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ZERMELO'S CANTORIAN THEORY OF SYSTEMS OF INFINITELY LONG PROPOSITIONS

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Abstract. In papers published between 1930 and 1935, Zermelo outlines a foundational program, with infinitary logic at its heart, that is intended to (1) secure axiomatic set theory as a foundation for arithmetic and analysis and (2) show that all mathematical propositions are decidable. Zermelo's theory of systems of infinitely long propositions may be termed "Cantorian" in that a logical distinction between *open and closed domains* plays a signal role. Well-foundedness and strong inaccessibility are used to systematically integrate highly transfinite concepts of demonstrability and existence. *Zermelo incompleteness* is then the analogue of the Problem of Proper Classes, and the resolution of these two anomalies is similarly analogous.

In four papers published between 1930 and 1935, Zermelo outlines a foundational program having two principal aims. First, there is his professed goal of supplanting finitary by infinitary concepts of language and proof. But a second, implicit, goal is the systematic integration of mathematical notions of existence and proof. The first of the four papers, from 1930, enjoys landmark status, containing the first extended discussion of the cumulative hierarchy of sets. The other three, published between 1931 and 1935, have had a less enthusiastic reception: the ideas contained therein have been taken to lead nowhere [22] and to be vague [12]. Even sympathetic commentary

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has sometimes failed to represent Zermelo's ideas with any accuracy [1], and the role of strong inaccessibility within the theory of proposition systems has never been recognized.

We shall show that Zermelo's four papers are closely related, the technical and philosophical achievements of the 1930 paper being the direct source of the theory of proposition systems (*Satzsysteme*) outlined in the three later ones. Moreover, although important questions remain unanswered, the two theories outlined in the four papers—one a theory of mathematical existence and the other an "isomorphic" theory of proof—are fully intelligible from a realist standpoint, provided that one takes into account Zermelo's tendency to make liberal assumptions regarding the expressive capabilities of languages left largely unspecified.

As for content, Zermelo sketches a theory of proposition systems according to which adequate formulations of mathematical theories are infinitary in almost every respect, finitary approximations to such formulations serving as convenient vehicles of mathematical practice. Further, solutions to important foundational problems are obtained by elucidating the relations holding among the terms of unbounded sequences of models/logics—an idea elevated to the level of a general context principle. Finally, Zermelo's theories may be termed "Cantorian" to the extent that a logical distinction between open and closed domains is central.

§1. Five theses concerning the infinite in mathematics. Although he published almost nothing between 1915 and 1928, Zermelo developed his alternative foundational program starting around 1920. The earliest pertinent document in the *Nachlaß* is a single-page typescript entitled *Thesen über das Unendliche in der Mathematik* [29]. In the present section, we examine, in order, each of Zermelo's five theses. They offer a convenient means of organizing our discussion. This approach also has the merit of putting first things first.

Thesis I is metaphysical in character and concerns the general nature of mathematical propositions.

> *Thesis I*. Every genuine mathematical proposition has infinitary character, i.e., it makes appeal to an infinite domain and must be understood as a collecting together [*Zusammenfassung*] of infinitely many elementary propositions [*Elementarsätze*].

What we have here is an assertion concerning the meaning of mathematical propositions. It is no doubt also intended as a normative proposal. We consider these two issues in turn.

That proposition (1) below has infinitary character is clear. But the claim is hardly obvious in the case of (2).

1. Given any natural number n, the sum of n and 0 is n.
2. The sum of 3 and 0 is 3.

One might conclude that, as Zermelo sees things, proposition (2) is not sufficiently general to be considered mathematical. This is certainly what Thesis I seems to be saying on the face of it. A more satisfying understanding of Zermelo's intention, given all the available textual evidence, is that (2) is a genuine mathematical proposition only to the extent that it presupposes a general theoretical context, namely, that of the entire system of natural numbers. In saying this, we mean to suggest that Thesis I is related to a certain context principle that is not among the five theses. That principle would assert that mathematical propositions presuppose contexts comprising infinite domains. Later on, Zermelo does present an argument for such a principle (see §8 below), and we view that argument as lending support to Thesis I.

That mathematical practice involves ontological presuppositions is an essential component of any realist account of mathematical truth. But Thesis I incorporates a related claim regarding the logical form of mathematical propositions, and this second claim is not to be inferred from realism. By an elementary proposition, Zermelo means one stating that some k-adic relation R holds of elements $a_1, \ldots, a_k$ of some fixed domain (see (1) in §6 and also Thesis V below). In other words, *Elementarsätze* would be the propositional correlates of atomic sentences. Zermelo's claim regarding logical form would have us paraphrase an elliptical proposition (2) as

2′. From proposition (1) it follows that the sum of 3 and 0 is 3.

where the premise in (2′), viz., (1), is an infinite conjunction.

Zermelo's treatment elsewhere of elementary propositions over arbitrary infinite domains indicates that domain elements $a_1, \ldots, a_k$, as well as k-adic relation R, are propositional components as in Russell [17]. As for propositions that are not elementary, Zermelo's use of the term *Zusammenfassung* would prefigure his later iterative characterization whereby propositions are constructed in a succession of stages. To the extent that this usage brings to mind Cantor's 1895 definition, there is the suggestion that (nonelementary) propositions are sets, of which more below.

In Thesis II we find a statement of mathematical Platonism—but not the usual one.

> *Thesis II.* The infinite is in reality given to us neither physically nor psychologically. It must be grasped, and posited, as Idea in the Platonic sense.

Thesis II is not a claim about infinite domains. Rather, it appears to concern a denotation for symbol ∞.

Thesis III concerns mathematical theories and proofs of their consistency.

> *Thesis III.* Since infinitary propositions can never be derived [*abgeleitet*] from finitary ones, it follows that the axioms of every mathematical theory must also be infinitary. Further, the

> consistency of such a theory can be proved only through the presentation of a corresponding, consistent, system of infinitely many elementary propositions.

Apparently, a demonstration of consistency would amount to giving a truth-value assignment to elementary components according to which the whole comes out true. The first sentence here might seem puzzling in light of a rule such as $\vee$-introduction. But, almost surely, what Zermelo has in mind is the standard stricture: if the conclusion deduced from premises is to be general in applying to all members of some infinite domain, then at least one premise must be general as well. This would suggest that he takes infinite conjunctions of elementary propositions to be paradigmatic (cf. Thesis I and §6 below).

The distinction between finitary and infinitary propositions has, in other more strictly philosophical contexts, been subject to debate. Thus, to adapt an example from Hempel, the *prima facie* finitary proposition "3 is a prime number" is logically equivalent to "All composite numbers are different from 3," which is expressible by an infinite conjunction. This has appeared to blur the distinction in the opinion of some. That Zermelo, in Thesis III and elsewhere, can assume the clarity of the distinction probably reflects a belief on his part that it is to be derived from the distinction between finite and infinite sets.

Thesis IV is the only one among the five that concerns logic in a narrower sense.

> *Thesis IV*. Traditional Aristotelian logic is inherently finitary and hence unsuitable as a foundation for mathematical science. There follows the necessity of a more encompassing infinitary, or Platonistic, logic, based on a sort of infinitary intuition, e.g., with regard to the issue of the Axiom of Choice. Paradoxically, [such a logic] is rejected out of force of habit by, of all people, the intuitionists.

The reference to the Axiom of Choice implies intuition of a realm of sets and choice functions. Thesis IV prefigures Zermelo's later description of infinite thought-experiments involving what appears to be *de re* intuition of transfinite hierarchies of propositions (see §5 below). And yet elsewhere Zermelo is unwilling to grant intuition more than a limited role in justifying mathematical knowledge (see [23] as well as Zermelo's editorial comments in [4], p. 377).

Finally, Thesis V expands upon an idea introduced in Thesis I.

> *Thesis V*. Every mathematical proposition is to be understood as a collecting together of (infinitely many) elementary propositions, the fundamental relations, through conjunction, disjunction, and

> negation. Every derivation of one proposition from other propositions, in particular, every proof, is nothing but a regrouping of the underlying elementary propositions.

Zermelo's characterization of proofs as regroupings is related to later published accounts wherein understanding a proof is taken to involve a mental scan of some transfinite hierarchy. The suggestion is that mathematics is of tautological character; specifically, a theorem is a truth-functional consequence of some collection of axioms, and its proof amounts to our coming to recognize this. Thesis V allows for distinct ways of collecting propositions into a whole. Specifically, we have infinite disjunctions. To the degree that every proposition is said to be a collecting together of elementary propositions, as opposed to *more* elementary propositions, the notion of "collecting together" being used here is extremely general. (Compare the manner in which a set is a collecting together of all the urelements including $\emptyset$ in its transitive closure.)

The set concept enters into Theses I–V by way of talk of domains and of operations of collecting together and regrouping. Ordinal notions, so central to Cantor's philosophy of the infinite, are not mentioned. Nonetheless, we shall see that the theory of ordinals directly informs Zermelo's views regarding mathematical existence and mathematical proof. Although there is no evidence of it in the Five Theses, Zermelo, influenced by his reading of Kant and Cantor, comes to concentrate on the dialectical nature of human thought generally and of mathematics in particular (see again [4], p. 377). We witness this dialectic whenever the world of mathematical objects, or some conceptual domain within it, e.g., the sequence of ordinal numbers, is apprehended simultaneously as limited (closed) and unlimited (open). Zermelo takes misunderstanding of this dialectic to be the surest source of the "ultrafinite paradoxes."

§2. Foundations of mathematics around 1930. Zermelo's engagement with foundational issues in this period is selective. Certain issues that greatly concern others, and that might be expected to fascinate Zermelo, appear to be of little interest to him. In any case, each of (1)–(3) below constitutes an important aspect of foundational disputes during this period. And each represents a facet of the debate that is of paramount importance for Zermelo and one to which he does react directly.

1. A salient feature of any model $\mathfrak{M}$ of the axioms of Zermelo–Fraenkel set theory is the nonexistence of a universal set. Expressed differently, the domain of $\mathfrak{M}$ is a proper class. But this domain would appear to be a mathematical object nonetheless, as indicated by metamathematical discussions concerning the structural properties of $\mathfrak{M}$ and its inner models. So the fact that the domain of $\mathfrak{M}$ is not a set is incompatible with objects-reductionism, according to which all mathematical objects are, in reality, sets. Those

who, like Skolem, take seriously this anomalous feature of *ZF* speak of the Problem of Proper Classes. They see reductionism as thereby refuted: axiomatic set theory is just another mathematical theory, like group theory or number theory, having no claim to foundational status.

2. In [18] it is shown that any countable, consistent set of first-order sentences has a countable model. (This is the downward Löwenheim–Skolem Theorem.) It follows that there exist models of first-order *ZF* in which the continuum is merely countable, on the face of it contradicting a well-known result of Cantor. One speaks of the Skolem Paradox. To be sure, there is no real contradiction. Rather, the appearance of paradox is an artifact of an emergent model-theoretic point of view. On the other hand, the ambient discussion can do nothing but further undermine objects-reductionism. After all, supposing for the moment that mathematical objects are just the sets described by the Zermelo–Fraenkel axioms and given that those axioms admit such an impressive variety of models—models that are not unique up to isomorphism—what sort of foundation for mathematics has been provided?

3. Starting in the twenties, a number of logicians—notably Skolem again—begin to suggest that Cantor's Continuum Hypothesis is undecided by current set theories. Such a view is expressed in Church's [5] and in the introduction to Gödel's dissertation [9].

The absence, from the list (1)–(3), of issues raised by Gödel's results deserves comment. In [10] it is shown that any consistent first-order theory T of sufficient strength is deductively incomplete in the sense that one can describe true arithmetic sentences of T that are neither provable nor refutable in T. The result applies to first- or second-order *ZF* in particular and, hence, threatens the standing of either of those theories as a foundation for mathematics. Since the undecidable sentences express mathematical truths concerning the natural numbers, should they not be provable in any theory that purports to provide such a foundation? Despite the reasonableness of this question, Zermelo never shows any real interest in the work of Gödel, which is puzzling. A partial explanation is Zermelo's having come, independent of Gödel, to view mathematical systems generally—both formal systems in Hilbert's sense and his own infinitary systems—as inherently incomplete descriptions of (some part of) the mathematical realm.

