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REVIEWS

The Association for Symbolic Logic publishes analytical reviews of selected books and articles in the field of symbolic logic. The reviews were published in ***The journal of symbolic logic*** from the founding of the JOURNAL in 1936 until the end of 1999. The Association moved the reviews to this BULLETIN, beginning in 2000.

The Reviews Section is edited by Herbert Enderton (Coordinating Editor), Geoffrey Hellman, Thomas Jech, Wolfram Pohlers, Philip Scowcroft, and Alasdair Urquhart (Managing Editor). Authors and publishers are requested to send, for review, copies of books to the *Association for Symbolic Logic, Box 742, Vassar College, 124 Raymond Avenue, Poughkeepsie, NY 12604, USA*.

In a review, a reference "JSL XLIII 148," for example, refers either to the publication reviewed on page 148 of volume 43 of the JOURNAL, or to the review itself (which contains full bibliographical information for the reviewed publication). Analogously, a reference "BSL VII 376" refers to the review beginning on page 376 in volume 7 of this BULLETIN, or to the publication there reviewed. "JSL LV 347" refers to one of the reviews or one of the publications reviewed or listed on page 347 of volume 55 of the JOURNAL, with reliance on the context to show which one is meant. The reference "JSL LIII 318(3)" is to the third item on page 318 of volume 53 of the JOURNAL, that is, to van Heijenoort's *Frege and vagueness*, and "JSL LX 684(8)" refers to the eighth item on page 684 of volume 60 of the JOURNAL, that is, to Tarski's *Truth and proof*.

References such as 65***1*** or 247***8*** are to entries so numbered in *A bibliography of symbolic logic* (the JOURNAL, vol. 1, pp. 121–218).

J. P. MAYBERRY. ***The foundations of mathematics in the theory of sets.*** Encyclopedia of mathematics and its applications, vol. 82. Cambridge University Press, Cambridge 2000, New York 2001, etc., xx + 424 pp.

Part One, consisting of Chapter 1 on the idea of foundations of mathematics and Chapter 2 primarily on the notion of *arithmos* in Greek arithmetic, sets the agenda. In the first chapter: "To expound those foundations [of mathematics] systematically, one must provide three things:" an account of "the fundamental *concepts* of mathematics [and] the *objects* that fall under those concepts," an account of "its *axioms*," and an account of its "*logic*" (pp. 8–9). The author takes as "a straightforward observation" that "the foundations of all of mathematics, including mathematical logic and the axiomatic method, now lie in the theory of sets" (p. xiii). So the foundations of mathematics are in fact the foundations of set theory, and, to expound this foundation, we must give an account of the basic concept of set, of sets, and of the logic of set theory. It seems to follow then that the logic of set theory is not mathematical logic and set theory is not itself an axiomatic theory; and indeed that is so from the author's point of view.

So which propositions about the universe of all sets are true is not determined by the axioms; rather the axioms themselves are subject to a requirement that, in some prior sense, they be true. Chapter 2 is to provide the content of these propositions, on the basis of which they become true or false. The author notes that the term "arithmos" in classical Greek mathematics meant something like finite set and it is this notion that he wants to identify

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with the concept of set. This may seem entirely too restrictive, since nowadays we like to consider infinite sets. But the ancient Greek terms for finite and infinite were "peras" and "apeiron," which need not simply mean finite and infinite in our sense (although that is one of the senses that Aristotle distinguishes in the *Physics*), but have a family of meanings, including the meaning of bounded and unbounded, determinate and indeterminate, and limit and unlimited. The author utilizes this ambiguity (as I, but presumably not he, would call it) as a bridge between the classical Greek theory of the cardinality relations among (roughly) finite sets (as in Euclid's *Elements*) and Cantor's theory of transfinite sets. "Euclidean finitism," characterized by Euclid's Common Notion 5 (applied to arithmoi), "The whole is greater than the part" (p. 61), is rejected by Cantor and replaced by "Cantorian finitism," which seems to be characterized by the proposition that the "species of (von Neumann) natural numbers is finite, and therefore forms a set" (p. 62).

Part Two, comprising Chapters 3 and 4, presents basic set theory, i.e., what is independent of the choice between Euclidean and Cantorian finitism. Chapter 3 further discusses the notion of arithmos or "number." Mayberry wants to construe sets as certain arithmoi on what seems to be ontological grounds: he believes that arithmoi have as unproblematic an existence as their units. For example, the arithmos of sheep in the flock has the same concrete existence as the sheep themselves. The major step that had to be taken in the development of set theory was to admit arithmoi some of whose units are themselves arithmoi. (As a matter of fact, in the *Parmenides*, Plato already took this step: to show that there are infinitely many distinct cardinalities, he started with two things, then took their pair, so that there were three things, etc.) But arithmoi are things that are multiplicities, and these do not include null things (there are no nothings) or singletons (everything is one, even if it is also a multiplicity). So, to talk about sets in the contemporary sense, we must introduce two objects, call them *ghosts*, and take sets to be arithmoi among whose units are both ghosts. The elements of a set are the units of the underlying arithmos other than the ghosts.

Although this is not the place to argue the point, I feel that in connection with the notion of set (as indeed with the notion of finite), the author shows too little respect for the centuries of gradual clarification that took place and attributes more clarity to the ancient Greek idea of arithmos than it had.

Undoubtedly, an important part of his motivation and rationale for his point of view is in his treatment of proper classes. Contrasted with the notion of an arithmos is that of the extension of a concept, which he calls a *species*. If we understand sets to be extensions of concepts, then there is the problem, already noted by Cantor in his review (65***1***) of Frege's ***Die Grundlagen der Arithmetik***, that the extensions of some concepts, such as the concept "x is a transfinite number," are not sets. Armed with the notion of arithmos, we can make the distinction between those species that are coextensive with sets and those that are not. Namely, if the species is "finite" in the author's sense, then it is an arithmos and so (with the addition of the ghosts) is a set. On the author's conception, species that do not determine sets and "global (second-order) functions," such as $x \mapsto \{x\}$ or ordinal successor, do not really exist: they correspond to open formulas and open terms, and speaking of them is only a convenience: in mathematical propositions they disappear.

For the conception of set theory as a theory about the universe of all sets, proper classes are indeed an embarrassment; but an alternative view (held by Zermelo) is that set theory *is* an axiomatic theory (in second-order logic) whose models, to within isomorphism of the form $\langle R(\kappa), \in\rangle$ ($R(\alpha)$ being the set of sets of rank less than α), are defined by its (consistent) categorical extensions and so it has no universal model. The proper classes relative to one of its models are sets in another. On this conception, there is no absolute notion of proper class and, moreover, impredicative higher-order logic in set theory makes perfectly good sense.

Chapter 4 works out the axioms of basic set theory. It turns out that the basic existence axioms of set theory, other than the axiom of infinity, generate finite (determinate) sets—including the axioms of replacement and power set, which in the latter case may seem surprising to those who doubt the "determinateness" of the power set operation. So basic set theory is ZFC without the axiom of infinity.

Part Three is a careful development of the elements of Cantorian set theory from the axiom of infinity. At the end of Chapter 5, the author argues, contra Zermelo and Gödel, that there are no grounds for the existence of cardinal numbers beyond those implied by ZFC. He considers the case of Mahlo cardinals or more precisely of regular closure cardinals μ for a global ordinal function σ ($\alpha < \mu$ implies $\sigma(\alpha) < \mu$). (If such a μ exists for each normal function σ, then the totality of all ordinals is Mahlo.) But the argument for the existence of μ that he rejects is not the argument that Gödel himself found convincing. The argument that Gödel *did* find convincing is a straightforward generalization of what seems like the most convincing argument for the axiom of replacement: if for no ordinal α in the initial segment X of ordinals does $R(\alpha)$ have a certain property and $R(X)$ does have the property, then X should be admitted as an ordinal: it is "finite" in the author's sense because it is determined by the condition of being the least ordinal with the property. The properties in question are those of satisfying some formula $\varphi(A)$, where A is a class. (From the point of view of $R(\alpha)$, A is $A \cap R(\alpha)$.) Since A is a species and so infinite on the author's view, he would perhaps want to reject this kind of argument. But in the case of the axiom of replacement, where, say, $\varphi(A)$ expresses that A is a function defined on some $\alpha \in X$ and whose values are in X, it seems to me that this is the best argument there is for the "finiteness" of X, of why we should, on the author's grounds, accept it as a set.

Part Four develops Euclidean set theory, based on the axiom that every set is Dedekind finite. In Chapter 10, there is an attractive development of a Euclidean theory of simply infinite systems. A simply infinite system based on the global function σ and the object a is a species of all linear orderings with first member a and in which each member other than the first is obtained from its predecessor by means of σ. σ and a determine a simply infinite system iff for no linear ordering r in it is the value of σ applied to the last element of r in the field of r. One should note that, in Quine's ***Set theory and its logic*** (JSL XXXVII 768) (which the author justly criticizes for its common-sense-is-bankrupt approach to set theory), there is another quite elegant treatment of simply infinite systems that does not depend on the axiom of infinity (or on its negation). On Quine's conception the elements of the system are the x that satisfy the condition: $\forall z[x \in z \wedge \forall y \in z(\sigma^*(y) \subseteq z) \longrightarrow a \in \sigma]$, where $\sigma^*(y)$ is the set of pre-images of y. Quine treats the special case of $\sigma(x) = x \cup \{x\}$ and $a = 0$, but the treatment generalizes.

The book under review is a very lively book, filled with striking theses, for the most part clearly formulated and argued for in detail. I found, reading it, that my head was continually shaking, sometimes positively and sometimes negatively. I confess that on philosophical matters, it probably shook more often no; but, I always felt engaged and—such is the nature of philosophy—it would not surprise me if other readers found themselves more in agreement with the author.

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Oliver Aberth. ***Computable calculus.*** Academic Press, San Diego, London, etc., 2001, xiii + 192 pp. + CD-ROM.

