained in the paper under review have been proved by Woodin (*Supercompact cardinals, sets of reals, and weakly homogeneous trees*, JSL LVII 1132), where he shows that the existence of a supercompact cardinal implies that every set of reals in $L(\mathbb{R})$ is the projection of a weakly homogeneous tree, a very strong property that implies Lebesgue measurability, Baire property, and so forth. The result also follows from the existence of infinitely many Woodin cardinals with a measurable cardinal above them all. Finally, Martin and Steel [M-S] proved that the existence of infinitely many Woodin cardinals implies that all projective sets are determined and, by Woodin's result, that if there exist infinitely many Woodin cardinals with a measurable cardinal above them all, then all sets of reals in $L(\mathbb{R})$ are determined.

JOAN BAGARIA

Institució Catalana de Recerca i Estudis Avançats (ICREA) and Universitat de Barcelona, Baldiri Reixac, s/n. 08028 Barcelona, Spain. bagaria@trivium.gh.ub.es.

D. A. MARTIN and J. R. STEEL. *Iteration trees*. ***Journal of the American Mathematical Society***, vol. 7 (1994), pp. 1–73.

The main result announced in this paper is the construction, assuming the existence of a Woodin cardinal, of an inner model with a Woodin cardinal and a Δ^1_3 well-ordering of the reals; however, this statement does not fully explain the continuing importance of the paper. The model itself is unsatisfactory (for example, it is not known whether it satisfies the GCH) and is now largely obsolete, but the techniques used to construct it represent a major breakthrough and provide the foundation for most of the explosive growth of inner model theory since the paper's publication in 1994.

The development of inner model theory leading to this paper, and the problems that it solved, are very nicely described in the introduction which should be of general interest even to those who are not interested in the technical details. Briefly, recall that the success of Gödel's L, the original inner model, depends largely on the following crucial property: given any two well-founded models of the sentence "$V = L$," one of the models is an initial segment of the other. In order to obtain inner models with a measurable cardinal, and later to obtain models with sequences of measurable cardinals, it was sufficient to modify this property to state that any two such models have iterated ultrapowers, one of which is an initial segment of the other. In order to construct models approaching a Woodin cardinal these straightforward linear iterated ultrapowers must be replaced by *iteration trees*, in which an extender E_α may be applied not to the model M_α of which it is a member, but instead to some earlier model M_{α^*} of the iteration. This greatly complicates the problem of showing that a model is iterable: instead of simply showing that the unique limit of any linear iteration is well founded, one must provide a strategy that selects, for any iteration tree, a cofinal branch with a well-founded limit (assuming, of course, that this same strategy was used to choose the limit points of the tree).

The definition of the strategy is dealt with in §2 by proving the uniqueness result that, in the absence of a Woodin cardinal, no iteration tree has more than one cofinal branch with a well-founded limit. This result is more important than it may appear, and in fact it is the only result in the paper that is widely cited in its original form.

One application of this basic result appears in §§3–4 of the paper, which use a bookkeeping technique depending on the fact that any initial part $\mathcal{T} \upharpoonright \nu$ of a tree $\mathcal{T}$ has a known well-founded branch, namely the one capped by the νth node of the tree; hence every other branch of $\mathcal{T} \upharpoonright \nu$ has an ill-founded limit.

Another application, discovered later, provides one way of getting around the fact that the iterability techniques of §§3–4 apply only to countable trees: given an uncountable tree $\mathcal{T}$, one uses a generic collapse map to obtain a model in which $\mathcal{T}$ is countable and hence has a well-founded branch. The uniqueness theorem of §2 implies that the branch is independent of the collapse map, and hence it must lie in the ground model.

The proof of iterability in §§3–4 is the heart of the paper: §4 presents the basic techniques in illustrative special cases, and the full proof in §4 adds the bookkeeping needed for arbitrary countable trees. The iterability proof given in this paper applies only to the model of §6, but the ideas are much more general and many later papers consist largely of paraphrases of these two sections adapting the proof to more sophisticated models.

The remainder of the paper consists of some open problems in §5, preceding the construction of the promised model in §6. The problems in §5 ask for the elimination of some of the limitations to the techniques of this paper, and later developments have shown that they are overly optimistic: all of the main questions have been answered negatively. Much progress has been made, however, in extending these techniques to better models and to larger cardinals, by using methods that evade or modify some of the limitations. These limitations (the recognition and acceptance of which are partly responsible for the success of this paper) include the assumption that there are no smaller inner models with a Woodin cardinal, in order to use the result of §2; the restriction to countable iteration trees; and the requirement that each extender in the model have a background extender that is a somewhat stronger extender in the real world. It may be that another, comparable breakthrough will be needed to extend this work further in order to construct models containing cardinals in the region of a superstrong cardinal.

William Mitchell

Department of Mathematics, University of Florida, Gainesville, FL 32611, USA.
mitchell@math.ufl.edu.

Steve Jackson. *A new proof of the strong partition relation on ω_1.* ***Transactions of the American Mathematical Society***, vol. 320 (1990), pp. 737–745.

Steve Jackson. *Admissible Suslin cardinals in $L(\mathbf{R})$.* ***The journal of symbolic logic***, vol. 56 (1991), pp. 260–275.

Steve Jackson. ***A computation of $\boldsymbol{\delta}^1_5$.*** Memoirs of the American Mathematical Society, no. 670. American Mathematical Society, Providence 1999, viii + 94 pp.

In his 1983 UCLA dissertation, Steve Jackson solved the most important open problem in Cabal mathematics, i.e., descriptive set theory under determinacy axioms. The problem was to calculate $\boldsymbol{\delta}^1_5$. It is easy to explain the precise meaning of "calculating $\boldsymbol{\delta}^1_5$"; it will be explained, below. But this explanation is misleading, since "calculate $\boldsymbol{\delta}^1_5$" was, in part, a test question. It was believed that in order to calculate $\boldsymbol{\delta}^1_5$, one had to develop a very detailed and delicate analysis of all sorts of set-theoretic topics: ultrafilters, sets of ordinals, functions on ordinals, definability properties of all of the above, and infinite-exponent partition properties. So "calculating $\boldsymbol{\delta}^1_5$" was a shorthand phrase meaning the development of an extremely technical theory (now called the *very fine structure of* $L(\mathbb{R})$), a theory good enough to enable one to calculate $\boldsymbol{\delta}^1_5$. This technical theory is not easy to explain; this review cannot do it justice.

Throughout this review, we work in $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD} + V = L(\mathbb{R})$.

Let us begin with a quick summary of pre-Jackson descriptive set theory. A beautiful theory of projective sets had been developed. There was a reasonably complete understanding of the analytic aspects of projective sets: both regularity properties, such as the Baire property, and structural properties, such as uniformization. The theory exhibited a periodicity of order two. That is, the pointclasses $\boldsymbol{\Pi}^1_1$, $\boldsymbol{\Pi}^1_3$, $\boldsymbol{\Pi}^1_5$, etc. shared the same structural properties, as did $\boldsymbol{\Pi}^1_2$, $\boldsymbol{\Pi}^1_4$, $\boldsymbol{\Pi}^1_6$, etc. It was known that there existed surjections from the reals onto very large ordinals (e.g. weakly Mahlo cardinals), and hence questions about sets of ordinals pulled back to questions about sets of reals. So descriptive set theory also had a set-theoretic aspect: cardinality, cofinality, c.u.b. sets, etc. This set-theoretic aspect of the subject was not understood nearly as well as the analytic aspect. Furthermore, even the analytic theory had one major gap. This theory involved the *projective ordinals*, $\boldsymbol{\delta}^1_n$, but it was not known what these ordinals are.