Reductionism is a doctrine to which Zermelo is very much drawn, even if his stated position regarding it remains agnostic (see [19] for our description of Zermelo's "programmatic reductionism"). We take set-theoretic reductionism to encompass the claims that all mathematical objects are sets, that all mathematical concepts are definable in terms of membership, and that all mathematical truths are truths about sets. As such, reductionism ascribes to set theory, and to the Zermelo–Fraenkel axioms in particular, a definite foundational role. If one cannot say with surety that Zermelo is a reductionist, nonetheless the foundational status that reductionism would confer upon the Zermelo–Fraenkel axioms is one that he appears to assume. At the very least, set theory is held to ground mathematical thought regarding number as well as the infinite.[1] (And such grounding of mathematical thought will necessarily involve giving an account of its dialectical character.) Surely, such views are harder to defend in the presence of (1)–(3). Consequently, each of (1)–(3) poses a challenge to Zermelo, and his responses—qualified embrace of (1), elimination (by technical means) of (2), and outright rejection of (3)—are found in the four papers of 1930–35. While this much is true, Zermelo's interest in infinitary, and ultimately set-theoretic, notions of logic and proof, as outlined in [36] et al., is not merely reactive. As evidenced by Theses I–V, this interest is part of a quest, extending over many years, for understanding of the role of infinity in mathematics.

Zermelo's views on the foundations of mathematics set him apart from each of the major intellectual currents of the period, despite some sharing of opinions. Among the *logicists*, at least Carnap remains open to the possible use of higher-order and infinitary logic in the development of mathematics. But, as Thesis IV already indicates, Zermelo makes infinitary logic the very touchstone of his theory of systems of propositions, and one objective of the present paper is to explain why. Nor is Zermelo's concept of proof, as described in §5 below, conventionalist in nature, despite Thesis V. This is because strong mathematical axioms will fix the lengths of allowable inference chains, and these axioms can hardly form part of the syntax of language.

In opposition to the *intuitionists*, Zermelo takes a conservative position with respect to mathematical practice: he states explicitly that all of classical mathematics is to be preserved (see [36]). He rejects constructivist ideology, readily accepting the Laws of Contradiction and Excluded Middle and nonconstructivist versions of the Axiom of Choice. But his attraction to some doctrine of universal definability may qualify his opposition to constructivism (see our [19]). Also, like Hilbert and unlike Brouwer, Zermelo holds that all mathematical problems are solvable (see [12]); but, in Zermelo's case, there are no grounds for interpreting this as confidence in reason in the usual sense, given his highly transfinite conception of demonstrability. In the end, the theory of proposition systems that Zermelo proposes would perhaps entail few revisions of mathematical practice, although there would undoubtedly be changes in the way one conceptualizes that practice. Zermelo's view that the mathematical realm can be described completely by no one axiomatic system is, on the one hand, merely to give Gödel's results their due. (We emphasize, however, that the source of this view is not Gödel but, rather, the technical results of [34].) On the other hand,

[1] With respect to number, see [33], where the set concept is said to "precede" that of number. (*Die Axiomatik der Mengenlehre soll doch dem Anzahlbegriff vorausgehen.*)

Zermelo's pronouncements regarding the incapability of axiomatic systems, even collectively, to more than approximate a full description of that realm bear comparison with Brouwer's negative assertions (see also Gödel's [8]). In particular, Zermelo's infinitary concept of proof, described below, is not unrelated to that of Brouwer apparently, who remarks at one point that proofs are transfinite iterations of basic inference steps (see [2], fn. 8).

Finally, Zermelo's interest in infinitary logic, which lacks a full-fledged syntactic component, means that mathematics cannot amount to mere symbol manipulation, as *Hilbertian formalists* claim. (The formulation of Thesis III suggests a reaction to Hilbert.) Whereas the later Hilbert has no interest in model theory, the study of models of *ZF* provides an important paradigm for Zermelo's foundational program. And if the primary purpose of Hilbert's early (as opposed to later) foundational program is the elucidation of one's starting points, Zermelo sets his sights, from the beginning, on certainty and never deviates from this more ambitious goal (see e.g., [27], p. 1).

Ignoring points of contact, Zermelo considers his views on the foundations of mathematics to be squarely at odds with the leading tendencies in the period around 1930. This leads him to begin the development of what he sees as a new foundational program. We shall describe that program, filling in details when necessary, and, to a lesser degree, assess it. Zermelo characterizes his program as proof-theoretic, which suggests Hilbert. But it is easy to cite obvious differences between Zermelo's proof theory and that of Hilbert. In addition to the former's commitment to infinitary logic, his interest in consistency proofs never rises to the level of Hilbert's despite Thesis III.[2] In general, Zermelo's program involves coming to think about mathematical theories in such a way that the philosophical and technical challenges reviewed in (1)–(3) do not, in fact, undermine the foundational status of the Zermelo–Fraenkel axioms. To be sure, like Hilbert's, Zermelo's program is a technical program—one whose goal is to extend Klein's *Erlanger Programm* through the application of invariant-theoretic techniques to logic.

[2]One could expect Gödel's Second Theorem to intrigue Zermelo as further evidence that finitary formulations of mathematical theories impose unnatural limitations. But Zermelo never once mentions the Second Theorem—neither in what has been published nor in the unpublished materials constituting the *Nachlaß*. Although Zermelo does take up Gödel's proof of the First Theorem (see [6] and [12]), the results of Gödel appear to hold no great interest for Zermelo, whose target is more often Skolem. We have already suggested that this is because Zermelo's conclusions in [34] to some extent anticipate Gödel. Indeed, Zermelo could regard Gödel as an ally in a shared struggle against finitary strictures, but, in general, Zermelo does not seem to see Gödel in this way. (In this connection, see [14], fn. 22.) Having said this, we note that the penultimate sentence of Zermelo's first letter to Gödel, dated September 21, 1931, can be construed as an invitation to an alliance: "Correctly interpreted, precisely your line of proof would contribute a great deal to [overcoming what I call the "finitistic prejudice"] and could thereby render a substantial service to the cause of truth" ([6], p. 69).

But Zermelo incorporates a philosophy of mathematics as well—one involving a reconceptualization of mathematical systems and of proofs occurring in them.

Important model-theoretic results, reported in [34], provide the impetus for Zermelo's program and for the philosophical speculation regarding mathematical knowledge that he associates with it (see §9 below). It makes sense to consider these results next.

§3. Models of second-order *ZF*: exploiting noncategoricity. In [34] Zermelo investigates what he calls "normal domains," i.e., the domains of models of $ZF^2 - Inf$ comprising urelements and sets.[3] (As a matter of general policy, Zermelo does not wish to rule out non-well-founded sets. But since Foundation is among his axioms in [34], normal domains have none.) Normal domains are related to what are now called $\in$-models with the difference that the latter involve no urelements.

Zermelo defines a cumulative hierarchy $\{P_\alpha\}_{\alpha\in On\setminus\{0\}}$ of urelements and sets essentially as follows.

$$(*)\quad P_\alpha = \begin{cases} \text{undefined} & \text{if } \alpha = 0\\ U & \text{if } \alpha = 1\\ P_\beta \cup \left[\wp\left(\bigcup_{0<\gamma<\alpha} P_\gamma\right) \setminus \bigcup_{0<\gamma<\alpha} P_\gamma\right] & \text{if } \alpha = \beta+1 \text{ and } \beta \neq 0\\ \bigcup_{0<\beta<\alpha} P_\beta & \text{if } \alpha \text{ is a limit ordinal}\end{cases}$$

Here U is an assumed collection of urelements, and variable α ranges through the nonzero ordinals.[4] With the proof of his Third Development Theorem, Zermelo shows in effect that P_α, for α any strongly inaccessible cardinal, is a normal domain.[5] His Second Isomorphism Theorem then implies that

[3]Throughout this paper we shall write ZF^2 for axioms distinguishing urelements from sets by way of a unary predicate $\mathscr{U}$, defined by $\mathscr{U}(x) \equiv (\forall y)(y \notin x) \wedge x \neq \emptyset$, whose negation is used to relativize certain first-order quantifiers in Extensionality, Power Set, and so forth. In designating models, we shall sometimes write $\langle U, V, E\rangle$ when we wish to emphasize the extension $U \subset V$ of $\mathscr{U}$.

[4]In the development of [34], terms P_α are used for initial segments of a noncumulative hierarchy, whereas terms Q_α denote mutually disjoint levels. Setting $U = \emptyset$, Zermelo's P_α have since come to be designated V_α. (For finite α only, we have $V_\alpha = P_{\alpha+1}$.) Zermelo's Q_α are characterizable as $Q_0 = U$ and $Q_\alpha = \{x \mid x \subseteq \bigcup_{\beta<\alpha} Q_\beta \wedge x \notin \bigcup_{\beta<\alpha} Q_\beta\}$ for $\alpha \geq 1$, after which we have $P_\alpha = \bigcup_{\beta<\alpha} Q_\beta$ for $\alpha \geq 1$. Alternatively, we define $\{P_\alpha\}_{\alpha>0}$ as at (*) and then let $Q_0 = U$ and $Q_\alpha = P_{\alpha+1} \setminus P_\alpha$ for $\alpha > 0$. Where $x \in Q_\alpha$ one says that x has *rank* α. Every urelement has rank 0 while $\emptyset$ has rank 1.

[5]The notion of strong inaccessibility will be important for the theory of systems of propositions discussed beginning in §5 below. Given ordinals $\alpha \geq \beta$, we have that α is *cofinal with* β provided that, for some strictly monotone function $f : \beta \longrightarrow \alpha$, every element of α is either itself in the range of f or is exceeded by some ordinal in that range. The *cofinality* of α is then the smallest such β. An infinite cardinal that is its own cofinality is said to be regular and is otherwise singular. Intuitively, the regular cardinals are those that are not accessible, from below, by some ordinal mapping f. Accordingly, a regular cardinal of the form $\aleph_\alpha$, with

two distinct normal domains built up from the same U must be of the form P_α and P_β, with $\alpha < \beta$ and $P_1 = U$. But in that case, by (*), P_α is a set within $\langle P_\beta, \in|P_\beta\rangle$. Moreover, as it implied, it can be shown that α and β here will be strongly inaccessible. Consequently, Zermelo's normal domains are seen to be precisely those corresponding to so-called "natural models." (Zermelo appears to assume, in addition, that $card(P_1) = card(U) < \alpha$ since otherwise Replacement fails in $\langle P_\alpha, \in|P_\alpha\rangle$.)

Zermelo presupposes an inexhaustible fund of strongly inaccessibles—in the metatheory, as it were—and argues that meditation on the sequence $\{\langle P_\alpha, \in|P_\alpha\rangle\}_{Inac(\alpha)}$ eliminates the Problem of Proper Classes. To examine this claim, we consider some $\langle P_\beta, \in|P_\beta\rangle$ with β strongly inaccessible. As mentioned above, we have that $\langle P_\beta, \in|P_\beta\rangle$ is itself a model of ZF^2. However, domain P_β is, relative to $\langle P_\beta, \in|P_\beta\rangle$ itself, a proper class. Happily, P_β will be an element of the domain of any later term $\langle P_\gamma, \in|P_\gamma\rangle$ in the sequence $\{\langle P_\alpha, \in|P_\alpha\rangle\}_{\alpha>0}$. In other words, P_β is a proper class in $\langle P_\beta, \in|P_\beta\rangle$ but a set in $\langle P_\gamma, \in|P_\gamma\rangle$. Of special interest is the case where γ also is strongly inaccessible, in which case $\langle P_\gamma, \in|P_\gamma\rangle$ is a model of ZF^2. As Zermelo sees matters, the Problem of Proper Classes is thereby resolved.

> The unbounded extensibility of the series of transfinite numbers permits a corresponding *representation of set theory* [*Darstellung der Mengenlehre*] as a similarly unbounded sequence of clearly differentiated models. And it is precisely the explicit differentiation of these different models of our noncategorical axiom system that ensures a satisfying resolution of the ultrafinite paradoxes: the proper classes of any one model resurface as full-fledged sets in the next, as well as in every later, model in the sequence. ([34], p. 29 [emphasis added])

Zermelo here speaks of the sequence $\{\langle P_\alpha, \in|P_\alpha\rangle\}_{Inac(\alpha)}$ as a collective representation of set theory. This reflects the situation whereby the collection of normal domains has turned out to be identical with the sequence $\{P_\alpha\}_{Inac(\alpha)}$. (Zermelo uses the term *darstellen* when speaking of the representation of set theory both by a single model and by an infinite sequence of models.) On the side of syntax, a transfinite sequence $\{\mathcal{S}_\alpha\}_{Inac(\alpha)}$ of axiom systems is suggested (see §4 below).