The title of this interesting and unusual book refers to calculus based on interval arithmetic and carried through by using a model of computation equivalent to a Turing machine. As anyone familiar with computability theory will realise, there are many differences between calculus carried out in this way and the standard treatment. For one, there is the problem of comparing two computable real numbers. To understand this problem, we need to appreciate that, according to Aberth, a computable real number is a program that, when fed a positive integer k as input, produces two rational numbers: an approximation m_k to x, and a non-negative error bound ε_k. Aberth requires that for two positive integers k, k' the intervals

$$a(k) = [m_k - \varepsilon_k, m_k + \varepsilon_k] \quad \text{and} \quad a(k') = [m_{k'} - \varepsilon_{k'}, m_{k'} + \varepsilon_{k'}]$$

intersect, and that for any rational $\varepsilon > 0$ and for all sufficiently large k, the length of $a(k)$ is less than ε. The underlying intuitive idea is that all the intervals $a(k)$ have a single point in common: namely, the real number under construction.

Now consider any increasing binary sequence $(\lambda_k)_{k=1}^{\infty}$. If $\lambda_k = 0$, set $m_k = 0$ and $\varepsilon_k = 1/k$; if $\lambda_k = 1 - \lambda_{k-1}$, set $m_j = 1/(k-1)$ and $\varepsilon_j = 0$ for each $j \geqq k$. According to Aberth's definitions of $>$ and $\geqq$, this process defines a non-negative computable real number x such that if $x = 0$, then $\lambda_k = 0$ for all k, and if $x > 0$, then there exists (and we can compute) k such that $\lambda_k = 1$ and $\lambda_{k-1} = 0$. Thus if for each non-negative computable real number x we could decide whether $x > 0$ or $x = 0$, then we could decide whether a given computable sequence λ of this sort ever takes on the value 1 or not. Since this is equivalent to the decidability of the halting problem, we cannot necessarily compare two real numbers within Aberth's framework. For similar reasons we cannot decide whether an arbitrary computable real is rational or irrational.

Another undecidable problem for computable reals is that of weak comparison: If we could decide, for an arbitrary computable real number x, which of the alternatives $x \geqq 0$, $x \leqq 0$ holds, then again we run afoul of the undecidability of the halting problem. In consequence, if we have a continuous function $f : [0,1] \to [-1,1]$ that is computable in the sense that it maps computable inputs to computable outputs, and if $f(0)f(1) < 0$, then we cannot be sure of finding a computable real number c between 0 and 1 such that $f(c) = 0$. More generally, as we move along a line segment from inside a set S in $\boldsymbol{R}^N$ towards a point outside S, we cannot be certain of finding a point where that line segment meets the boundary of S.

Aberth discusses these and related issues of computable mathematics very clearly. But he does much more: he shows how to develop a positive portion of calculus in a systematically computable way. For example, he shows that a uniformly continuous computable function $f : [0,1] \to \boldsymbol{R}$ is bounded and has a computable supremum, and explains clearly why one cannot expect, in general, to find a (computable) point where the supremum is attained. In fact, using the famous theorem of Specker on bounded monotone sequences of computable reals, he constructs a computable uniformly continuous function $s : [0,1] \to [0,1]$ such that $s(x) < \sup s$ for each computable real $x \in [0,1]$.

Among the topics studied in Aberth's later chapters are differentiation, the Riemann integral, and differential equations. After a careful presentation of the Picard iteration for solving initial value problems in the presence of a Lipschitz condition, he proves that for a certain uniformly continuous $f : [0,1] \times [-1,1] \to \boldsymbol{R}$, the initial value problem $y' = f(x,y)$ with initial condition $y(0) = 0$ has no computable solution on the interval $[0,h]$ for any $h > 0$.

Altogether, this book provides a fascinating presentation of calculus done in a novel way and of the necessary computational tools, including chapters on the ideal computer, and ideal computer simulation. Added value is provided by the CD sold with it, that simulates the ideal computer and provides programs enabling the student to carry through various tasks associated with the material in the text.

Nevertheless, although the book is to be recommended to anyone thinking of presenting a course on computable mathematics, it is not without fault or flaw. In particular, I take issue with the author's comment, on page 5, that "the Lebesgue integral … is not a constructive integral." This may be partially correct, in that the standard presentations of the Lebesgue integral are non-constructive, but it is certainly possible to give a constructive and computable development of the Lebesgue integral: see Bishop and Bridges, ***Constructive analysis*** (JSL LII 1047), Chapter 6.

This links to my final criticism. As in his earlier book on computable mathematics, ***Computable analysis*** (JSL XLIX 988), Aberth nowhere mentions the substantial body of work carried out since 1967 (when Bishop's ***Foundations of constructive analysis***, JSL XXXVII 744, appeared) in Bishop-style constructive mathematics, in which intuitionistic logic, rather than any formal notion of ideal computer, is the key. All Bishop-style constructive mathematics can be adapted, in a relatively straightforward way, to lie within Aberth's framework. While I am not afraid to admit a bias in this matter, I feel that the omission of any indication that there is a large body of alternatively-developed constructive mathematics compatible with Aberth's work is potentially misleading and could prevent students of computable mathematics from viewing the broad picture of constructivity.

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JOHN ARNOLD KALMAN. ***Automated reasoning with Otter.*** With a foreword by Larry Wos. Rinton Press, Princeton 2001, xv + 536 pp. + CD-ROM.

John Kalman's ***Automated reasoning with Otter*** is a complete user's guide for OTTER. William McCune's OTTER is the most widely used automated theorem-proving program. It is the current offering of the Argonne automated reasoning group led by L. Wos. OTTER and its resolution-based variants are often beaten in the annual CADE theorem-proving contests, but it has the most mathematical results of any prover (although the most famous theorem-prover result, the completeness of Robbins's axiomatization of Boolean algebras, belongs to McCune's EQP). This book's author, John Kalman, is noted for solving open problems about equivalential calculus and condensed detachment using OTTER and OTTER's predecessor ITP.

McCune's *OTTER 3.0 reference manual and guide* (report ANL-94/6, available through www.mcs.anl.gov/AR/otter) is the best reference manual for OTTER; it is concise and finding information about flags and parameters is easy. For the beginner seeking a theorem-proving overview, ***Automated reasoning, Introduction and applications*** by Wos et al. (JSL LI 464, LIX 1437) is easier reading but it is not specific to OTTER. But Kalman's book is the best user's guide (a reference describes what the commands do; a user's guide shows how to use the commands). It has 705 examples and exercises which cover the needs of beginning and advanced users. It thoroughly covers OTTER's inference rules (such as binary resolution and hyperresolution, paramodulation), OTTER's theorem-proving strategies (such as set-of-support and term orderings), OTTER's many flags and parameters, and the techniques (such as Skolemization) needed to convert a problem into an equiconsistent set of OTTER-readable clauses (universal disjunctive sentences). The CD included with the book contains version 3.04 of the OTTER program and a 3MB Acrobat file with the complete contents of the text.

Kalman includes advanced and specialized topics involving Skolem function symbols, term ordering, recursive path orderings, dynamic demodulation, and the many paramodulation flags. He includes topics not found elsewhere such as how to use the passive list to handle inequality and how subsumption can lead to derivational incompleteness. He does not cover fringe features such as Veroff's linked resolution, hot lists, ancestor subsumption, and conditional demodulation.

Kalman's book is also a fairly comprehensive guide to OTTER's accomplishments and the type of problems it can solve. About a quarter of the book consists of listings of input and output files for a wide variety of problems OTTER can solve. In areas such as combinatory logic and propositional axiomatizations, OTTER has found publishable results. But for most mathematical problems of medium difficulty (such as the quadratic formula), OTTER and all other theorem provers usually wander astray in a combinatorial explosion of irrelevancies no matter how the flags and parameter are set.

The book's completeness (no prerequisites are needed) and many exercises make it suitable for use as a text on theorem proving. But theorem proving is usually part of an AI course that gives equal time to logic programming, neural nets, and genetic algorithms. At 551 pages and $88, it is too big and expensive for a supplemental reading text.

This book has extensive coverage of the logical foundation of theorem proving but usually without proofs. Kalman does not prove J. A. Robinson's theorem that most general unifiers always exist but he shows how they are generated, why symbol clashes or occurs checks imply non-unifiability, and how to get OTTER to print out most general unifiers. He states but does not prove the many refutational completeness proofs involving resolution, paramodulation, and set of support. The Knuth–Bendix completion procedure is well presented but without the usual theorems about critical pairs, confluence, and termination. Of the texts that cover resolution-based theorem proving with formal proofs, Chang and Lee's ***Symbolic logic and mechanical theorem proving*** (Academic Press, 1973) is the classic. Alexander Leitsch's ***The resolution calculus*** (Springer, 1997) has the most recent references.

What proofs there are are done precisely and in full detail—Kleene style. But the mathematics is not elegant. This can be blamed as much on the subject as the author. Automated reasoning does not have the depth, maturity, and beauty of complexity theory. Wos was one of the first to realize that it is primarily an experimental science. Progress depends as much on writing programs to determine what works as on writing papers on what might work. Workers in the field are more impressed with results (such as McCune's solution of Robbins's problem) than with theoretical arguments.

There are almost no errors. "6," "7," "8" should be "f," "g," "h" on pages 434–435. On page 510 "max-weight" should be "max_weight." The other errors (pages 116, 363, 377) are harmless.

The index is incomplete. It has "paramodulation" but neither "resolution" nor "binary resolution." It has "substitution" but not "replacement." On page 82 Kalman abbreviates "inferred-clause-processing procedure" to "icpp." This will cause problems for a non-linear reader who encounters this un-indexed abbreviation later in the book. The reason for complaining is that while most readers will be advanced OTTER users, the book is far too verbose to be read in its entirety by anyone except beginners.

If you need a user's guide to OTTER that goes beyond McCune's sixty-page reference manual, this is your only choice.

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MELVIN FITTING and RICHARD L. MENDELSOHN. ***First-order modal logic.*** Synthese library, vol. 277. Kluwer Academic Publishers, Dordrecht, Boston, and London, 1998, xii + 287 pp.