The ordinal $\boldsymbol{\delta}^1_n$ is defined to be the supremum of the lengths of the $\boldsymbol{\Delta}^1_n$ prewellorderings of $\mathbb{R}$. Now $\boldsymbol{\delta}^1_1 = \aleph_1$, so $\boldsymbol{\delta}^1_3$, $\boldsymbol{\delta}^1_5$, etc. are the generalizations of $\aleph_1$ to higher levels of the projective hierarchy. One example of such a generalization must suffice. For κ a cardinal, the *κ-Borel* sets form the smallest collection of sets that includes the open sets and is closed under unions of λ-many sets, for all $\lambda < \kappa$. Suslin's theorem states that a set is $\boldsymbol{\Delta}^1_1$ iff it is $\boldsymbol{\delta}^1_1$-Borel. This generalizes: For odd n, a set is $\boldsymbol{\Delta}^1_n$ iff it is $\boldsymbol{\delta}^1_n$-Borel. It was known that all of the $\boldsymbol{\delta}^1_n$'s were cardinals. But which cardinals? To fill the gap in the analytic theory of projective sets, one needed to calculate the $\boldsymbol{\delta}^1_n$'s in terms of the $\aleph$ function. It was known that $\boldsymbol{\delta}^1_{2n+2}$ is the successor of $\boldsymbol{\delta}^1_{2n+1}$, so only the odd levels needed to be calculated. The first two cases were known: $\boldsymbol{\delta}^1_1 = \aleph_1$ and $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$. Therefore, the first open question was $\boldsymbol{\delta}^1_5$.

From previous work of Kunen and others, it was believed that the calculation of $\boldsymbol{\delta}^1_{2n+1}$ was related to calculating the lengths of the ultrapowers of $\boldsymbol{\delta}^1_{2n-1}$ with respect to ultrafilters on $\boldsymbol{\delta}^1_{2n-1}$, and that the ultrafilters in question were produced via infinite-exponent partition properties, which is a type of partition property inconsistent with AC. Specifically, for κ a regular cardinal and λ an ordinal with $\lambda \leqq \kappa$, $\kappa \to (\kappa)^\lambda$ means that any partition of the increasing λ-sequences from κ into two pieces has a homogeneous set of size κ. We say that κ satisfies the *strong partition property* if $\kappa \to (\kappa)^\kappa$ and the *weak partition property* if $\kappa \to (\kappa)^\lambda$ for all $\lambda < \kappa$.

In 1983, the author showed that $\boldsymbol{\delta}^1_5 = \aleph_\alpha$, where $\alpha = \omega^{\omega^\omega} + 1$. A few years later he was able to calculate the other projective ordinals as well: $\boldsymbol{\delta}^1_{2n+1} = \aleph_{[f(2n-1)+1]}$, where $f(-1) = 0$ and $f(m+1) = \omega^{f(m)}$. These results were announced in his note, *AD and the very fine structure of* $L(\mathbb{R})$, ***Bulletin of the American Mathematical Society***, n.s. vol. 21 (1989), pp. 77–81. The proof is by induction, and the induction requires calculating the $\boldsymbol{\delta}^1_n$'s and proving partition properties simultaneously. The calculation fills the gap in the analytic part of the theory of projective sets, and the proof gives a detailed understanding of the set-theoretic part of that theory. An expository paper on this topic will soon be published (S. Jackson, *Structural consequences of AD*, ***Handbook of set theory***, Kluwer, to appear in 2003). While that paper is by no means easy reading, it is considerably easier than reading the actual proofs, and it is where those with a serious interest in this subject should begin. The third publication under review contains a complete proof of the first step in the induction: Assuming the strong partition property for $\boldsymbol{\delta}^1_1$ and the weak partition property for $\boldsymbol{\delta}^1_3$, the author calculates $\boldsymbol{\delta}^1_5$ and proves the strong partition property for $\boldsymbol{\delta}^1_3$ and the weak partition property for $\boldsymbol{\delta}^1_5$. The proof for $\boldsymbol{\delta}^1_7$ and higher has still not been published.

At the bottom levels of the projective hierarchy, deep set-theoretic results had been proved prior to Jackson's work, but these results relied on the theory of indiscernibles for $L(x)$ (x a real), and indiscernibility theory does not generalize to higher levels. So part of Jackson's task was to get new proofs that do not involve indiscernibility. Consider the strong partition property for ω_1, which was first proved by Martin. To prove it, one needs a coding by reals of functions from ω_1 into ω_1 that satisfies a complicated form of the boundedness theorem. AD implies that all such functions are in $L(x)$ for some x, so Martin's coding was simply the real $x^\#$ together with a Skolem term. In the first paper under review, the author gives a new proof by finding a different coding of functions; he analyzes ultrafilters on ω_1 and uses this analysis to get a coding that works. This new proof does generalize, and it constitutes part of the calculation of the projective ordinals.

We have considered projective sets and projective ordinals, but it is natural to go on to more complicated pointsets and larger ordinals. A set of reals is *κ-Suslin* if it is the projection of a tree on $\omega \times \kappa$, and κ is a *Suslin cardinal* if there is a set that is κ-Suslin but not λ-Suslin for any $\lambda < \kappa$. The Suslin cardinals are c.u.b. in their supremum. This concept is the proper generalization of projective ordinals: The first ω Suslin cardinals are the predecessors of the projective ordinals. Thus $\boldsymbol{\delta}^1_{2n+1}$ is a Suslin cardinal and a set is $\boldsymbol{\delta}^1_{2n+1}$-Suslin iff it is $\boldsymbol{\Sigma}^1_{2n+2}$. The generalization of "calculate $\boldsymbol{\delta}^1_n$," calculating a Suslin cardinal κ in terms of the $\aleph$ function, will often be either meaningless or trivial; for example, if κ is sufficiently closed, then $\kappa = \aleph_\kappa$. Nevertheless, it is meaningful to generalize very fine structure up to κ: to analyze ultrafilters and functions on ordinals, to prove partition properties, etc. It seems likely that the methods used for the projective ordinals should handle all the successor Suslin cardinals, and that other known methods will handle singular limit Suslin cardinals. That leaves the case of inaccessible Suslin cardinals.

To see where the problem lies, consider the normal measures on a regular Suslin cardinal, κ. If κ is $\boldsymbol{\delta}^1_{2n+1}$, then for regular $\lambda < \kappa$, the λ-c.u.b. filter on κ is a normal measure, and these are the only normal measures. But if κ is weakly Mahlo, there are stationary sets of regular cardinals, and, as shown by Kleinberg, these stationary sets are associated with normal measures; furthermore, there is a well-ordering of (equivalence classes of) stationary sets similar to the Mahlo order, whose order-type we denote $o(\kappa)$. If $o(\kappa)$ is not too large, the projective ordinal theory generalizes, but if $o(\kappa)$ is too large, there are serious problems which have not yet been overcome. But does there exist a κ such that $o(\kappa)$ is "too large"? In the second paper under review, the author proves that such κ's do exist. This shows that current techniques for developing the very fine structure of $L(\mathbb{R})$ break down at some point (and, in fact, break down at $\kappa^{\mathbb{R}}$, the ordinal of the inductive sets). While today's techniques do not succeed in extending the theory, it is likely that in the future more sophisticated techniques will succeed, and that Jackson's very fine structure theory will extend to all of $L(\mathbb{R})$.

Howard S. Becker

Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA. becker@math.sc.edu.

Itay Neeman and Jindřich Zapletal. *Proper forcings and absoluteness in $L(\mathbb{R})$*. ***Commentationes mathematicae Universitatis Carolinae***, vol. 39 (1998), pp. 281–301.

Itay Neeman and Jindřich Zapletal. *Proper forcing and $L(\mathbb{R})$*. ***The journal of symbolic logic***, vol. 66 (2001), pp. 801–810.

It is by now a standard fact in set theory, due to W. Hugh Woodin, that if there exist certain large cardinals—a proper class of Woodin cardinals suffices—then the theory of the inner model $L(\mathbb{R})$, the smallest model of Zermelo–Fraenkel set theory containing the reals and the ordinals, cannot be changed by set forcing, even if one adds a constant for each real number. The two papers under review extend this result to the theory of $L(\mathbb{R})$ with parameters for real numbers and ordinals, under the restriction that the set forcing in question is *reasonable*.