As formulated by Zermelo himself, the philosophical payoff of the investigation in [34] is clarification of the foundational status of ZF^2. This clarification takes the form of a certain picture of the world of sets, namely,

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α a limit ordinal, is said to be *weakly inaccessible*. If, in addition, $card(\wp(x)) < \aleph_\alpha$ whenever $card(x) < \aleph_\alpha$, then $\aleph_\alpha$ is said to be *strongly inaccessible*. Zermelo himself uses the term *Grenzzahl* (boundary number) for what are in effect strongly inaccessible cardinals (see [13]). One difference is that Zermelo includes ω as the smallest of his boundary numbers. If we were to follow Zermelo in counting ω as strongly inaccessible, then strongly inaccessible levels of the cumulative hierarchy would model the axioms of ZF^2 without the Axiom of Infinity.

its characterization as the unbounded sequence $\{P_\alpha\}_{Inac(\alpha)}$ of normal domains. Each normal domain is then capable of mathematical determination, albeit delayed.

> The ultrafinite antinomies of the theory of sets ... depend ultimately on a failure to distinguish between set theory itself, as determined noncategorically by its axioms, and the individual models [*darstellende Modelle*] representing it. What appears in one model as an ultrafinite nonset is a full-fledged set with cardinal number and order type in the next model and, indeed, provides the cornerstone for construction of the new domain. To the unbounded sequence of Cantorian ordinals there corresponds a likewise unbounded companion sequence of clearly distinguishable set-theoretic models, each of which gives expression to the entire classical theory. ([34], pp. 46–47)

Zermelo concludes that the noncategoricity of ZF^2 has turned out to be philosophically useful. For it is noncategoricity that is reflected in the Second Isomorphism Theorem, and it, together with the Third Development Theorem, rids us of the Problem of Proper Classes. (Incidentally, since second-order logic admits no downward Löwenheim–Skolem Theorem, Zermelo's second-order construal of his axioms blocks the Skolem Paradox.)

A key element in Zermelo's foundational program, of which [34] is an integral part, is the use of set theory as a source of conceptual clarification, where the concepts to be clarified include those of number, function, and proof. In some sense, talk of sets should ground talk of numbers and functions—perhaps because numbers and functions are species of sets. Similarly, it should ground talk of proof. But surely set theory can furnish no conceptual clarification in the absence of a clear picture of the world of sets it describes. We have such a picture in the sequence $\{P_\alpha\}_{Inac(\alpha)}$, which, in its dialectical aspect, gives expression to our sense that the world of sets is both limited (closed) and also unlimited (open). We shall speak of the *Dialectical Principle of Open and Closed Domains* when referring to Zermelo's view that we must fully exploit the dialectic obtaining between successive terms of the sequence $\{P_\alpha\}_{Inac(\alpha)}$. (We discuss Zermelo's distinction between open and closed domains in §4. For the moment, the reader needs to know only that any normal domain is closed.) Accordingly, normal domains take up positions within *sequences* of normal domains, and it is reflection upon the relations holding among the corresponding models of ZF^2 that secures that theory's foundational status. Ancillary to Zermelo's Dialectical Principle we have the following adjunct: foundational criticism of axiomatic set theory may be answered by citing relations holding within the collection of *all* normal domains (*all* natural models of ZF^2). Both the Dialectical Principle and its adjunct will come into play within Zermelo's theory of systems of propositions (see §7 below).

The Problem of Proper Classes affords us an opportunity to develop insight into the dialectical nature of the mathematical realm—specifically, the sense in which it is simultaneously open and closed. It is open to the extent that it can be extended indefinitely. But there are interim points of closure as well, and the paradoxes appear problematic only when we become fixated on the latter, failing to appreciate the back-and-forth movement between individual representing structures and entire sequences of such representations.

> The two opposing tendencies of the life of the mind—on the one hand, the concept of creative progress [*schöpferischer Fortschritt*] and, on the other, that of inclusive closure [*zusammenfassender Abschluß*]—are the roots of the Kantian antinomies; both tendencies find their symbolic representation as well as their symbolic resolution in the transfinite number sequence, which is grounded in the well-ordering concept. The transfinite number sequence, in its turn, exhibits no genuine closure in its unbounded advance but, rather, only interim stopping points, namely, those strongly inaccessibles that mark off the higher models from the lower ones. And thus the set-theoretic antinomies, properly understood, lead not to any narrowing or truncation but, rather, to an unsurveyable unfolding and enrichment of the science of mathematics. ([34], p. 47)

Zermelo's discussion brings to the fore the philosophical benefit of the investigation of models of set theory: we expose the relations holding among those models, which in turn effects a profound change in our thinking about the Problem of Proper Classes—a solution of sorts.

That solution amounts to a representation of set theory as $\{P_\alpha\}_{Inac(\alpha)}$—a technical analysis, and an unqualified embrace, of the dialectical character of our knowledge of sets and domains. There is no solution at all in the customary sense whereby one might hope to overcome the Problem of Proper Classes, once and for all, through the introduction of a universal set, to take an example. As depicted by Zermelo, the anomalous circumstance constituting the Problem of Proper Classes is resolved but then reappears in an unending cycle from which there can be no escape. Alluding to Kant's antinomies, Zermelo tells us that

> Human reason, by virtue of its inner nature, finds itself pressed to view the world simultaneously as limited and as unlimited, as finite and as infinite. And no mathematical theory will rid us of this fact ([4], p. 377).

Indeed, Zermelo is saying that "this fact" is one that any truly foundational program must embody.

§4. Zermelo's philosophy of open and closed domains and its Cantorian origin. This distinction is introduced in Zermelo's 1929 Warsaw lectures in conjunction with the Dialectical Principle.

> According to Skolem, all of set theory is supposedly represented by one *countable* model so that, for him, the problem of the cardinality of the continuum, for instance, loses its real meaning. . . . Every normal domain is "closed" and can hence be construed as a "set" in some "higher" model. Every initial sequence of the (transfinite) number sequence is a closed domain, but following any "set" or any closed domain of ordinals there are always more ordinals: the unbounded number sequence is itself an "open" domain. No closed normal domain can represent all of set theory in that every strongly inaccessible corresponds to an initial sequence of the number sequence and, hence, no normal domain contains all strongly inaccessibles. Set theory in its entirety is representable only by the "open" totality of all normal domains. ([23], p. 133)

Zermelo's distinction would appear to be a gloss on the model-theoretic results of [34]. The choice of terminology reflects the fact that P_α for α inaccessible is a normal domain and thus *closed* under the several set-forming operations, e.g., the power set operation, involved in the axioms of ZF^2. Similarly, in the third sentence Zermelo is saying that any initial sequence of ordinals up to a *limit*, we must assume, is closed under the successor operation.

But Zermelo's distinction is destined to transcend its set-theoretic origins, taking on logical character.

> A "closed domain" is one that is determined or ordered by a categorical system of postulates.[6] ([30], p. 1)

By categoricity Zermelo means just what we mean: any two models are isomorphic. This notion of closed domain should be regarded as logical since it involves the use of formal systems to fix structures (up to isomorphism). Zermelo here shifts from talk of normal domains to talk of domains *simpliciter*. Consequently, his remarks must be construed as general in character, a domain being the extension of a concept like *integer* or *rational number* or, significantly for what is to follow, *proposition*.

Zermelo views his distinction as Cantorian in character, for, immediately following the last-quoted sentence, we read that

> [A closed domain] is precisely what Cantor actually intended with his famous definition of a "set." It can be consistently treated as a set in the context of any purely mathematical observations or demonstrations. Every closed domain can be well ordered and possesses a cardinal as well as an ordinal number.[7] ([30], p. 1)

[6] *Ein „geschlossener Bereich" ist ein solcher, der durch ein* kategorisches *Postulatsystem bestimmt oder geordnet werden kann.*

[7] [*Ein geschlossener Bereich*] *ist genau das, was Cantor mit seiner bekannten Definition einer „Menge" eigentlich gemeint hat, und kann bei allen rein mathematischen Betrachtungen und*

The reference would be to Cantor's 1883 definition according to which a set is any "totality of definite elements that can be united into a whole by means of a law" [3]. As for open domains, they are those that are not closed, it seems, judging from the remark about Cantor. But Zermelo says more.

> An "open domain" is a well-ordered sequence of nested *closed* domains with the property that each of its closed [proper] subdomains can be extended [so as to obtain another closed proper subdomain]. It cannot itself be a "closed" domain certainly and cannot be ordered by any categorical postulate system. In contrast, each of its *initial segments* is closed and capable of being well ordered, and even the entire open domain can be well ordered in such a way that all elements of an earlier level precede all elements of any later one. The open domain possesses *neither* a cardinal nor an ordinal number.[8] ([30], p. 1 [square brackets not in original])

The second sentence here might indicate that Zermelo is taking an algebraic characterization (closure under formation operations) to be the actual definition of a closed domain. But $\bigcup_{Inac(\alpha)} P_\alpha = \bigcup_{\alpha>0} P_\alpha$ would seem to be closed in the algebraic sense, whereas Zermelo says that it is open. On the other hand, $\bigcup_{Inac(\alpha)} P_\alpha$ is conceivably not characterizable by a categorical set of postulates, which encourages one to regard Zermelo's logical characterization as primary. But, in that case, levels P_α, for α an inaccessible limit of inaccessibles, are now anomalous (see below).

His technical results encourage Zermelo to exploit strong inaccessibility by applying his distinction to the cumulative hierarchy. He appears convinced that each inaccessible level P_α is categorically characterizable by some axiom system $\mathcal{S}_\alpha$ and is hence closed.

> If there exist any strongly inaccessibles $\pi > \omega$ whatever, then among them there is a smallest π_1. Obviously neither its existence nor its nonexistence follows from the Zermelo–Fraenkel axioms, precisely because e.g., strongly inaccessible ω exists in the "Cantorian," but not in the "finitary," domain. In other words, the question receives different answers in the various models of set theory and is thus not decided by the axioms alone. Our axiom system is not categorical. ... Naturally, if one so desires, one

Deduktionen überall und widerspruchsfrei als Menge behandelt werden. Jeder geschlossene Bereich kann wohlgeordnet werden und besitzt sowohl eine Mächtigkeit wie eine Ordnungszahl.

[8] *Ein „offener Bereich“ ist eine wohlgeordnete Folge successiv einander umschließender* geschlossener *Bereiche von der Beschaffenheit, daß in ihm jeder geschlossene Unterbereich noch erweitert werden kann. Er kann also selbst gewiß* kein *„geschlossener“ Bereich sein und durch kein kategorisches Postulatsystem geordnet werden. Dagegen ist jeder seiner* Abschnitte *geschlossen und wohlordnungsfähig, und auch der ganze, offene Bereich kann in der Weise wohlgeordnet werden, daß alle Elemente einer vorangehenden Schicht allen Elementen jeder folgenden vorangehen. Der offene Bereich hat* weder *eine Mächtigkeit noch eine Ordnungszahl.*

> can always force categoricity artificially by introducing additional axioms. ([34], p. 45)

We shall refer below to Zermelo's *Assumption Regarding Categorical Characterization*. (See also the quotation from [26] given at the beginning of §6.) Note that P_α, for α an inaccessible limit of inaccessibles, is closed by the assumption but open by the indefinite extensibility of its closed proper subdomains (see quoted definition of "open domain").

Zermelo's logical notion of closed domain remains vague to the extent that, quite characteristically, he never stipulates the language in which characterizing axioms are to be formulated. But circumstantial evidence suggests a countable language with second-order quantifiers ranging over propositional functions. In that case, a serious problem arises. Namely, since there are only denumerably many second-order sentences, there can be only denumerably many (finite sets of) second-order axioms. But doubtless Zermelo also believes that, for every ordinal κ, there is a unique strongly inaccessible cardinal θ_κ. It follows that there will not be enough axioms to go around: at most denumerably many inaccessible levels can be fixed categorically, contrary to Categorical Characterization. (If infinite sets of axioms are permitted, change "denumerably many" to "continuum many.")