This is one of the few recent texts, if not the only one, devoted to modal predicate logic. The text contains two treatments of the logic, side by side. There is a detailed formal treatment, and a detailed philosophical treatment of the features and alternatives generated by the formal logic. The text is pitched at upper-level undergraduates or honours students.

The text contains an introductory treatment of the normal modal propositional logics, and then a most detailed formal treatment of the normal modal predicate logics with identity, and then *extends* the predicate logic with predicate abstraction. The favoured proof method is semantic tableau. The text emphasises possible-worlds relational semantics. This emphasis makes the tableau proof methods a "natural" for the systems. Although there is considerable metalogic in the volume, it is motivated by a detailed treatment of a sequence of philosophical problems. So the metalogic is not an end in itself. This volume is unlike recently published modal logic texts where the metalogic has become the main centre of interest, and modal logic is seen as mainly providing an "internal" perspective on metalogical structures of mathematical interest.

The book is rich in philosophical logic discussion. The discussions cover many topics including Aristotle's essentialism, the future sea battle, modality *de dicto* and *de re*, conditionals and necessity, the objects of quantification in modal logic, Quine's criticisms of modal logic, the morning star and the evening star, existence and quantification, non-being, reference and existence, predicate abstraction, rigid designation, unactualised possibles, fiction, and definite descriptions. I can look briefly at two things out of dozens. They are quantification and possible worlds, and singular terms and predicate abstraction.

First we look at the treatment of quantification and possible worlds. The authors show that within any normal modal logic there are many modal predicate logics. The variety has to do with the way the domain of quantification varies from possible world to possible world. They show how the various arrangements are formally related to each other.

In the light of the range of possibilities, the authors consider the relationship between quantification and existence. They abandon some of the twentieth-century obsessions about existential import. Their treatment of existence shows that it really does not matter formally whether we take existence to be a predicate or take quantifiers to have existential import. They wish to deal seriously with the notion that some entities which exist in one world, might not exist in another. If one uses temporal modal logic in which possible worlds are successive states of the universe, then we expect that some things will cease to exist and some will come into existence.

The authors show that if one pegs this change to changing domains of quantification from world to world, then the logic is far more complex than if one makes the domain of quantification a constant domain of possible existents. "Now $(\exists x)$ ... can be read, 'there is something, x, that could exist, such that ... ' " (page 106). Existence can then be treated formally as a predicate. The nature of the predicate, whether it is a "real" predicate or not (as Kant contended) is open for unprejudiced debate. Their opting for possible existents might be said to continue the orthodoxy of existential import, but in the weakest way possible.

Discussions of quantification into modal contexts are often couched in terms of the *de dicto–de re* distinction. They show how a careful consideration of quantification makes clear that the *dicto–de re* distinction has no real philosophical substance when considered simply in terms of the contrast between, say, $\Box(\exists x)Fx$ and $(\exists x)\Box Fx$ as possible translations of the ambiguous "Something is necessarily F." The former "says, *In every possible world, something is F*. [The latter] says, *Something is, in every possible world, F*. ... [The latter] requires that there be some one thing (at least)—let's call it a—which is, in every possible world, F. So, a must be F in world Γ, a must be F in world Δ, and so on. [The former] requires only that in each world there be something that is F, but does not require that it be the same thing in each world" (page 87).

This leads us to our second topic. The authors argue that the *dicto–de re* distinction gets its real grip when one is looking at singular terms in sentences such as "The number of planets is necessarily odd." Then, they claim, one has to look at predicate abstraction. They say, "where t is a singular term (say, 'the number of planets') we will distinguish the *de dicto* $\Box\langle\lambda x.x \text{ is odd}\rangle(t)$ which says, 'It is necessary *that t* is odd,' from the *de re* $\langle\lambda x.\Box(x \text{ is odd})\rangle(t)$ which says, 't is such that it is necessary *of it* that it is odd.' The *de dicto* reading is false" (page 87).

Another way of looking at this use of predicate abstraction is to take $\Box Fc$ as being ambiguous about the scope of the box. The scope might be just the F, or the whole Fc.

In the case of the former, the box is a predicate modifier and not a propositional operator; it is *de re* rather than *de dicto*. They say, "On the one hand, we can regard the constant as primary: c designates an object in the actual world and that object is said, in every possible world, to be F. On the other hand, we can regard the modal operator as primary: in every possible world, the object designated by c in that world is said to be F" (page 190 f).

I think the distinctions they make can be made otherwise than with predicate abstraction, and that they run descriptive singular terms and names too much together. But, arguments about this are far too extended for this review. The important point is that they offer a set of well-motivated solutions to many difficult problems, and future debate will take place in the shadow of their arguments.

This text is an excellent and most useful volume. It is pitched correctly; the exercises are just right. This text brings modal logic back to its origins in philosophy and philosophical logic. Even where one disagrees with the line being argued, the onus is to show a better line of argument. The logical and philosophical communities are fortunate to have such an excellent text focussed wholly on this area. It sets a high standard for anything following. It is to be highly recommended.

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Proof theory, History and philosophical significance, edited by Vincent F. Hendricks, Stig Andur Pedersen, and Klaus Frovin Jørgensen, Synthese library, vol. 292, Kluwer Academic Publishers, Dordrecht, Boston, and London, 2000, xii + 244 pp.—therein:

VINCENT F. HENDRICKS, STIG ANDUR PEDERSEN, and KLAUS FROVIN JØRGENSEN. *Introduction*. Pp. 1–7.

SOLOMON FEFERMAN. *Highlights in proof theory*. Pp. 11–31.

LEO CORRY. *The empiricist roots of Hilbert's axiomatic approach*. Pp. 35–54.

DAVID ROWE. *The calm before the storm: Hilbert's early views on foundations*. Pp. 55–93.

WILFRIED SIEG. *Toward finitist proof theory*. Pp. 95–114.

DIRK VAN DALEN. *The development of Brouwer's intuitionism*. Pp. 117–152.

MORITZ EPPLE. *Did Brouwer's intuitionistic analysis satisfy its own epistemological standards?* Pp. 153–178.

SOLOMON FEFERMAN. *The significance of Weyl's Das Kontinuum*. Pp. 179–194

ERHARD SCHOLZ. *Hermann Weyl on the concept of continuum*. Pp. 195–217.

SOLOMON FEFERMAN. *Relationships between constructive, predicative and classical systems of analysis*. Pp. 221–236.

This collection of papers is based on a conference with the same title held at the University of Roskilde, Denmark, in 1997. Three systematic lectures by Feferman form the backbone of the volume. Their texts maintain the spontaneity of a lecture without sacrificing accuracy of detail and are a pleasure to read. The first one traces the path from Hilbert's old axiomatic proof theory to Gentzen's structural proof theory as based on sequent calculi, and ends up with a brief review of infinitary systems of proof.

Feferman's first essay is the only one dealing with developments in proof theory from the beginning of the 1930's on, except for a brief mention of Gödel and Gentzen by Sieg. Despite its title and the wish of the editors to "trace the development of proof theory and illuminate its growth into a mature theory" (*Introduction*, p. 1), the contents of the volume might be described better as "Hilbert's old proof theory and the intuitionism of Brouwer and Weyl." All the essays except the first and last ones, both by Feferman, fall entirely within this category. As an indication of the limited coverage of proof theory, it can be noted that such a great success of proof-theoretical research of the last decades as the computational semantics of intuitionistic or constructive logic, also known as the "Curry–Howard isomorphism," does not get as much as a mention in the volume.

The foundational work of Hilbert, Weyl, and Brouwer has received a large amount of attention ever since the "Grundlagenstreit" of the 1920's. It should be safe to say that there is a general understanding of the positions of the main protagonists of the early foundational debates. Beyond a general understanding, detailed historical-foundational work is an entirely different matter and requires, besides specialized knowledge in logic and mathematics, experience in the methods of historical research. Such detailed work for the period in question is still largely to be done; for example, work on the Hilbert school of the 1920's is quite recent, and is exemplified in this volume by the essay of Wilfried Sieg.

The editors open their *Introduction* by describing proof theory as "a technical sub-field of mathematical logic." This should hardly be the attitude of most proof theorists who view their work as part of a distinctly foundational enterprise: the study of the structure of mathematical proofs. Classic examples of foundational results, in contrast to "technical" (whatever that may be), include the full mastery of the structure of proofs in first-order logic through Gentzen's sequent calculus, and the consistency of arithmetic and the determination of its proof-theoretic strength. There is any amount of subsequent work along similar lines, "technical" only insofar as that is considered a synonym for difficult.

This volume has a thematic unity that places it above most other books originating in conference papers. Its usefulness for those interested in the earlier phases of the foundational debate is further increased by a detailed index, for which the editors deserve a sincere thanks.

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Sergei N. Artemov. *Explicit provability and constructive semantics*. ***The bulletin of symbolic logic***, vol. 7 (2001), pp. 1–36.

The meaning of the intuitionistic connectives is often explained, informally, in terms of the Brouwer–Heyting–Kolmogorov (BHK) interpretation: a proof of $A \wedge B$ consists of a proof of A paired with a proof of B, a proof of $A \to B$ consists of a procedure for transforming a proof of A into a proof of B, and so on. The simplicity and intuitive appeal of this explanation suggests that there should be a formal semantics lurking underneath. But, after surveying the existing semantics for intuitionistic logic (including Kripke and Beth models, algebraic and topological semantics, realizability, and various syntactic interpretations), Artemov concludes that none of them fit the bill. He then undertakes the challenge of providing one that does.