Roughly speaking, a forcing construction P is proper if for every condition p in P and every countable elementary substructure X (containing p and P) of a suitably large initial segment of the universe, there is a condition $q \leqq p$ in P that is generic for X, i.e., that forces every maximal P-antichain in X to intersect the generic filter at a condition in X. A forcing P is reasonable if for each p in P there exists a stationary set of such X. Note that reasonable forcings need not be semi-proper, as the standard forcing to build a club disjoint from a given costationary subset of ω_1 by initial segments is reasonable. For absoluteness with ordinal parameters, some restriction of the class of forcings is required: if α is any uncountable ordinal in the ground model then there is a partial order that adds a surjection from ω onto α, making α countable in the $L(\mathbb{R})$ of the extension. Woodin has provided another counterexample, showing that Shelah's semi-proper forcing to make the non-stationary ideal on ω_1 saturated can change the value of the ordinal $\boldsymbol{\delta}^1_2$ (the supremum of the lengths of the $\boldsymbol{\Delta}^1_2$ prewellorderings of the reals, all of which is computed in $L(\mathbb{R})$).

The two main results of the two papers are stated in terms of a cardinal κ satisfying a certain large-cardinal property, a reasonable forcing P of cardinality less than κ, and a V-generic filter $G \subset P$. The first result, the embedding theorem, says that given such objects there exists an elementary embedding from $L(\mathbb{R})^V$ to $L(\mathbb{R})^{V[G]}$ that fixes the ordinals. The second, the anticoding theorem, states that if A is a set of ordinals, then $A \in L(\mathbb{R})^V$ if and only if $A \in L(\mathbb{R})^{V[G]}$.

The two papers derive these facts by different methods and from different hypotheses. The first paper presumes that κ is a weakly compact Woodin cardinal and uses the stationary tower, specifically the tower $\mathbb{Q}_{<\kappa}$ consisting of stationary sets of countable sets. The paper also applies several unpublished theorems of Woodin (and provides their proofs, in itself a service to the community). In particular, a theorem of Woodin showing that certain sets of reals are $<\delta$-weakly homogeneously Suslin (for a given Woodin cardinal δ) is presented, and then used to prove the image theorem: if κ is a weakly compact Woodin cardinal and A is a bounded subset of $\Theta^{L(\mathbb{R})}$, then $A \in L(\mathbb{R})$ if and only if, for some set B, every condition in $\mathbb{Q}_{<\kappa}$ forces that $j(A) = B$, where j is the induced embedding. The key property of reasonable forcings is the following fact: if κ is a Woodin cardinal and $P \in V_\kappa$ is reasonable, then it is possible, in a set forcing extension of V, to have elementary embeddings $j : V \to M$ and $j^* : V[G] \to N$, where $V[G]$ is an extension of V by P, j is induced by the $\mathbb{Q}_{<\kappa}$ of V, and $j^* \restriction V = j$. (In fact, the first paper deals with proper forcing instead of reasonable forcing, but this fact can easily be shown for reasonable forcing, and then the rest of the proof of the embedding theorem is easily modified.) The last important bit of machinery needed is a result of Woodin saying that if the axiom of determinacy holds in $L(\mathbb{R})$, then $L(\mathbb{R})$ is a symmetric extension of its version of HOD, the inner model consisting of the hereditarily ordinal definable sets. It is shown furthermore that the forcing producing this extension is homogeneous, in the sense that HOD can compute the theory of its symmetric extension with parameters from the ground model (slightly more than this is needed: the forcing in question is an iteration, and the intermediate models can similarly compute the theory of their common extension). To prove the embedding theorem, then, one shows that $L(\mathbb{R})^{V[G]}$ is a symmetric extension of $\mathrm{HOD}^{L(\mathbb{R} \cap V)}$ by the same forcing. Similar methods are used to prove the anticoding theorem, although modifying the written proof to account for reasonable forcings appears to be more difficult. Essentially the same arguments go through with a small increase in complexity (and the need to consider ill-founded ultrapowers) if κ is assumed to be a limit of Woodin cardinals below a measurable cardinal. The final section of the paper shows that it is consistent that there is a subset A of ω_1 not in $L(\mathbb{R})$ and a forcing preserving stationary subsets of ω_1 forcing A into $L(\mathbb{R})$, thereby showing that the restriction to reasonable forcings is necessary for the anticoding theorem, even if one restricts to subsets of ω_1. Woodin had previously shown this without this restriction, again using the fact that Shelah's forcing can change $\boldsymbol{\delta}^1_2$.

The second paper proceeds from the existence of certain inner models in which some ordinal countable in V is a limit of Woodin cardinals. These models are also assumed to be *uniquely iterable* for iteration trees of length up to and including κ^+. This means first of all that these models are iterable, in the sense that the good player wins the corresponding full iteration game of length $\kappa^+ + 1$, and furthermore that any corresponding iteration tree of size $< \kappa^+$ has a unique cofinal branch such that the resulting limit model is itself iterable. The existence of such models has consistency strength below that of infinitely many Woodin cardinals below a measurable cardinal. The proofs here rely on results of Woodin using a construction sometimes (though not here) called the extender algebra; the proofs of these results are given in an eprint version of the paper, posted at an address given in the references. Roughly, these results say that in any sufficiently iterable inner model M—the models considered here suffice—there is a forcing construction P in M such that for any real x (in any model) there is an iteration $\pi : M \to N$ such that x is (or induces) an N-generic filter for $\pi(P)$. To prove the embedding theorem then one produces for any given real z an iterate M_∞ of M containing z such that $\mathbb{R} \cap V$ and $\mathbb{R} \cap V[G]$ are both the reals of symmetric extensions of M_∞ by (different) generic filters contained in the same homogeneous forcing, showing that $L(\mathbb{R})^V$ and $L(\mathbb{R})^{V[G]}$ satisfy the same sentences with parameters for ordinals and for z. To show that this symmetric extension works, one needs that the required extender algebras in M_∞ (really, their power-sets in M_∞) are countable in V. To ensure that the reals of $V[G]$ are also a symmetric extension of M_∞, an iteration given by a theorem of Woodin is modified to produce a suitable countable version by collapsing a countable elementary submodel (of a sufficiently large initial segment of V) in V whose existence is given by the fact that P is reasonable. Again, similar, though more involved, methods are used to prove the anticoding theorem.

That iteration trees and the stationary tower can often reproduce one another's results is by now a well-established (though unexplained) phenomenon, although this is perhaps the first time that it has been displayed so explicitly. The authors note as well that the embedding theorem can also be derived from work of Foreman and Magidor, in conjunction with unpublished results of Woodin.

Paul B. Larson

Department of Mathematical Sciences, University of Toronto, Toronto, ON M5S 3G3, Canada. larson@math.toronto.edu.

James Cummings. *A model in which GCH holds at successors but fails at limits*. ***Transactions of the American Mathematical Society***, vol. 329 (1992), pp. 1–39.

James Cummings. *Strong ultrapowers and long core models*. ***The journal of symbolic logic***, vol. 58 (1993), pp. 240–248.

James Cummings. *Coherent sequences versus Radin sequences*. ***Annals of pure and applied logic***, vol. 70 (1994), pp. 223–241.

James Cummings, Matthew Foreman, and Menachem Magidor. *Squares, scales and stationary reflection*. ***Journal of mathematical logic***, vol. 1 (2001), pp. 35–98.

The papers under review represent a selection of James Cummings's work in large cardinals and forcing. Throughout, κ and λ will be used to represent uncountable cardinals.

The first of Cummings's papers is related to the singular cardinals hypothesis (SCH), which is one of the most venerable hypotheses in the history of set theory. It can be stated in numerous ways, but for convenience, and in keeping with Cummings's presentation in this paper, we will let SCH be the assertion that if κ is singular, then the size of 2^κ is the least cardinal $\lambda \geqq |2^{<\kappa}|$ such that $\mathrm{cof}(\lambda) > \kappa$.