Having said this, it is still possible to axiomatize at least a "few" inaccessible levels P_{θ_κ}, and we sketch an approach. The corresponding $\mathcal{S}_{\theta_\kappa}$ could consist of the axioms of ZF^2 augmented by a characteristic axiom $\mathcal{Z}_{\theta_\kappa}$ effectively postulating the existence of precisely those levels of the cumulative hierarchy indexed by ordinals up to, but not including, θ_κ. For some values, e.g., $\kappa = \omega$ or κ the first measurable cardinal, we have that κ is definable in $\langle P_{\theta_\kappa}, \in|P_{\theta_\kappa}\rangle$ by a first-order formula $F(x)$ such that $F(x)$ picks out κ uniformly in any $\langle P_{\theta_\alpha}, \in|P_{\theta_\alpha}\rangle$ for $\theta_\alpha > \kappa$. For such a κ, axiom $\mathcal{Z}_{\theta_\kappa}$ can then say that (1) for every $\beta < \kappa$, there is a set of inaccessibles of order type β that is *closed downward* (any inaccessible less than a member is itself a member) and (2) there is no such set of order type κ. In other words, $\mathcal{Z}_{\theta_\kappa}$ asserts the existence of a κ-sequence of inaccessibles containing all inaccessibles. Again, this approach works, but it works at most denumerably (continuum) many times, whereas Zermelo seems to assume that something like this is possible for every θ_κ.[9] Indeed, uniform definability at strongly inaccessible levels must fail already for some countable κ.

We wish eventually to extend Zermelo's concept of a closed domain to the systems of propositions discussed beginning in §5. To advance toward this

[9]Categorical Characterization would hold in a model of the axioms of Zermelo set theory (without Replacement) together with "there are arbitrarily large strongly inaccessible cardinals," "the collection of inaccessible cardinals has order type ω," "P_α exists for all α," and "every set belongs to some P_α." $\langle P_\kappa, \in|P_\kappa\rangle$, for κ the (singular) supremum of the first ω inaccessible cardinals, would be such a model. This proposal is not put forth as a way of reading Zermelo.

end, we introduce some preliminary considerations. Suppose that an infinite domain $\mathcal{W}$ structured by well-founded relation $\prec$ is given such that $\mathcal{W}$ is describable as cumulative hierarchy $\bigcup_{\alpha>0} P'_\alpha$ with (1) $P'_1 = Q'_0 = U'$ for some *basis* $U' \subset \mathcal{W}$ and (2) $P'_\alpha \subset P'_\beta$ whenever $\alpha < \beta$. Further, suppose that injective $\phi \colon \langle P'_1, P'_\alpha, \prec|P'_\alpha\rangle \longrightarrow \langle P_1, P_\alpha, \in|P_\alpha\rangle$ is an isomorphic embedding of inaccessible level $P'_\alpha \subseteq \mathcal{W}$ into an inaccessible level of a cumulative hierarchy of sets. By *Aussonderung*, $\phi(P'_\alpha) \subseteq P_\alpha$ is a set in $\langle P_1, P_\beta, \in|P_\beta\rangle$ for inaccessible $\beta > \alpha$, and this may already be enough to legitimize $\langle P'_1, P'_\alpha, \prec|P'_\alpha\rangle$ in Zermelo's eyes. But his remarks in [30] indicate that a categorical characterization of $\phi(P'_\alpha)$ is required if P'_α is to be closed, and the general argument for the existence of such a characterization is not clear to us: even granting that $\langle P_1, P_\alpha, \in|P_\alpha\rangle$ itself is fixed categorically, nonetheless, by cardinality again, the same can be said of at most denumerably (continuum) many of its substructures. So our reconstruction of Zermelo's thinking concerning closed domains will involve additional assumptions beyond Categorical Characterization.

Despite these reservations, we believe that Zermelo regards (1)–(2) below as immediate consequences of his logical characterization of open and closed domains.

1. For arbitrary inaccessible α, any subdomain $\langle P'_1, P'_\alpha, \prec|P'_\alpha\rangle$ of well-founded domain $\langle U', \mathcal{W}, \prec\rangle = \left\langle P'_1, \bigcup_{\beta>0} P'_\beta, \prec\right\rangle$ is closed by virtue of the fact that $\langle P'_1, P'_\alpha, \prec|P'_\alpha\rangle$ can be isomorphically embedded in level $\langle P_1, P_\alpha, \in|P_\alpha\rangle$ of some cumulative hierarchy with $P'_1 \subseteq P_1$.
2. $\langle U', \mathcal{W}, \prec\rangle$ is an open domain since it can be isomorphically embedded only in some $\left\langle P_1, \bigcup_{\alpha>0} P_\alpha, \in\right\rangle$, which, again, is assumed incapable of categorical characterization.

Zermelo's distinction surely dates from the period of his Warsaw lectures, since there is no mention of open or closed domains in what may be the best extant record of his thought during the immediately preceding years.[10] Although Zermelo's distinction is *logical* in character, we choose to speak of his *philosophy* of open and closed domains. One reason for doing this is that Zermelo's primary use of his technical distinction will be in philosophy.

§5. Well-foundedness and mathematical proof. Well-foundedness plays an essential role in [34]. Foundation ensures that, where P_α is a normal domain, $\langle P_\alpha, \in|P_\alpha\rangle$ involves no infinitely descending $\in$-chains. Well-foundedness figures prominently in [36], [37], and [38] as well. Zermelo first uses it to

[10]Zermelo and his American protege Marvin Farber planned a joint work on logic based on their conversations during the years 1923–27. The project seems not to have gotten very far, but Farber's notes, now housed in his Buffalo archive, may prove to be of interest. (Marvin Farber [1901–80], who, like Zermelo in 1923, attended lectures of Husserl in Freiburg, is well known in phenomenology circles as founding editor of the journal ***Philosophy and phenomenological research*** [1940–].)

describe proposition systems themselves and then characterizes the notion of proof relative to such systems. The latter analysis in turn serves as the core idea in an argument for universal decidability. The significance of well-foundedness for the theory of proposition systems is most vigorously expressed in an early draft of [38].

> On page 31 of my [34], mentioned above, I formulated the "Axiom of Foundation" for the $\in$-relation as follows: every subdomain T (of the entire domain of sets) contains at least one element t_0 that, regarded as a set, possesses no element t in T. With the addition of this axiom I was able, by banning "circular" and "infinitely descending" sets, to fully characterize those domains that are truly useful within set theory. But this concept of "well-foundedness" has greater scope, as I mentioned at that time. Indeed, as indicated already in my [36] and as I intend to elaborate here, this concept appears to me to constitute the most natural and most secure foundation for a general "infinitary," and hence truly mathematical, logic of proposition systems.[11] ([25])

Zermelo introduces the concept of a well-founded hierarchy of propositions in [38]. One starts with a collection of atomic propositions and obtains more complex ones by iterating unary negation and infinitary conjunction and disjunction. (Again, Zermelo appears to assume a Russellian, nonreductionist, conception whereby individuals and relations holding between them are the components of atomic propositions.) The immediate subcomponent relation is well-founded on the collection of all propositions and enables one to speak of a well-founded hierarchy of propositions on analogy with the cumulative hierarchy of sets: if α is the least ordinal such that $\mathcal{A}_1$ and $\mathcal{A}_2$ are both found within initial segment P^*_α of the hierarchy of propositions, then proposition $\mathcal{A}_1 \wedge \mathcal{A}_2$ is found at level Q^*_α and hence within initial segment $P^*_{\alpha+1}$. (We shall say that $\mathcal{A}_1 \wedge \mathcal{A}_2$ has *rank** α.) Well-foundedness means that, although propositions can be infinitely "long," they cannot be infinitely "deep:" there will be no infinitely descending subcomponent chains.

Zermelo's informal idea is that of a maximal iterative concept of proposition: at each level Q^*_α, "all possible" (new) conjunctions $\bigwedge K$, with $K \subseteq P^*_\alpha$, are formed. (Letting K be empty gives $\top$ in Q^*_1, and K is permitted to be

[11] *In der oben genannten Fundamenta-Note* (F.M. Bd. XVI) *habe ich auf S. 31 für die* $\in$*-Relation als „Axiom der Fundierung" das folgende formuliert: Jeder Teilbereich* T *(des ganzen Mengen-Bereiches"* (sic) *enthält wenigstens ein Element* t_0, *das (als Menge betrachtet) kein Element* t *in* T *besitzt. Durch Hinzufügung dieses Axioms* F) *war es mir möglich, durch Ausschließung der „zirkelhaften" und „abgründigen" Mengen die in der Mengenlehre wirklich brauchbaren Mengenbereiche vollständig zu charakterisieren. Dieser „Fundierungs"-Begriff reicht aber weiter, als ich damals zum Ausdruck brachte, ja er scheint mir, wie ich schon in meinem Elster-Vortrag andeutete und wie ich hier weiter auszuführen gedenke, die natürlichste und sicherste Grundlage für eine allgemeine „infinitistische" und damit wahrhaft mathematische Syllogistik der Satzsysteme zu bilden.*

a singleton.) An alternative conception would be the following. We assume that each P^*_α can be well ordered. Conjunctions in Q^*_α are then (new) well-ordered "concatenations" $\mathcal{A}_0 \wedge \mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_\xi \wedge \ldots$ of propositions from P^*_α, where the propositions are arranged in order as ξ ranges through the ordinals less that some β with $0 < \beta < \alpha$. We shall not pursue this second, more restrictive, conception.

A hierarchy of propositions in Zermelo's sense is not itself a mathematical theory. But, relative to a fixed domain D, any theory T, understood as a collection of propositions, can be situated within some cumulative $\{P^*_\alpha\}_{\alpha>0}$ or noncumulative $\{Q^*_\alpha\}_{\alpha\geq 0}$. (This is another example of the domain-relative nature of Zermelo's thinking about mathematical theories.) Assuming that the axioms of T implicitly define some single k-adic R, basis Q^*_0 will consist of all propositions $Ra_1 \ldots a_k$, where $a_1, \ldots, a_k$ are arbitrary elements of D. As pertinent examples of well-founded hierarchies, we propose (1)–(2) below (cf. [38], p. 142).

1. Let D be the collection of permutations of symmetric group S_n for fixed n. Base level Q^*_0 comprises all propositions of the form $\lceil \psi_i \circ \psi_j = \psi_k \rceil$. We have that $|D| = n!$ so that there will be $(n!)^3$ propositions in Q^*_0. Of these, precisely $(n!)^2$ will be true—one for each entry in an assumed multiplication table.[12]
2. In the case of set theory, Q^*_0 will contain all and only propositions of the form $\lceil a$ is a member of $b\rceil$, where a and b are arbitrary elements of some given domain D of urelements and (well-founded) sets built up from them. (It is best to assume here that D itself is a set.) If we assume the relations implicit in some cumulative hierarchy, then Q^*_0 will consist of $|D|^2$ propositions, most of them false. For instance, for arbitrary a, there will be exactly one b such that both (i) $\lceil a$ is a member of $b\rceil$ is true and (ii) proposition $\lceil c$ is a member of $b\rceil$ is false for every $c \in D$ distinct from a. In other words, for every a, the singleton of a exists and is unique.

At higher levels we have propositions obtained from those of Q^*_0 by conjunction, disjunction, and negation. Thus, in (2), if $\lceil a$ is a member of $b\rceil$ and $\lceil b$ is a member of $c\rceil$ are propositions in Q^*_0, we have propositions $\neg\lceil a$ is a member of $b\rceil$, $\lceil a$ is a member of $b\rceil \wedge \lceil b$ is a member of $c\rceil$, and $\lceil a$ is a member of $b\rceil \vee \lceil b$ is a member of $c\rceil$ in Q^*_1. Again, conjunctions and disjunctions may be infinite—indeed, already at level Q^*_1. (The situation would be different if we were to adopt the concatenation conception of conjunction formation.)

[12] The hierarchy described in (1) is not a representation of the general theory of finite permutation groups but, rather, that of the theory of a particular permutation group S_n for n fixed. Concerning the general theory, Zermelo says only that "the situation is somewhat different" ([38], p. 142). We speculate that the hierarchy induced by that general theory would have in Q^*_0 all propositions of the forms $\lceil \psi$ is an n-permutation$\rceil$ and $\lceil \psi_i \circ \psi_j = \psi_k \rceil$.

By a *truth partition* of a hierarchy of propositions, Zermelo means an arbitrary assignment of truth values to the propositions of basis Q_0^*; propositions at higher levels then acquire truth values in accordance with the classical definitions of the relevant connectives. A proposition $\mathcal{A}$, assumed to belong to some hierarchy of propositions, is tautologous (contradictory) provided that every truth partition of basis Q_0^* results in $\mathcal{A}$'s being assigned value true (false). Proposition $\mathcal{A}$ logically implies proposition $\mathcal{B}$ if any truth partition making $\mathcal{A}$ true makes $\mathcal{B}$ true as well. Propositions $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent if each logically implies the other. Much of [36], [37], and [38] is given over to unexceptional insights such as these. But Zermelo's ultimate goal is to relate his transfinite hierarchies of propositions to the twin issues of realism and decidability (see §10 below). His transfinite concept of proof provides the link.