Artemov's solution involves interpreting intuitionistic propositional logic in a logic LP of propositions and proofs. Consideration of similar interpretations will provide some useful context. Many researchers in constructive logic take the Curry–Howard (or "propositions as types") isomorphism, formally represented by deductive type theories like Martin-Löf's, to offer a formal explication of the BHK interpretation. But Artemov objects that this does not go far enough: such type theories *name* the associated proof objects, but do not supply the linguistic resources to reason about them as proofs per se. As a logic of provability, Gödel's modal logic S4 fares better in this respect: for any proposition, p, one can also express the proposition, $\Box p$, that p is provable. Indeed, Gödel's 1933 interpretation of Heyting's propositional calculus in S4 shows that intuitionistic logic can be understood in terms of such a notion of provability. But Gödel also pointed out that one cannot interpret the S4 box operator in terms of a proof predicate for an arithmetic deductive system, since, under such an interpretation, the S4 theorem $\Box(\Box\bot \to \bot)$ asserts that the system can prove its own consistency. So the challenge amounts to showing how one can understand the S4 notion of "provability" in terms of actual "proofs."

Artemov's system LP draws from both the modal and type-theoretic approaches, and, given the constraints he has set up, provides a natural solution. Proof objects are represented by *proof polynomials*, which can be concatenated and applied to one another. The logic is classical, with the usual propositional variables and connectives; but whenever t is a proof polynomial and φ is a formula, $t : \varphi$ is a new proposition, intended to denote that t is a proof of φ. To interpret the S4 axiom $\Box p \rightarrow \Box\Box p$, Artemov introduces one additional operator: whenever t is a proof of φ, $!t$ is intended to denote a proof that t is a proof of φ. In other words, for each t and φ, LP has axioms of the form $t : \varphi \rightarrow !t : t : \varphi$.

Artemov shows that the system works as advertised. Any derivation in LP can be projected "forgetfully" to a derivation in S4, and, conversely, any formula that can be derived in S4 has an explicit realization that can be derived in LP. Together with Gödel's interpretation, this shows that a formula is derivable in intuitionistic logic if and only if an appropriate realization is derivable in LP. Artemov also shows how to give LP an arithmetic interpretation relative to any standard proof predicate for, say, Peano arithmetic, and he proves soundness and completeness with respect to such interpretations. This in turn yields cut-elimination theorems for suitable sequent formulations of LP.

The historical notes and references are exceptionally thorough, and this paper will serve not only as a standard reference to the various attempts to come to formal terms with the BHK interpretation, but also more generally as a useful source of information for the semantics of intuitionistic and modal logic. Artemov is admirably clear in laying out the motivations and relevant background information, and I have little to add to his exposition.

Because Artemov takes his interpretation to offer a "semantics" for intuitionistic logic, I may, perhaps, indulge in a brief reflection on this notion. I can think of three reasons that one might seek a formal semantics for a deductive system that is already in hand: (1) one might want to explicate the meaning of the deductive formalism in terms that are intuitively prior, or independently interesting; (2) one might want to have a tool for studying the deductive system itself, such as for proving independence or establishing other metamathematical properties; or (3) one may intend the semantics itself (perhaps coupled with the associated deductive system) to have applications to areas outside logic, such as mathematics, linguistics, or computer science.

Artemov's work succeeds with respect to (1). For example, the simple reflective rules of LP (and the use of a fixed-point lemma in obtaining arithmetic interpretations) clarifies the circularity needed to make sense of the S4 notion of provability. With respect to (2), it is not clear whether LP can offer anything that cannot be obtained by using, say, Kripke structures or cut elimination; in any event, this line is not pursued. With respect to (3), it is possible that LP, viewed as a term calculus, can help provide a foundational framework for reasoning about functional programming languages in which programs are equipped to construct pieces of their own code. Artemov raises the possibility of similar applications in the realm of formal verification. It will be interesting to see if such hopes are borne out.

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GAISI TAKEUTI. *Incompleteness theorems and S_2^i versus S_2^{i+1}*. ***Logic Colloquium '96, Proceedings of the colloquium held in San Sebastián, Spain, July 9–15, 1996***, edited by J. M. Larrazabal, D. Lascar, and G. Mints, Lecture notes in logic, no. 12, Springer, Berlin, Heidelberg, New York, etc., 1998, pp. 247–261.

GAISI TAKEUTI. *Gödel sentences of bounded arithmetic*. ***The journal of symbolic logic***, vol. 65 (2000), pp. 1338–1346.

"Bounded arithmetic" subsumes different theories, which in their expressive power and strength are usually included in the fragment $I\Delta_0 + \exp$ of Peano arithmetic in which induction is restricted to bounded formulas and exponentiation is a total function. In the reviewed papers, bounded arithmetic (henceforth BA) means the theory $I\Delta_0 + \Omega_1$ (Ω_1 expresses a growth rate strictly smaller than exponentiation, namely that $2^{|x|^2}$ exists for all x, where $|x|$ is the length of the binary representation, i.e., an integer-valued logarithm, of x) and its stratification into levels S_2^i that were introduced by Buss (***Bounded arithmetic***, JSL LVI 759). The theories S_2^i stand in close relationship to levels of the polynomial time hierarchy PH. Like the separation of PH, or the P =? NP question—one of the most challenging questions nowadays in mathematics—the analogous question, whether the hierarchy S_2^i collapses, remains open after more than sixteen years. The collapse of BA theories implies the (provable) collapse of PH (LXIV 1821). The reverse implication is unknown, but many people working in this area think that the separation of BA theories and of polynomial time hierarchy levels needs similar methods.

In the reviewed papers, Takeuti joins two main ideas: He describes a strategy to attack P =? NP by separating certain extensions of BA theories, and he discusses several kinds of Gödel sentences and connected consistency statements that could be helpful for separating those theories. Let me explain the latter aspect first.

One natural approach to the separation of BA theories is Gödel's incompleteness theorems. Buss in his book (op. cit.) already showed that S_2^1 can formalize syntax and prove Gödel's incompleteness theorems for S_2^i for several notions of formalized derivability like the "usual" (unrestricted) one, and also derivability where all occurring formulas are bounded. All right, that much is good. The next step towards a separation should be to prove in some stronger fragment of BA, say the whole BA, one of the consistency notions of a weaker fragment, say S_2^1. The notion of derivability and consistency mentioned first, the "usual" one, is much too strong. Wilkie and Paris showed that already the ("usual") consistency of Robinson's Q is not provable in $I\Delta_0 + \exp$ (*On the scheme of induction for bounded arithmetic formulas*, ***Annals of pure and applied logic***, vol. 35 (1987), pp. 261–302). For the second notion, bounded derivability and consistency, Pudlák proved that BA cannot prove the bounded consistency of S_2^1 (*A note on bounded arithmetic*, ***Fundamenta mathematicae***, vol. 136 (1990) pp. 85–89). The reason bounded consistency is too strong is that feasibly arithmetized terms can represent values exponentially larger than their Gödel number. Therefore, the obvious "evaluation strategy" for proving consistency—evaluate all terms in a derivation and use a partial truth predicate to show the validity of all sequents in the derivation—cannot work here as exponentiation is not a total function in BA.

In the reviewed papers, Takeuti tries to go one way around these obstacles. Call a number x small if 2^x exists. The rough idea behind Takeuti's approach is that if the Gödel number of a derivation of contradiction is small then the values of all occurring terms in that derivation can be evaluated in BA, and thus the above-described evaluation strategy for proving consistency might work. Of course, to be able to evaluate terms inside BA and to have partial truth predicates at hand, Takeuti has to consider a very tight form of derivability in S_2^i, which he calls "strictly i-normal proofs" and which he already studied in an earlier paper (*Bounded arithmetic and truth definition*, ***Annals of pure and applied logic***, vol. 39 (1988), pp. 75–104).

Takeuti's second main idea is a strategy to attack P =? NP by separating theories: Extend S_2^i by finitely many function symbols for polynomial time computable functions plus simple defining axioms for them and other simple true statements (here "simple" means expressible by sequents without use of logical symbols)—call this a p-extension and the resulting theory again S_2^i. If for all such p-extensions the hierarchy S_2^i can be separated then P $\neq$ NP. This is because if P = NP then we can express this by simple axioms, hence in the resulting p-extension every (bounded) property collapses to a P-property, hence this hierarchy collapses.

Together with his approach on Gödel sentences, this leads Takeuti to several conjectures that would imply P $\neq$ NP. For example, BA proves a weak consistency of S_2^i, which remains true for all p-extensions. Takeuti conjectures that all p-extensions of S_2^i cannot prove their own weak consistency. But then P $\neq$ NP. Concerning the feasibility of his conjecture Takeuti remarks (in the first reviewed paper, pp. 256–257) that the results by Baker, Gill, and Solovay

(*Relativizations of the $\mathcal{P} =?\mathcal{NP}$ question*, JSL LI 1061), which state that there are oracles α, β relativizing P and NP such that $\mathrm{P}^{\alpha} = \mathrm{NP}^{\alpha}$ and $\mathrm{P}^{\beta} \neq \mathrm{NP}^{\beta}$, cannot be applied to his setting.

I have one general concern: If we consider a certain entity of Gödel numbers T (any kind of Gödelization) of closed terms in the language of BA and another entity T_x of Gödel numbers of terms in one fixed variable x, and if we further consider an evaluation function $val : T \to \mathbb{N}$ and a substitution function $sub : T_x \times \mathbb{N} \to T$, then faithfulness of the arithmetization implies that all polynomial growth rates can be represented in the system (T, T_x, val, sub). For proving Gödel's diagonalization lemma we need the substitution function *sub* to have a polynomial growth rate, and for showing the correctness of derivations in BA by the evaluation strategy we need the evaluation function *val* to have a polynomial growth rate. But this is impossible, because then $f(a,b) := val(sub(a,b))$ has polynomial growth rate, i.e., $|f(a,b)| \leqq |\max(a,b)|^c$ for some $c \in \mathbb{N}$. By faithfulness there is some $t_{c+1} \in T_x$ such that $|m|^{c+1} \leqq |val(sub(t_{c+1}, m))| \leqq |\max(t_{c+1}, m)|^c = |m|^c$ for m big enough, a contradiction. I think this indicates that in order to obtain a separation via Gödel sentences we need further ideas that differ strongly from the evaluation strategy.