Much is known about SCH. A very brief discussion of some of the relevant results will now be given. In the 1970's, Silver showed in his paper *On the singular cardinals problem* (JSL XLVI 864) that if the generalized continuum hypothesis (GCH) fails at a singular cardinal κ of uncountable cofinality, then GCH must fail on an unbounded set (in fact, on a closed, unbounded set) of cardinals below κ. On the other hand, Magidor, also in the 1970's, showed using strong hypotheses that the analogue of Silver's result was not true for cardinals of countable cofinality. In particular, Magidor showed in his paper *On the singular cardinals problem II* (JSL XLVI 864), starting from a ground model in which $\kappa < \lambda$ are such that κ is supercompact and λ is huge, that it is possible to construct a generic extension in which $2^{\aleph_n} = \aleph_{n+1}$ for every natural number n, yet $2^{\aleph_\omega} = \aleph_{\omega+2}$. Later, in the late 1980's and early 1990's, Gitik showed in two papers (*The negation of the singular cardinal hypothesis from* $o(\kappa) = \kappa^{++}$, JSL LVI 344, and *The strength of the failure of the singular cardinal hypothesis*, JSL LX 340) that the exact consistency strength of $2^{\aleph_n} = \aleph_{n+1}$ for every natural number n and $2^{\aleph_\omega} = \aleph_{\omega+2}$ is a cardinal κ of Mitchell order κ^{++}, thereby weakening considerably the hypotheses Magidor used.

The first of Cummings's papers provides a global version of Magidor's and Gitik's results. Starting from a ground model containing a $\wp^3(\kappa)$-hypermeasurable cardinal (in modern terminology, such a cardinal is called a $\kappa + 3$ strong cardinal), Cummings constructs a submodel of a generic extension in which, as his title suggests, $2^\kappa = \kappa^+$ if κ is a successor cardinal, yet $2^\kappa = \kappa^{++}$ if κ is a limit cardinal. Cummings's proof is intricate and quite involved, using as it does a generalized version of Radin forcing (to be discussed at greater length below). His proof also relies to a large extent on techniques and analyses due to Woodin.

The second of Cummings's papers considers a question posed by Steel, namely whether it is possible to have a model V in which there is a measurable cardinal κ carrying a normal measure $\mathcal{U}$ such that $\wp(\kappa^+) \subseteq V^\kappa/\mathcal{U}$. (Note that it is always the case that if κ is a measurable cardinal in V and $\mathcal{U} \in V$ is a normal measure over κ, then $\wp(\kappa) \subseteq V^\kappa/\mathcal{U}$.) Cummings shows in this paper that the existence of such a model is equiconsistent with the existence of a $\wp^2(\kappa)$-hypermeasurable, i.e., $\kappa + 2$ strong, cardinal. As in his first paper, the forward direction of Cummings's proof, namely the construction of the model in which this unusual measurable cardinal exists, uses forcing techniques and analyses due to Woodin. The reverse direction, in which it is shown that if one starts with a model where such a measurable cardinal κ exists, then κ is $\wp^2(\kappa)$-hypermeasurable in an inner model, uses the theory of core models for non-overlapping extender sequences.

The third of Cummings's papers is a technical study on Radin forcing, which was introduced by Radin in his paper *Adding closed cofinal sequences to large cardinals*, ***Annals of mathematical logic***, vol. 22 (1982), pp. 243–261. Radin forcing provides a means of adding a closed, unbounded set to a large cardinal (such as a measurable cardinal of high Mitchell order or a supercompact cardinal) via a sequence of ultrafilters called a Radin sequence of measures which is frequently defined by using some sort of sufficiently closed elementary embedding. This closed, unbounded set is usually called a Radin sequence. Depending upon the exact nature of the definition of the Radin forcing, the cardinal κ to which the Radin sequence has been added either becomes a singular cardinal, retains its regularity, or retains both its regularity and the large cardinal property it originally exhibited. Radin forcing has found numerous applications, such as in Mitchell's paper *How weak is a closed unbounded ultrafilter?* (***Logic Colloquium '80***, Studies in logic and the foundations of mathematics, vol. 108, North-Holland Publishing Company, 1982, pp. 209–230), Gitik's paper *Regular cardinals in models of ZF* (JSL LIX 668), the first of Cummings's papers reviewed herein, and unpublished work due to Woodin.

In the paper under review, Cummings starts with the assumption that $V = L[\vec{E}]$, where $\vec{E}$ is a certain coherent sequence of extenders having $(\kappa, \kappa^{++}) \in \mathrm{dom}(\vec{E})$. He then lets $E = \vec{E}(\kappa, \kappa^{++})$ and takes $j_E : V \to M_E$ as the associated elementary embedding and $\langle u_\alpha : \alpha < \kappa^{++}\rangle$ as the Radin sequence of measures definable from j_E. Under these circumstances, he characterizes the measures u_α in terms of the elementary embedding generated by a certain portion of E, where α satisfies a particular technical condition that occurs unboundedly often below κ^{++}. He then defines a forcing notion $\mathbb{R}(\vec{F}, \vec{\gamma})$ via a coherent sequence of extenders $\vec{F}$ and a suitable sequence of ordinals $\vec{\gamma}$ and shows that, depending upon the exact nature of $\vec{F}$ and $\vec{\gamma}$, $\mathbb{R}(\vec{F}, \vec{\gamma})$ corresponds either to Radin's version of Radin forcing as given in his paper mentioned above or to Mitchell's version of Radin forcing as given in his paper mentioned above.

The fourth of Cummings's papers, written jointly with Foreman and Magidor, is an extensive article discussing different versions of Jensen's combinatorial principle square ($\square$) and their relationships to scales as found in Shelah's pcf theory and various kinds of stationary reflection principles. It is divided into fourteen sections. I will try to summarize and highlight their contents in what follows, although for brevity, there will be omissions made. Sections 1 and 2 are introductory, containing a discussion on background information, basic terminology, and some known results. Sections 3 and 4 discuss relationships among different principles due to Shelah, weak forms of square, and various reflection properties. Section 5 discusses an improved version of square and limitations on its consistency. Sections 6 and 8 discuss how one forces various kinds of square sequences, with Section 8 containing the surprising result that under certain circumstances, it is possible to do a small forcing that adds a square sequence. Section 7 discusses how to construct a model by forcing in which a certain weak version of square holds at $\aleph_\omega$ yet another weak version of square fails at $\aleph_\omega$. Section 9 shows that even though there are limitations to the kinds of square sequences that can exist above a supercompact cardinal, it is still possible to force and obtain a certain square principle that can hold above a supercompact cardinal. Section 10 contains another discussion about square and reflection principles at $\aleph_\omega$. Section 11 discusses the kind of reflection and scale principles found after Prikry forcing has been done at the measurable cardinal κ. Section 12 shows how to force a strong form of diamond ($\Diamond$) due to Jensen and how to use it to obtain another result about square and reflection principles at $\aleph_\omega$. Section 13 shows how to force over a ground model for a strong form of Martin's Maximum to obtain a model in which one reflection principle holds yet another one fails. Section 14 concludes the paper with a list of open problems and a diagram indicating the relationships among the different square, scale, and stationary reflection principles discussed.

Cummings and his coauthors provide lucid and well-written presentations of important and difficult theorems in the theory of large cardinals and forcing, touching also upon relevant related results in inner model theory and pcf theory. Persistent and dedicated readers of these papers will be richly rewarded for their efforts.

Arthur W. Apter

Department of Mathematics, Baruch College of CUNY, New York, NY 10010, USA. awabb@cunyvm.cuny.edu.

Andreas Blass. *Simple cardinal characteristics of the continuum*. ***Set theory of the reals***, edited by Haim Judah, Israel mathematical conference proceedings, vol. 6, Gelbart Research Institute for Mathematical Sciences, Bar-Ilan University, Ramat-Gan 1993, distributed by the American Mathematical Society, Providence, pp. 63–90.