Mathematical proof is analyzed in [38] in terms of truth partitions of an assumed hierarchy of propositions. For example, proving implication

$$\bigwedge K \Longrightarrow \mathcal{A} \tag{**}$$

where $\mathcal{A}$ and all members of K are in P_α^*, involves consideration of all possible truth partitions of P_α^* in order to ascertain that every partition making all members of K true also makes $\mathcal{A}$ true. (As will be seen in §6, ordinal α is best thought of as a strongly inaccessible cardinal.)

> A mathematical "proof" is, after all, nothing but a *well-founded system of propositions such that it is not possible, without violating the rules of logic, to partition the system into two classes, one consisting of truths and the other of falsehoods, whereby the premises belong to the former while the proposition to be proved belongs to the latter*. ([36], p. 87)

Although Zermelo refers, here and elsewhere, to the system itself, it is also clear that comprehension of such a proof of (**) would involve an infinite *Gedankenexperiment* (our term)—or better, a possibly infinite collection of infinite *Gedankenexperimente*, one for each truth partition of Q_0^*.[13] The collection of all truth partitions will be infinite just in case Q_0^* is. Individual *Gedankenexperimente* will be infinite if propositions of infinite complexity occur among premises and conclusion or if there are infinitely many premises. We take (**) to involve a single premise of infinite complexity.

By identifying proofs with "systems" Zermelo would preserve intersubjectivity while perforce sacrificing communicability. Also, there is a question of computational complexity even in the case of a finitary proposition $\mathcal{A}$: worst case, the number of computation steps required to complete a proof

[13]This reading of Zermelo has a history. In [15] Lakatos writes of "well-known discussions about the admissibility of Zermelo's and Gentzen's thought-experiments" (p. 52, fn. 2). Whatever the nature of these discussions, our inquiries indicate that they are no longer well known, at least as they bear on Zermelo.

of $\mathcal{A}$ will be exponential in n, where n is the number of atomic propositions involved in $\mathcal{A}$. One might conclude that new infinitary rules of proof, on the model of mathematical induction (see §9 below), are needed so that infinite *Gedankenexperimente* can be abbreviated and their results communicated. This seems to be the force of the following remark, coming toward the end of [38].

> An [infinitary] "proof" will contain *infinitely many* intermediate propositions in most cases, and it is not yet clear to what extent and through what auxiliary means it can be made intelligible for our *finite* understanding. At root, *any* mathematical proof, e.g., proof by complete induction, is thoroughly infinitary, and nonetheless we are capable of grasping it. On the face of it, there seem to be no fixed bounds of intelligibility. ([38], pp. 144–45)

§6. A Cantorian theory of systems of infinitely long propositions. The connection between the technical results concerning cumulative hierarchies of sets, published in [34], and the cumulative hierarchies of propositions of [38] is cited in the latter.

> Proceeding upward, a well-stratified system of propositions can be closed off at will, e.g., at a well-defined strongly inaccessible π (cf. [34]), and then possesses all the attributes of a set. ([38], p. 141–42)

Thus we mentioned in §5 an assumption that P^*_α is well ordered. Zermelo's intentions with respect to strong inaccessibility are more forcefully expressed in an unpublished summary of [38]:

> The hierarchical system of propositions occurring below any strongly inaccessible level is a domain of sets [*Mengenbereich*] upon which we shall *make the same demands as in the case of the axiomatized domains* of [34].[14] (emphasis added) ([26], p. 1)

At the very least, these passages urge that set theory and logic be developed within a common framework, namely, ZF^2 together with an axiom asserting the existence of sufficiently many strongly inaccessible cardinals. This common framework is also implied by the emphasis on well-foundedness in both [34] and [38]. But Zermelo nowhere elaborates, and the present section of this paper, as well as §7 that follows, constitute an expansion of his elliptical remarks. In particular, the inspired philosophical passages of [34] §5, which give Zermelo's technical results there such interest (see §3 above), have obvious counterparts in the context of [38], and we shall attempt to provide

[14] *Das Folgesystem der Sätze unterhalb einer Stufe (Grenzzahl) ist ein Mengenbereich, an den man dieselben Forderungen stellen wird wie an die in der früheren Abhandlung „Grenzzahlen und Mengenbereiche" des Verf.* (Fund. math. 16 (1930) 29–47; F.d.M _ (sic)) *axiomatisierten Bereiche.*

these. It is this "common framework" reading of [38] that we investigate below, giving particular attention to the role of strong inaccessibility. But our reading of Zermelo is conservative: as hinted in Thesis I, it is not unlikely that, in Zermelo's view, complex propositions are *sets* and proofs are special configurations ("systems") of sets.

Zermelo envisions a theory of proposition systems driven by his doctrine of open and closed domains. The general outline of this theory is described, very freely, in (1)–(6) below. In considering our reconstruction, the reader should be aware that Zermelo's own papers contain relatively few symbols.

1. To begin, Zermelo's conception of a mathematical theory T differs from the modern conception according to which T is a class of sentences, of some quantificational language, closed under semantic consequence. Specifically, Zermelo's starting point is invariably some nonempty, typically infinite, *fundamental domain* D of objects in the role of subject matter. What theory T will be able to say about these objects is determined by some nonempty collection $\mathcal{R} = \{R_1^{n_1}, \ldots, R_j^{n_j}\}$ of relations over D (see fn. 21). Ultimately, T comprises (a) fundamental domain D, (b) the collection $C_{\mathcal{R}}$ of all *fundamental relations* over D involving members of $\mathcal{R}$, where a fundamental relation is an atomic proposition of the form $R_i^{n_i} a_1 \ldots a_{n_i}$ with $R_i^{n_i}$ in $\mathcal{R}$ and $a_1, \ldots, a_{n_i}$ arbitrary members of D, and (c) a nonempty collection Π of truth partitions of $C_{\mathcal{R}}$—intuitively, precisely those making every axiom of T true. (Letting T be the triple composed of fundamental domain D, *collection* $\mathcal{R}$ *of relations*, and truth partitions Π would be the more elegant choice perhaps. But Zermelo consistently starts with (b).)

2. A hierarchy $\mathcal{H}_{D,\mathcal{R}}$ of propositions corresponds to any theory $T = \langle D, C_{\mathcal{R}}, \Pi \rangle$. Collection $C_{\mathcal{R}}$ serves as basis Q_0^*. Each cumulative level P_α^*—each *system* of propositions—then represents an "order (*Stufe*) of quantification" relative to T.

3. Application of the algebraic distinction between open and closed domains to systems of propositions requires that we identify some characteristic formation operation. We choose formation of conjunction $\bigwedge K$ for K a subcollection of a proper initial segment of the system in question. Given Zermelo's second-order reading of *Aussonderung*, this just amounts to K's being a set of propositions belonging to the system.

4. In one sense, the fullest representation of theory T is the class Val_Π of all and only those propositions within $\mathcal{H}_{D,\mathcal{R}}$ that are Π-*valid*, i.e., that come out true according to each and every member of Π. But, even if $Val_\Pi \cap Q_0^*$ is null, we can argue that Val_Π, like $\mathcal{H}_{D,\mathcal{R}}$ itself, is an open domain and (the inverse image of) a set on pain of contradiction. (Cf. Cantor's assertion of the inconsistency of the domain of everything thinkable [*Inbegriff alles Denkbaren*].)

5. Each initial segment $Val_\Pi \cap P_\alpha^*$, for arbitrary $\alpha > 0$, makes for a system of propositions that is closed in the algebraic sense, given (3). Even

Categorical Characterization is not enough to yield closure in the logical sense, however.

6. Finally, the significance of strong inaccessibility having been established in [34], levels P^*_α with α inaccessible are of special interest. Corresponding domains $Val_\Pi \cap P^*_\alpha$ will be closed in both the algebraic and logical senses, assuming Categorical Characterization (see below). Such domains are partial representations or *closed approximations* of T. On the other hand, even $Val_\Pi \cap P^*_\alpha$ represents a restricted point of view such that it is possible to "say more." Specifically, such a closed domain points beyond itself in that it will not contain the conjunction $\bigwedge(Val_\Pi \cap P^*_\alpha)$ of all its members.

Let $\ll$ be the relation holding between two propositions $\mathcal{A}$ and $\mathcal{B}$ just in case $\mathcal{A}$ is an immediate subcomponent of $\mathcal{B}$. (Thus, $\mathcal{A} \ll \neg\mathcal{A}$ and so forth.) Given a mathematical theory $T = \langle D, C_\mathcal{R}, \Pi\rangle$, we project some hierarchy $\mathcal{H}_{D,\mathcal{R}}$ of propositions relative to fundamental domain D and collection $C_\mathcal{R}$ of finitary relations (cf. (1) and (2) in §5). We wish to show that, while $\mathcal{H}_{D,\mathcal{R}}$ is an open domain, denumerably (continuum) many of its inaccessible levels $\langle P^*_\alpha, \ll|P^*_\alpha\rangle$ may be categorically characterizable and hence closed. In accordance with (1) at the end of §4, we show that such a $\langle P^*_\alpha, \ll|P^*_\alpha\rangle$ can be isomorphically embedded in a certain $\langle P_\alpha, \in|P_\alpha\rangle = \langle U, P_\alpha, \in|P_\alpha\rangle$. In fact, this can be shown in any number of ways.

For simplicity, we drop $\vee$ as primitive and let $\mathcal{H}^{\wedge,\neg}_{D,\mathcal{R}} = \left\langle \bigcup_{\alpha>0} P^*_\alpha, \ll \right\rangle$ be the hierarchy of propositions that results from first setting $Q^*_0 = C_\mathcal{R}$ and then closing under arbitrary negations and conjunctions (of distinct propositions) only. One small problem is that Zermelo has $\mathcal{A} \wedge \mathcal{B}$ and $\mathcal{A} \vee \mathcal{B}$ at one and the same level of any hierarchy, whereas our device will put $\phi(\mathcal{A} \vee \mathcal{B}) = \phi(\neg[\neg\mathcal{A} \wedge \neg\mathcal{B}])$ two levels higher than $\phi(\mathcal{A} \wedge \mathcal{B})$. Otherwise, ϕ will be rank-preserving.

Injective $\phi\colon \left\langle C_\mathcal{R}, \bigcup_{\alpha>0} P^*_\alpha, \ll \right\rangle \longrightarrow \left\langle C_\mathcal{R} \cup \{f\}, \bigcup_{\alpha>0} P_\alpha, \in \right\rangle$ defined below isomorphically embeds $\mathcal{H}^{\wedge,\neg}_{D,\mathcal{R}} = \left\langle C_\mathcal{R}, \bigcup_{\alpha>0} P^*_\alpha, \ll \right\rangle$ within the cumulative hierarchy of sets whose urelements are the members of $C_\mathcal{R}$ together with distinguished f.

i. $\phi(\mathcal{A}) = \mathcal{A}$ if $\mathcal{A}$ is in $Q^*_0 = C_\mathcal{R}$
ii. $\phi(\mathcal{A}) = \emptyset$ if $\mathcal{A}$ is empty conjunction $\top$
iii. $\phi(\mathcal{A}) = \{f, \phi(\mathcal{B})\}$ if $\mathcal{A}$ is of the form $\neg\mathcal{B}$
iv. $\phi(\mathcal{A}) = \{\phi(\mathcal{B}) \mid \mathcal{B} \text{ in } K\}$ if $\mathcal{A}$ is of the form $\bigwedge K$, where K is a nonempty set of propositions

That ϕ is not onto is clear from the fact that a set x containing f is the image under ϕ of a proposition in $\mathcal{H}^{\wedge,\neg}_{D,\mathcal{R}}$ just in case $card(x) = 2$. Also, f is not the image of any proposition. The restriction of ϕ to Val_Π is then an isomorphic embedding of that collection within the cumulative hierarchy of sets.