The first reviewed paper summarizes all necessary notions and results on strictly i-normal proofs and partial truth definitions for them. Let $\mathrm{Prf}^i(w, \ulcorner\varphi\urcorner)$ denote "w is the Gödel number of a strictly i-normal proof of φ," and let $|x|_k$ denote the k-times iterated "integer-logarithm" $|\cdot|$ applied to x. Takeuti shows that BA can prove weak consistency statements like $\forall w \neg \mathrm{Prf}^i(|w|, \ulcorner\to\urcorner)$ (the empty sequent "$\to$" representing contradiction) and weak forms of reflection like $\exists w\, \mathrm{Prf}^i(|w|_2, \ulcorner\forall x\, \varphi(x)\urcorner) \to \forall x\, \varphi(|x|)$ for simple φ. Of course, Gödel's second incompleteness theorem can only be proved for the "usual" consistency statement $S_2^i \nvdash \forall w \neg \mathrm{Prf}^i(w, \ulcorner\to\urcorner)$. Takeuti defines Gödel sentences φ_k^i satisfying $S_2^1 \vdash \varphi_k^i \leftrightarrow \forall x \neg \mathrm{Prf}^i(|x|_k, \ulcorner\varphi_k^i\urcorner)$. They have similar properties to the above-described variations of consistency. By Gödel's diagonalization lemma, Takeuti obtains $S_2^i \nvdash \varphi_k^i$ for all k, but on the other hand $S_2^{i+1} \vdash \forall x \neg \mathrm{Prf}^i(|x|_{k+1}, \ulcorner\varphi_k^i\urcorner)$ for $k \geqq 2$. He formulates two conjectures in the above-described form that would imply $\mathrm{P} \neq \mathrm{NP}$.

The second reviewed paper extends this line of research and contrasts it with Gödel sentences based on the usual (unrestricted) derivability. That is, let $\mathrm{PRF}^i(w, \ulcorner\varphi\urcorner)$ denote the formalized notion of "w is the Gödel number of an (unrestricted) S_2^i-proof of φ," and define Gödel sentences Φ_k^i satisfying $S_2^1 \vdash \Phi_k^i \leftrightarrow \forall x \neg \mathrm{PRF}^i(|x|_k, \ulcorner\Phi_k^i\urcorner)$. Takeuti compares pairs $(\mathrm{PRF}^i, \Phi_k^i)$ with $(\mathrm{Prf}^i, \varphi_k^i)$, obtaining results like $I\Delta_0 + \exp \nvdash \forall x \neg \mathrm{PRF}^i(|x|_{k+1}, \ulcorner\Phi_k^i\urcorner)$ for all k, in contrast to the above-obtained results for $(\mathrm{Prf}^i, \varphi_k^i)$. He further establishes relationships between different Gödel sentences, and between Gödel sentences and weak consistency statements.

The reviewed papers are top-level research papers in the field of applications of Gödel sentences to the separation problem of BA. This implies that for the study of them one needs some background in BA (at least parts of the Buss book). Like all papers of Takeuti that I have read, I found the ones under review inspiring, showing new ideas to the separation problem of BA.

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ANDREAS WEIERMANN. *How is it that infinitary methods can be applied to finitary mathematics? Gödel's T: a case study*. ***The journal of symbolic logic***, vol. 63 (1998), pp. 1348–1370.

The paper under review is concerned with Gödel's system T of primitive recursive functionals of finite type, which is sufficient for providing a functional interpretation of first-order Peano arithmetic PA. The main technical achievement of the paper is a novel proof of the strong normalization of T by defining a new assignment $[\ ]_0 : T \to \omega$ which decreases under one-step reduction $\rhd^1$ of terms in T, i.e., $t \rhd^1 t'$ entails $[t']_0 < [t]_0$. Moreover, $[\ ]_0$ is an ε_0-recursive function. As a byproduct, optimal bounds for the natural fragments T_n of T are obtained.

As the title of the paper suggests, this new analysis of Gödel's T has to be seen as a specific case study of a more general program, which has its roots in infinitary proof theory. There the general concept of *miniaturization* and *concretion* of certain (large) cardinals has long been central in the development of ordinal notation systems. Further, as has been shown by Weiermann, it is often possible to miniaturize or project down the ordinal analysis of a (strong) system in order to obtain a perspicuous proof-theoretic analysis of a corresponding weaker system. For example, he has shown in *How to characterize provably total functions by local predicativity* (***The journal of symbolic logic***, vol. 61 (1996), pp. 52–69) how to pin down the standard local predicativity treatment (due to Pohlers) of the theory of one inductive definition ID_1 to a technically smooth analysis of Peano arithmetic PA. In this miniaturization, the collapsing function $D : \varepsilon_{\Omega+1} \to \Omega$ is replaced by the miniaturized collapsing function $\psi : \varepsilon_0 \to \omega$. The treatment of PA thus obtained indeed also produces optimal bounds for the fragments $I\Sigma_{n+1}$ of PA.

The treatment of Gödel's T in the paper under review can be seen as a miniaturization of Howard's analysis of bar recursion of type zero and is inspired by the local predicativity approach to pure proof theory. In particular, the definition of the assignment $[\ ]_0$ mentioned above makes crucial use of the miniaturized collapsing function ψ. Whereas previous treatments of the subsystems T_n of T (in which the recursors have type level less than or equal to $n+2$) used the ordinal bound ω_{n+3} (e.g. in Weiermann's paper *A proof of strongly uniform termination for Gödel's T by methods from local predicativity*, ***Archive for mathematical logic***, vol. 36 (1997), pp. 445–460—a precursor of the article under review), the present paper yields the optimal strong normalization bound ω_{n+2} for T_n.

This is a very important paper both from the technical as well as from the conceptual point of view. In a long and sparkling introduction, the author outlines his vision of how to apply a certain kind of infinitary methods to questions of finitary mathematics, and he discusses exciting connections of his results with term rewriting theory, hierarchy theory, and computational complexity. The paper is largely self-contained and, due to its extensive motivation and conceptual discussion, it should be accessible to a wide readership.

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ULRICH KOHLENBACH. *Relative constructivity*. ***The journal of symbolic logic***, vol. 63 (1998), pp. 1218–1238.

In this article, the author continues his investigations of the rate of growth of functions definable in certain subsystems of analysis in finite types. In previous papers (e.g. *Mathematically strong subsystems of analysis with low rate of growth of provably recursive functionals*, BSL VII 280) the classical systems G_nA^ω were analyzed and proof-theoretic tools such as monotone functional interpretation (introduced by the author) were used. In the paper under review, Kohlenbach studies the intuitionistic versions $G_nA_i^\omega$ of these systems and analyzes them via a new monotone realizability interpretation. He shows that the addition of certain strong non-constructive and even classically refutable analytical and logical principles has no impact on the rate of growth of definable functions.

The main result of the paper is summarized in the following.

Main theorem. Let E-$G_nA_i^\omega$ be the system $G_nA_i^\omega$ plus extensionality axioms in all finite types, the full axiom of choice AC in all finite types, and the independence of premiss principle $IP_\neg$ for negated formulas. Furthermore, let $\mathcal{A}$ consist of the following (classically valid) principles: (1) Attainment of the maximum of $f \in C([0,1]^d, \boldsymbol{R})$. (2) Mean value theorem for integrals. (3) Cauchy–Peano existence theorem. (4) Brouwer's fixed point theorem for continuous functions $f : [0,1]^d \to [0,1]^d$. (5) The generalization of weak (i.e., binary) König's lemma for sequences of binary trees. (6) The "double negation shift" DNS: $\forall x^\rho \neg\neg A(x) \to \neg\neg\forall x^\rho A(x)$ (lower case Greek letters denote arbitrary finite types). (7) The "lesser limited principle of omniscience" LLPO, which is intuitionistically equivalent to the decidability of $\leqq$ on the real numbers. (8) Comprehension for negated formulas. Then from a proof of a closed formula

$$\forall u^1, k^0\, \forall v \leqq_\tau tuk\, (\neg C \to \exists w^\gamma D(u,k,v,w))$$

in $\text{E-G}_n\text{A}_i^\omega + \text{AC} + \text{IP}_\neg + \mathcal{A}$, one can extract a bound Ψ such that

$$\forall u^1, k^0\, \forall v \leqq_\tau tuk\, \exists w \leqq_\gamma \Psi uk\, (\neg C \to \exists D(u,k,v,w))$$

is provable in $\text{G}_n\text{A}^\omega + \text{AC} + \mathcal{A}$ (and therefore classically true).

The bound Ψ is explicitly definable from the first $n+1$ branches of the Ackermann function and the functional $\lambda f^1 \lambda n^0.\max(f0, \ldots, fn)$. The formulas C and D are arbitrary (whereas in similar theorems concerned with classical logic D had to be quantifier free).

The proof of this theorem rests mainly on the fact that all formulas in $\mathcal{A}$ can be expressed in the form

$$(*) \qquad \forall x^\delta (A \to \exists y \leqq_\rho sx\, \neg B)$$

and the fact that all formulas of this form have a very simple monotone realizability interpretation. In general Kohlenbach defines the statement "t^* satisfies the monotone realizability interpretation of A" (roughly) as the formula

$$\exists x (t^* \text{ s-maj } x \wedge x \text{ mr } A)$$

where s-maj is Bezem's strong variant of Howard's majorizability relation and mr is Kreisel's modified realizability (in fact modified realizability with truth is used here). It is fairly easy to see that any strong majorant s^* of s satisfies the monotone realizability interpretation of $(*)$, provably from $(*)$.

If in $\mathcal{A}$ the axioms (6) and (8) are replaced by comprehension for $\exists$-free formulas and certain classically refutable principles such as a formula F stating (roughly) that every type-2 functional is bounded on a compact set, the extraction of the bound Ψ is still possible. This is surprising, since the classical system $\text{E-G}_n\text{A}^\omega + \text{AC} + F$ is inconsistent. The point is that, since the axioms are now of an even more restricted form than $(*)$, the fact that Ψ is a bound can now be proved from axioms that, although false in the full set-theoretic type structure $\mathcal{S}^\omega$, hold in the model $\mathcal{M}^\omega$ of strongly majorizable functionals. This implies that Ψ is a bound in $\mathcal{M}^\omega$, and hence also in $\mathcal{S}^\omega$, because Ψ is of type level 2.