The paper investigates some of the common cardinal characteristics with respect to the complexity of their definitions in terms of descriptive set theory.

Let Γ denote a pointclass in the sense of descriptive set theory, boldface or lightface or even the whole OD$\mathbb{R}$. The reader should be aware that in the print in the proceedings volume the boldface Greek letters appear as lightface, and either think harder or get a version of the article where the TeX comes out as originally intended (e.g. the version posted on http://www.math.lsa.umich.edu/~ablass).

The following definitions are the starting point: An uncountable cardinal κ is a Γ-characteristic if there is a family of κ sets, each in Γ, such that ${}^\omega\omega$ is covered by the family but not by any subfamily of cardinality less than κ. An uncountable cardinal κ is a uniform Γ-characteristic if there is a binary relation R on ${}^{\omega}\omega$ such that $R \in \Gamma$ and such that κ is the minimum cardinality of a family $\mathcal{X} \subseteq {}^{\omega}\omega$ such that for every $y \in {}^{\omega}\omega$ there exists $x \in \mathcal{X}$ with $R(x, y)$.

In §2 it is shown that numerous well-known cardinal characteristics are uniform Π^0_2-characteristics or uniform Σ^0_2-characteristics and that the tower number $\mathfrak{t}$ is a $\mathbf{\Pi}^0_2$-characteristic. It is shown that every (uniform) $\mathbf{\Sigma}^0_{n+1}$-characteristic is a (uniform) $\mathbf{\Pi}^0_n$-characteristic and that there are no $\mathbf{\Sigma}^0_1$-characteristics. Some related implications are proved. In some cases the given complexity is sharp, though there is not (yet) any notion of completeness or universality for a complexity class of characteristics.

In §3 it is proved that $\mathbf{cov}(B)$ is a lower bound for all $\mathbf{\Pi}^0_1$-characteristics.

In §4 the following theorem on the arbitrariness in $\mathbf{\Pi}^0_2$-characteristics is proved: Assume GCH and let C be a closed set of uncountable cardinals containing $\aleph_1$, containing all uncountable cardinals $\leqq |C|$, and containing the immediate successors of all its members of cofinality ω. Then there is a notion of forcing satisfying the c.c.c. and forcing that $\mathfrak{c} = \max(C)$ and that both the set of $\mathbf{\Pi}^0_2$-characteristics and the set of OD$\mathbb{R}$-characteristics are equal to C.

Each $\kappa \in C$ becomes a $\mathbf{\Pi}^0_2$-characteristic in the forcing extension because a maximal almost-disjoint family of size κ is added. That $\lambda \notin C$ is not even an OD$\mathbb{R}$-characteristic is proved by using automorphisms of the forcing partial order P. Only some of the premisses on C are shown to be necessary, whereas the necessity of the others is still open, which is also mentioned in Question 4 in the open question section, §7. The issue reminds the reviewer of pcf and in particular of Theorem 1.3 and the remarks and announcements thereafter in Shelah and Thomas, *The cofinality spectrum of the infinite symmetric group*, ***The journal of symbolic logic***, vol. 62 (1997), pp. 902–916.

The corresponding question on the collection of $\mathbf{\Pi}^0_1$-characteristics was posed by Blass at the Bar-Ilan conference and answered by Goldstern and Shelah in *Many simple cardinal invariants*, ***Archive for mathematical logic***, vol, 32 (1993), pp. 203–221.

In §5 some intricacies of dualizing cardinal characteristics are carefully discussed. A relation R is called invariant if changing finitely many digits in x and in y does not change the truth of $R(x, y)$. A dual form of the result of §3 is proved: If κ is a uniform $\mathbf{\Pi}^0_2$-characteristic that is determined by an invariant $\mathbf{\Pi}^0_2$-relation, then $\kappa \leqq \mathbf{unif}(B)$.

In §6 the partition characteristics **par** and **hom** and relatives are defined and lower and upper bound are derived.

In §7 open questions are listed. Meanwhile, by August 2002, some of them have been (partially) answered. About the complexity of $\mathfrak{h}$ and $\mathfrak{g}$: Zapletal introduced a notion of tameness, that is a wider notion than being uniform Γ for some projective Γ. Using deep work by Zapletal based on large cardinals and results by Woodin, Mildenberger and Zapletal showed that $\mathfrak{g}$ and $\mathfrak{h}$ are not tame. See Appendix B.2 of Zapletal's forthcoming monograph, ***Cardinal invariants and descriptive set theory***, http://www.math.ufl.edu/~zapletal. The questions on the complexities of $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{a}$, $\mathbf{cof}(B)$, and others are still open to the reviewer's knowledge. (Some of them are tame.) The question whether $\mathbf{hom} = \mathbf{hom}_3$ was answered affirmatively by Brendle, who proved $\mathbf{hom}_3 = \max(\mathfrak{r}_\sigma, \mathfrak{d})$ in J. Brendle, *Strolling through paradise*, ***Fundamenta mathematicae***, vol. 148 (1995), pp. 1–25.

The appendix to the article contains an elegant simplification of Shelah's proof of $\mathfrak{d} \leqq \mathfrak{i}$.

The paper is characteristically well written. It motivates further research by its results, by asking questions, and by giving streamlined proofs to known results and thus making them better accessible.

HEIKE MILDENBERGER

University of Vienna, Institut für formale Logik, Währinger Str. 25, 1090 Wien, Austria. heike@logic.univie.ac.at.

B. BALCAR and F. FRANEK. *Independent families in complete Boolean algebras*. ***Transactions of the American Mathematical Society***, vol. 274 (1982), pp. 607–618.

BOHUSLAV BALCAR, JAN PELANT, and PETR SIMON. *The space of ultrafilters on N covered by nowhere dense sets*. ***Fundamenta mathematicae***, vol. 110 (1980), pp. 11–24.

BOBAN VELICKOVIC. *OCA and automorphisms of $\mathcal{P}(\omega)/fin$*. ***Topology and its applications***, vol. 49 (1993), pp. 1–13.

The three papers under review, though they treat markedly different subjects, have in common that they mark important points in the developments of these subjects: the Balcar–Franek paper marks an end, the other two mark a beginning.

The main theorem from the Balcar–Franek paper says that a complete Boolean algebra has an independent family of maximum possible size, i.e., if B is complete then it has a subset A of the same cardinality such that

$$\bigwedge F \setminus \bigvee G > 0,$$

whenever F and G are finite disjoint subsets of A. It marks an end because it solves a specific problem and settles a host of questions on complete Boolean algebras (and, dually, extremally disconnected spaces) that depend on the completeness and the cardinality of the algebras alone—some might even argue that it settles all such problems. An instructive example is provided by homomorphic images: if B and C are complete and $|C| \leqq |B|$, then C is a homomorphic image of B; to prove this, map some independent family in B onto C and apply injectivity of C to extend this map to a homomorphism—the dual version states, given two extremally disconnected spaces X and Y, one can embed X into Y iff $w(X) \leqq w(Y)$.

The Balcar–Pelant–Simon paper should have been called *$\mathcal{P}(\omega)/fin$ has a dense tree* (or, for topologists, *ω^* has a tree π-base*) because to many working on this algebra and its Stone space ω^*, this is one of the most important structural facts discovered about these objects. To be precise: there is a dense $2^{\aleph_0}$-branching tree in $\mathcal{P}(\omega)/fin$ whose height equals the distributivity number of the algebra (this cardinal is the same as the minimum cardinality of a family of nowhere dense sets in ω^* with a dense union). This tree has become an important tool in constructions and it is, naturally, useful when one tries to recognize algebras that have the same completion as $\mathcal{P}(\omega)/fin$.