Where $\mathcal{H}^{\wedge,\neg}_{D,\mathcal{R}} = \left\langle C_\mathcal{R}, \bigcup_{\alpha>0} P^*_\alpha, \ll \right\rangle$ is any hierarchy of propositions, as just described, we shall refer to $\langle C_\mathcal{R}, P^*_\alpha, \ll|P^*_\alpha\rangle \subset \mathcal{H}^{\wedge,\neg}_{D,\mathcal{R}}$, with α fixed, as a *system* of propositions. Apparently, any such system is determined up to isomorphism by (1) its *index*, i.e., by the cardinality of the domain of fundamental relations at level $Q_0^* = C_{\mathfrak{R}}$, and (2) its *characteristic*, i.e., by the permissible level of nesting of connectives (see [34], First Isomorphism Theorem). As for the role of strong inaccessibility, any system $\langle C_{\mathfrak{R}}, P_\alpha^*, \ll|P_\alpha^*\rangle$ with α inaccessible (and hence any $\langle C_{\mathfrak{R}} \cap Val_\Pi, P_\alpha^* \cap Val_\Pi, \ll|P_\alpha^* \cap Val_\Pi\rangle$) will be isomorphically embeddable within a natural model of ZF^2 having urelements $C_{\mathfrak{R}} \cup \{f\}$ and characteristic α, as indicated in (i)–(iv). Thus we have that each term $\langle P_\beta^*, \ll|P_\beta^*\rangle$ in the sequence $\{\langle P_\alpha^*, \ll|P_\alpha^*\rangle\}_{\alpha>0}$ can be extended to a later term $\langle P_\gamma^*, \ll|P_\gamma^*\rangle$ with γ inaccessible such that $\langle \phi(P_\gamma^*), \in|\phi(P_\gamma^*)\rangle$ is a submodel of a natural model of ZF^2, where both submodel and extension have characteristic γ.

As noted at the end of §4, we seek a categorical characterization of $\phi(P_\alpha^*) = \{\phi(\mathcal{B}) \mid \mathcal{B} \text{ in } P_\alpha^*\}$, for fixed inaccessible α. To this end, we present an axiom system $\mathcal{S}_\alpha^A$ that categorically determines the simple extension $\langle C_{\mathfrak{R}} \cup \{f\}, P_\alpha, \in|P_\alpha, A\rangle$, where A is a new class constant. We assume, by Categorical Characterization, that $\langle C_{\mathfrak{R}} \cup \{f\}, P_\alpha, \in|P_\alpha\rangle$ is categorically axiomatized by $\mathcal{S}_\alpha$ (see §4). Our $\mathcal{S}_\alpha^A$ will consist of $\mathcal{S}_\alpha$ augmented by the following five second-order axioms.

I. $(\forall x)\left[(\mathscr{U}(x) \wedge x \neq f) \rightarrow x \in A\right]$ (cf. (i) and fn. 3)
II. $\emptyset \in A$ (cf. (ii))
III. $(\forall x)\left[x \in A \rightarrow \{f, x\} \in A\right]$ (cf. (iii))
IV. $(\forall x)\left[(x \subseteq A \wedge (\exists y) y \in x) \rightarrow x \in A\right]$ (cf. (iv))
V. An axiom saying that A is the smallest class satisfying (I)–(IV)

Axiom system $\mathcal{S}_\alpha^A$ categorically characterizes $\phi(P_\alpha^*)$ as proper class A in $\langle C_{\mathfrak{R}} \cup \{f\}, P_\alpha, \in|P_\alpha, A\rangle$ and hence categorically characterizes its inverse image P_α^* as a cumulative level of $\mathcal{H}_{D,\mathfrak{R}}^{\wedge,\neg}$. Again, if this is to work in general, Categorical Characterization would have to be true—among other things. Otherwise, only denumerably (continuum) many inaccessible levels of $\mathcal{H}_{D,\mathfrak{R}}^{\wedge,\neg}$ will be categorically axiomatizable. This limitation might be taken to suggest that Zermelo is allowing for axiomatizations based on infinitary logic. The textual evidence does not support this, however. Moreover, such a move trivializes the notion of characterization: what have we achieved if characterizing $\mathcal{S}_\alpha^A$ is no smaller than characterized P_α^*? (But Zermelo may see things very differently.) A way out would be to assume only countably (denumerably) many inaccessibles.

One last step is required. Namely, having categorically characterized P_α^*, with α strongly inaccessible, it remains to see that $Val_\Pi \cap P_\alpha^* = T_\alpha^*$ is similarly characterizable and is hence a closed approximation of T. The idea would be to supplement the categorical characterization of $P_\alpha^* = \phi^{-1}(A)$ with (1) new second-order axioms, expressed in a language with first-order variables of several sorts, describing the collection Π of truth partitions of $C_{\mathfrak{R}}$, (2) first-order axioms corresponding to the classical understanding of $\wedge$ and $\neg$, as well as (3) a single axiom stating that if $\mathcal{A}$ is in P^*_α and $\pi(\mathcal{A}) = true$ for all $\pi \in \Pi$, then $\mathcal{A}$ is in T^*_α. But to suppose that such supplementation is possible in general is, once again, to make strong assumptions regarding the availability of characterizations of the sort (1)–(3)—assumptions that, again just by cardinality, do not seem to be true.[15]

Categorical characterization of a system of propositions involves fixing both its index and a strongly inaccessible characteristic, and we have focused on the latter. So before moving on, we mention another limitation of our approach, this time concerning the domain D of any theory T. Suppose, for example, that $T = \langle D, C_{\mathfrak{R}}, \Pi \rangle$ is the theory of a particular infinite group, which means that D is infinite. In that case, the collection $C_{\mathfrak{R}}$ of fundamental relations will also be infinite with $|D| = |C_{\mathfrak{R}}|$, and, again, only a "few" such collections will be categorically characterizable. One can readily grant that, for almost all currently interesting groups, we will have either $|D| = |C_{\mathfrak{R}}| \leq \omega$ or $|D| = |C_{\mathfrak{R}}| = c$ where c is any one of denumerably (continuum) many definable constants denoting cardinals beyond ω. So, for almost any group anyone is likely to care about, its index can be fixed categorically. Nonetheless, if the theories of infinite groups generally are to be describable as sequences of categorically characterizable structures $\langle C_{\mathfrak{R}}, P^*_\alpha, \ll | P^*_\alpha \rangle$, then, as the reader will by now have guessed, strong assumptions regarding characterization of $C_{\mathfrak{R}}$ are required at the outset.

In this section we have tried to reconstruct the thinking that lies behind Zermelo's remark that one must "make the same demands" on inaccessible levels within hierarchies of propositions. We recommend thinking of our ϕ in (i)–(iv) as the embodiment of a "transfer principle" (*Übertragungsprinzip*) in an old sense, translating "logics" into closed domains of sets.

§7. Zermelo incompleteness. Based on something like our discussion in §6, Zermelo concludes that, at a minimum, set theory grounds our concepts of logic and proof. He invites us to regard every proposition above level $Q^*_0 = C_{\mathfrak{R}}$ as (the inverse image of) a set. Axiomatic set theory, augmented by large cardinal axioms, is the final arbiter of what (nonatomic) propositions there are, of what closed systems of propositions there are, and of which proofs are possible. So regarded, Zermelo's proof theory constitutes an application of set theory whereby fundamental relations $R^{n_i}_i a_1 \dots a_{n_i}$ play the role of urelements.

[15]Here is a counterexample, as pointed out by Professor Solovay. It is easy to see that there are continuum many isomorphism classes of countable abelian groups: for any set S of primes, it can be shown that there exists a countable abelian group the orders of whose elements are precisely the members of S. Each of these isomorphism classes induces a theory in Zermelo's sense. So if supplementations (1)–(3) are finite, then there will not be enough to go around.

If we assume set-theoretic reductionism for the moment, viz., the doctrine that some cumulative hierarchy of sets just is the mathematical realm, then we can go on to make a further claim. We have seen that transfinite systems of infinitely long propositions can be described that can then be embedded isomorphically in natural models of ZF^2. But Zermelo insists that these closed systems are the only adequate logical background for mathematical theories describing the various aspects of the mathematical realm, e.g., its group-theoretic aspects, number-theoretic aspects, and so forth. This in turn leads us to formulate, on Zermelo's behalf, a certain *Isomorphism Thesis: the structure of the collection of propositions purporting to describe the mathematical realm is isomorphic to a substructure of that realm.* In other words, our "logics" are isomorphically embeddable in natural models of ZF^2, where a logic is any system $\langle C_{\mathfrak{R}}, P^*_\alpha, \ll|P^*_\alpha\rangle$ with α strongly inaccessible.[16]

Corresponding to the transfinite sequence $\{\langle C_{\mathfrak{R}} \cup \{f\}, P_\alpha, \in|P_\alpha\rangle\}_{Inac(\alpha)}$ of models of ZF^2 we have a transfinite sequence $\{\langle C_{\mathfrak{R}}, P^*_\alpha, \ll|P^*_\alpha\rangle\}_{Inac(\alpha)}$ of logics, each term in the latter sequence being isomorphically embeddable in the corresponding term of the former. Expressed differently, our logics stack precisely in the manner of models of ZF^2. Since any proof in Zermelo's sense will have one of these logics as its context, Zermelo's theory should be seen as a systematic integration of mathematical concepts of existence and proof. On analogy with sequence $\{\langle P_\alpha, \in|P_\alpha\rangle\}_{Inac(\alpha)}$ of concepts of mathematical existence, Zermelo introduces a sequence $\{\langle P^*_\alpha, \ll|P^*_\alpha\rangle\}_{Inac(\alpha)}$ of highly transfinite, nested concepts of mathematical demonstrability. The general concept of mathematical demonstrability is then represented both by the individual terms of this sequence and by the sequence in its entirety. In particular, what is a "nonproof" relative to one proof concept may become a full-fledged proof relative to the next one.

It seems likely that Zermelo regards incompleteness as the proof-theoretic counterpart—proof-theoretic in his own, very special, sense—of the Problem of Proper Classes. Where $T = \langle D, C_{\mathfrak{R}}, \Pi\rangle$ is any theory whatever, incompleteness is an intrinsic property of its closed approximations $T_\alpha = Val_\Pi \cap P^*_\alpha$, and we will be surprised by such incompleteness only if we lose sight of the dialectic according to which only Val_Π in its entirety can fully

[16] In [1] Barwise commemorates a number of early papers on infinitary logic, and his citation of Zermelo in this connection is most apt. However, [38] is inaccurately described as an investigation of "the propositional part of $L_{\omega_1\omega}$." For one thing, according to Zermelo's maximal iterative conception of his systems of propositions, some conjunctions and disjunctions already at level Q^*_1 will be of uncountable length if the collection Q^*_0 of fundamental relations happens to be uncountable. If anything, Zermelo's logics can be related to languages $L_{\theta\theta}$, with θ strongly inaccessible, by letting D, and hence $Q^*_0 = C_{\mathfrak{R}}$, be countably infinite. Briefly, where κ and λ are infinite cardinals, $L_{\kappa\lambda}$ is infinitary first-order logic with the following restrictions: (1) any disjunction (conjunction) has fewer than κ disjuncts (conjuncts); (2) κ individual variables are available; and (3) the quantified variables of any formula comprise a set of fewer than λ members.

represent T. Proposition $\bigwedge(Val_\Pi \cap P^*_\alpha)$ is an example of a true but unprovable proposition—a *Zermelo sentence*, if you will—by which we mean unprovable relative to the concept of demonstrability corresponding to logic $\langle P^*_\alpha, \ll|P^*_\alpha\rangle$ (cf. (5) in §6 above). Resolution is to be had in the provability of $\bigwedge(Val_\Pi \cap P^*_\alpha)$ within logics higher up.[17] This mirrors the situation whereby $\{\phi(\mathcal{A}) \mid \mathcal{A}$ in $Val_\Pi \cap P^*_\alpha\}$ is a proper class in $\langle C_\mathfrak{R} \cup \{f\}, P_\alpha, \in|P_\alpha\rangle$ but a set at higher levels. Herein we see the application of the Dialectical Principle to systems of propositions, as is suggested by the following quotation from Zermelo's second letter to Gödel, dated October 29, 1931.

> These undecidable sentences [*Sätze*] become "decidable" in a "higher" system. But this "higher" system differentiates itself from the initial system not as the result of adding *new sentences* but, rather, through new means of proof. ([12], p. 302)

The new means of proof would, on one reading, be *Gedankenexperimente* of next higher, inaccessible order type. (Note, however, that we are adding new sentences.)