The intuitionistic system $\text{E-G}_n\text{A}_i^\omega + \text{AC} + F$ proves the following (classically refutable) principles: (9) "Dini's theorem without monotonicity and continuity assumptions," i.e. every pointwise convergent sequence of real functions on $[0,1]^d$ converges uniformly. (10) Heine–Borel property for $[0,1]^d$ and sequences of arbitrary (not necessarily open) balls. This means that valid bounds Ψ with low rate of growth are extractable also from proofs using these principles.

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Wilfried Buchholz. *Notation systems for infinitary derivations*. ***Archive for mathematical logic***, vol. 30 no. 5–6 (1991), pp. 277–296.

Wilfried Buchholz. *Explaining Gentzen's consistency proof within infinitary proof theory*. ***Computational logic and proof theory, 5th Kurt Gödel colloquium, KGC '97, Vienna, Austria,***

August 25–29, 1997, Proceedings, edited by Georg Gottlob, Alexander Leitsch, and Daniele Mundici, Lecture notes in computer science, vol. 1289, Springer, Berlin, Heidelberg, New York, etc., 1997, pp. 4–17.

SERGEI TUPAILO. *Finitary reductions for local predicativity, I: recursively regular ordinals.* ***Logic Colloquium '98, Proceedings of the annual European summer meeting of the Association for Symbolic Logic, held in Prague, Czech Republic, August 9–15, 1998***, edited by Samuel R. Buss, Petr Hájek, and Pavel Pudlák, Lecture notes in logic, no. 13, Association for Symbolic Logic, Urbana, and A K Peters, Natick, Mass., 2000, pp. 465–499.

The three papers under review form a suite that presents a new view of a divergence in ordinal analyses: the finitary approach by G. Gentzen on one side and the infinitary one by K. Schütte on the other, and their extensions to stronger theories. The papers reveal the fact that the divergence is on the surface.

Let $T \supseteq I\Sigma_1$ and let d be a derivation of a Σ^0_1-sentence in T. In the finitary analyses, one devises reduction steps $d \mapsto r(d)$ for reducible d, where the end-formula of $r(d)$ is the same as that of d. Then one assigns an ordinal (term) $o(d)$ in such a way that $o(r(d)) < o(d)$. These r and o are defined in a finitary way, i.e., primitive recursively. Thus r terminates in a finite number of steps, provided there is no primitive recursive descending chain of ordinal terms. This yields the 1-consistency of T, the main concern in the finitary analyses, since it is easy to see that the 1-consistency of T is equivalent to the normal form theorem for the rewriting rule r.

On the other side, the subjects in infinitary analyses are infinitary derivations. Consider infinitary derivations in one-sided sequent calculi with the cut rule: derive Γ, Δ from $\Gamma, \neg C$ and C, Δ. For a sequent Γ, let $\vdash^\alpha_\rho \Gamma$ mean that there exists an infinitary derivation of Γ such that its tree height is $\leqq \alpha$, and the (logical) depth $\mathrm{dp}(C) < \rho$ for all cut formulas C in it. The main technical lemma (Main Lemma) in infinitary analyses says that if $\vdash^\alpha_\rho \Gamma, \neg C$ and $\vdash^\beta_\rho C, \Delta$ with $\mathrm{dp}(C) \leqq \rho$, then $\vdash^{\alpha\#\beta}_\rho \Gamma, \Delta$ for the natural sum $\alpha\#\beta$. From this it is easy to see that if $\vdash^\alpha_{\rho+1} \Gamma$, then $\vdash^{2^\alpha}_\rho \Gamma$ (Cut Elimination Lemma). Therefore one can eliminate all cut inferences so that the height of the resulting cut-free derivation can be controlled in terms of the height and cut-degree of the original derivation. This yields an upper bound for the order types of provably well-founded recursive relations in T, and yields the 1-consistency proof for T via encoding of recursive infinitary derivations.

The cut-elimination procedure together with the growth of ordinal heights of derivations in the infinitary analyses is fairly standard at least up to the *predicative* theories, while the finitary analyses have suffered from complaints: the lack of the significance of reductions r and ordinal assignments o. The first paper by Buchholz resolves the uneasy feeling.

A term generated from constants for function symbols in some term structures denotes an infinitary object. For example, an ordinal term generated from 0 with functions $+$ and $\lambda\alpha.\omega^\alpha$ denotes an ordinal $< \varepsilon_0$. One can view, by the same token, finitary derivations as notations of infinitary ones.

The first paper by Buchholz gives a notation system for infinitary derivations for first-order arithmetic PA: it extends terms for finitary derivations by introducing function symbols such as $\mathsf{R}_C(d_0, d_1)$ *denoting* the result of applying the Main Lemma to derivations (denoted by) d_0, d_1, and $\mathsf{E}(d)$ the result of applying the Cut Elimination Lemma to d. For the extended terms d, then, let d^∞ be the infinitary derivation denoted by d. Buchholz defines in a primitive recursive way a finitary term $d[n]$ denoting the nth immediate subderivation of the d^∞, and an ordinal $o(d)$ which gives the (standard upper bound for the) ordinal height of d^∞, and shows $o(d[n]) < o(d)$. These yield terminating reduction steps $r'(d) = d[0]$ stemming from infinitary analyses. Such an idea was implicit in G. Mints, *A new reduction sequence for arithmetic* (***Selected papers in proof theory***, Bibliopolis, 1992, pp. 77–96), and even in the transition period 1936–1938 of Gentzen. Buchholz, however, presents the idea rigorously. Furthermore the termination of r' yields the 1-consistency of T readily, i.e., without heavy coding of infinitary derivations. The second paper by Buchholz shows that the approach by r' is essentially the same as Gentzen's r. Thus the paper explains precisely the meaning of Gentzen's reduction r and ordinal assignment o in the light of infinitary analyses.

Now for *impredicative* theories, there are several approaches to infinitary analyses. Here only two of them will be mentioned, both of which are due to Buchholz: the Ω-rule and operator-controlled derivations as an essence of the local predicativity method by W. Pohlers. Although the latter one by controlled derivations is the more perspicuous and flexible, neither seems to receive the position of being standard. Moreover one has to devise larger ordinal notation systems simultaneously with analyses. W. Buchholz (*Explaining the Gentzen–Takeuti reduction steps: a second-order system*, ***Archive of mathematical logic***, vol. 40 (2001), pp. 255–272) explains G. Takeuti's finitary analysis of Π^1_1-CA by means of the Ω-rule.

The paper by Tupailo is based on the reviewer's finitary analysis of the theory of recursively regular ordinals, which in turn was inspired by G. Jäger's infinitary analysis by the local predicativity method. Tupailo gives a notation system for infinitary derivations occurring in the method, as Buchholz did similarly for PA. He has to overcome some technical difficulties such as, in defining complexity measures of formulas, a subtlety in uniformity of infinitary derivations induced by finitary ones, and function symbols to extend finitary derivations. This includes E_γ (lower the cut-degree by ω^γ), and B, D for the bounding and collapsing lemmas, respectively. Thus Tupailo shows that the work initiated by Buchholz can be extended to analyses based on the local predicativity method.

Since finitary analyses can be derived from infinitary ones as these papers show, one might ask: Can we discard finitary ones? Yes in principle, I think. But the finitary analyses may help us to devise ordinal notation systems more easily. For example Takeuti's ordinal diagrams express more or less directly his reduction steps. I have in mind the fact that Takeuti had obtained a finitary analysis of impredicative theory prior to the advent of admissible ordinals by Kripke and Platek, and I suppose the precedence was due to not only his ability but also his approach.

Besides the papers mentioned above, now two papers, G. Mints, *Reduction of finite and infinite derivations*, ***Annals of pure and applied logic***, vol. 104 (2000), pp. 167–188, and W. Buchholz, *Finitary treatment of operator controlled derivations*, ***Mathematical logic quarterly***, vol. 47, (2001), pp. 363–396, have been published on the line of work to connect finitary and infinitary analyses in proof theory. Future work would include finding a relationship between a termination proof of the ε-substitution method by W. Ackermann and the proof by G. Mints via cut-elimination of infinitary derivations, and comparing proofs of Mints's normal form theorem in G. Mints, *A normal form for logical derivations implying one for arithmetic derivations*, ***Annals of pure and applied logic***, vol. 62 (1993), pp. 65–79, and in T. Arai and G. Mints, *Extended normal form theorems for logical proofs from axioms*, ***Theoretical computer science***, vol. 232 (2000), pp. 121–132. One may export a product in one analysis to the other; for example, when formulating a finitary analysis related to β-proofs by J.-Y. Girard.

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Jeremy Avigad. *A realizability interpretation for classical arithmetic*. ***Logic Colloquium '98, Proceedings of the annual European summer meeting of the Association for Symbolic Logic, held in Prague, Czech Republic, August 9–15, 1998***, edited by Samuel R. Buss, Petr Hájek, and Pavel Pudák, Lecture notes in logic, no. 13, Association for Symbolic Logic, Urbana, and A K Peters, Natick, Mass., 2000, pp. 57–90.

It is well known that intuitionistic Heyting arithmetic, HA, and classical Peano arithmetic, PA, have the same provably recursive functions, i.e., prove the same Σ_1-formulas. Most of the numerous proofs of this fact also provide methods for extracting witnessing terms from classical proofs of Σ_1-formulas, either via a translation of PA into HA, or by directly giving a constructive interpretation for PA.

The paper under review introduces a new method of interpreting proofs in PA constructively, based on a new kind of formalized realizability for a Tait-style, one-sided sequent-calculus. This interpretation is quite appealing as it gives the rules of this calculus a very direct and natural computational meaning, and might therefore be of practical interest for program extraction from classical proofs.