The third paper has in common with the second one that it offers researchers an important handle on the structure of $\mathcal{P}(\omega)/fin$; in this case for dealing with homomorphisms between it and related algebras. In it one finds two things: (1) a proof, building on Veličković's *Definable automorphisms of $\mathcal{P}(\omega)/fin$* (***Proceedings of the American Mathematical Society***, vol. 96 (1986), pp. 130–135), that an automorphism of $\mathcal{P}(\omega)/fin$ with a close-to-definable lifting must be trivial (meaning that it is induced by bijection between cofinite subsets of ω); and (2) a proof from OCA + MA that any automorphism of $\mathcal{P}(\omega)/fin$ must have such a lifting. This can be seen as a synthesis of earlier proofs of the consistency that all automorphisms of $\mathcal{P}(\omega)/fin$ are trivial (Shelah, ***Proper forcing***, JSL L 237, and Shelah and Steprāns, *PFA implies all automorphisms are trivial*, ***Proceedings of the American Mathematical Society***, vol. 104 (1988), pp. 1220–1225).

The methods used in this paper were versatile enough to cause a small flood of results on triviality and non-existence for homomorphisms between $\mathcal{P}(\omega)/fin$ and other algebras; the notes and comments in Farah's ***Analytic quotients, Theory of liftings for quotients over analytic ideals on the integers*** (Memoirs of the American Mathematical Society, no. 702, 2000) give a better overview of these than would be possible in this space.

KLAAS PIETER HART

Section Algebra and Geometry, Faculty of Information Technology and Systems, TU Delft, Postbus 5031, 2600 GA Delft, The Netherlands.
k.p.hart@its.tudelft.nl.

The entwining snake is reborn

Richard Laver. *The left distributive law and the freeness of an algebra of elementary embeddings*. ***Advances in mathematics***, vol. 91 (1992), pp. 209–231.

Richard Laver. *A division algorithm for the free left distributive algebra*. ***Logic Colloquium '90, ASL summer meeting in Helsinki***, edited by J. Oikkonen and J. Väänänen, Lecture notes in logic, no. 2, Springer-Verlag, Berlin, Heidelberg, New York, etc., 1993, pp. 155–162.

Richard Laver. *On the algebra of elementary embeddings of a rank into itself*. ***Advances in mathematics***, vol. 110 (1995), pp. 334–346.

Richard Laver. *Braid group actions on left distributive structures, and well orderings in the braid groups*. ***Journal of pure and applied algebra***, vol. 108 (1996), pp. 81–98.

Patrick Dehornoy. *An alternative proof of Laver's results on the algebra generated by an elementary embedding*. ***Set theory of the continuum***, edited by H. Judah, W. Just, and H. Woodin, Mathematics Sciences Research Institute publications, vol. 26, Springer-Verlag, New York, Berlin, Heidelberg, etc., 1992, pp. 27–33.

Patrick Dehornoy. *Braid groups and left distributive operations*. ***Transactions of the American Mathematical Society***, vol. 345 (1994), pp. 115–150.

Patrick Dehornoy. *A normal form for the free left distributive law*. ***International journal of algebra and computation***, vol. 4 (1994), pp. 499–528.

Patrick Dehornoy. *From large cardinals to braids via distributive algebra*. ***Journal of knot theory and its ramifications***, vol. 4 (1995), pp. 33–79.

J. R. Steel. *The well-foundedness of the Mitchell order*. ***The journal of symbolic logic***, vol. 58 (1993), pp. 931–940.

Randall Dougherty. *Critical points in an algebra of elementary embeddings*. ***Annals of pure and applied logic***, vol. 65 (1993), pp. 211–241.

Randall Dougherty. *Critical points in an algebra of elementary embeddings, II*. ***Logic: from foundations to applications, European logic colloquium***, edited by Wilfrid Hodges, Martin Hyland, Charles Steinhorn, and John Truss, Clarendon Press, Oxford University Press, Oxford, New York, etc., 1996, pp. 103–136.

Randall Dougherty and Thomas Jech. *Finite left-distributive algebras and embedding algebras*. ***Advances in mathematics***, vol. 130 (1997), pp. 201–241.

Many of the most fascinating and rewarding mathematical results are associated with connecting disparate concepts and areas. The links Richard Laver establishes in [L1] and [L2] clearly fulfill this criterion. (The papers under review will be cited as [L1]–[L4], [D1]–[D4], [St], [Do1]–[Do2], and [DJ] in the order listed above.) While algebra and set theory nowadays yield many interactions, Laver's coupling of large cardinals with discrete (and even finite) algebraic binary objects seems to remain unchallenged in its uniqueness.

On a naïve level there is even a temptation to regard the connections as something nearly mystical: on one hand there are non-trivial elementary embeddings of a rank R_λ into itself—a faithful copy of the (mathematical) reality within its own limits; on the other hand there stands the left distributive law

$$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z),$$

a strikingly simple algebraic expression of self-similarity. Indeed, it is natural to regard self-similarities as mappings, and a binary operation that embodies the self-similarity can be expected to exhibit a great diversity of such mappings that can be derived from it in a direct way. Self-similarities are mathematically endomorphisms, and the binary operation $\cdot$ is left distributive if and only if the left translation $x \mapsto a \cdot x$ is an endomorphism for every a.

Patrick Dehornoy connects free left distributive systems to the braid group B_∞, and this made the subject even more prone to mystical interpretation: the entwining snake that is reborn.

Technically, elementary embeddings give rise to self-distributivity by applying one of them, say $j : R_\lambda \to R_\lambda$, to another, say $k : R_\lambda \to R_\lambda$, setting

$$j \cdot k = \bigcup_{\gamma<\lambda} j(k \upharpoonright R_\gamma).$$

It is not difficult to see that such an operation is left distributive. Laver proves [L1] that its closure, starting from a single non-trivial elementary embedding j, is isomorphic to a free monogenerated left distributive binary system. Such a system can be viewed as the quotient of the set of all binary terms in a variable x over the equivalence $\equiv$ that is obtained by transitive applications of mutual exchanges of subterms $r \cdot (s \cdot t)$ and $(r \cdot s) \cdot (r \cdot t)$. Every term in one variable can be identified with a certain binary tree, and it seems natural to expect that a tree will never be $\equiv$-equivalent to any of its proper subtrees. It turned out, however, that such a property is not easy to verify, even when restricted to left subtrees (subtrees with roots that can be reached by traveling from the top only to the left). This property was for some time called the irreflexivity conjecture, and it was not known if it is possible to prove (in ZFC) the existence of monogenerated binary LD-systems that satisfy it (such systems are then necessarily free). Laver derives [L1], by a short argument, the irreflexivity conjecture from the assumption of the existence of a non-trivial elementary embedding.

The larger part of [L1] has little to do with the set theory and deals with the proof of the existence of a normal form for the free LD-system generated by x. There was not much upon which Laver could have built: while left distributive operations are quite common in algebra (simply think about group conjugation), they are nearly always idempotent ($x \cdot x = x$). Laver could, however, use the fact that elementary embeddings can be composed, and this led him to the laws of the LD-monoid (Dehornoy's term):

$$\begin{aligned} (x \circ y) \circ z &= x \circ (y \circ z), & x \cdot (y \circ z) &= (x \cdot y) \circ (x \cdot z), \\ x \circ y &= (x \cdot y) \circ x, & x \cdot (y \cdot z) &= (x \circ y) \cdot z. \end{aligned}$$

What Laver does, in fact, is to determine a normal form for LD-monoids (free LD-systems embed into free LD-monoids, but the normal form of a term using only $\cdot$ can involve $\circ$). I shall say more about the nature of Laver's normal form(s) later.

Laver's normal form can be used to decide the word problem. But the proof of its existence is technically rather demanding. Dehornoy suggests [D1] an alternative way to decide if two binary terms are LD-equivalent. He proves that $u \equiv v$ holds if and only if v expands to some $\partial^k u$. Now, an expansion means replacement of a subterm $r \cdot (s \cdot t)$ by $(r \cdot s) \cdot (r \cdot t)$, and $\partial^k u$ refers to an iterative application of the operator ∂. One can describe ∂u as an expansion of u at every possible subterm position, but without iterations on the same position. Formally, $\partial x = x$, $\partial(r \cdot s) = \partial r * \partial s$, where $*$ stands here for $t * x = t \cdot x$ and $t * (r \cdot s) = (t * r) \cdot (t * s)$. Since u always expands to ∂u, it expands to its kth iterate as well. Hence u and v have a common expansion if $u \equiv v$, and it can be proved that k can be chosen to be $\leqq$ the number of elementary changes needed to convert u to v. This does not seem to suffice to decide the equivalence of u and v, since we do not know how to estimate k. Two additional principles are needed; both of them can be easily verified.