Before moving on, we return briefly to the issue of the nature of Zermelo's propositions. Are they sets? First of all, nonatomic propositions, finite and infinite, are *set-like* in Zermelo's view: the intended analogy with the cumulative hierarchy of sets is surely strongest if (1) conjuncts are unordered and (2) no conjunct is ever repeated.[18] Recalling Thesis I, it is reasonable to suppose that nonatomic propositions *are* sets according to him. The quoted passage from [26] (see p. 497 above) extends this to atomic propositions. The appeal of this view is clear: since mathematicians reason about propositions, they, like all mathematical objects, should be reducible to sets. In that case, Zermelo's infinitary proof theory amounts to an elaboration of set-theoretic reductionism. But the strength of Zermelo's commitment here is difficult to judge. It is only in the unpublished [26] that initial segments $\langle P^*_\alpha, \ll|P^*_\alpha\rangle$ with α inaccessible are termed *Mengenbereiche*. In the published version [38] we read, instead, that they "possess all the attributes of a set." There is no way of knowing which is the earlier, but, in any case, the two characterizations

[17] If this is a plausible reconstruction of Zermelo's own view, one can still question its applicability to arithmetic, for example. Namely, where $T = \langle \mathbb{N}, C_{\{R_+, R_\times\}}, \Pi\rangle$ and Π consists of a single truth partition making $R_+(2,2,4)$ and $R_\times(5,6,30)$ true, one can reasonably ask, What is to be gained, from the point of view of extending our knowledge of the natural numbers, by proceeding beyond $L_{\omega_1\omega}$ (cf. $Val_\Pi \cap P^*_{\omega_1}$)? After all, already in $L_{\omega_1\omega}$ it is possible to give a categorical characterization of the arithmetic of natural numbers. Whatever truth is expressed by Zermelo sentence $\bigwedge(Val_\Pi \cap P^*_\alpha)$, for α inaccessible, it adds nothing to our understanding of the natural numbers.

[18] If Zermelo's propositions are really sets, then certain propositions, e.g., $\mathcal{A} \vee \mathcal{B}$ and $\mathcal{B} \vee \mathcal{A}$, normally regarded as distinct but equivalent, would now be identical. And this is exactly how Zermelo sees things (cf. [38], pp. 140–41), while emphasizing that $\mathcal{A}$ and $\neg\neg\mathcal{A}$ are distinct. Note, however, that, for $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ in Q^*_0, conjunction $(\mathcal{A} \wedge \mathcal{B}) \wedge (\mathcal{C} \wedge \mathcal{D})$ is in Q^*_2, whereas $\mathcal{A} \wedge \mathcal{B} \wedge \mathcal{C} \wedge \mathcal{D}$ is in Q^*_1.

are hardly equivalent. Does this signal a change of heart, or has Zermelo merely clarified his idea?

§8. Making the case for infinitary logic. Although systems for arithmetic, analysis, geometry, and set theory are mentioned in passing, Zermelo's focus in [36], [37], and [38] is logic and the general context of mathematical theorizing. He emphatically rejects the use of finitary first-order formal systems of the sort to which the techniques of Skolem and, by implication, Gödel are applicable. It is possible to discern four arguments against finitary restrictions. These arguments are distributed over Zermelo's published and unpublished writings from the period 1929–35.

Zermelo's attitude regarding the justification of mathematical axioms may be termed pragmatic. For instance, in justifying the Axiom of Choice, Zermelo emphasizes its fruitful use within demonstrations of a wide variety of mathematical propositions. Furthermore, he shows that mathematicians have regularly assumed the Axiom of Choice without being aware of doing so. Closer to home, it seems certain that Zermelo takes the technical and philosophical results of his [34] to provide support for some postulate asserting the existence of strongly inaccessible cardinals beyond ω. Both the appeals to fertility and the prominent role given to community practice mark Zermelo's views as pragmatic. That mathematical axioms may be justified by their success in applications amounts to an empirical justification of those axioms. There may be no incompatibility with Zermelo's Platonism if we can assume that widespread use of an axiom entails widely shared intuitive evidence (see [19]).

In any case, Zermelo's rejection of finitary restrictions is similarly pragmatic: just as appeals to fertility justify Choice, so undesirable consequences of finitary formulations militate against adopting *them* as the standard (see [36]). The considerable advantages of first-order logic, say, are outweighed in Zermelo's eyes by a paradoxical relativism and the bane of undecidability (syntactic incompleteness). We might refer to this as the *Argument from Undesirable Consequences of Finitary Formulations*.

Mathematical induction serves as the starting point for a second argument in support of infinitary logic. In [38] proof by mathematical induction is taken to be the application of an infinitary proof rule. (Recall that Poincaré regards an argument by induction as an infinite sequence of syllogisms.) Although not explicitly presented as such, Zermelo intends an argument here—an argument for the adoption of infinitary formulations, in contrast to finitary formalizations of whatever order. Given that mathematical induction is indispensable to mathematics and given its infinitary character, infinitary proof rules should be accepted generally. Note the analogy with Zermelo's earlier, pragmatic defense of the Axiom of Choice on the grounds that it has been widely used, in many cases tacitly, within the mathematical community. In that case, it is a certain principle being defended by virtue of its widespread, but largely unacknowledged, use. In the present case, we have a widely used proof rule whose infinitary character has been underappreciated.

Zermelo would probably hope to extend this argument, which we might term the *Argument from the Wide Use of Implicitly Infinitary Proof Rules*, by citing other, plausibly infinitary, rules of current interest. Reformulations of König's Tree Lemma and an infinitary version of Ramsey's Theorem present themselves as possibilities. Similarly, Zermelo might see the makings of such an (infinitary?) proof rule in the process of inferring the truth of Gödel sentences for formal systems. Zermelo alludes to such a rule at the very end of his second letter to Gödel.

> What is to be understood by 'proof'? In complete *generality*, one understands thereby *a system of sentences such that, by assuming the premises, the validity of the conclusion can be made clear.* And the question then is, What counts as 'clear'? Certainly not merely the sentences of some finitistic scheme, since, [as the proof of the First Incompleteness Theorem shows, such a system] can always be 'extended'." ([12], p. 303)

The point of the Argument from Wide Use is that infinitary proof rules are already being applied within mathematics. Hence mathematicians should recognize that their practice already commits them to infinitary logic. The argument brings us back to an important, unresolved issue. Namely, what would be the effect upon practice of the adoption of infinitary logic? The Argument from Wide Use would suggest only that mathematicians might now have additional proof rules at their disposal. Moreover, by implication, we may take Zermelo to be saying that the new techniques would not really be so different from techniques already in use: adding them would be to go on with the investigation of the traditional subject matter of mathematics in much the traditional manner. It would not, despite appearances, involve any really radically new modes of inquiry. In other words, the adoption of infinitary logic would be at once conservative and expansive.

A third argument in favor of infinitary systems is presented in the Warsaw lectures [28] and intimated in Thesis I. The starting point for this argument is what Zermelo takes to be a salient feature of the meaning of mathematical statements. Most often, he claims, the domains of the intended interpretations of formal theories are infinite. Moreover, even theories having only finite models, e.g., the theory of finite groups, acquire mathematical significance only as the result of a general instantiability throughout more encompassing infinite structures, e.g., the natural numbers or the integers.

> All of traditional arithmetic is grounded in the assumption of an unbounded number sequence as defined by the Peano postulates. Even the theory of finite groups, of finite sets, of congruence, and

> of finite fields of prime characteristic acquire their real mathematical, and nontrivial, significance only by virtue of the circumstance that they can be applied to finite domains of arbitrary cardinality. Moreover, such applications are in fact always developed only in the context of an all-encompassing, indeed infinite, arithmetic. A purely "finitistic" mathematics would no longer be mathematics in any true sense: one would really have nothing to prove in that everything could merely be verified by examination of some finite model. Genuine mathematics is, by its very nature, infinitistic and grounded in the assumption of infinite domains; accordingly, it may be designated as the "logic of the infinite." ([28], p. 136)

Finite group theory, albeit the study of finite structures, is informed by the unbounded propagation of these finite structures throughout an infinite background structure. (This idea is exemplified, in a contemporary setting, by finite model theory, wherein one studies the asymptotic properties of families of finite structures.) As an illustration of the thought expressed in the third and fourth sentences of the quoted passage, consider the theorem stating that if a permutation of n letters is expressed as the product of disjoint cycles, then its order is the least common multiple of the lengths of those cycles. Any proof will appeal to the commutativity of arbitrary disjoint cycles. But if mathematicians were to restrict themselves to feasible values of n, then there would be no need to appeal to any general principle. Instead, one could, for each such n, verify computationally that the order of every n-permutation, presented as a product of disjoint cycles, is the least common multiple of their lengths.

We read Zermelo as saying that, since the meaning of any mathematical proposition is, in the manner described, intrinsically infinitary (recall Thesis I), to restrict mathematics to finitary languages introduces a certain disanalogy between meaning and, for lack of a better term, syntax. (Or, as was suggested in §7 above, the disanalogy would be that between the structure of the mathematical realm and the structure of the collection of all our descriptions of individual mathematical objects within it.) Zermelo's point may be that every informal mathematical theory, even the theory of finite groups, concerns, if not all of the cumulative hierarchy, at least transfinite initial segments of it. Consequently, in order to avoid structural disanalogy, all mathematical theories must be formulated using infinitary logic. We might refer to the *Argument from the Inherently Infinitary Character of Mathematical Theories*.[19]

[19] In a section of Farber's notes headed "Truth and Validity" we find the following representative passage, reminiscent of Thesis I: "*Really* math. (in number sense), is the ∞ *systems*, where ∞ domains occur; where $\therefore$ the validity of the derived prop's. cannot be verified (must be proved). E.g., Fermat prop. is really math. But not a simple number-formula, meaning ref. to a finite no."

But why is it so crucial to preserve an analogy between meaning and syntax? Even if we grant that, collectively, mathematical propositions do presuppose an infinite ontology, why must their internal structure mirror this ontology? Apparently, Zermelo is assuming something like our Isomorphism Thesis (see §7 above), which, if accepted without further argument, begs the issue.

To be sure, Zermelo is interested enough in applying mathematical ideas to finite domains. For instance, in [32] he establishes necessary and sufficient conditions for White's having a winning chess strategy from a given position q: a certain finite collection $\overline{U}_\tau(q)$ of sequences of positions, each representing up to τ legal moves beginning from q, must be nonempty, where τ is determined solely by the total number of possible positions. Such a result concerning an unfeasibly large, but finite, state space should count as mathematics surely. For Zermelo this means viewing it as a specific instance of a general, truly mathematical, assertion about finite games.

Zermelo has one more argument for infinitary logic. We delay our discussion to our final §10 and turn now to the issue of mathematical knowledge.

§9. A Cantorian theory of mathematical knowledge. In giving an account of Zermelo's epistemology of mathematics, a starting point must be his remarks regarding self-evidence. He holds that mathematical axioms such as the Axiom of Choice are justified "by analyzing the modes of inference that in the course of history have come to be recognized as valid and by pointing out that the principles are intuitively evident and necessary for science" [31]. The German original speaks of *die anschauliche Evidenz der Prinzipien*. Apparently, in Zermelo's view, judgments of self-evidence are delivered by mathematical intuition.

The limitations of mathematical intuition are, much later, the topic of an important philosophical remark.

> In general, the true subject matter of mathematics is not sign combinations, as many assume, but, rather, conceptually ideal relations [*begrifflich–ideale Relationen*] between elements of a conceptually fixed infinite manifold [*Mannigfaltigkeit*]. In this regard our notation systems are ever but incomplete aids, which vary according to circumstances, for our finite understanding in its at least stepwise approximation [*schrittweise Annäherung*] to mastery of the infinite, which we are unable, by means of direct intuition, to either survey or grasp [*erfassen*] in its entirety.[20] ([36], p. 85)

[20]Some remarks of Cantor would provide the model for the quoted passage, which is not to deny some obvious differences:

> I have now come to consider it my task not merely to investigate, mathematically, the relations holding between the transfinite numbers but, most significantly, to determine where they arise in nature. For me there can be no doubt that, in this

Zermelo's use of the term *begrifflich–ideal* would hearken to a traditional distinction between the real objects of the natural sciences (*Realwissenschaften*) and the *formal*, *geistig*, or *ideal* objects of mathematical science (*Formalwissenschaft*). The first sentence here must be construed as an endorsement of Platonism. Clearly it is the contrast between Hilbert's views and his own that exercises Zermelo here, and that fact would have influenced the manner in which he chooses to express himself. Assuming that the elements of Zermelo's manifold have an objective existence, it is natural to ask whether the relations holding between them do as well. Famously, Gödel will come to express just such an opinion, and Zermelo may hold a similar view.[21]

Roughly put, Zermelo's claim in his second sentence is that mathematicians, through the judicious use of language, acquire approximate knowledge of the infinite. It is interesting, and perhaps significant, that the grammatical subject of this sentence is notation systems, because Zermelo consistently avoids committing himself to any particular symbolism, always expressing axioms for sets in a German largely devoid of mathematical symbols. We concentrate on this second sentence, and yet our interpretation of it will be influenced by the content of the first.