The author also introduces a variant of the Gödel–Gentzen double-negation translation of PA into HA with respect to which his new realizability interpretation corresponds to Kreisel's modified realizability followed by the Dragalin–Friedman translation. This means that the witnessing term for a PA-proof of a Σ_1-formula is the same as the one extracted via modified realizability from the translated proof. Furthermore, he shows that the same witnesses are obtained by applying suitable cut reduction rules to the classical PA-proofs directly, which, as a side result, implies that certain strategies for eliminating cuts and extracting witnesses are insensitive to the order in which the reductions are applied.

The author continues with investigations of finitary and infinitary cut elimination procedures *á la* Gentzen and Mints. Among other results, it is shown that the witnesses obtained via these methods are, again, the same as those obtained by the methods discussed above.

It is also pointed out in the paper that there are similarities between this work and direct computational semantics for classical proofs investigated by Murthy, Parigot, and others. The reviewer would like to add that it would be interesting to clarify the precise relationship between these methods and also the relations to other methods, based, for example, on Gödel's Dialectica interpretation.

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Albert Visser. *Submodels of Kripke models*. ***Archive for mathematical logic***, vol. 40 (2001), pp. 277–295.

When one tries to translate a classical result to an intuitionistic setting, one often arrives at a more complicated statement with a more complicated proof. Nonetheless, as this article by Visser illustrates, this does not necessarily imply that the result is less elegant.

The author investigates the intuitionistic analogue of the Łoś–Tarski theorem for classical predicate logic CQC which states that the formulas of first-order predicate logic that are closed under taking (classical) submodels are precisely the purely universal formulas. He arrives at the following intuitionistic analogue: the formulas of the intuitionistic predicate calculus IQC that are closed under taking submodels are precisely the semi-positive formulas. Here a submodel of a Kripke model is the result of restricting the frame of the original model. The semi-positive formulas are the formulas in which only implications of the form $(P \to A)$ occur, where P is an atomic formula. Note that both notions are superfluous in a classical setting: any class of classical models is (as a class of Kripke models) closed under submodels, and any first-order formula is classically equivalent to a semi-positive formula.

As the author points out in the introduction, the theorem and its proof are exactly as one would expect them to be. They can be seen as an extension of a similar result in intuitionistic propositional logic by Visser, van Benthem, de Jongh, and Renardel de Lavalette, which states that the (propositional) formulas preserved under subformulas are precisely the NNIL-formulas. An NNIL-formula is a formula in which only implications of the form $(p \to A)$ occur, where p is a propositional variable.

The paper closes with three other results on submodels of Kripke models: one in the context of IQC and the other two in the context of Heyting arithmetic HA. All these theorems, except the last one, are again as expected, but the proofs are not at all straightforward. They will be discussed individually.

When one defines models for IQC, one has some choices for the connection between nodes k, k' for which $k \preceq k'$ holds. In a quite general definition one requires only that the domain D_k of k can be embedded in $D_{k'}$ and that identity is locally interpreted as an equivalence relation. These models are called eq$^+$-models by the author. There are two natural ways to strengthen this definition: by requiring (a) that D_k is a subset of $D_{k'}$ or (b) that identity is locally true identity. The former models are called eq-models, the latter are called Kripke models in the article (in the literature the name Kripke model is often used for what is here called an eq-model). It is easy to see that every eq$^+$-model is isomorphic to a Kripke model. Visser shows that every eq$^+$-model is also isomorphic to an eq-model. The question whether every eq$^+$-model is isomorphic to a model that satisfies both (a) and (b) makes sense only for theories with decidable identity (theories, such as HA, in which $x = y \vee x \neq y$ is derivable). Visser gives a simple example that settles this question in the negative. The author provides us with a nice alternative, however, by showing that in the presence of decidable identity every eq$^+$-model is bisimilar to a model that satisfies both (a) and (b). Although there occur already similar statements in the literature, this one seems to be the first with a correct proof. In fact, in all previous constructions the resulting model is isomorphic to the eq$^+$-model, and as mentioned above, this cannot be the case.

The other two theorems are about the Kripke models of HA. The first one $(*)$ states that HA is not axiomatizable by semi-positive formulas. The second one answers, in the negative, the question (by Sam Buss) whether there is a classical theory T such that HA is complete with respect to the class of models for which the structures associated to the nodes are classical models of T. Both proofs follow from the fact that no class of models that is sound and complete with respect to HA can be closed under taking submodels (in the first case one uses the main theorem of the paper). In the proof of this fact the author uses the de Jongh translation. Visser provides another proof of $(*)$ using realizability. In another article (*Substitutions of Σ^0_1-sentences: explorations between intuitionistic propositional logic and intuitionistic arithmetic*, ***Annals of pure and applied logic***, vol. 114 (2002), pp. 227–271) Visser also gives two different proofs of a property of HA, where one uses realizability and the other uses the de Jongh translation. It would be interesting to know if the connection between the de Jongh translation and realizability could be made more apparent.

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LASZLO FUCHS and SAHARON SHELAH. *Kaplansky's problem on valuation rings*. ***Proceedings of the American Mathematical Society***, vol. 105 (1989), pp. 25–30.

PAUL C. EKLOF. *A transfer theorem for nonstandard uniserials*. ***Proceedings of the American Mathematical Society***, vol. 114 (1992), pp. 593–600.

PAUL C. EKLOF and SAHARON SHELAH. *On a conjecture regarding nonstandard uniserial modules*. ***Transactions of the American Mathematical Society***, vol. 340 (1993), pp. 337–351.

P. C. EKLOF and S. SHELAH. *Explicitly non-standard uniserial modules*. ***Journal of pure and applied algebra***, vol. 86 (1993), pp. 35–50.

Shelah's discovery that the Whitehead problem (asking whether any Abelian group A with $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$ is free) is undecidable (*Infinite abelian groups, Whitehead problem and some constructions*, JSL LI 1068) ushered in a new era in Abelian group theory and the theory of modules over valuation domains, in which the structure of large objects was explored from varying model-theoretic angles. The cross-fertilization between algebra and model theory occurring in the wake led to the resolution of further long-standing problems—purely algebraic at first sight—such as Kaplansky's problem on valuation rings (see below). The negative answer given in the mid-80's by Fuchs and Salce is based on the existence of certain non-standard uniserial modules—a subject that had puzzled the algebraic community for quite some time. In light of the above articles, it is now known to be linked to the underlying set-theoretic model as well. On the side, let me mention that the study of uniserial modules (modules with totally ordered submodule lattice) has recently again moved into the limelight, this time by dint of results showing that, while direct sums of uniserials do not have the Krull–Schmidt property, an alternate uniqueness theory—pioneering in its conceptual layout—governs them instead (A. Facchini, *Krull–Schmidt fails for serial modules*, ***Transactions of the American Mathematical Society***, vol. 348 (1996), pp. 4561–4575, and N. V. Dung and A. Facchini, *Weak Krull–Schmidt for infinite direct sums of uniserial modules*, ***Journal of algebra***, vol. 193 (1997) pp. 102–121).

A valuation ring is a commutative ring that is a uniserial module over itself; such a ring is called a valuation domain if it has no zero divisors. The investigation of such rings prompted Kaplansky to ask whether every valuation ring can be obtained as an epimorphic image of a valuation *domain*. In 1985, Fuchs and Salce answered this question in the negative (***Modules over valuation domains***, Marcel Dekker, 1985, p. 151). In particular, they observed that any divisible non-standard uniserial module U over a valuation domain R yields a counterexample (a uniserial R-module is non-standard if it is not isomorphic to a quotient J/I where $I \subset J$ are R-submodules of the field of fractions of R); in this situation the trivial extension of R by U can readily be seen to be a valuation ring that fails to be a factor ring of a valuation domain. The first existence result for the requisite non-standard uniserials is due to Shelah (***Around classification theory of models***, Lecture notes in mathematics, vol. 1182, Springer-Verlag, 1986, pp. 135–150), who started his proof with a forcing argument and subsequently derived a corresponding existence theorem in ZFC (Zermelo–Fraenkel with choice). Shortly afterwards, Fuchs and Salce (loc. cit.) gave an existence proof with a more algebraic flavor which, however, was limited to the constructible universe ($V = L$). The first of the articles listed above then provided an alternate proof for the existence of divisible non-standard uniserials in arbitrary models of ZFC, by bridging the gap with the help of a model-theoretic absoluteness theorem. This line of approach, namely first establishing non-standard uniserials in a specific model of ZFC and then following with an absoluteness argument extending existence, was rounded off by Eklof's general transfer principle—the subject of the second of the articles under review: Given the existence—in ZFC together with Jensen's diamond principle—of non-standard uniserial modules $U_1, \dots, U_m$ satisfying certain first-order sentences $\varphi_1, \dots, \varphi_m$, respectively, this transfer scheme guarantees analogous uniserials in ZFC without supplements. It enabled Eklof to prove each of a gamut of classes of non-standard uniserial modules (ranging from uniserial modules all of whose factors are non-standard, to uniserial modules having only standard proper factors) to be non-empty in ZFC. The subdivision of non-standard uniserials in terms of their factor modules is due to S. Bazzoni and L. Salce (*On non-standard uniserial modules over valuation domains and their quotients*, ***Journal of algebra***, vol. 128 (1990), pp. 292–305), who made use of the diamond principle to produce candidates in each of the resulting classes.