The first one states that whenever a term s can be expanded (in an iterative way) to a term t, then each left subterm of s can be expanded (in an iterative way) to a left subterm of t. The second principle states that $t \cdot x^{[n]} \equiv x^{[n+1]}$ whenever t is a term of height $\leqq n$. Here $x^{[n]}$ means $x \cdot (x \cdot (\dots (x \cdot x)))$, where x appears n times. The height of $x^{[n]}$ is n.

The number k of elementary changes that are needed to convert $t \cdot x^{[n]}$ to $x^{[n+1]}$ can be bounded by some $f(n)$, and we see that t can be expanded to a left subterm of $\partial^{f(n)} x^{[n]}$. If m is the maximum height of u and v, then both u and v can be expanded to left subterms of $\partial^{f(m)} x^{[m]}$. An expansion enhances the term size, and hence these subterms can be identified in a finite number of steps. Write $u <_{\mathrm{L}} v$ if there exist $u' \equiv u$ and $v' \equiv v$ such that u' is a proper left subterm of v'. The irreflexivity conjecture guarantees that $u <_{\mathrm{L}} v$ and $u \equiv v$ are mutually exclusive. The method here described lets us decide if $u <_{\mathrm{L}} v$ or $u \equiv v$ or $v <_{\mathrm{L}} u$. We also see that $<_{\mathrm{L}}$ induces a linear ordering $<$ of the monogenerated LD-system.

Associate with each expansion (or contraction) at a certain term position a syntactic operator. A detailed analysis of identities between compositions of syntactic operators that can occur when $t \cdot x^{[n]}$ is being converted to $x^{[n+1]}$ leads Dehornoy [D2] to a certain infinite group which can be factorized to the braid group B_∞. Using this connection Dehornoy gives a proof of the irreflexivity conjecture without resorting to elementary embeddings. His results imply that one can define a free monogenerated LD-system within B_∞, by setting

$$\alpha \cdot \beta = \alpha s(\beta)\sigma_1 s(\alpha^{-1}),$$

where s is the shift endomorphism $\sigma_i \mapsto \sigma_{i+1}$. (Using this definition Larue gave a shorter proof of the irreflexivity.)

It was mentioned above that for every term u there exists $v \equiv u$ which is a left subterm of some $\partial^k x^{[n]}$. The first occurrence of such v (with respect to k and n) can be regarded as a normal form of u. Such a normal form does need the operation $\circ$. It is very wasteful, however, with respect to the term sizes, and the normal form described in [D3] is a result of efforts to make it more economical by applying (roughly speaking) ∂^{-1} whenever possible. This means, as a first approximation, to utilize the fact that left subterms of ∂t are equivalent to (certain) cuts of t. A cut is induced by every position of a variable in t by neglecting the part of t that is to the right of the link between the considered position and the root of the term, and by reconnecting the remaining part in the (only) natural way.

Let us now turn to Laver's normal form(s). He uses various orderings of terms, but most of them, while originally defined in a specific way on a specific subset of terms, are then proved to be a manifestation of $<_{\mathrm{L}}$. To avoid details and subtleties, let us write $<$ for all of them.

Every term t can be expressed (in a finite number of ways) as

$$(((u_0u_1)u_2)\dots u_{n-1}) * u_n,$$

where $*$ can mean both $\cdot$ and $\circ$. The term t is said to be p-normal if $u_0 = p$, if the terms $u_1, \dots, u_n$ are p-normal, and if $u_{i+2} \leqq ((u_0u_1)\dots)u_i$. If $*$ is $\circ$ and $n \geqq 2$, then one requires, in addition, $u_n < ((u_0u_1)\dots)u_{n-2}$. The ordering $<$ is obtained recursively by lexicographically comparing the associated sequences. The associated sequence of $((u_0u_1)\dots)u_n$ starts by $u_0, u_1, \dots, u_n$, and nothing is added when $*$ equals $\cdot$. In the case of $\circ$, the element u_n gets replaced by the infinite sequence of finite iterates $I_k(r, s)$, $k \geqq 0$, where $r = ((u_0u_1)\dots)u_{n-1}$ and $s = u_n$. (The finite iterates are defined by $I_0(r, s) = s$, $I_1(r, s) = r$, and $I_{k+2}(r, s) = I_{k+1}(r, s) \cdot I_k(r, s)$, and can be regarded as a sequence of approximations of $r \circ s$, which is its limit.)

If u_0 is allowed to equal $p^{[i]}$, $i \geqq 1$, and if $u_1 < u_0$ is required, then we get the definition of the p-normal form. The uniqueness of an expression by this form is easy to establish (by using the irreflexivity), but to prove that for every $v \geqq p$ there exists $u \equiv v$ in a p-normal form, a complicated induction argument seems to be needed [L1]. The argument proceeds simultaneously for all bases p and resorts to changes of the base. The case $p = x$ gives "the" normal form for the monogenerated LD-monoids. Dehornoy suggests calling it the left normal form (in contrast to his right normal form).

In [L2] Laver does not say that a term is p-normal, instead he says it is in the p-division form (or in p-DF); the technical difference consists in assuming that each $q < p$ is in p-DF. He establishes the existence of p-DF for all p and explains why he calls it the division form: given a $q > p$, there is a $<$-greatest a_0 with $pa_0 < q$. Proceeding further, one gets the expression of q as $(((pa_1)a_2)\dots a_{n-1}) * a_n$, and each a_i is subjected to the same division process. At the end one obtains a tree of terms which are $\leqq p$.

Many results about systems with one generator can be generalized to systems with more variables without difficulties, by considering the so-called variable clashes. The generic situation for a variable clash arises when terms u and v have left subterms that coincide at every variable position except the rightmost one. Such terms cannot be compared by $<_{\mathrm{L}}$, but they become comparable when a linear ordering of the variable set gets involved. In this way one can linearly order every free LD-monoid.

Such an ordering is used by Laver in [L4] when he defines p-DDF terms by a slight alteration of the definition of a p-normal term (here he requires, in addition, $u_1 < u_0$ and $u_n < ((u_0u_1)\dots)u_{n-2}$ in all cases). A DDF-term is defined to be a p-DDF term, where p is a variable. Not every term is equivalent to a DDF-term, but DDF-terms over variables $x_0, x_1, \dots$ can be used as a carrier for a faithful and order-preserving partial action of B_∞, which is induced by $x_i^{\sigma_i} = x_{i-1}$, $x_{i-1}^{\sigma_i} = x_{i-1}x_i$, and $x_j^{\sigma_i} = x_j$, if $j \notin \{i-1, i\}$.

Let me now describe another action of B_∞. While the former action was inspired by Artin's action on free groups, the present action has its roots in knot theory, where idempotent LD-systems are often considered. The carrier of the action is formed by infinite sequences $c_0, c_1, c_2, \dots$ of elements of an LD-system. The action by σ_i modifies the sequence by changing c_{i-1}, c_i to $c_{i-1}c_i, c_{i-1}$. If the LD-system is left cancellative and left divisible, then one gets a full action. If it is only left cancellative, a partial action of B_∞ is obtained. This is the case of free LD-systems, and it is not difficult to see that the common definition range of any two $\alpha, \beta \in B_\infty$ is non-empty. Dehornoy observes that there exists a linear ordering $<$ of B_∞ such that for any $c_0, c_1, c_2, \dots$ of this common range one has $\alpha < \beta$ if and only if the α-image of $c_0, c_1, c_2, \dots$ is lexicographically less than the β-images of $c_0, c_1, c_2, \dots$ (with respect to $<_{\mathrm{L}}$). This ordering of B_∞ can be used to get new results within the theory of braids (first steps appear already in [D2]). See also [D4], which is an expository paper that explains in detail a number of aspects that are here mentioned only briefly. Laver uses Dehornoy's ordering of braids in his paper [L4], and proves that the ordering extends the earlier partial ordering of Elrifai and Morton.