To begin, in the two sentences quoted—henceforth *Stepwise Approximation*—Zermelo is making two philosophical claims. One is Platonism. The other concerns the necessary incompleteness of our mathematical knowledge. The relations that are the subject of Zermelo's first sentence would be those of the form $R_i^{n_i} a_1 \dots a_{n_i}$, serving collectively as basis Q_0^* for some hierarchy of propositions. And this is related to another feature of Stepwise Approximation about which we can say something quite definite: since it is the incompletability of mathematics that appears to be the issue, the source of Zermelo's idea would be the technical and philosophical breakthroughs of [34]. We have seen how each succeeding term in the sequence $\{P_\alpha, \in|P_\alpha\}_{Inac(\alpha)}$ models the axioms of ZF^2. Without any doubt this se-

> manner, we shall continue to advance, without ever encountering an insurmountable barrier, and yet never achieve even an approximate conception [*angenäherten Erfassen*] of the Absolute. The Absolute can be recognized [*anerkannt*] but never apprehended [*erkannt*] even approximatively [*annähernd*]. ([3], p. 205, note 2))

[21]The following passage, from the beginning of an envisioned book on set theory, is pertinent.

> The starting point of mathematical reasoning is invariably a domain of objects with given relations. The collection of postulated relations should be regarded as a domain component of an importance equal to that of the collection of objects. ([27], p. 2)

The German original reads: *Ausgang mathematischer Schlußfolgerungen sei stets ein Bereich von Dingen mit gegebenen Beziehungen. Hierbei sollen die Gesamtheit der aufgestellten Beziehungen als ebenso wesentlicher Bestandteil des Bereiches aufgefaßt werden wie die Gesamtheit seiner Dinge.*

quence of models is the prototype for the manner in which we extend our mathematical knowledge.

The second sentence asserts that the subject matter of mathematics transcends any particular formalization and suggests that mathematical knowledge can be given no final, exhaustive form. (The characterization of our understanding as finitary should not be taken to imply that Zermelo is talking about finitary formalizations exclusively.) In the first instance, it is the realm of mathematical objects, or at least a part of it, that is held to be beyond our grasp—beyond our limited ability to "collect together." But, by implication, the collection of truths about those objects is then not completable by us.

In Stepwise Approximation, Zermelo contrasts our "finite understanding" with the "infinite manifold" of mathematics; the former at best "approximates" the latter. His use of the term *Mannigfaltigkeit* would be intended to emphasize that the mathematical realm in its entirety is an open domain, signaling the Cantorian roots of Zermelo's epistemology.[22] To this extent, the emphasis in the quoted passage on the finite/infinite dichotomy is misleading. For Zermelo's point is surely that approximation, in contrast to categorical characterization, is the only possibility not because the domain in question is infinite but, rather, because it is open. Having said that, the prominence given to the dichotomy probably means that something else is also going on here. Two related ideas spring to mind.

First, the mathematical realm is infinite in that it contains infinitely many objects; it cannot be grasped in its entirety. Second, individual mathematical objects, e.g., real numbers or infinite sets, may be infinite; such an object cannot be grasped in its entirety. Clearly, it is the first of these ideas that engages Zermelo here. But the second idea is close at hand. One possible elaboration of this second idea would involve definability, a concept that figures prominently in Zermelo's thinking.

An impredicative specification of an infinite set S of reals can be supplanted by an infinite disjunction of (denumerably) infinite conjunctions of fundamental relations of the form $\ulcorner$the nth fractional digit is $d\urcorner$, each of which characterizes one digit in the decimal expansion of some real number. If such a formula can never be completed, it is also true that extending any finite part of it yields a closer, discrete approximation of S. An infinite sequence of ever-longer finite approximations to an infinitely long, but well-founded, definition would serve to mirror an incremental process of coming to grasp a real number or an infinite set. By implication, definition replicates understanding, i.e., the process of coming to know, and we witness an analogy between (interpreted) syntax and epistemology.

[22]Zermelo's usage follows a philosophical tradition going back to Kant and beyond according to which a manifold is a multiplicity lacking unity. Ironically, this usage is opposed to that of Cantor, who uses *Mannigfaltigkeit* and *Menge* as synonyms. Zermelo certainly does not mean to suggest that the mathematical realm is a set.

To extend this analogy, progress in mathematics might be described as stepwise advance toward a categorical axiomatization of the mathematical realm. Since that realm is beyond mathematical determination, however, we shall never achieve that full description using the languages and logics that are the products of our finitary understanding. Zermelo is saying that ultimate mastery is unattainable, which is one way to see the link between his open domains and Cantor's Absolute. Genuine mastery extends to closed domains only.

Finally, we return to an issue raised at the very beginning of this paper. Specifically, is Zermelo an objects-reductionist? Elsewhere (see [19]) we have labeled his position agnostic. But Stepwise Approximation may reveal something regarding Zermelo's thinking, assuming, as we do, that it is an epistemological corollary of the technical results of [34]. For Stepwise Approximation then says that the mathematical realm is a structural analogue of the cumulative hierarchy of sets and is investigated in a similarly stepwise fashion. There are a number of possible explanations for the postulated structural similarity. On one reading, the overall structure of the mathematical realm is like that of the cumulative hierarchy just because the latter is a (proper) part of the former. But another possible reading is that their structural similarity is a consequence of the fact that the world of mathematical objects and the world of sets are one and the same open domain.

§10. Infinitary logic and mathematical realism. Zermelo states outright that every mathematical proposition is decidable (see [37]).[23] We might take this to mean that either (i) every mathematical proposition is provable or refutable within a system whose axioms have some claim to credibility based upon self-evidence or fruitfulness or (ii) every mathematical proposition has a determinate truth value. (The claim in (ii) is known as mathematical realism). In Zermelo's case, unlike Hilbert's, it is reasonable to take 'decidability' as a cover term for both (i) and (ii).

Some link between infinitary logic and the decidability of the Continuum Hypothesis is implicit in [38] and elsewhere. As a point of departure, consider (1)–(3) below.

1. To begin, we take the Continuum Hypothesis to be a certain infinite conjunction—call it CH.[24]

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[23]With respect to Cantor's Continuum Hypothesis in particular, we mention [24], where Zermelo writes that "the Axiom of Choice in conjunction with the other axioms suffices to make something like the problem of the cardinality of the continuum into a categorical problem." (German original: *Das Auswahlprinzip in Verbindung mit den übrigen Axiomen genügt, um z.B. das „Kontinuum-Problem" zu einem kategorischen zu machen.*) See also [23].

[24]We can formulate the Continuum Hypothesis as $\forall x \exists f[(x \subseteq \wp(\omega) \wedge \mathit{Inf}(x)) \rightarrow (f : x \xrightarrow{1-1} \omega \vee f : \wp(\omega) \xrightarrow{1-1} x)]$. Any generalization would correspond to an infinite

2. Next, as in (2) of §5 above, we let Q_0^* consist of all propositions of the form $\ulcorner a$ is a member of $b \urcorner$. Then, if π is an arbitrary truth partition of basis Q_0^*, we have π saying in effect which pairs $\langle a, b \rangle$ instantiate the membership relation.
3. Finally, CH will be either true or false relative to π.

It is impossible to construe (1)–(3) as an argument for mathematical realism. After all, realism is the claim that arbitrary mathematical propositions have truth values—not that they have truth values relative to some assignment of truth values to their atomic components.

Instead, we shall take Zermelo to be making a claim of "transcendental" character: if one wishes to explain how realism could be true, then one has no choice but to adopt infinitary logic. The argument we impute to him falls into two parts. First, one describes an infinite hierarchy of propositions and shows that CH has a determinate truth value, relative to any truth partition π of its basis Q_0^*, even if we cannot presently know what that truth value is. The second part of Zermelo's argument, as we formulate it, will focus on one particular truth partition.

Let $CHUS = \{P_\alpha\}_{\alpha>0}$ be some cumulative hierarchy of urelements and sets. Assume further that an infinitary description of $CHUS$ is available, saying in effect how many urelements there are and how many sets there are of each cardinality at each level. Let us use $Gr(CHUS)$ to denote this description or *graph*, regarded as a single infinite axiom for set theory. (It seems that our single axiom here is a conjunction, the conjuncts of which together constitute [the inverse image of] a proper class.) Next, assume some hierarchy $\mathcal{H}_{CHUS,\{\in\}}$ of propositions, along the lines of (2) in §5. As mentioned already, conjunction CH, occurring at some level of hierarchy $\mathcal{H}_{CHUS,\{\in\}}$, will be assigned a truth value by *any* partition of basis Q_0^*. But realism requires more: we must be able to distinguish some particular truth partition π_0 as the true description of the world of sets, and the point of realism is then that CH will be assigned some truth value by π_0. In other words, CH coming out true (false) under π_0 would amount to CH actually being true (false). Which partition would be the distinguished one? That is easy to say if one assumes a Platonic realm of urelements and sets corresponding to what we are calling $CHUS$. *In that case, the distinguished partition π_0 of $\mathcal{H}_{CHUS,\{\in\}}$ would be the one that just happens to describe the relations holding in CHUS, that is, the one making $Gr(CHUS)$ true*. Something like the foregoing reasoning may motivate the following remarks.

> Suppose that the [collection Q_0^* of] fundamental relations is now partitioned arbitrarily into "true" and "false:" $Q_0^* = \mathcal{T} \cup \mathcal{F}$,

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conjunction (of conjunctions of disjunctions) $(\forall\alpha \in \beta)\forall x \exists f[(x \subseteq \wp(\aleph_\alpha) \wedge card(x) \geq \aleph_\alpha) \rightarrow (f : x \xrightarrow{1-1} \aleph_\alpha \vee f : \wp(\aleph_\alpha) \xrightarrow{1-1} x)]$, where inaccessible β is fixed.

> whereby such a "truth partition" corresponds to any subset of Q_0^* whatever. (In the case of a [single] dyadic relation, this truth partition takes the form of a two-dimensional matrix.) Then, in accordance with the usual logical rules, this partitioning carries over to every higher level and hence to the entire [well-founded] system $\mathcal{H}$ of propositions [*abgeleitete Relationen*]. ... Should a proposition $\mathcal{A}$ of $\mathcal{H}$ be assigned "true," then $\mathcal{A}$ is thereby also *proved* under the assumption of the postulated matrix of the fundamental relations. Moreover, $\mathcal{A}$ is proved at the same level [*Quantifikationsstufe*] at which it is defined. ([36], pp. 86–87 [different symbols in original])

Truth partition π_0 making $Gr(CHUS)$ true is relevant to the issue of the decidability of CH in the sense of (i), as is indicated by the passage just quoted. But (ii) is Zermelo's real concern. We recommend thinking of $Gr(CHUS)$ as the complete catalog of the world of mathematics (relative to a choice of urelements). In that case, decidability of CH amounts to the claim that the appropriate *Gedankenexperiment* would reveal that either (a) partition π_0 making each and every conjunct of $Gr(CHUS)$ true also makes CH true or (b) partition π_0 making each and every conjunct of $Gr(CHUS)$ true makes CH false. (It seems that $Gr(CHUS)$ cannot itself be in $\mathcal{H}_{CHUS,\{\in\}}$, although each of its conjuncts will be.) In this sense, the propositions of $\mathcal{H}_{CHUS,\{\in\}}$ that are true under π_0 constitute a "complete" system.

> [A complete system of propositions is one that] contains every logical consequence of its propositions. With respect to certain issues, consideration of such systems can be preferred to any axiomatic formulation. ([35])

Decidability of CH is surely among the "issues" Zermelo has in mind here. He appears to concede that the Continuum Hypothesis is not decided by ZF^2, say.

Our argument, which may also be Zermelo's, is not an interesting argument for mathematical realism since it takes Platonism as premise. But it is possible to see it as an interesting elaboration of the relation between realism (or universal decidability) and Platonism. And since systems of infinitary propositions are part of this elaboration, it may be that Zermelo has come to see the argument as lending support to infinitary logic as an explanatory tool.

Universal decidability is definitely not the genesis of Zermelo's turn to infinitary logic since only the latter is mentioned in the Five Theses of 1921. Rather, this is a case of core doctrine reinforcing antecedent conviction. Zermelo's view seems to be that infinitary logic is the sole vehicle for formulating mathematical theories that enables the Platonist to see that there are no absolutely undecidable propositions. Just why Zermelo regards infinitary logic as an essential element in this story is harder to understand. But the important role of well-founded hierarchies of infinitely long propositions in integrating the mathematical world of sets and the logical world of propositions would have a lot to do with it.

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