The next generation of ideas was triggered by a conjecture of B. Osofsky (*Constructing nonstandard uniserial modules over valuation domains*, ***Azumaya algebras, actions, and modules***, Contemporary mathematics, vol. 124, American Mathematical Society, 1992, pp. 151–164); it was directed at deciding whether a given valuation domain has non-standard uniserials. The criterion she suggested points to the intimate connection between non-standard uniserial modules and the existence of families of ideals L of the underlying valuation domain R—indexed by "sufficiently dense" subsets of ω_1—with the property that the ring R/L fails to be complete in a certain linear topology. Osofsky's work also provided a comparatively elementary construction principle for non-standard uniserial modules in ZFC, which replaced the original model-theoretic reasoning with standard ordinal and cardinal theory. So it looked as though the subject could be divorced from logic after all. That this is not possible was subsequently shown by Eklof and Shelah (in the third of the listed papers): The validity of Osofsky's conjecture is in fact undecidable. The fourth paper, a sequel to the third, refined and supplemented the results of the former. We start with a loose sketch of the decisive concepts of these two papers—more precision can be found in the final paragraph. The *type* of an arbitrary uniserial module U over a valuation domain R is a certain standard uniserial module associated with U (the attribute "standard" meaning "isomorphic to a subfactor of the quotient field of R, viewed as an R-module"). Inspired by Osofsky's work, Eklof and Shelah assign to any such type an invariant living in a Boolean algebra of equivalence classes of subsets of ω_1, the equivalence relation and partial order being induced by the filter of closed unbounded subsets of ω_1, via "$S_1 \cap C = S_2 \cap C$" and "$S_1 \cap C \subseteq S_2 \cap C$" respectively, for some closed unbounded C. This invariant associated with a uniserial type measures the wealth of certain ideals L in R having the property that the factor ring R/L is not complete in a type-dependent linear topology. Very roughly, one might say "the richer the supply of such ideals, the better the chance of finding non-standard uniserial R-modules of the corresponding type."

To provide meaningful detail concerning some key results of the last two papers, we need to sharpen these concepts; for simplicity, we will restrict our attention to uniserial domains R of cardinality $\aleph_1$. Given an R-subfactor J/I of the quotient field Q of R, a uniserial module U is said to be of type J/I if $J/I \cong D(u)/\operatorname{Ann}_R(u)$ for some non-zero element $u \in U$, where $\operatorname{Ann}_R(u)$ is the annihilator of u in R and $D(u)$ is the R-submodule of Q generated by the inverses r^{-1} of those elements $r \in R \setminus \{0\}$ that satisfy $u \in rU$ (this construction is independent of the choice of $u \in U \setminus \{0\}$); then U is non-standard precisely when $U \ncong J/I$. The corresponding invariant $\Gamma(D(u)/\operatorname{Ann}_R(u))$ is as follows: If $D(u) = \bigcup_{\mu<\omega_1} r_\mu^{-1} R$ with $r_\mu \mid r_\nu$ whenever $\mu < \nu$, the value of $\Gamma(D(u)/\operatorname{Ann}_R(u))$ in the described Boolean algebra is the equivalence class of the set

$$S = \{\alpha \mid \alpha \text{ is a limit ordinal } < \omega_1 \text{ such that } R/\Big(\bigcap_{\mu<\alpha} r_\mu R\Big) \text{ is not complete}\}.$$

If 1 denotes the largest element of our Boolean algebra, Osofsky's conjecture takes on the form, "R has a non-standard uniserial module of type J/I precisely when $\Gamma(J/I) = 1$" (still restricted to valuation domains of cardinality $\aleph_1$). Eklof and Shelah confirm it under the negation of the continuum hypothesis. Moreover, they show that $\Gamma(J/I) = 1$ always guarantees the existence of a non-standard uniserial of type J/I. On the other hand, the converse is independent of ZFC with the generalized continuum hypothesis. As for Γ-invariants at the opposite end of the spectrum, $\Gamma(J/I) = 0$ excludes the existence of a non-standard uniserial of type J/I. As might be expected, the constructible universe again provides a "clean" outcome: For $V = L$, the existence of non-standard uniserial R-modules of type J/I is equivalent to $\Gamma(J/I) \neq 0$.

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Four papers related to the normal Moore space conjecture

Peter J. Nyikos. *A provisional solution to the normal Moore space problem*. ***Proceedings of the American Mathematical Society***, vol. 78 (1980), pp. 429–435.

William G. Fleissner. *If all normal Moore spaces are metrizable, then there is an inner model with a measurable cardinal*. ***Transactions of the American Mathematical Society***, vol. 273 (1982), pp. 365–373.

Alan Dow, Franklin D. Tall, and William A. R. Weiss. *New proofs of the consistency of the normal Moore space conjecture I*. ***Topology and its applications***, vol. 37 (1990), pp. 33–51.

Zoltán Balogh. *On collectionwise normality of locally compact, normal spaces*. ***Transactions of the American Mathematical Society***, vol. 323 (1991), pp. 389–411.

A Moore space is a space satisfying the first three parts of one of the axioms in R. L. Moore's characterization of the Euclidean plane. The normal Moore space conjecture, abbreviated NMSC, states that all normal Moore spaces are metrizable. This question was first raised in print by F. B. Jones in 1933, and from then until the time of the first two papers under review here, was one of the most intensively researched problems in set-theoretic topology. I will not define Moore space here, because after a seminal paper of R. H. Bing in 1951, work centered on its practical, if not logical, equivalent: If X is first-countable and normal, then it is collectionwise normal. (X is *collectionwise normal* if any discrete collection $\{H_\alpha : \alpha < \kappa\}$ of closed sets can be "separated" by a pairwise-disjoint collection $\{U_\alpha : \alpha < \kappa\}$ of open sets, i.e., $H_\alpha \subset U_\alpha$ for all α.)

Consistent counterexamples to NMSC (e.g. from MA_{ω_1}) had been known since the late 1960's. Nyikos was the first to obtain a general positive consistency result, although to do it he needed a statement, specifically the product measure extension axiom (PMEA), that implied the existence of measurable cardinals. PMEA says that the product measure on 2^κ (i.e., the obvious generalization of the standard Haar measure on 2^ω) can be extended to a measure on all subsets of 2^κ. That PMEA is consistent modulo a strongly compact cardinal was a then-unpublished result of K. Kunen. How is PMEA relevant to normal Moore spaces? That is Nyikos's important insight. Once one notices the connection, the mechanics of a proof that PMEA implies normal first-countable spaces are collectionwise normal are shockingly simple!

Consistency of a strongly compact cardinal implying the consistency of NMSC having been established, Fleissner's paper essentially "settles" the normal Moore problem by showing that NMSC in turn implies the consistency of measurable cardinals. Fleissner does this by showing that if there is a strong limit cardinal κ of cofinality ω such that $2^\kappa = \kappa^+$, along with a non-reflecting stationary subset E of $\{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \omega\}$, then a non-metrizable normal Moore space can be constructed. Note that the continuum hypothesis implies that ω is such a cardinal. By work of Jensen and Dodd, the *non-existence* of such a cardinal implies that there is an inner model with a measurable cardinal; in fact, Mitchell established inner models with a proper class of measurables.

The space Fleissner constructs from the stated set-theoretic assumptions looks like a big set of isolated points to which the so-called Baire zero-dimensional metric space E^ω, where E is given the discrete topology, is attached. It is an intricate and difficult construction which builds on ideas in a number of prior examples of the normal, not collectionwise normal, type due to Bing, C. Navy, and Fleissner himself. Information coded into the isolated points in clever ways is key to proving both the normality and the non-collectionwise normality of the space, with combinatorics of what Fleissner terms "stafull" subsets of E^n a main ingredient in the latter.

After Nyikos's breakthrough, F. Tall championed a project inspired by the idea that measure theory surely has rather little to do with the normal Moore space problem, and that a proof of NMSC that avoided measure might yield additional related results which could not be derived from PMEA. Assisted by Weiss and (later) Dow, paper three above is the first installment of the result of this successful project. It replaces the argument from PMEA by forcing and reflection. The main result is that normal Moore spaces are metrizable in a model obtained by adding κ-many Cohen reals, where κ is supercompact. (It is also indicated how this proof can be done by using a strongly compact cardinal instead of a supercompact, and with random reals instead of Cohen reals; but the supercompact Cohen real argument seems handier for generalizations.) The rough idea is first to show that there are no "small" counterexamples, i.e., of size less than κ, by proving that if a Cohen generic partition into two pieces of a closed discrete collection can be separated by disjoint open sets, then the ground model representation of this separation induces a separation of the whole collection. It is here that the germ of the measure-theory idea in Nyikos's proof remains in the Dow–Tall–Weiss argument, in the form of an "endowed" poset. A poset P is *endowed* if for each positive integer n the following holds: For each maximal antichain A one can assign a finite subset $L(A)$ of A such that, for any $p \in P$ and maximal antichains $A_1, \dots, A_n$, there are $q_i \in L(A_i)$ and $r \leqq p$ such that $r \leqq q_i$ for each $i = 1, \dots, n$. Clearly a measure algebra is endowed; what is exploited here is the fact, due to Dow, that Cohen posets are endowed.

Having established the non-existence of "small" counterexamples, the paper then uses supercompact reflection to show that if there were any counterexample at all, there would be a counterexample of size less than κ, a contradiction. The authors develop their forcing-reflection machinery in sufficiently general terms to obtain other interesting applications, most of which appear in the sequel to the paper under review (pages 115–129 in the same volume).

Balogh's paper, in which he solves an important natural outgrowth of the normal Moore space problem, is an outstanding illustration of the flexibility of the Dow–Tall–Weiss approach over PMEA. Balogh proves that if supercompact many Cohen reals or random reals are added to a model, then in the extension, every normal locally compact space is collectionwise normal. (The result is stated in a more general form which implies NMSC.) To help with the reflection arguments, a structure called a (*normalized*) $\mathcal{A}$-*structure* is defined which codes the combinatorics of a closed discrete collection of subsets of a (normal) locally compact space. The Dow–Tall–Weiss argument was assisted by the fact that $j''X$, where X is a normal Moore space and j is an elementary embedding related to the supercompact, was a sub*space* of $j(X)$, thanks to first-countability, and hence a corresponding discrete collection still satisfied a normality condition. Here, sub*structures* of normalized $\mathcal{A}$-structures need not be "normalized," but (by a topological argument) sufficiently many ((κ, μ)-c.u.b. many in Balogh's terminology) are, to make the reflection argument work.

One thing remains unclear about NMSC: Where does it fit exactly in the hierarchy between a proper class of measurables and a strongly compact cardinal? Nevertheless, the four papers under review represent a beautiful culmination of long-standing efforts on the part of these and many other researchers on a central problem (or in Balogh's case, a closely related problem) in set-theoretic topology.

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