While free left distributive systems are now well rooted in ZFC, some of the properties of finite tables A_n are still dependent on the existence of non-trivial elementary embeddings.

To define A_n, consider a table with rows and columns $1, 2, \dots, 2^n$, put $2, 3, 4, \dots, 2^n, 1$ in the first column and $1, 2, \dots, 2^n$ in the last row. Regard now the table as the multiplication table of a binary operation $\cdot$, and fill it from the bottom, using the relation $a \cdot (b \cdot 1) = (a \cdot b) \cdot (a \cdot 1)$ to compute the row a. The rows a, $1 \leqq a \leqq 2^n$, exhibit periodic behavior, with periods of length 2^t, and one denotes t by $o_n(a)$. (The starting information can be restricted, in fact, just to the first column, as $2^n \cdot 1 = 1$ allows us to derive $2^n \cdot b = b$ from a special case of our formula, i.e., from $2^n \cdot (b \cdot 1) = (2^n \cdot 1) \cdot 1$.)

Laver proves [L3] that $\lim o_n(1) = \infty$, and this is the result for which the ZFC-proof still seems elusive.

Let us turn to the set-theoretical construction of A_n. For a non-trivial elementary embedding $k : R_\lambda \to R_\lambda$, denote by $\mathrm{cr}(k)$ the least ordinal γ with $k(\gamma) \neq \gamma$. Then $k(\gamma) > \gamma$ and k yields a non-decreasing injective mapping of ordinals below λ.

The Laver–Steel theorem [L3] considers an infinite sequence of elementary embeddings $k_0, k_1, k_2, \dots$ and demonstrates that λ is equal to the supremum of the critical ordinals $\mathrm{cr}((((k_0k_1)k_2)\dots)k_n)$. Laver uses Steel's arguments [St] of well-foundedness and argues in terms of extenders. The version of Laver's proof given by Dougherty [Do2] exploits only the well-foundedness and Kunen's theorem on cofinality of the critical sequence of a non-trivial elementary embedding. (The critical sequence of k is the one formed by $\mathrm{cr}(k)$, $k(\mathrm{cr}(k))$, $k(k(\mathrm{cr}(k)))$, ... ; when $k = j$, then its members are denoted by $\kappa_0, \kappa_1, \kappa_2, \dots$.)

The LD-monoid generated by j is countable, and hence we can order all critical ordinals that are yielded by all k from this LD-monoid. Denote them by $\gamma_0, \gamma_1, \gamma_2, \dots$. Laver [L3] proves that $\gamma_i = \mathrm{cr}(j_{[2^i]})$, where $j_{[1]} = j$, $j_{[i+1]} = j_{[i]} \cdot j$, and that $\mathrm{cr}(j_{[m]}) = \mathrm{cr}(j_{[2^n]})$, where n is the greatest integer such that 2^n divides m.

He also shows that A_n can be obtained by factorizing over the equivalence $\stackrel{\alpha}{=}$, with $\alpha = \gamma_n$. Now, $k_1 \stackrel{\alpha}{=} k_2$ if and only if $k_1 \stackrel{*}{\cap} R_\alpha = k_2 \stackrel{*}{\cap} R_\alpha$, with

$$k \stackrel{*}{\cap} R_\alpha = \{(x, y) \in R_\alpha \times R_\alpha : y \in k(x)\}.$$

Sets $k \stackrel{*}{\cap} R_\alpha$ were used already in Laver's proof of irreflexivity [L1]. Equivalences $\stackrel{\alpha}{=}$ are compatible not only with the binary operation $\cdot$, but also with $\circ$, and so A_n can be regarded as an LD-monoid, with $a \circ b = (a \cdot (b + 1)) - 1$.

It is usual to replace 2^n in A_n by 0. Each A_n is a homomorphic image of all A_m, $m \geqq n$, by the reduction modulo 2^n, and we see that one can build an inverse limit of algebras A_n that is formed by infinite sequences $(c_1, c_2, \dots)$, where $c_i \in A_i$ and $c_{i+1} \equiv c_i \bmod 2^i$. Denote by A_∞ the subsystem of this LD-system that is generated by $(1, 1, \dots)$. The limit of ordinals γ_n is equal to λ, and hence A_∞ has to be free. But the freeness of A_n follows from the set-theoretical interpretation, and it is not clear how to prove it within ZFC.

It is not difficult to find various conditions equivalent to the freeness of A_∞, and one of them is $\lim o_n(1) = \infty$. Further ones appear in [DJ].

Define $F(n)$ by $\kappa_n = \gamma_{F(n)}$. Dougherty proves in [Do1] that $F(n)$ grows faster than any primitive recursive function. His result was obtained before Laver discovered tables A_n and uses the language of critical sequences. But because $\mathrm{cr}(k) \geqq \gamma_n$ means that the projection of k to A_n is 0 (where k is interpreted as a term over j and j is mapped to 1), such results can be easily expressed in terms of A_n. In particular we see that $F(n)$ measures how much m has to grow in order to increase the logarithm $n = o_m(1)$ of the period. Of course, $F(n)$ is defined for all n if and only if A_∞ is free. The value of $F(4)$ is still not known and [Do1] contains a proof that it must be a huge integer. The arithmetic of tables A_n is very difficult and would need a review of its own. Hence I only mention that [Do2] contains certain computations that seem to point to a method for getting an explicit formula for all $F(n)$, but that generalizations of these computations seem to face severe difficulties.

The main contribution of [DJ] consists in an abstraction from the set-theoretical context. Dougherty and Jech start from the identities

$$\mathrm{cr}(k_1 \cdot k_2) = k_1(\mathrm{cr}(k_2)) \quad \text{and} \quad \mathrm{cr}(k_1 \circ k_2) = \min(\mathrm{cr}(k_1), \mathrm{cr}(k_2)),$$

and they look at an elementary embedding k (which is generated by j) as if it were an increasing function on $0, 1, 2, \dots$ (which are replacing $\gamma_0, \gamma_1, \gamma_2, \dots$). A set of strictly increasing functions equipped with a left distributive operation that satisfies $\mathrm{cr}(a \cdot b) = a(\mathrm{cr}(b))$ is called an embedding algebra. Dougherty and Jech prove that A_∞ is free if and only if there exists a non-trivial embedding algebra. (The critical point of a strictly increasing function f is the least n with $f(n) > n$.)

The proof involves definition of an (extended) two-sorted embedding algebra, which contains concepts that have been abstracted from the environment of elementary embeddings in order to model all relevant arguments about ordinals. In the process of the proof the authors are even able to verify the uniqueness, and hence we know that the embedding algebra constructed from A_∞ (assuming it is free) is the only embedding algebra in which every number is critical for some of its functions.

They also show that one cannot prove the freeness of A_∞ in primitive recursive arithmetic PRA, and contrast this result (which is rendered in a somewhat more precise form) with the situation of free LD-systems and free LD-monoids, where all of the proofs take place within PRA.

The commentary here is restricted to papers I was asked to review. I tried to give a coherent perspective, and hence some of the papers were mentioned only very briefly. Such cases illustrate the limits of my perspective. I did not include references to any other relevant papers, both older and newer. There are quite a few of them, and the relevance of some of them is comparable to the relevance of some of the papers reviewed. None of the missing papers would, however, alter the scene in a principal way. (This work was supported by institutional grant MSM 113200007 and by the Grant Agency of the Czech Republic, grant 201/02/0594.)

ALEŠ DRÁPAL

MFF, Charles University, Sokolovská 83, 18675 Prague 8, Czech Republic.
drapal@karlin.mff.cuni.cz.